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Fujita's approximation theorem in positive characteristics

By

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Introduction

Let X be a projective variety of dimension n over an algebraically closed field k.

For any line bundle L on X, we define the *volume* of L to be:

$$\mathrm{vol}_X(L) := \limsup_{m\to\infty} \frac{h^0(X, L^{\otimes m})}{m^n/n!}$$

It is known that this function vol_X can be extended to a homogeneous, continuous real valued function on $\mathrm{N}^1(X)_{\mathbb{R}}$.

The main theorem of this paper is as follows:

Theorem 0.1 (Fujita's approximation theorem). *Let $\xi \in \mathrm{N}^1(X)_{\mathbb{Q}}$ be a rational big class. Then, for an arbitrary small real number $\epsilon > 0$, there exist a birational morphism $\pi : X' \to X$ of projective varieties and a decomposition*

$$\pi^*\xi = \alpha + e$$

in $\mathrm{N}^1(X')_{\mathbb{Q}}$, which satisfy the following conditions:

(i) *α is an ample class and e is effective.*

(ii) $\mathrm{vol}_{X'}(\alpha) > \mathrm{vol}_X(\xi) - \epsilon$.

Note that the characteristic of the base field k could be positive. The proof of Fujita's approximation theorem in the original paper [3] uses Hironaka's desingularization theorem. Other proofs, obtained by Lazardsfeld [11] and Nakamaye [13], also uses the desingularization theorem. In this sense, Fujita's approximation theorem can be applied only to the case where characteristic is zero. In this paper, we verified this theorem in positive characteristics. The idea is to use de Jong's alteration theorem (see Theorem 2.1) as a substitute for Hironaka's theorem.

This paper consists of 3 sections.

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In Section 1, we discuss the general facts about the divisorial sheaves. In Section 2, we prove Fujita's approximation theorem in arbitrary characteristic. As an application of Fujita's approximation theorem we discuss the behaviour of the dimension of the space of global sections of line bundles on a polarized projective variety in Section 3. The result in this chapter will be used in future references.

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Notation and conventions

In this paper, any scheme is seperated, of finite type over its base field. A variety is a geometrically integral scheme. We use Snapper's definition of the intersection theory.

Since we mainly work on projective varieties, we use the notion of line bundles and Cartier divisors interchangeably.

For a projective scheme X, we denote the Picard group by $\mathrm{Pic}(X)$, the group of line bundles numerically equivalent to 0 by $\mathrm{Num}(X)$, and the Néron-Severi group of X by $\mathrm{N}^1(X)$: $\mathrm{N}^1(X) = \mathrm{Pic}(X)/\mathrm{Num}(X)$. $\mathrm{N}^1(X)_{\mathbb{Q}}$ (resp. $\mathrm{N}^1(X)_{\mathbb{R}}$) is defined by $\mathrm{N}^1(X)_{\mathbb{Q}} = \mathrm{N}^1(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ (resp. $\mathrm{N}^1(X)_{\mathbb{Q}} = \mathrm{N}^1(X) \otimes_{\mathbb{Z}} \mathbb{R}$). $\rho(X) := \mathrm{rank}\, \mathrm{N}^1(X)$ is the Picard number of X, which is finite (cf. [12]). Similarly, we call the linear combination of 1-codimensional subvarieties with rational (resp. real) coefficients "$\mathbb{Q}$-divisor" (resp. "$\mathbb{R}$"-divisor). We sometimes use the term"$\mathbb{Z}$-divisor" for the usual Weil divisor (i.e. with integral coefficients). $\overline{\mathrm{Eff}}(X) \subset \mathrm{N}^1(X)_{\mathbb{R}}$ is the closure of the convex cone spanned by the classes of effective $\mathbb{Q}$-divisors. The elements in $\overline{\mathrm{Eff}}(X)$ are called pseudo-effective. If X is integral, then we denote the function field (or, the constant sheaf associated to the function field) by $\mathrm{Rat}(X)$.

1. Behavior of divisorial sheaves

Throughout this section, X is an n-dimensional normal variety over an algebraically closed field k.

First, let us summerize basic properties of divisorial sheaves.

Let F be an coherent sheaf on X. *The dual of* F is defined as $F^\vee := \mathscr{H}om_{\mathcal{O}_X}(F, \mathcal{O}_X)$. Further, *the double dual of* F is $F^\wedge := (F^\vee)^\vee$. F is *reflexive* if the natural map $F \to F^\wedge$ is an isomorphism. F is *divisorial* if F is reflexive and of rank 1.

The set of isomorphism classes of divisorial sheaves has a natural group structure by taking the multiplication as $L \cdot M := (L \otimes M)^\wedge$.

Let $D = \sum_\Gamma n_\Gamma \Gamma$ be a Weil divisor on X, where Γ runs over all the prime divisors on X, and $n_\Gamma \in \mathbb{Z}$ are zero except finitely many Γ's.

The divisorial sheaf $\mathcal{O}_X(D)$ *associated to* D is defined as:

$$\mathcal{O}_X(D) : U \mapsto \{f \in \mathrm{Rat}(X) \mid v_\Gamma(f) \geq -n_\Gamma \ (\Gamma \cap U \neq \emptyset)\},$$

where U is any open subset of X, and v_Γ is the valuation associated to Γ. Note that this definition coincides with that of the line bundle associated to a Cartier divisor.

For any two Weil divisors D and E, We have:

$$(\mathcal{O}_X(D) \otimes \mathcal{O}_X(E))^\wedge \simeq \mathcal{O}_X(D+E).$$

Moreover, if either D or E is Cartier, then we need not take the double dual:

$$\mathcal{O}_X(D) \otimes \mathcal{O}_X(E) \simeq \mathcal{O}_X(D+E).$$

We denote by $\mathrm{WPic}(X)$ the group of isomorphism classes of divisorial sheaves on X. Given a dominant morphism $\pi : Y \to X$ of normal varieties, define $\pi^! : \mathrm{WPic}(X) \to \mathrm{WPic}(Y)$ by $\pi^! L := (\pi^* L)^\wedge$.

Proposition 1.1. *Let L be a divisorial sheaf on X. If $\pi : X' \to X$ is a projective birational morphism of normal varieties, then the natural map $f : L \to \pi_* \pi^! L$ induced by $L \to \pi_* \pi^* L$ is an isomorphism. In particular, $H^0(X', \pi^! L) \simeq H^0(X, L)$.*

Proof. Let X_0 be the set of smooth points of X such that π is isomorphism on X.

This is an open subset of X, and $X \backslash X_0$ has codimension ≥ 2 via Zariski's main theorem. Since f is injective, $f(L)$ is divisorial. On the other hand, f is isomorphic on X_0. Thus, f is surjective, since $\pi_* \pi^! L$ is torsion free. □

Proposition 1.2. *Let L and M be divisorial sheaves on X. If $\pi : X' \to X$ is a projective birational morphism of normal varieties, then we have a natural injection*:

$$(\pi^*(L \otimes M))^\wedge \hookrightarrow (\pi^! L \otimes \pi^! M)^\wedge.$$

Proof. We have a natural map

$$\pi^*(L \otimes M) \to \pi^! L \otimes \pi^! M.$$

Taking the double dual, we obtain

$$(\pi^*(L \otimes M))^\wedge \to (\pi^! L \otimes \pi^! M)^\wedge.$$

The above map is isomorphic on the generic point of X', so it is injective, since the right hand side is torsion free. □

Remark 1.3. Note that $\pi^!$ is *not* a group homomorphism in general. Even worse, we cannot construct a map between $\pi^!((L \otimes M)^\wedge)$ and $(\pi^! L \otimes \pi^! M)^\wedge$ in general.

Let $S := \{L_i\}_i$ be a set of divisorial sheaves on X. A projective birational morphism $\pi : X' \to X$ of normal varieties is *a Cartierization of* S if $\pi^! L_i$ is an invertible sheaf on X' and the natural map $\pi^* L_i \to \pi^! L_i$ is surjective for all $L_i \in$

S. If S consists of one divisorial sheaf L, then we simply say *a Cartierization of* L instead. For a set of Weil divisors $\mathcal{D} := \{D_i\}_i$, *a Cartierization of* $\mathcal{D}$ is a Cartierization of $\{\mathcal{O}(D_i)\}_i$. By virtue of Raynaud's flattening theorem [14], there exists a Cartierization of a divisorial sheaf. By repeating the operation, there exists a Cartierization of finite set of divisorial sheaves.

Proposition 1.4. *Let H be an ample divisor on X, and D a Weil divisor on X. Let $\pi : X' \to X$ be a Cartierization of D. Then we have*

$$(H^{n-1}.D) = (\pi^* H^{n-1} \cdot \pi^! \mathcal{O}(D)).$$

Proof. Using the projection formula and the Proposition 1.1 above,

$$\begin{aligned}(\pi^* H^{n-1} \cdot \pi^! \mathcal{O}(D)) &= (H^{n-1} \cdot \pi_* \pi^! \mathcal{O}(D)) \\ &= (H^{n-1}.\mathcal{O}(D)) = (H^{n-1}.D).\end{aligned}$$

□

Proposition 1.5. *Let L be a diviorial sheaf on X, and $\pi : X' \to X$ a Cartierization of L. Then, for any projective birational map $f : Z \to X'$ of normal varieties, $\pi \circ f$ is a Cartierization of L.*

Proof. From the surjective map $\pi^* L \to \pi^! L \to 0$, we have a surjective map $(\pi \circ f)^* L \to f^* \pi^! L \to 0$. Since $f^* \pi^! L$ is invertible, this gives another surjective map $(\pi \circ f)^! L \to f^* \pi^! L \to 0$. $(\pi \circ f)^! L$ is torsion free, so that this is an isomorphism. □

Let L be a divisorial sheaf on X, and $s \in H^0(X, L)$ be a non-zero global section of L. This induces a homomorphism $\mathcal{O}_X \to L$. We denote by $\mathrm{div}(s)$ the Weil divisor induced from s.

Let (L, s) and (L', s') be pairs of a divisorial sheaf and its global section, respectively. We say that (L, s) and (L', s') are *isomorphic* if there exists an isomorphism $\varphi : L \to L'$ which satisfies $s' = \varphi \circ s : \mathcal{O}_X \to L'$:

$$\begin{tikzcd} \mathcal{O}_X \arrow[d, "s"'] \arrow[dr, "s'"] & \\ L \arrow[r, "\varphi"] & L' \end{tikzcd}$$

Note that in the above situation, $(L, s) \simeq (L', s')$ if and only if $\mathrm{div}(s) = \mathrm{div}(s')$.

Let D be an effective Weil divisor on X. $\mathcal{O}(D)$ can be regarded as a $\mathcal{O}_X$-submodule of $\mathrm{Rat}(X)$ in a natural way. $1 \in \mathrm{Rat}(X)$ defines a section $1_D \in H^0(X, \mathcal{O}_X(D))$. We call this section *canonical section of* $\mathcal{O}_X(D)$. Note that if L is a divisorial sheaf, $s \in H^0(X, L) \backslash \{0\}$, and $D := \mathrm{div}(s)$, then:

$$(L, s) \simeq (\mathcal{O}(D), 1_D).$$

Let L be a divisorial sheaf on X and s be a global section of L. Let $\pi : Y \to X$ be a generically finite morphism of normal varieties. Then $\pi^!(s) \in$

$H^0(X, \pi^! L)$ is defined by the composition:

$$\mathcal{O}_Y \xrightarrow{\pi^*(s)} \pi^* L \to \pi^! L.$$

$\pi^!(s)$ is not zero if and only if s is not zero. It is easy to see that if $(L, s) \simeq (L', s')$, then $(\pi^!(L), \pi^!(s)) \simeq (\pi^!(L'), \pi^!(s'))$. Therefore, for any effective Weil divisor $D = \mathrm{div}(s)$, $\pi^!(D) := \mathrm{div}(\pi^!(s))$ is well defined. We call this *the naive pull back of the Weil divisor D*. Note that $\pi^! D$ is defined only for effective divisors.

Proposition 1.6. *Let D, D' be two non-zero effective Weil divisors on X, and let $\pi : Y \to X$ be a generically finite morphism of normal varieties. Then the following holds*:

(1) *If π is proper, then $\pi_*(\pi^!(D)) = (\deg \pi) D$. Here, π_* is the push out in the sense of the intersection theory.*

(2) *If D and D' are linearly equivalent, then so are $\pi^!(D)$ and $\pi^!(D')$.*

(3) *If D' is Cartier, then*

$$\pi^!(D + D') = \pi^!(D) + \pi^*(D').$$

Proof. (1) Since X is normal, and π is generically finite, there exists an Zariski open subset $X_0 \subset X$ which satisfies:

(a) The codimension of $X \backslash X_0$ is not less than 2.

(b) $\pi_0 : \pi^{-1}(X_0) \to X_0$ is finite.

(c) X_0 is smooth.

Then,

$$\pi_*(\pi^!(D))|_{X_0} = \pi_{0*}(\pi^!(D)|_{\pi^{-1}(X_0)}) = \pi_{0*}(\pi_0^*(D|_{X_0})) = (\deg \pi) D|_{X_0}.$$

Condition (a) shows that, $\pi_*(\pi^!(D)) = (\deg \pi) D$.

(2) Since D and D' are linearly equivalent, there exists a divisorial sheaf L and its non-zero sections s, s' which satisfies $D = \mathrm{div}(s)$, $D' = \mathrm{div}(s')$. Hence $\mathrm{div}(\pi^!(s))$ is linearly equivalent to $\mathrm{div}(\pi^!(s'))$, which shows that $\pi^!(D) \sim \pi^!(D')$.

(3) Let $L := \mathcal{O}(D)$ and $L' := \mathcal{O}(D')$, and $s := 1_D$ and $s' := 1_{D'}$ be the canonical section of L, L', respectively. Then $\pi^!(L \otimes L') = \pi^!(L) \otimes \pi^!(L')$ and

$$\pi^!(D) + \pi^!(D') = \mathrm{div}(\pi^!(s) \otimes \pi^*(s'))$$
$$\pi^!(D + D') = \mathrm{div}(\pi^!(s \otimes s')).$$

Here, $\mathrm{div}(\pi^!(s) \otimes \pi^*(s'))$, $\mathrm{div}(\pi^!(s \otimes s'))$ are both non-zero section of $\pi^!(L \otimes L')$ $= \pi^!(L) \otimes \pi^*(L)$, and they coincide on $\pi^{-1}(X_0)$, where X_0 is the smooth locus of X. Since $\pi^!(L \otimes L')$ is torsion free, $\mathrm{div}(\pi^!(s) \otimes \pi^*(s')) = \mathrm{div}(\pi^!(s \otimes s'))$, which shows that

$$\pi^!(D + D') = \pi^!(D) + \pi^!(D').$$

$\square$

In order to calculate the cohomology of divisorial sheaves, we must study the property of the cokernel of the map between them:

Proposition 1.7. *Let D be a Weil divisor on X, and P a prime divisor. Then we have*:

(1) $Q := \mathrm{coker}(\mathcal{O}_X(D-P) \to \mathcal{O}_X(P))$ *is a coherent $\mathcal{O}_P$-module, and torsion free as a $\mathcal{O}_P$-module. We denote this module Q by $(\mathcal{O}_X(D);P)$.*

(2) *If D' is a Cartier divisor on X, then*

$$(\mathcal{O}_X(D+D');P) = \mathcal{O}_X(D')|_P \otimes (\mathcal{O}_X(D);P).$$

Proof. (1) The problem is local, so it is sufficient to check at each point x on X. If $x \notin P$, then the assertion is obvious, so that we may assume that $x \in P$.

Step 1: Q_x is a coherent $\mathcal{O}_P$-module.

It is sufficient to show that $a\varphi \in \mathcal{O}_{X,x}(D-P)$ for all $a \in \mathcal{O}_{X,x}(-P)$ and $\varphi \in \mathcal{O}_{X,x}(D)$. Let $D = \sum_\Gamma n_\Gamma \Gamma$. It is sufficient to check each valuation associated to height 1 prime divisors: We have

$$v_\Gamma(a\varphi) = v_\Gamma(a) + v_\Gamma(\varphi) \geq -n_\Gamma$$

for all $\Gamma \neq P$. Also,

$$v_P(a\varphi) \geq 1 - n_P,$$

which shows that $a\varphi \in \mathcal{O}_X(D-P)$. Coherency is obvious.

Step 2: Q_x is a torsion free $\mathcal{O}_P$-module. It is sufficient to show that

$$a\varphi \in \mathcal{O}_{X,x}(D-P) \Rightarrow \varphi \in \mathcal{O}_{X,x}(D-P)$$

for any $a \in \mathcal{O}_{X,x} \backslash \mathcal{O}_{X,x}(-P)$ and $\varphi \in \mathcal{O}_{X,x}(D)$. This is again easily seen by looking at each valuation associated to height 1 prime divisors.

(2) We have an exact sequence:

$$0 \to \mathcal{O}_X(D-P) \to \mathcal{O}_X(D) \to (\mathcal{O}_X(D);P) \to 0$$

Tensoring with $\mathcal{O}_X(D')$, we get:

$$0 \to \mathcal{O}_X(D+D'-P) \to \mathcal{O}_X(D+D') \to (\mathcal{O}_X(D);P) \otimes \mathcal{O}_X(D') \to 0.$$

The last term equals to $\mathcal{O}_X(D')|_P \otimes (\mathcal{O}_X(D);P)$ since D' is Cartier. $\square$

Lemma 1.8. *Let $\pi : Y \to X$ be a birational morphism of normal varieties, and let Q be a torsion free $\mathcal{O}_X$-module, and Q' a $\mathcal{O}_Y$-module. Assume that there exist a map $h : \pi^*Q \to Q'$ and a non-empty Zariski open set $X_0 \in X$ which satisfies*:

(*a*) $\pi : \pi^{-1}(X_0) \simeq X_0$.

(*b*) *h is isomorphic on $\pi^{-1}(X_0)$.*

Then the natural map

$$H^0(X,Q) \to H^0(Y,Q')$$

is injective.

Proof. Let $s \in H^0(X, Q)$ be the kernel of the above map. Then, by the assumption, $s|_{X_0} = 0$. Since Q is torsion free, we have $s = 0$. □

Proposition 1.9. *Let D be a Weil divisor on X, and P a prime divisor. Then we have the following*:

(1) *There is a natural injection*:

$$(\mathcal{O}_X(-D); P) \hookrightarrow \mathrm{Hom}_{\mathcal{O}_P}((\mathcal{O}_X(D); P), \mathcal{O}_P).$$

(2) *Let $\pi : P' \to P$ be a proper birational morphism such that P' is normal and $\pi^*((\mathcal{O}_X(D); P))/\mathrm{torsion}$ is an invertible $\mathcal{O}_{P'}$-module. Then, the above map induces an injective map*

$$\pi^!(\mathcal{O}_X(-D); P) \hookrightarrow (\pi^!(\mathcal{O}_X(D); P))^\vee.$$

Here, $\pi^!(\mathcal{O}_X(-D); P) := (\pi^*(\mathcal{O}_X(-D); P))^\wedge$, and $\pi^!(\mathcal{O}_X(D); P) := (\pi^*(\mathcal{O}_X(D); P))^\wedge$, and we are taking the dual (*resp. double dual*) *with respect to $\mathcal{O}_{P'}$.*

Proof. (1) By restricting the multiplication on $\mathrm{Rat}(X)$ to $\mathcal{O}_X(D)$ and $\mathcal{O}_X(-D)$, we get the natural coupling

$$\langle \cdot, \cdot \rangle : \mathcal{O}_X(D) \times \mathcal{O}_X(-D) \to \mathcal{O}_X.$$

It is easy to see the followings:

$$\begin{aligned} &\varphi \in \mathcal{O}_X(D - P), \psi \in \mathcal{O}_X(-D) \Rightarrow \langle \varphi, \psi \rangle \in \mathcal{O}_X(-P) \\ &\varphi \in \mathcal{O}_X(D), \psi \in \mathcal{O}_X(-D - P) \Rightarrow \langle \varphi, \psi \rangle \in \mathcal{O}_X(-P). \end{aligned}$$

Therefore, the coupling $\langle \cdot, \cdot \rangle$ induces

$$\langle \cdot, \cdot \rangle : (\mathcal{O}_X(D); P) \times (\mathcal{O}_X(-D); P) \to \mathcal{O}_P.$$

Therefore we have the map

$$(\mathcal{O}_X(-D); P) \to \mathrm{Hom}_{\mathcal{O}_P}((\mathcal{O}_X(D); P), \mathcal{O}_P).$$

On the other hand, this map is isomorphic at the generic point of P. Since $(\mathcal{O}_X(-D); P)$ is torsion free, it is injective.

(2) Note that

$$\pi^!(\mathcal{O}_X(D); P) = \pi^*(\mathcal{O}_X(D); P)/\mathrm{torsion},$$

since the right hand side is invertible. Pulling back the map of (1) induces

$$\pi^*(\mathcal{O}_X(-D); P) \to \pi^* \mathrm{Hom}_{\mathcal{O}_P}((\mathcal{O}_X(D); P), \mathcal{O}_P).$$

On the other hand, we have the natural map

$$\begin{aligned} \pi^* \mathrm{Hom}_{\mathcal{O}_P}((\mathcal{O}_X(D); P), \mathcal{O}_P) &\to \mathrm{Hom}_{\mathcal{O}_{P'}}(\pi^*(\mathcal{O}_X(D); P), \mathcal{O}_{P'}) \\ &\simeq \mathrm{Hom}_{\mathcal{O}_{P'}}(\pi^*(\mathcal{O}_X(D); P)/\mathrm{torsion}, \mathcal{O}_{P'}). \end{aligned}$$

Composing the above two, we have

$$\pi^*(\mathcal{O}_X(-D); P) \to \mathrm{Hom}_{\mathcal{O}_{P'}}(\pi^!(\mathcal{O}_X(D); P), \mathcal{O}_{P'}).$$

Note that the right hand side is equal to $(\pi^!(\mathcal{O}_X(D); P))^\vee$, in particular, invertible. Taking the double dual, we obtain

$$\pi^!(\mathcal{O}_X(-D); P) \to (\pi^!(\mathcal{O}_X(D); P))^\vee,$$

which is injective, since it is isomorphic on the generic point of P'. □

A *complete linear system* on X is defined as a set (may be empty) of all effective Weil divisors linearly equivalent to some given Weil divisor D. It is denoted by $|D|$. Just as in the case of Cartier divisors, $|D|$ has a natural structure of the set of closed points of the projective space $\mathbb{P}(H^0(X, \mathcal{O}(D)))$. Define *the base locus of* $|D|$ as:

$$\mathrm{Bs}\,|D| := \begin{cases} X & \text{if } |D| = \emptyset \\ \cap_{D' \in |D|} \mathrm{Supp}(D') & \text{if } |D| \neq \emptyset. \end{cases}$$

Similarly, for a divisorial sheaf L on X, define *the base locus of* $|L|$ as:

$$\mathrm{Bs}\,|L| := \begin{cases} X & \text{if } H^0(X, L) = 0 \\ \cap_{s \in H^0(X,L)\setminus\{0\}} \mathrm{Supp}(\mathrm{div}(s)) & \text{if } H^0(X, L) \neq 0. \end{cases}$$

Obviously, $\mathrm{Bs}\,|D| = \mathrm{Bs}\,|\mathcal{O}_X(D)|$.

Lemma 1.10. *Let D be a Weil divisor on X, and P be a prime divisor. If P is contained in the base locus of $|D|$, the the map*

$$H^0(X, \mathcal{O}_X(D - P)) \to H^0(X, \mathcal{O}_X(D))$$

is an isomorphism.

Proof. We have the short exact sequence

$$0 \to \mathcal{O}_X(D - P) \to \mathcal{O}_X(D) \to (\mathcal{O}_X(D); P) \to 0$$

and the deduced long exact sequence

$$0 \to H^0(X, \mathcal{O}_X(D - P)) \to H^0(X, \mathcal{O}_X(D)) \to H^0(P, (\mathcal{O}_X(D); P)).$$

Suppose that $H^0(X, \mathcal{O}_X(D-P)) \to H^0(X, \mathcal{O}_X(D))$ is not surjective. Then we have a section $s \in H^0(X, \mathcal{O}_X(D))$ which does not vanish in $H^0(X, (\mathcal{O}_X(D); P))$. Since $(\mathcal{O}_X(D); P)$ is torsion free, s is not zero at the generic point of P. This shows that $\mathrm{div}(s)$ does not contain P, which is a contradiction. □

Proposition 1.11. *Let D be a Weil divisor on X, and P a prime divisor. Fix a positive integer $m \in \mathbb{Z}_{>0}$. Let $\pi : P' \to P$ be a proper birational morphism which satisifies*:

(*a*) *P' is normal.*
(*b*) *$L_1 := \pi^*(\mathcal{O}_X(D); P)/\mathrm{torsion}$ and $L_2 := \pi^*(\mathcal{O}_X(mD); P)/\mathrm{torsion}$ are both invertible.*
Then $L_2 - mL_1$ is effective.

Proof. We have a natural map $\mathcal{O}_X(D)^{\otimes m} \to \mathcal{O}_X(mD)$. By this map, we have a commutative diagram

$$\begin{tikzcd}
\mathcal{O}_X(D)\otimes\ldots\otimes\mathcal{O}_X(D-P)\otimes\ldots\otimes\mathcal{O}_X(D) \arrow[r] \arrow[d] & \mathcal{O}_X(mD-P) \arrow[d] \\
\mathcal{O}_X(D)\otimes\ldots\otimes\mathcal{O}_X(D)\otimes\ldots\otimes\mathcal{O}_X(D) \arrow[r] & \mathcal{O}_X(mD)
\end{tikzcd}$$

from which we obtain a well defined map $(\mathcal{O}_X(D); P)^{\otimes m} \to (\mathcal{O}_X(mD); P)$. Pulling back by π and dividing by torsion parts, we obtain

$$L_1^{\otimes m} \to L_2.$$

This map is isomorphic at the generic point, so it is injective, because $L_1^{\otimes m}$ is torsion free. □

2. Big line bundles

Throughout this section, X is an n-dimensional projective variety defined over an algebraically closed field k.

Let L be a line bundle on X. The volume of L is defined by

$$\mathrm{vol}_X(L) := \limsup_{m\to\infty} \frac{h^0(X, L^{\otimes m})}{m^n/n!}$$

We will list basic properties of the volume function. We refer to [10, Chapter 2.2] for proofs. (Note that in the book [10], the auther assume that the base field is the complex number field. However, the following properties hold in any algebraically closed field.)

For a line bundle L on X,
(i) L is big if and only if $\mathrm{vol}_X(L) > 0$.
(ii) If L is nef, then $\mathrm{vol}_X(L) = (L^n)$.
(iii) $\mathrm{vol}_X(aL) = a^n \mathrm{vol}_X(L)$ holds for any line bundle L on X and any positive integer $a \in \mathbb{Z}_{>0}$.
(iv) If two line bundles L, L' are numerically equivalent, then $\mathrm{vol}_X(L) = \mathrm{vol}_X(L')$.
By the properties (iii) and (iv), we can extend vol_X uniquely to a homogeneous function on $\mathrm{N}^1(X)_{\mathbb{Q}}$. In this sense,
(v) vol_X is continuous on $\mathrm{N}^1(X)_{\mathbb{Q}}$.
By the property (v), we can extend vol_X uniquely to a continuous function on $\mathrm{N}^1(X)_{\mathbb{R}}$.
(vi) Volume is a birational invariant, i.e: Let $\pi : X' \to X$ be a projective birational morphism of varieties X'. Then, $\mathrm{vol}_{X'}(\pi^* L) = \mathrm{vol}_X(L)$.

The following Theorem 2.1, Proposition 2.2, Proposition 2.3, and Proposition 2.4 will be used to prove the main theorem.

Theorem 2.1 (de Jong, [8])**.** *For any closed subset Z of X, There exists a projective, generically étale morphism $\pi : Y \to X$ from a non-singular variety Y, with $\pi^{-1}(Z)$ a normal crossing divisor on Y.*

Proof. [8, Theorem 4.1]. □

Proposition 2.2 (Hodge Index Theorem)**.** *Let $H_1, \ldots, H_n \in \mathrm{N}^1(X)_{\mathbb{R}}$ be nef classes of line bundles. Then we have*:

$$(H_1^n) \ldots (H_n^n) \le (H_1 \cdot \ldots \cdot H_n)^n.$$

Proof. [10, Theorem 1.6.1]. □

Proposition 2.3. *Let $A \in \mathrm{N}^1(X)_{\mathbb{R}}$ be a nef class, and $E \in \mathrm{N}^1(X)_{\mathbb{R}}$ a pseudoeffective class. Assume that $B := A + E$ is nef. Then the following holds*:

(i) $(A^n) \le (A^{n-1} \cdot B) \le (A^{n-2} \cdot B^2) \le \ldots \le (A \cdot B^{n-1}) \le (B^n)$.

(ii) $(A^n) = (B^n)$ *holds if and only if* $(A^{r+1} \cdot B^{n-r-1}) = (A^r \cdot B^{n-r})$ *for some* $0 \le r \le n-1$.

Proof. (i) Considering perturbation by a small ample divisor, we may assume that E is represented by an effective $\mathbb{Z}$-divisor.

Since nefness is preserved under pull backs, we have

$$\begin{aligned}(A^r \cdot B^{n-r}) - (A^{r+1} \cdot B^{n-r-1}) &= (A^r \cdot B^{n-r-1} \cdot E)\\ &= ((A|_E)^r \cdot (B|_E)^{n-r-1}) \ge 0.\end{aligned}$$

(ii) The "only if" part follows from (i). Let us prove the "if" part. It suffices to show $(A^n) \ge (B^n)$.

Clearly, we may assume that A and B is represented by very ample $\mathbb{Z}$-divisors. Let Y be the subvariety of X, defined by a proper intersection of $n-r-1$ hyperplanes in $|B|$. Set $\bar{A} := A|_Y$ and $\bar{B} := B|_Y$. By Proposition 2.2, we have

$$(\bar{A}^{r+1}) = (\bar{A}^r \cdot \bar{B}) \ge (\bar{A}^{r+1})^{\frac{r}{r+1}} \cdot (\bar{B}^{r+1})^{\frac{1}{r+1}},$$

which shows that

$$(A^{r+1} \cdot B^{n-r-1}) = (\bar{A}^{r+1}) \ge (\bar{B}^{r+1}) = (B^n).$$

By the same argument as above (define Y as a proper intersection of r hyperplanes in $|A|$), we have $(A^n) \ge (A^r \cdot B^{n-r})$. Thus,

$$(A^n) \ge (A^r \cdot B^{n-r}) \ge (A^{r+1} \cdot B^{n-r-1}) \ge (B^n).$$

□

Proposition 2.4 (Kodaira's Lemma)**.** *Let D be a big Cartier divisor and F arbitrary effective Cartier divisor on X. Then*

$$H^0(X, \mathcal{O}_X(mD - F)) \neq 0$$

for all sufficiently large m satisfying $H^0(X, \mathcal{O}_X(mD)) \neq 0$.

Proof. [10, Proposition 2.2.6]. □

The following is our main theorem.

Theorem 2.5 (Fujita's approximation theorem, [3])**.** *Let $\xi \in \mathrm{N}^1(X)_{\mathbb{Q}}$ be a rational big class. Then, for an arbitrary small real number $\epsilon > 0$, there exist a projective birational morphism $\pi : X' \to X$ of projective varieties and a decomposition*

$$\pi^*\xi = \alpha + e$$

in $\mathrm{N}^1(X')_{\mathbb{Q}}$, which satisfy the following conditions:

(i) *α is an ample class, and e is effective.*

(ii) $\mathrm{vol}_{X'}(\alpha) > \mathrm{vol}_X(\xi) - \epsilon$.

The rest of this section will be devoted to the proof of this theorem.

Lemma 2.6. *Let $\xi \in \mathrm{N}^1(X)_{\mathbb{Q}}$ be a big class. Then there exists a decomposition $\pi^*\xi = \alpha + e$ in $\mathrm{N}^1(X)_{\mathbb{Q}}$ with α ample and e effective.*

Proof. There exists a positive integer $r \in \mathbb{Z}_{>0}$ such that $D := r\xi$ is represented by a $\mathbb{Z}$-divisor. By Kodaira's Lemma 2.4, for any ample Cartier divisor A, there exists an positive integer s such that $E := sD - A$ is effective. Hence,

$$\xi = A/rs + E/rs$$

is one of the decomposition satisfying the given condition. □

Lemma 2.7. *Let D, E be two effective $\mathbb{Z}$-divisors on X. Then, there exists a projective, generically étale morphism $\pi : Y \to X$ and effective Cartier divisors F, D_1, E_1 on Y which satisfy the following*:

(i) *Y is non-singular.*

(ii) $\pi^! D = F + D_1$, $\pi^! E = F + E_1$

(iii) $\mathrm{Supp}\, D_1 \cap \mathrm{Supp}\, E_1 = \emptyset$.

Proof. By using Theorem 2.1, we may assume that X is non-singular and $\mathrm{Supp}(D + E)$ is a normal crossing divisor.

We need some notation:

Let $D := \sum_i a_i \Gamma_i$ and $E := \sum_i b_i \Gamma_i$, where Γ_i runs through all prime Cartier divisors, and $a_i, b_i \in \mathbb{Z}_{>0}$. Let G be the greatest common divisor of D and E, i.e. $G := \sum_i \min\{a_i, b_i\}\Gamma_i$. Also, set

$$\bar{D} := D - G = \sum_i \bar{a}_i \Gamma_i,$$

$$\bar{E} := E - G = \sum_i \bar{b}_i \Gamma_i.$$

Define Z as the intersection of the support of $\bar{D}$ and $\bar{E}$. Let $Z = \cup_\lambda Z_\lambda$ be the decomposition of Z into irreducible components, and set

$$r(D,E) := \sup_\lambda \{\mathrm{mult}_{Z_\lambda}(\bar{D}) \cdot \mathrm{mult}_{Z_\lambda}(\bar{E})\}.$$

Note that $r(D,E) = 0$ is equivalent to $\mathrm{Supp}(\bar{D}) \cap \mathrm{Supp}(\bar{E}) = \emptyset$, i.e.

$$\begin{aligned} D &= G + \bar{D} \\ E &= G + \bar{E} \end{aligned}$$

is the required decomposition.

Set

$$S(D,E) := \{\lambda \mid \mathrm{mult}_{Z_\lambda}(\bar{D}) \cdot \mathrm{mult}_{Z_\lambda}(\bar{E}) = r(D,E)\}$$

and $m(D,E) := \#S(D,E)$. We prove the lemma by induction on $(r(D,E), m(D,E))$, with respect to the lexicographical order in $(\mathbb{Z}_{\geq 0})^2$.

Suppose $r(D,E) \neq 0$. Take any element from $S(D,E)$, say λ_1, and let $\mu : X' \to X$ be the blow up along Z_{λ_1}. Note that since $\mathrm{Supp}(D+E)$ is a normal crossing divisor, Z_{λ_1} is non-singular, hence X' is also non-singular. Set $D' := \mu^* \bar{D}$, and $E' := \mu^* \bar{E}$.

Claim 2.8.

$$(r(D',E'), m(D',E')) < (r(D,E), m(D,E)).$$

Let Γ'_i be the strict transform of Γ_i, and set $R := \mu^{-1} Z_{\lambda_1}$. Also, let Z'_λ be the strict transform of Z_λ, for all $\lambda \neq \lambda_1$. There exists a unique pair (j,l) which satisfies $Z_{\lambda_1} = \Gamma_j \cap \Gamma_l$ and $\bar{a}_j \neq 0$, $\bar{b}_l \neq 0$. Then we have:

$$\begin{aligned} D' &= \sum_i \bar{a}_i \Gamma'_i + \bar{a}_j R, \\ E' &= \sum_i \bar{b}_i \Gamma'_i + \bar{b}_l R. \end{aligned}$$

Assume $\bar{a}_j \geq \bar{b}_l$. Then the greatest common divisor G' of D' and E' is $\bar{b}_l R$. Set

$$\begin{aligned} \bar{D}' &:= D' - G' = \sum_i \bar{a}_i \Gamma'_i + (\bar{a}_j - \bar{b}_l) R, \\ \bar{D}' &:= D' - G' = \sum_i \bar{b}_i \Gamma'_i. \end{aligned}$$

Set $W_i := R \cap \Gamma'_i$. Then,

$$\mathrm{Supp}(\bar{D}') \cap \mathrm{Supp}(\bar{E}') = \left(\bigcup_{\lambda \neq \lambda_1} Z'_\lambda \right) \cup \left(\bigcup_{\substack{i \\ b_i \neq 0}} W_i \right).$$

Looking at the multiplicity on each components, we have

$$\begin{aligned} \mathrm{mult}_{W_i}(\bar{D}') \cdot \mathrm{mult}_{W_i}(\bar{E}') = (\bar{a}_j - \bar{b}_l) \cdot \bar{b}_i &< \bar{a}_j \cdot \bar{b}_l \\ &= r(D,E) \end{aligned}$$

and

$$\mathrm{mult}_{Z'_\lambda}(\bar{D}') \cdot \mathrm{mult}_{Z'_\lambda}(\bar{E}') = \mathrm{mult}_{Z_\lambda}(\bar{D}) \cdot \mathrm{mult}_{Z_\lambda}(\bar{E})$$

for $\lambda \neq \lambda_1$, which shows that

$$(r(D', E'), m(D', E')) < (r(D, E), m(D, E)).$$

For the case $\bar{a}_j < \bar{b}_l$ is proven similarly.

By the induction hypothesis, we have $\pi : Y \to X'$ and decompositions

$$\pi^* D' = F + D_1$$
$$\pi^* E' = F + E_1$$

where $\mathrm{Supp}(D_1) \cap \mathrm{Supp}(E_1) = \emptyset$. Then,

$$(\mu \circ \pi)^* D = ((\mu \circ \pi)^* G + F) + D_1$$
$$(\mu \circ \pi)^* E = ((\mu \circ \pi)^* G + F) + E_1$$

is the required decompositions. □

Lemma 2.9. *Assume X is normal. Let L be a $\mathbb{Z}$-Weil divisor, and E, G two effective $\mathbb{Q}$-Weil divisors which has no common components. Assume that $H := L - E + G$ is nef and big $\mathbb{Q}$-Cartier divisor, and $(H^{n-1}.E) = 0$. Then the support of E is contained in the base locus of $|L|$.*

Proof. We divide the proof into several steps.

Step 1: We may assume that E, G are effective $\mathbb{Z}$-Weil divisors. Choose a positive integer $l \in \mathbb{Z} > 0$ such that lE, lG are $\mathbb{Z}$-Weil divisors. Then we have

$$\mathrm{Supp}(lE) = \mathrm{Supp}(E) \subset \mathrm{Bs}\,|lL| \subset \mathrm{Bs}\,|L|.$$

Step 2: We may assume that L is an effective Cartier divisor, and $L = M + F$, where M is base point free effective Cartier divisor and F is a fixed part of F (We call M the moving part of L). Let $\nu_1 : X_1 \to X$ be the Cartierization of L, and $\nu_2 : X_2 \to X_1$ be the blow up along the base point of $|\nu_1^! L|$. Set $\nu := \nu_2 \circ \nu_1$. Then we have:

(a) $\nu^! L$ is invertible, and

(b) $\nu^! L = M + F$, where M is the moving part and F the fixed part.

Let $\tilde{E}$, $\tilde{G}$ be the strict transform of E, G respectively. Then there exists effective Weil divisors R_1, R_2 which satisfies:

(c) $\nu^*(H) - \nu^!(L) + \tilde{E} - \tilde{G} = R_1 - R_2$.

(d) R_1 and R_2 are exceptional with respect to ν, and they have no common components.

Set $E' := \tilde{E} + R_2$, and $G' := \tilde{G} + R_1$. Then $\nu^*(H) = \nu^!(L) - E' + G'$ and E' and G' has no common components. Moreover, we have

$$(\nu^*(H)^{n-1} \cdot E') = (H^{n-1} \cdot \nu_* E') = (H^{n-1} \cdot E) = 0.$$

So, by the hypothesis, $\mathrm{Supp}(E') \subset \mathrm{Bs}\,|\nu^!(L)|$. Hence,

$$\mathrm{Supp}(E) = \nu(\mathrm{Supp}(E')) \subset \nu(\mathrm{Bs}\,|\nu^! L|).$$

On the other hand, $\nu_*(\nu^!(L)) = L$, so that $\nu(\mathrm{Bs}\,|\nu^! L|)$ is contained in the base locus of $|L|$.

Step 3: Since E and G has no common components, it suffices to prove that $G + F - E$ is effective as a $\mathbb{Z}$-Weil divisor.

Assume $G + F - E$ is not effective.

Since H is nef and big, there exists an effective Cartier divisor D such that $H - (1/m)D$ is ample for any sufficiently large positive integer $m \in \mathbb{Z}_{>0}$. Since $G + F - E$ is not effective, there exists a positive integer m such that $(1/m)D + G + F - E$ is not effective and $H - (1/m)D$ is ample. Also, we may assume that mH is effective.

By Lemma 2.7, we have a generically étale projective morphism $\pi : Y \to X$ which satisfies the following condition:

(i) Y is smooth.

(ii) There are effective divisors E_0, E_1, F_1 on Y such that

$$\pi^!(mE) = E_0 + E_1,$$
$$\pi^!(D + mG + mF) = E_0 + F_1$$

and $\mathrm{Supp}(E_1) \cap \mathrm{Supp}(F_1) = \emptyset$.

Claim 2.10. $\pi_*(E_1) \neq 0$.

Assume that $\pi_*(E_1) = 0$. Then we have

$$(\deg \pi) mE = \pi_*(\pi^!(mE)) = \pi_*(E_0),$$
$$(\deg \pi)(D + mG + mF) = \pi_*(\pi^!(D + mG + mF)) = \pi_*(E_0) + \pi_*(F_1).$$

Therefore, we obtain

$$D + mG + mF - mE = \frac{1}{\deg \pi} \pi_* F_1,$$

which contradicts to the assumption $(1/m)D + G + F - E$ is not effective.

Set $A := (mH - D) + m(L - F)$, and $B := F_1 + \pi^* A$. Since $(mH - D)$ is ample and $L - F$ is base point free, A is ample.

Claim 2.11. $B \sim E_1 + 2m\pi^* H$.

Note that D and F are Cartier. Using Proposition 1.6, we obtain:

$$\begin{aligned} B - E_1 - 2m\pi^* H &= \pi^!(D + mG + mF) - \pi^!(mE) - \pi^*(mH) \\ &\quad - \pi^*(D) + \pi^*(mL) - \pi^*(mF) \\ &= \pi^!(mG) - \pi^!(mE) - \pi^*(mH) + \pi^*(mL) \\ &\sim \pi^!(mG + mL) - \pi^!(mH + mE) \end{aligned}$$

Note that $mL + mG = mH + mE$, so we obtain the result.

For any irreducible curve C on X', $C \not\subset \mathrm{Supp}(E_1)$ or $C \not\subset \mathrm{Supp}(F_1)$, because $\mathrm{Supp}(E_1) \cap \mathrm{Supp}(F_1) = \emptyset$. This means that either $C \cdot E_1 \geq 0$ or $C \cdot F_1 \geq 0$. Since π^*A and π^*H are nef, we have that $C \cdot B \geq 0$, hence B is nef.

Since π^*H is nef, we have

$$\begin{aligned} 0 &= (\deg \pi)(H^{n-1} \cdot mE) = (\pi^*H^{n-1} \cdot \pi^!(mE)) \\ &= (\pi^*H \cdot E_0) + (\pi^*H^{n-1} \cdot E_1) \geq (\pi^*H^{n-1} \cdot E_1) \geq 0 \end{aligned}$$

i.e. $(\pi^*H^{n-1} \cdot E_1) = 0$. Thus, $((2m\pi^*H)^n) = ((2m\pi^*H)^{n-1} \cdot B)$. By Proposition 2.3, we have $(2m\pi^*H \cdot B^{n-1}) = (B^n)$, i.e. $(B^{n-1} \cdot E_1) = 0$. Since $B = F_1 + \pi^*A$ and π^*A is nef, Proposition 2.3 yields

$$0 = ((\pi^*A)^{n-1} \cdot E_1) = (\deg \pi)(A^{n-1} \cdot \pi_*E_1).$$

Note that π_*E_1 is not zero, so this is a contradiction. □

Let L be a line bundle on X. Define the Δ-*genus of* (X, L) as:

$$\Delta(X, L) := n + (L^n) - h^0(X, L).$$

It is well known that, if L is a nef and big line bundle, then we have $\Delta(X, L) \geq 0$ (cf. [4, §1]).

The following lemma is crucial for the proof of Fujita's approximation theorem.

Lemma 2.12. *We assume that X is normal. Let B be a nef and big $\mathbb{Z}$-Cartier divisor on X, and E an effective $\mathbb{Z}$-Cartier divisor on X. If $A := B+E$ is nef, then we have*:

$$h^0(X, \mathcal{O}(A)) - h^0(X, \mathcal{O}(B)) \leq n((A^n) - (B^n)).$$

Proof. Set $E = \sum_i \mu_i E_i$, where E_i's are mutually distinct prime divisors. Let $I^+ := \{i \in I \mid ((A+B)^{n-1} \cdot E_i) > 0\}$ and $m_i := \max_{a+b=n-1}(A^a \cdot B^b \cdot E_i)$. Note that since A and B are nef, $i \in I^+$ if and only if $m_i > 0$. Define a sequence of effective $\mathbb{Z}$-divisors $\{S_j\}_j$ inductively as follows:

(i) $S_0 := E$.

(ii) Suppose $S_{j-1} = \sum_i \nu_i E_i$ is defined. Let $r_j := \max_i \nu_i/\mu_i$. Choose one $i(j) \in I$ such that $\nu_{i(j)}/\mu_{i(j)} = r_j$. Also, choose $i(j)$ from I^+ if it is possible. Then let $S_j := S_{j-1} - E_{i(j)}$.

Claim 2.13. $h^0(X, \mathcal{O}(B + S_{j-1})) - h^0(X, \mathcal{O}(B + S_j)) \leq nm_{i(j)}$.

We prove the claim by dividing into two cases.

Case $i(j) \notin I^+$: Let $q_j := \max_{i \in I^+} \nu_i/\mu_i$. Note that $q_j < r_j$. There exist two effective $\mathbb{Q}$-divisors F, G which have no common components and $S_{j-1} - q_jE = F - G$. Let $H_j := q_jA + (1 - q_j)B (= B + q_jE)$. This is nef and big. Since

$$S_{j-1} - q_jE = \sum_i \mu_i(\nu_i/\mu_i - q_j)E_i,$$

if E_i is a component of $\operatorname{Supp} F$, then $i \notin I^+$, which shows that $(H_j^{n-1} \cdot E_i) = 0$. Since $H_j = B + S_{j-1} - F + G$, we can apply Lemma 2.9 which shows that any component of $\operatorname{Supp} F$ is contained in the fixed part of $|B + S_{j-1}|$. Since $E_{i(j)}$ is a component of $\operatorname{Supp} F$, we have

$$h^0(X, \mathcal{O}(B + S_{j-1})) = h^0(X, \mathcal{O}(B + S_j))$$

from the exact sequence

$$(*) \quad 0 \to \mathcal{O}_X(B + S_j) \to \mathcal{O}_X(B + S_{j-1}) \to (\mathcal{O}_X(B + S_{j-1}); E_{i(j)}) \to 0.$$

Case $i(j) \in I^+$:

Choose an positive integer $m \in \mathbb{Z}_{>0}$ such that mr_j is an integer. Let $Q_1 := (\mathcal{O}_X(B + S_{j-1}); E_{i(j)})$ and $Q_2 := (\mathcal{O}_X(m(B + S_{j-1})); E_{i(j)})$. These are coherent $\mathcal{O}_{E_{i(j)}}$-modules. There exists a projective birational morphism $\pi : \tilde{E} \to E_{i(j)}$ which satisfies:

(a) $\tilde{E}$ is normal.

(b) $\pi^!(Q_1) := \pi^* Q_1/\text{torsion}$ and $\pi^!(Q_2) := \pi^* Q_2/\text{torsion}$ are invertible $\mathcal{O}_{\tilde{E}}$-modules, and $\pi^!(Q_1) = H_1 + F_1$, $\pi^!(Q_2) = H_2 + F_2$, where H_i's are the moving parts and F_i's are the fixed parts respectively.

To obtain this, use the flattening ([14, Chap.4, Theorem 1]). Note that Q_i's are torsion free rank 1 coherent $\mathcal{O}_{E_{i(j)}}$-modules, hence the flattening of Q_i's become invertible. From now on, we will identify $\pi^!(Q_i)$ with a divisor representing $\pi^!(Q_i)$.

Claim 2.14. *$H_2 - mH_1$ is effective.*

From Proposition 1.11, we see that $\pi^!(Q_2) - m\pi^!(Q_1)$ is effective. Let H' be the moving part of $m\pi^! Q_1$. Then $H' - mH_1$ is effective. On the other hand, $H_2 - H'$ is effective, so we obtain the claim.

Let $\rho : \tilde{E} \to \mathbb{P}^N$ be the morphism induced by the linear system $|H_1|$. Let W be the image of ρ, and let d, ω be the dimension of W, the degree of W, respectively. Let Y be a general fiber of $\tilde{E} \to W$. Since $i(j) \in I^+$, $A + B$ is nef and big on $E_{i(j)}$, thus so is on Y. Therefore, there exists non-negative integers a, b with $a + b = n - 1 - d$ such that $(A^a \cdot B^b \cdot Y) > 0$. Here, so $(A^a \cdot B^b \cdot Y) \geq 1$ because the intersection number is an integer. Let $P_j := (r_j A + (1 - r_j)B)|_{\tilde{E}} (= (B + r_j E)|_{\tilde{E}})$. This is nef.

Claim 2.15. *$P_j - H_1$ is effective.*

By Proposition 1.9(2), we have a injective map

$$\pi^!(\mathcal{O}_X(-m(B + S_{j-1})); E_{i(j)}) \hookrightarrow \pi^!(Q_2)^\vee.$$

Tensoring with $\pi^* \mathcal{O}_X(m(r_j A + (1 - r_j)B))|_{E_{i(j)}}$, it yields

$$\pi^* \mathcal{O}_X(m(r_j A + (1 - r_j)B))|_{E_{i(j)}} \otimes \pi^!(\mathcal{O}_X(-m(B + S_{j-1})); E_{i(j)}) \\ \hookrightarrow \pi^* \mathcal{O}_X(m(r_j A + (1 - r_j)B))|_{E_{i(j)}} \otimes \pi^!(Q_2)^\vee.$$

The left hand side equals to

$$\pi^!(\mathcal{O}_X(m(r_jA+(1-r_j)B-B-S_{j-1}));E_{i(j)})$$
$$= \pi^!(\mathcal{O}_X(m(\textstyle\sum_i(\mu_i r_j - \nu_i)E_i));E_{i(j)}),$$

which contains $\mathcal{O}_{\tilde{E}}$, since $\mu_i r_j - \nu_i \geq 0$. Note that A and B are Cartier. The right hand side equals to $\mathcal{O}_{\tilde{E}}(mP_j - H_2 - F_2)$. Hence $mP_j - H_2$ is effective. On the other hand, we have $H_2 - mH_1$ from the above Claim, so $m(P_j - H)$ is effective.

We have

$$\begin{aligned}
\omega \leq \omega \cdot (A^a \cdot B^b \cdot Y) &= (A^a \cdot B^b \cdot H^d \cdot \tilde{E}) \leq (A^a \cdot B^b \cdot P_j^d \cdot \tilde{E}) \\
&= \sum_{l=0}^{d} \binom{d}{l} r_j^l (1-r_j)^{d-l} (A^{a+l} \cdot B^{b+d-l} \cdot E_{i(j)}) \\
&\leq \sum_{l=0}^{d} \binom{d}{l} r_j^l (1-r_j)^{d-l} m_{i(j)} = m_{i(j)}.
\end{aligned}$$

Since $\Delta(W, \mathcal{O}(1)) \geq 0$, we have that

$$h^0(E_{i(j)}, \pi^!(Q_1)) \leq d + \omega \leq n - 1 + m_{i(j)} \leq nm_{i(j)}.$$

Using Lemma 1.8, we have $h^0(E_{i(j)}, Q_1) \leq h^0(\tilde{E}, \pi^!(Q_1)) \leq nm_{i(j)}$.
Now the result follows along the long exact sequence arising from (*).

From the above claim, we see that

$$\begin{aligned}
h^0(X, \mathcal{O}(A)) - h^0(X, \mathcal{O}(B)) &\leq \sum_j (h^0(X, \mathcal{O}(B+S_{j-1})) - h^0(X, \mathcal{O}(B+S_j))) \\
&\leq \sum_j nm_{i(j)} = n \sum_i \mu_i m_i \\
&\leq n \sum_i \mu_i \left(\sum_{a+b=n-1} (A^a \cdot B^b \cdot E_i) \right) \\
&= n \sum_{a+b=n-1} (A^a \cdot B^b \cdot E) \\
&= n \sum_{a+b=n-1} ((A^{a+1} \cdot B^b) - (A^a \cdot B^{b+1})) \\
&= n((A^n) - (B^n)).
\end{aligned}$$

□

Proof of Theorem 2.5. We divide the proof into two steps.
Step 1: We may replace the statement "α is ample" by "α is nef and big".

We assume that there exist a projective birational morphism $\pi : X' \to X$ of varieties and a decomposition

$$\pi^*\xi = \alpha + e$$

in $\mathrm{N}^1(X)_{\mathbb{Q}}$ satisfying:

(i) α is nef and big, and e is effective.

(ii) $\mathrm{vol}_{X'}(\alpha) > \mathrm{vol}_X(\xi) - \epsilon/2$.

Since α is nef and big, there exists an effective class $\eta \in \overline{\mathrm{Eff}}(X)_{\mathbb{Q}}$ such that $\alpha - t\eta$ is ample for any sufficiently small $t > 0$. By the continuity of the volume, there is a sufficiently small $t_0 > 0$ which satisfies

$$\mathrm{vol}_{X'}(\alpha - t_0\eta) > \mathrm{vol}_{X'}(\alpha) - \epsilon/2.$$

Then

$$\pi^*\xi = (\alpha - t_0\eta) + (t_0\eta + e)$$

is the decomposition we desired.

Step 2: Let d be the volume of $\xi \in \mathrm{N}^1(X)_{\mathbb{Q}}$. Assume that there exists a positive number $\epsilon > 0$ such that for any projective birational $\pi : X' \to X$ and any decomposition $\pi^*\xi = \alpha + e$ in $\mathrm{N}^1(X')_{\mathbb{Q}}$ (α is a nef and big class, and e is effective), we have $\mathrm{vol}_{X'}(\alpha) \leq d - \epsilon$. Take the supremum value of the ϵ which satisfies the above condition. Note that $d - \epsilon > 0$ by Lemma 2.6. Then, for arbitrary small $\delta > 0$ (we assume that $d - \epsilon - \delta > 0$), there exists a projective birational $\pi_0 : X_0 \to X$ and a decomposition $\pi_0^*\xi = \alpha_0 + e_0$ in $\mathrm{N}^1(X_0)_{\mathbb{Q}}$ (α_0 is ample and e_0 is effective), with $\mathrm{vol}_{X_0}(\alpha_0) > d - \epsilon - \delta$. We will identify ξ with $\pi_0^*\xi$ and X_0 with X in the proceedings. There exists a positive integer $r \in \mathbb{Z}_{>0}$ such that $A := r\alpha_0$ and $E := re_0$ are both represented by $\mathbb{Z}$-divisors where A is globally generated and E is effective. Then $L := A + E$ represents $r\xi$.

For any positive integer $s \in \mathbb{Z}_{>0}$, let $X_s \to X$ be the blow up along the base point of sL, and let H_s be the moving part of sL on X_s (H_s is nef). We may assume that X_s is normal. Note that $H_s - sA|_{X_s}$ is effective, because sA is globally generated. By the previous condition, we have

$$(H_s^n) = \mathrm{vol}_{X_s}(H_s) \leq s^n r^n (d - \epsilon).$$

By Lemma 2.12, we have:

$$\begin{aligned} h^0(X, \mathcal{O}(sL)) \leq h^0(X_s, \mathcal{O}(H_s)) &\leq h^0(X_s, \mathcal{O}(sA)) + n((H_s^n) - ((sA)^n)) \\ &\leq h^0(X_s, \mathcal{O}(sA)) + ns^n r^n (d - \epsilon - \mathrm{vol}(\alpha_0)) \\ &< h^0(X_s, \mathcal{O}(sA)) + ns^n r^n \delta. \end{aligned}$$

Hence, $\mathrm{vol}_X(L) \leq \mathrm{vol}_{X_s}(A) + n \cdot n! r^n \delta$. Dividing both sides by r^n, we have:

$$d = \mathrm{vol}_X(\xi) \leq \mathrm{vol}_X(\alpha_0) + n \cdot n!\delta \leq d - \epsilon + n \cdot n!\delta.$$

Taking $\delta \to 0$ leads us to a contradiction. $\square$

Corollary 2.16. *Let $\xi \in \mathrm{N}^1(X)_{\mathbb{R}}$ be a big class. Then, for arbitrary small $\epsilon > 0$, there exists a projective birational morphism $\pi : X' \to X$ of varieties and a decomposition*

$$\pi^*\xi = \alpha + e$$

in $\mathrm{N}^1(X')_{\mathbb{R}}$, which satisfies:

(i) *α is an ample class, and e is effective.*

(ii) $\mathrm{vol}_{X'}(\alpha) > \mathrm{vol}_X(\xi) - \epsilon$.

Remark 2.17. $\xi \in \mathrm{N}^1(X)_{\mathbb{R}}$ is *big* if $\mathrm{vol}_X(\xi) > 0$. The set of big classes form an open cone in $\mathrm{N}^1(X)_{\mathbb{R}}$. We call this the "big cone".

$\xi \in \mathrm{N}^1(X)_{\mathbb{R}}$ is *effective* if ξ is an $\mathbb{R}$-linear combination of classes of effective line bundles. Note that the cone generated by effective classes need not be closed, i.e. there is a slight but crucial difference between the notion of effective class and pseudoeffective class.

Lemma 2.18. *Let V be a finite dimensional vector space over $\mathbb{Q}$, and let $V_{\mathbb{R}} := V \otimes_{\mathbb{Q}} \mathbb{R}$. (we attach the usual topology to $V_{\mathbb{R}}$.) Let U be an open subset of $V_{\mathbb{R}}$. Then, for all $v \in U$, there exist $v_1, \ldots, v_l \in U \cap V$ and $r_1, \ldots, r_l \in \mathbb{R}$ which satisfy*:

(i) $0 \le r_i \le 1$ *for all i and* $\sum_{i=1}^{l} r_i = 1$.

(ii) $\sum_{i=1}^{l} r_i v_i = v$.

Proof. Since V is dense in $V_{\mathbb{R}}$ and U is open, there exist $v_1, v_2, \ldots, v_l \in U \cap V$ such that v is contained in the convex polytope the vertex of which is $v_1, \ldots, v_l$. Then it is easy to see that there exist $r_1, \ldots r_l \in \mathbb{R}$ which satisfy the above conditions. □

Proof of 2.16. Since the volume function is continuous, Lemma 2.18 shows that there exist rational big classes $\xi_1, \ldots, \xi_l \in \mathrm{N}^1(X)_{\mathbb{Q}}$ satisfying:

(i) $\xi = \sum_{i=1}^{l} r_i \xi_i$, where r_i's are real numbers satisfying $0 \le r_i \le 1$ and $\sum_i r_i = 1$.

(ii) $\mathrm{vol}_X(\xi_i) > \mathrm{vol}_X(\xi) - \epsilon/2$ holds for all i.

Then we have a projective birational morphism $\pi : X' \to X$ and decompositions $\pi^*\xi_i = \alpha_i + e_i$ which satisfies:

(iii) α_i is nef and big, and e_i is effective.

(iv) $\mathrm{vol}_{X'}(\alpha_i) > \mathrm{vol}_X(\xi_i) - \epsilon/2$ for all i.

Note that this is possible, because nefness and bigness are preserved under pull backs. Set $\alpha := \sum_i r_i \alpha_i$ and $e := \sum_i r_i e_i$. Then we have a decomposition $\pi^*\xi = \alpha + e$. Since α is nef, we have (using the Hodge Index Theorem 2.2):

$$\mathrm{vol}_{X'}(\alpha) = \frac{1}{n!}(\alpha^n) \ge \frac{1}{n!}\min_i(\alpha_i^n) = \min_i \mathrm{vol}_{X'}(\alpha_i) > \mathrm{vol}_X(\xi) - \epsilon.$$

□

Corollary 2.19. *Let $H \in \mathrm{N}^1(X)_{\mathbb{R}}$ be a nef and big class, and $\delta \in \overline{\mathrm{Eff}}(X)$ a pseudoeffective class. Then we have the following inequality*:

$$\mathrm{vol}_X(\delta) \le \frac{(H^{n-1}.\delta)^n}{(H^n)^{n-1}}.$$

Proof. The statement clearly holds if $\mathrm{vol}_X(\delta) = 0$. Suppose $\mathrm{vol}_X(\delta) > 0$. Then for arbitrary small $\epsilon > 0$, there exists an projective birational morphism $\pi : X' \to X$ and a decomposition $\pi^*\delta = \alpha + e$, where α is ample, e is effective, and $\mathrm{vol}_{X'}(\alpha) \geq \mathrm{vol}_X(\delta) - \epsilon$. Then, by Proposition 2.2,

$$\begin{aligned}\mathrm{vol}_X(\delta) &\leq \mathrm{vol}_{X'}(\alpha) + \epsilon = (\alpha^n) + \epsilon \\ &\leq \frac{(\pi^*H^{n-1}.\alpha)^n}{(\pi^*H^n)^{n-1}} + \epsilon \leq \frac{(\pi^*H^{n-1}.\pi^*\delta)^n}{(\pi^*H^n)^{n-1}} + \epsilon \\ &= \frac{(H^{n-1}.\delta)^n}{(H^n)^{n-1}} + \epsilon.\end{aligned}$$

Taking $\epsilon \to 0$, the result follows. □

3. Uniform convergence of cohomology

Throughout this section, (X, H) is an n-dimensional $\mathbb{Q}$-polarized projective variety over an algebraically closed field k.

Theorem 3.1 (Fujita's Vanishing Theorem). *For any coherent sheaf F on X, there exists an integer $c \in \mathbb{Z}_{>0}$ such that*

$$H^i(X, F \otimes L \otimes \mathcal{O}(cH)) = 0$$

for all $i > 0$ and all nef line bundle L, where $\deg_H L = (H^{n-1}.L)$.

Proof. [5, Theorem (1)]. □

Theorem 3.2. *For arbitrary small $\epsilon > 0$, there exists $d_0 > 0$ such that*

$$\frac{h^0(X, L)}{(\deg_H L)^n} < \frac{1}{n!}\mathrm{vol}_X\left(\frac{L}{\deg_H L}\right) + \epsilon$$

for all line bundle L satisfying $\deg_H L > d_0$.

This theorem tells us that the difference between the dimension of the space of global sections and the volume can be ignored uniformly with respect to the degree. Hence we call it "uniform convergence".

The rest of this chapter is devoted for the proof of this theorem.

Lemma 3.3. *Let C be a closed cone contained in the nef and big cone of $\mathrm{N}^1(X)_{\mathbb{R}}$. Let M be a nef and big class. Then, for arbitrary small $\epsilon > 0$, there exists $d_0 > 0$ such that*

$$\frac{h^0(X, L)}{(M^{n-1} \cdot L)^n} < \frac{1}{n!}\mathrm{vol}_X\left(\frac{L}{M^{n-1} \cdot L}\right) + \epsilon$$

for all line bundles L with $[L] \in C \cap \mathrm{N}^1(X)$ and $(M^{n-1} \cdot L) > d_0$, where $[L]$ is the class of L in $\mathrm{N}^1(X)$.

Proof. Fix a norm $\|\cdot\|$ on the finite dimensional space $\mathrm{N}^1(X)_{\mathbb{R}}$. By Fujita's vanishing Theorem 3.1, there exists an effective $\mathbb{Z}$-Cartier divisor D such that

$$H^i(X, L + D) = 0$$

for all $[L] \in \mathrm{N}^1(X) \cap C$ (From now on, a Cartier divisor is identified with its associated line bundle, and the multiplication is denoted by $+$). Thus, Grothendieck-Riemann-Roch shows that

$$h^0(X, L + D) = \frac{1}{n!}(L^n) + O(\|L\|^{n-1})$$

for all $L \in \mathrm{N}^1(X) \cap C$. Since C is in the nef and big cone, we have

$$0 < (M^n)^{n-1}(L^n) \leq (M^{n-1} \cdot L)^n$$

for all $L \in C - \{0\}$. Since C is closed, $\{L \in C \mid (L^n) = 1\}$ is compact, hence so is $\{L \in C \mid (M^{n-1} \cdot L) = 1\}$. Thus, there is a positive constant $c > 0$ such that

$$c^{-1}\|L\| \leq (M^{n-1} \cdot L) \leq c\|L\|,$$

i.e. $(M^{n-1} \cdot \cdot)$ behaves like a norm on C. Combining this with the above evaluation, we obtain

$$h^0(X, L + D) = \frac{1}{n!}(L^n) + O((M^{n-1} \cdot L)^{n-1})$$

Since C is in the nef cone, we have $\mathrm{vol}_X(L) = (L^n)$ for all $L \in C$. Thus,

$$\frac{h^0(X, L)}{(M^{n-1} \cdot L)^n} \leq \frac{h^0(X, L + D)}{(M^{n-1} \cdot L)^n} = \frac{1}{n!}\,\mathrm{vol}_X\left(\frac{L}{(M^{n-1} \cdot L)}\right) + O((M^{n-1} \cdot L)^{-1}).$$

Taking $(M^{n-1} \cdot L) \to \infty$, we obtain the result. $\square$

Let us start the proof of Theorem 3.2.

Assume that there exists a positive number $\epsilon > 0$, and a infinite sequence of effective line bundles $L_1, L_2, \ldots$ on X which satisifies:

(a) $d_i := \deg_H L_i > 0$ increases to the infinity.

(b) $\dfrac{h^0(X, L_i)}{d_i^n} - \dfrac{1}{n!}\,\mathrm{vol}_X\left(\dfrac{L_i}{d_i}\right) \geq \epsilon$ for all i.

Let $\tilde{L_i} := L/d_i \in \mathrm{N}^1(X)_{\mathbb{Q}}$. Since H is ample, Kleiman's criterion implies that $\overline{\mathrm{Eff}}(X) \cap \{\deg_H = 1\}$ is compact. Hence, there is a subsequence of $\{\tilde{L_i}\}_i$ which converges. We may assume that $\{\tilde{L_i}\}_i$ converges to $\tilde{L_0} \in \overline{\mathrm{Eff}}(X) \cap \{\deg_H = 1\}$.

Case 1: Assume that $\tilde{L_0}$ is on the boundary of $\overline{\mathrm{Eff}}(X)$. In this case, $\mathrm{vol}_X(\tilde{L_0}) = 0$. Fix a sufficiently large integer t. Let $\pi : X' \to X$ be the blow up along the base locus of $L_i + tH$, and set $\pi^*(L_i + tH) = A + E$, where A is the moving part and E the fixed part. Since we have taken sufficiently large t, we may assume A is big. Then, using the positivity of the Δ-genus,

$$\begin{aligned} h^0(X, L_i) &\leq h^0(X, L_i + tH) \\ &= h^0(X', A) \\ &\leq n + (A^n) \\ &= n + \mathrm{vol}_{X'}(A) \\ &\leq n + \mathrm{vol}_X(L_i + tH). \end{aligned}$$

Hence we have

$$\frac{h^0(X, L_i)}{d_i^n} \leq \frac{n}{d_i^n} + \mathrm{vol}_X\left(\tilde{L_i} + \frac{tH}{d_i}\right).$$

Since $\tilde{L_i} + \dfrac{tH}{d_i}$ converges to $\tilde{L_0}$ and vol_X is a continuous function, the right hand side converges to 0 as i increases. This contradicts to the condition (b).

Case 2: Assume that $\tilde{L_0}$ is in the interior of $\overline{\mathrm{Eff}}(X)$. In this case, $\tilde{L_0}$ is big, i.e. $\mathrm{vol}_X(\tilde{L_0}) > 0$. Fix a sufficiently small $\delta > 0$. Take $\tilde{M}_1, \ldots, \tilde{M}_{n-1} \in \overline{\mathrm{Eff}}(X) \cap \{\deg_H = 1\} \cap \mathrm{N}^1(X)_{\mathbb{Q}}$ which satisifies:

(c) Let C be the closed convex cone generated by the $\tilde{M}_i$'s. Then $\tilde{L}_0 \in C$.

(d) $|\,\mathrm{vol}_X(\tilde{M}_i) - \mathrm{vol}(\tilde{L})| < \delta$ for all i and all $\tilde{L} \in C \cap \{\deg_H = 1\}$.

Fujita approximation theorem tells that there exists a projective birational morphism $\pi : X' \to X$ and decompositions $\pi^* \tilde{M}_i = \tilde{H}_i + E_i$ in $\mathrm{N}^1(X)_{\mathbb{Q}}$, which satisfies:

(e) $\tilde{H}_i$ is nef, and E_i is effective.

(f) $\mathrm{vol}_{X'}(\tilde{H}_i) \geq \mathrm{vol}_X(\tilde{M}_i) - \delta$.

Since δ is sufficiently small, $\tilde{H}_i$ is big. We may assume $\mathrm{vol}_X(H_1)$ is minimal among $\mathrm{vol}_X(H_i)$'s. There exists a positive integer $r \in \mathbb{Z}_{>0}$ such that $M_i := r\tilde{M}_i$ and $H_i := r\tilde{H}_i$ are in $\mathrm{N}^1(X)$. Set $\Gamma := \{\sum m_i M_i \mid m_i \in \mathbb{Z}_{\geq 0}\}$.

Claim 3.4. *There exists $d_0 > 0$ such that*

$$\frac{h^0(X, L)}{(\deg_H L)^n} < \frac{1}{n!}\,\mathrm{vol}_X\left(\frac{L}{\deg_H L}\right) + \epsilon/2$$

for all line bundle L with $[L] \in \Gamma$ and $\deg_H L > d_0$.

Let $L = \sum m_i M_i$. Let $\pi^* L = A + E$, where A is the moving part of L and E is the fixed part of L. Note that $A - B_L$ is effective, where $B_L := \sum m_i H_i$. Then, by Lemma 2.12, we have

$$\begin{aligned} h^0(X, L) \leq h^0(X', \pi^* L) &= h^0(X', A) \\ &\leq h^0(X', B_L) + n((A^n) - (B_L^n)) \\ &\leq h^0(X', B_L) + n(\mathrm{vol}_X(L) - (B_L^n)). \end{aligned}$$

On the other hand,

$$\begin{aligned} (B_L^n) &= \left(\textstyle\sum_i m_i H_i\right)^n \geq \left(\textstyle\sum_i m_i (H_i^n)^{1/n}\right)^n \\ &\geq \left(\textstyle\sum_i m_i (H_1^n)^{1/n}\right)^n \geq \left(\textstyle\sum_i m_i\right)^n (H_1^n) \\ &= (\deg_H L)^n (\tilde{H}_1^n). \end{aligned}$$

Hence,

$$\begin{aligned}
\frac{h^0(X,L)}{(\deg_H L)^n} &\leq \frac{h^0(X',B_L)}{(\deg_H L)^n} + n\left(\mathrm{vol}_X\left(\frac{L}{\deg_H L}\right) - \frac{(B_L^n)}{(\deg_H L)^n}\right) \\
&\leq \frac{h^0(X',B_L)}{(\pi^* H^{n-1}.B_L)^n} + n\left(\mathrm{vol}_X\left(\frac{L}{\deg_H L}\right) - \mathrm{vol}_{X'}(\tilde{H}_1)\right) \\
&\leq \frac{h^0(X',B_L)}{(\pi^* H^{n-1}.B_L)^n} + 2n\delta.
\end{aligned}$$

By Lemma 3.3, we have

$$\limsup_{\deg_H L\to\infty} \frac{h^0(X',B_L)}{(\pi^* H^{n-1}.B_L)^n} \leq \frac{1}{n!}\left(\mathrm{vol}_X\left(\frac{L}{\deg_H L}\right)+\delta\right).$$

Since we have taken δ sufficiently small, the claim follows.

Let us go back to the proof of Theorem 3.2. Since Γ is a lattice in C, there is a finite set S of effective Cartier divisors such that for all $L \in C$, there exists $D \in S$ such that $L + D \in \Gamma$. If $\deg_H L > d_0$ (d_0 is the positive number of the above claim),

$$\begin{aligned}
\frac{h^0(X,L)}{(\deg_H L)^n} &\leq \frac{h^0(X,L+D)}{(\deg_H L)^n} \\
&\leq \frac{h^0(X,L+D)}{(\deg_H(L+D))^n}\cdot\frac{(\deg_H(L+D))^n}{(\deg_H L)^n} \\
&\leq \frac{1}{n!}\left(\mathrm{vol}_X\left(\frac{L+D}{\deg_H(L+D)}\right)+\frac{\epsilon}{2}\right)\cdot\frac{(\deg_H(L+D))^n}{(\deg_H L)^n}.
\end{aligned}$$

Since S is finite, $(\deg_H(L+D))^n/(\deg_H L)^n$ converges uniformly to 1 as $\deg_H L \to \infty$. This contradicts to (b). $\square$

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