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Well-posedness for hyperbolic higher order operators with finite degeneracy

By

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Abstract

We consider a class of higher order hyperbolic equations with finite degeneracy.

We give sufficient conditions in order the Cauchy problem to be well-posed in $\mathcal{C}^\infty$ and in Gevrey classes.

1. Introduction

In this paper we consider the Cauchy problem:

$$\begin{cases} L(t,\partial_t,\partial_x)u(t,x) = M(t,\partial_t,\partial_x)u(t,x) & (t,x)\in[-T,T]\times\mathbb{R}^n \\ \partial_t^j u(t_0,x) = u_j(x)\,, & x\in\mathbb{R}^n, \quad j=0,\dots,m-1\,, \end{cases} \tag{1.1}$$

where L is an homogeneous differential operator of order m:

$$L(t,\partial_t,\partial_x) = \partial_t^m + \sum_{\substack{\alpha_0+|\alpha|=m \\ \alpha_0<m}} a_{\alpha_0,\alpha}(t)\partial_t^{\alpha_0}\partial_x^\alpha\,,$$

while $M = \sum_{j=0}^{m-1} M_j$ is a differential operator of order $\le m-1$:

$$M_j(t,\partial_t,\partial_x) = \sum_{\alpha_0+|\alpha|=j} b_{\alpha_0,\alpha}(t)\partial_t^{\alpha_0}\partial_x^\alpha\,, \quad j=0,1,\dots,m-1\,.$$

We assume that the coefficients of the principal symbol belong to $\mathcal{C}^m$, and the lower order terms are continuous in t.

Let $\gamma^s = \gamma^s(\mathbb{R}^n)$ be the space of Gevrey functions, that is the functions $f\in\mathcal{C}^\infty(\mathbb{R}^n)$ such that for any compact K there exists C_K such that

$$\sup_{x\in K}|\partial_x^\alpha f(x)| \le C_K\,|\alpha|!^s\,, \qquad \forall\alpha\in\mathbb{N}^n\,.$$

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We say that the Cauchy problem (1.1) is well-posed in $\mathcal{C}^\infty$ (resp. γ^s), if for any $u_j \in \mathcal{C}^\infty(\mathbb{R}^n)$ (resp. $u_j \in \gamma^s(\mathbb{R}^n)$), $j = 0, \dots, m-1$, and any $f \in \mathcal{C}\big([-T,T]\,;\mathcal{C}^\infty(\mathbb{R}^n)\big)$ (resp. $f \in \mathcal{C}\big([-T,T]\,;\gamma^s(\mathbb{R}^n)\big)$), (1.1) admits a unique solution $u \in \mathcal{C}^m\big([-T,T]\,;\mathcal{C}^\infty(\mathbb{R}^n)\big)$ (resp. $u \in \mathcal{C}^m\big([-T,T]\,;\gamma^s(\mathbb{R}^n)\big)$). It is well known that in order the Cauchy problem (1.1) to be well-posed in $\mathcal{C}^\infty$ or in Gevrey spaces, the operator L needs to be *hyperbolic*, that is the characteristic roots $\tau_1(t,\xi), \dots, \tau_m(t,\xi)$ (the solutions in τ of the characteristic equation: $L(t,\tau,\xi) = 0$) are all real [12], [14], [15]. On the other side if L is *strictly hyperbolic*, i.e. the characteristic roots are distinct, then (1.1) is well-posed in $\mathcal{C}^\infty$ and in all Gevrey classes. But if L is *weakly hyperbolic*, i.e. the characteristic roots may coincide, then the Cauchy problem (1.1) is well-posed in γ^d, for $1 < d < d_B = \dfrac{r}{r-1}$, where r is the largest multiplicity of the characteristic roots [2]. The bound d_B is in general sharp, unless one assume further conditions on the principal symbol and on the lower order terms. The following example gives a good description of the problem. Consider the operator

$$P(t,\partial_t,\partial_x) = \partial_t^2 - t^{2K}\partial_x^2 + t^l\partial_x\,.$$

In [10] it is proved that the Cauchy problem for P is well-posed in $\mathcal{C}^\infty$ if, and only if, $l \geq K-1$; moreover if $l < K-1$, the Cauchy problem for P is well-posed in γ^s, for $s < s_0 := \dfrac{2K-l}{K-l-1}$, and is not well-posed for $s > s_0$.

Another interesting example is given in [7] where they constructed a function $a \in \mathcal{C}^\infty$ verifying $a(t) > 0$ for $t \neq 0$ and vanishing at infinite order in 0 such that the Cauchy problem for $\partial_t^2 - a(t)\partial_x^2$ is not well-posed in $\mathcal{C}^\infty$.

In order to control the degeneracy of the principal symbol, we assume that it has *finite degeneracy*, that is its discriminant

$$\Delta(t,\xi) = \prod_{j<k}\big(\tau_j(t,\xi) - \tau_k(t,\xi)\big)^2$$

vanishes only at finite order.

Finite degeneracy implies that the characteristic roots may coincide only at isolated points, say $t_1 < \ldots < t_N$ and they are simple for $t \notin \{t_1, \dots, t_N\}$. Without loss of generality, we can assume that $N = 1$. Indeed, choosing $t'_0, t'_1, \dots, t'_n$ such that $-T = t'_0 \leq t_1 < t'_1 < t_2 < \cdots < t'_{n-1} < t_n \leq t'_n = T$, we can consider the Cauchy problem in each interval $[t'_{j-1}, t'_j]$, with $j = 1, \dots, n$, so that we are reduced to the case of one multiple point (the solution on the whole interval $[-T,T]$ can be obtained gluing together the solutions in each interval $[t'_{j-1}, t'_j]$).

However, finite degeneracy seems still too general to handle. Indeed we have to do a more precise assumption.

Before to state our assumptions, we introduce some notations.

Notations. Let $f(t)$ and $g(t)$ be defined in $[-T,T]$, we will write $f \lesssim g$ to mean that there exists a positive constant C such that

$$f(t) \leq C\,g(t)\,, \quad \forall t \in [-T,T]\,.$$

Similarly, we will write $f \approx g$ to mean that $f \lesssim g$ and $g \lesssim f$.

Analogously, if $f(t,\xi)$ and $g(t,\xi)$ are symbols defined in $[-T,T] \times \mathbb{R}^n$, we will write $f \lesssim g$ (resp. $f \approx g$) to mean that there exists a positive constant C such that

$$f(t,\xi) \leq C\, g(t,\xi) \qquad \Big(\text{resp } C^{-1}\, g(t,\xi) \leq f(t,\xi) \leq C\, g(t,\xi)\Big)\,,$$

for $(t,\xi) \in [-T,T] \times \mathbb{R}^n$.

Assumption 1. There exists $\kappa_1 \leq \kappa_2 \leq \cdots \leq \kappa_m$, such that:

$$|\tau_j(t,\xi) - \tau_k(t,\xi)| \approx |t-t_1|^{\kappa_j}\, |\xi|\,, \qquad \text{if } 1 \leq j < k \leq m\,. \tag{1.2}$$

Note that the condition (1.2) implies that $\tau_1, \ldots, \tau_{m-1}$ vanish at finite order in t_1, τ_m may vanish at infinite order in t_1.

Since the characteristic roots coincide only for $t = t_1$, we can find a permutation π_+ (resp. π_-) of the indices such that:

$$\tau_{\pi_+(1)} \leq \tau_{\pi_+(2)} \leq \cdots \leq \tau_{\pi_+(m)} \quad \text{on } [t_1, T]$$

$\big($resp. $\tau_{\pi_-(1)} \leq \tau_{\pi_-(2)} \leq \cdots \leq \tau_{\pi_-(m)}$ on $[-T, t_1]\big)$.

Thanks to Bronšteĭn Lemma [1] (see also [13] and [17]), $\tau_{\pi_+(1)} \leq \tau_{\pi_+(2)} \leq \cdots \leq \tau_{\pi_+(m)}$ are Lipschitz continuous on $[t_1,T]$ (resp. $\tau_{\pi_-(1)} \leq \tau_{\pi_-(2)} \leq \cdots \leq \tau_{\pi_-(m)}$ are Lipschitz continuous on $[-T,t_1]$), hence $\tau_1, \tau_2, \ldots, \tau_m$ are Lipschitz continuous on $[-T,T]$.

In [6] it is considered the Cauchy problem for an homogeneous equation, that is $M_j \equiv 0$, $j = 0, 1, \ldots, m-1$, and it is proved that if L has finite degeneracy then the Cauchy problem (1.1) is $\mathcal{C}^\infty$ well-posed if, and only if,

$$|\tau_j(t,\xi)| \lesssim |\tau_j(t,\xi) - \tau_k(t,\xi)| \quad \forall t \in [-T,T],\ \forall \xi \in \mathbb{R}^n\,, \tag{1.3}$$

for any $j,k \in \{1,2,\ldots,m\}$ with $j \neq k$.

If L has finite degeneracy, then (1.3) is equivalent to:

$$|t-t_1|\,\big|\tau_j'(t,\xi)\big| \lesssim |\tau_j(t,\xi) - \tau_k(t,\xi)| \quad \forall t \in [-T,T],\ \forall \xi \in \mathbb{R}^n\,, \tag{1.4}$$

for any $j,k \in \{1,2,\ldots,m\}$ with $j \neq k$. As in [5] instead of (1.4), we assume a slightly more general condition:

Assumption 2. There exist $\Lambda_m \geq 0$ such that:

$$|t-t_1|^{1+\Lambda_m}\,\big|\tau_j'(t,\xi)\big| \lesssim |\tau_j(t,\xi) - \tau_k(t,\xi)| \quad \forall t \in [-T,T],\ \forall \xi \in \mathbb{R}^n\,, \tag{1.5}$$

for any $j,k \in \{1,2,\ldots,m\}$ with $j \neq k$.

With no loss of generality, we can assume $\Lambda_m \leq \kappa_{m-1} - 1$, since, by Bronšteĭn Lemma, τ_j' are bounded, hence Assumption 1 implies Assumption 2 with $\Lambda_m = \kappa_{m-1} - 1$.

Since the characteristic roots are simple for $t \neq t_1$, using implicit function theorem to express $\tau_j'(t)$, we can see that Assumption 2 is equivalent to:

$$\sup_{\substack{t\in[-T,T]\\ \xi\in\mathbb{R}^n}} \sum_{j\neq k} \left[\frac{|t-t_1|^{1+\Lambda_m}}{|\tau_j(t,\xi)-\tau_k(t,\xi)|} \frac{\big|\partial_t L\big(t,\tau_j(t,\xi),\xi\big)\big|}{\big|\partial_\tau L\big(t,\tau_j(t,\xi),\xi\big)\big|} \right]^2 < +\infty \,, \tag{1.6}$$

and (1.6) can be expressed in terms of the coefficient of $L(t,\tau,\xi)$.

The Cauchy problem for homogeneous equations in Gevrey spaces under Assumption 2 has been considered in [5] (also without the assumption of finite degeneracy), where they proved the γ^d well-posedness, for $1 < d < 1 + \dfrac{\kappa_1+1}{\Lambda_m}$.

We precise this result and extend it to non homogeneous equations, assuming the following condition on the lower order terms:

Assumption 3. There exist $\Lambda_j \geq 0$, $j = 1,\dots,m-1$, such that:

$$|t-t_1|^{\Lambda_j} \big|M_j\big(t,\tau_k(t,\xi),\xi\big)\big| \lesssim \big|\partial_\tau^{m-j} L\big(t,\tau_k(t,\xi),\xi\big)\big| \quad \forall t\in[-T,T],\ \forall\xi\in\mathbb{R}^n\,, \tag{1.7}$$

for $j = 1,\dots,m-1$ and $k = 1,2,\dots,j+1$.

Remark 1.1. Assumption 3 is equivalent to:

$$|t-t_1|^{\Lambda_j} \big|M_j\big(t,\tau_k(t,\xi),\xi\big)\big| \lesssim \prod_{l\in\{1,\dots,j+1\}\setminus\{k\}} |\tau_k(t,\xi)-\tau_l(t,\xi)| \,, \tag{1.8}$$

for $j = 1,\dots,m-1$ and $k = 1,2,\dots,j+1$, hence Assumption 1 implies:

$$\big|M_j\big(t,\tau_{j+1}(t,\xi),\xi\big)\big| \lesssim |t-t_1|^{[s_j-\Lambda_j]^+} |\xi|^j \,, \tag{1.9}$$

where $s_j = \kappa_1 + \kappa_2 + \dots + \kappa_j$ and $[a]^+$ is the positive part of $a \in \mathbb{R}$: $[a]^+ = \max(a,0)$.

In fact, since

$$L(t,\tau,\xi) = \big(\tau-\tau_1(t,\xi)\big)\cdots\big(\tau-\tau_m(t,\xi)\big)\,,$$

we have:

$$\partial_\tau^{m-j} L(t,\tau,\xi) = \sum_{1\leq k_1<k_2<\dots<k_j\leq m} \big(\tau-\tau_{k_1}(t,\xi)\big)\cdots\big(\tau-\tau_{k_j}(t,\xi)\big)\,,$$

and:

$$\partial_\tau^{m-j} L\big(t,\tau_k(t,\xi),\xi\big) = \sum_{\substack{1\leq k_1<k_2<\dots<k_j\leq m\\ k\notin\{k_1,\dots,k_j\}}} \big(\tau_k(t,\xi)-\tau_{k_1}(t,\xi)\big)\cdots\big(\tau_k(t,\xi)-\tau_{k_j}(t,\xi)\big)\,.$$

Now it's clear that

$$\left|\partial_\tau^{m-j} L\big(t, \tau_k(t,\xi), \xi\big)\right| \approx \prod_{l \in \{1,\dots,j+1\} \setminus \{k\}} |\tau_k(t,\xi) - \tau_l(t,\xi)| \ ,$$

and we get the equivalence between Assumption 3 and (1.8).

We set

(1.10)

$$d_m = \max\Big(2 + \frac{2}{\Lambda_m}, 1 + \frac{\kappa_1 + 1}{\Lambda_m}\Big) = \begin{cases} +\infty & \text{if } \Lambda_m = 0\,, \\ 1 + \dfrac{\kappa_1 + 1}{\Lambda_m} & \text{if } 0 < \Lambda_m \le \kappa_1 - 1\,, \\ 2 + \dfrac{2}{\Lambda_m} & \text{if } \kappa_1 - 1 < \Lambda_m \le \kappa_{m-1} - 1\,, \end{cases}$$

$$(1.11)\quad d_{j,k} = \begin{cases} +\infty & \text{if } \min(s_{j-k}, \Lambda_j) \le m - j\,, \\ \Big(1 - \dfrac{(m-j)(\kappa_h + 1)}{s_h + (m-k-h)\kappa_h - [s_{j-k} - \Lambda_j]^+}\Big)^{-1} & \\ & \text{if } \min(s_{j-k}, \Lambda_j) > m - j\,, \end{cases}$$

where

$$(1.12)\quad h = h(j,k) = \min\Big\{\ l \in \mathbb{N} \setminus \{0\} \ \Big|\ s_l + l \ge m - k + [s_{j-k} - \Lambda_j]^+ \ \Big\}\,,$$

for $1 \le j \le m-1$ and $0 \le k \le j-1$, and

$$d_0 = \min\Big(d_m\,, \min_{1 \le k \le j \le m-1} d_{j,k}\Big)\,.$$

Remark 1.2. d_m (resp. $d_{j,k}$) is a decreasing function with respect to Λ_m (resp. Λ_j).

Remark 1.3. Note that $\min(s_{j-k}, \Lambda_j) > m - j$, is equivalent to

$$s_{j-k} + j - k > m - k + [s_{j-k} - \Lambda_j]^+\,,$$

and hence it implies $h \le j - k$.

Then we have:

Theorem 1.1. *If Assumptions* 1, 2, *and* 3 *are satisfied, then the Cauchy problem* (1.1) *is* γ^d *well-posed for any* $1 < d < d_0$. *If moreover* $d_0 = +\infty$*, then the Cauchy problem* (1.1) *is also* $\mathcal{C}^\infty$ *well-posed.*

In particular, we have:

Corollary 1.1. *If Assumptions* 1, 2, *and* 3 *are satisfied, with* $\Lambda_j \le m - j$*, for* $j = 1, 2, \dots, m$*, then the Cauchy problem* (1.1) *is well-posed in all Gevrey classes and in* $\mathcal{C}^\infty$.

Remark 1.4. Assumptions 1 and 2 are not necessary, in general. In fact, let $\tau_1, \dots, \tau_m$ be *any* real smooth functions, the Cauchy problem for the operator

$$(\partial_t + \tau_1 \partial_x)(\partial_t + \tau_2 \partial_x) \cdots (\partial_t + \tau_m \partial_x)$$

is well-posed in $\mathcal{C}^\infty$ and in all Gevrey classes.

For results concerning operators with infinite degeneracy, we refer to [5], [8], [9], [11], and the references cited there.

Example 1.1. [10], [16]. Let

$$\begin{aligned} L(t, \partial_t, \partial_x) &= \partial_t^2 - a\partial_x^2 \\ M_1(t, \partial_t, \partial_x) &= b_0 \partial_t + b_1 \partial_x \\ M_0(t, \partial_t, \partial_x) &= c\,. \end{aligned}$$

L is weakly hyperbolic if, and only if $a \geq 0$. Assumption 1 means that $a \approx |t|^\kappa$, for some $\kappa \geq 1$. Assumption 2, and 3 are equivalent to:

$$|t|^{1+\Lambda_2} \left| a'(t) \right| \lesssim a(t)\,, \tag{1.13}$$

$$|t|^{\Lambda_1} \left| b_1(t) \right| \lesssim \sqrt{a(t)}\,. \tag{1.14}$$

Theorem 1.1 implies that the Cauchy problem is well-posed in γ^d, for

$$1 < d < d_0 = \min\Big(\frac{\Lambda_2 + \kappa + 1}{\Lambda_2}, \frac{\Lambda_1 + \kappa}{\Lambda_1 - 1} \Big)\,.$$

In particular, if $a(t) = t^{2K}$, and $b_1(t) = t^l$, then (1.13) and (1.14) hold true with $\Lambda_2 = 0$ and $\Lambda_1 = K - l$, and we get $d_0 = \dfrac{2K - l}{K - l - 1}$.

We discuss third order operators in Section 2.

We deduce from Theorem 1.1 the following result concerning the strong hyperbolicity in Gevrey classes (cf. [4]):

Theorem 1.2. *If Assumption* 1 *is satisfied then the Cauchy problem* (1.1) *is well-posed in* γ^d*, for any* $d < d^*$*, where*

$$d^* = \Big(1 - \frac{\kappa_h + 1}{s_h + (m - h)\kappa_h} \Big)^{-1},$$

and

$$h = \min\Big\{ l \;\Big|\; s_l + l \geq m \Big\}\,, \tag{1.15}$$

for any $M(t, \partial_t, \partial_x)$ *of order* $\leq m - 1$.

Example 1.2. If $m = 2$ then

$$d^* = \begin{cases} +\infty & \text{if } \kappa_1 = 1\,, \\ \dfrac{2\kappa_1}{\kappa_1 - 1} & \text{if } \kappa_1 \geq 2\,. \end{cases}$$

Example 1.3. If $m = 3$ (see [3]), then

$$d^* = \min\Big(\frac{3\kappa_1}{2\kappa_1 - 1}, \frac{\kappa_1 + 2\kappa_2}{\kappa_1 + \kappa_2 - 1}\Big) = \begin{cases} 2 + \dfrac{1}{\kappa_2} & \text{if } \kappa_1 = 1\,, \\ \dfrac{3\kappa_1}{2\kappa_1 - 1} & \text{if } \kappa_1 \geq 2\,. \end{cases}$$

Example 1.4. If $m = 4$, then

$$d^* = \min\Big(\frac{4\kappa_1}{3\kappa_1 - 1}, \frac{\kappa_1 + 3\kappa_2}{\kappa_1 + 2\kappa_2 - 1}\Big) = \begin{cases} \dfrac{3}{2} + \dfrac{1}{2\kappa_2} & \text{if } \kappa_1 = 1\,, \\ \dfrac{3\kappa_2 + 2}{2\kappa_2 + 1} & \text{if } \kappa_1 = 2\,, \\ \dfrac{4\kappa_1}{3\kappa_1 - 1} & \text{if } \kappa_1 \geq 3\,. \end{cases}$$

Remark 1.5. A closer look to the proof of Theorem 1.1 shows that the hypothesis (1.2) is used only for $j \leq h^* = \max h(j,k)$, where $h(j,k)$ is defined in (1.12). Hence, we can drop hypothesis (1.2) for $j > h^*$.

Similarly, we can drop hypothesis (1.2) for $j > h$ in Theorem 1.2, where h is defined in (1.15). Note that $h \leq \left\lceil \frac{m}{2} \right\rceil$, where $\lceil a \rceil = \min\{p \in \mathbb{N} | a \leq p\}$.

Example 1.5. If $\kappa_j = 1$, $j = 1, \dots, \left\lceil \frac{m}{2} \right\rceil$, then $d^* = \dfrac{m}{m-2}$.

Example 1.6. If $\kappa_1 \geq \left\lceil \frac{m}{2} \right\rceil$ then $d^* = \dfrac{m\kappa_1}{(m-1)\kappa_1 - 1}$.

Remark 1.6. Note that $d^* > \dfrac{m}{m-1}$ (cf. [2]).

The same technique we use to prove Theorem 1.1 can be adapted also to operators which does not verify Assumption 1. As an example, we consider a fourth order operator, whose characteristic roots verify the following Assumption instead of Assumption 1:

Assumption 1′. There exist $\kappa \leq \ell_1 \leq \ell_2$, such that:

$$\tau_1 - \tau_2 \approx t^\kappa\,, \quad \tau_3 - \tau_4 \approx t^\kappa\,, \quad \tau_1 - \tau_3 \approx t^{\ell_1}\,, \quad \tau_2 - \tau_4 \approx t^{\ell_2}\,.$$

Example 1.7. If

$$\tau_1 = t\,, \quad \tau_2 = -t\,, \quad \tau_3 = t + t^2\,, \quad \tau_4 = -t - t^2\,,$$

then L verifies Assumption 1′ ($\kappa = 1$, $\ell_1 = \ell_2 = 2$) and does not verify Assumption 1.

Example 1.8. If

$$\tau_1 = -t\,, \quad \tau_2 = t\,, \quad \tau_3 = 2t\,, \quad \tau_4 = t + t^2\,,$$

then L verifies both Assumption 1 ($\kappa_1 = \kappa_2 = 1$, $\kappa_3 = 2$) and 1′ ($\kappa = \ell_1 = 1$, $\ell_2 = 2$).

We define d_m as in (1.10), and

$$\widetilde{d}_{j,k} = \begin{cases} +\infty & \text{if } \min(s_{j-k}(k), \Lambda_j) \le m-j\,, \\ \Big(1 - \dfrac{(m-j)(\kappa_h+1)}{s_h(k) + (m-k-h)\kappa_h - [s_{j-k}(k) - \Lambda_j]^+}\Big)^{-1} & \\ & \text{if } \min(s_{j-k}(k), \Lambda_j) > m-j\,, \end{cases}$$

where

$$s_h(0) = \begin{cases} \kappa & \text{if } h=1 \\ 2\kappa & \text{if } h=2 \\ 2\kappa + \ell_2 & \text{if } h=3 \end{cases} \qquad s_h(k) = \begin{cases} \kappa & \text{if } h=1 \\ \kappa + \ell_1 & \text{if } h=2 \end{cases}, \quad \text{if } k>0\,.$$

Note that $\widetilde{d}_{j,k}$ is defined as $d_{j,k}$ in Theorem 1.1, but in $\widetilde{d}_{j,k}$ the s_h depend on k. We have:

Theorem 1.3. *Let* $\widetilde{d}_0 = \min\Big(d_3\,,\ \min\limits_{1\le k\le j\le 3} \widetilde{d}_{j,k}\Big)$.

If L verifies Assumptions $1'$, 2, *and* 3, *then the Cauchy problem* (1.1) *is* γ^d *well-posed for any* $1 < d < \widetilde{d}_0$.

If moreover $\widetilde{d}_0 = +\infty$, *then the Cauchy problem* (1.1) *is also* $\mathcal{C}^\infty$ *well-posed.*

Note that if $\kappa = \ell_1$ then Assumption $1'$ is equivalent to Assumption 1 with $\kappa_1 = \kappa_2$, and $d_0 = \widetilde{d}_0$.

The plan of this paper is the following. In order to make the proof of Theorem 1.1 more comprehensible and easier to digest, in Section 2 we state and prove Theorem 1.1 in a special case: we consider a third order operator with finite degeneracy verifying (1.3) (hence Assumption 2 is verified with $\Lambda_3 = 0$) and one space variable. In Section 3, we prove some Lemmas we need to prove Theorem 1.1; the core of the proof of Theorem 1.1 is an energy estimate and it will take place in Section 4. We prove Theorem 1.2 in Section 5 and Theorem 1.3 in Section 6.

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2. A special case of Theorem 1.1: third order operators

In this section we consider the third order Cauchy problem:

$$\begin{cases} Lu = M_2 u + M_1 u + M_0 u \\ u(0,x) = u_0 \quad u_t(0,x) = u_1 \quad u_{tt}(0,x) = u_2\,. \end{cases} \tag{2.1}$$

where:

$$L(t, \partial_t, \partial_x) = \partial_t^3 + a_1 \partial_t^2 \partial_x + a_2 \partial_t \partial_x^2 + a_3 \partial_x^3$$

$$M_2(t,\partial_t,\partial_x) = b_0\partial_t^2 + b_1\partial_t\partial_x + b_2\partial_x^2$$
$$M_1(t,\partial_t,\partial_x) = c_0\partial_t + c_1\partial_x$$
$$M_0(t,\partial_t,\partial_x) = d\,.$$

We assume that L has finite degeneracy and it satisfies (1.3) (and hence (1.4)).

To fix the notations, let κ, ℓ be such that:

$$|\tau_1(t,\xi)-\tau_2(t,\xi)| \approx |\tau_1(t,\xi)-\tau_3(t,\xi)| \approx |t|^\kappa\,|\xi|, \quad |\tau_2(t,\xi)-\tau_3(t,\xi)| \approx |t|^\ell\,|\xi|,$$

for some $1 \le \kappa \le \ell < +\infty$.

We assume that the lower order terms satisfy:

$$|b_1| \lesssim |t|^\lambda\ , \quad |b_2| \lesssim |t|^\eta\ , \quad |c_1| \lesssim |t|^\mu\ . \tag{2.2}$$

Let

$$d_\lambda = \begin{cases} \dfrac{2\kappa-\lambda}{\kappa-\lambda-1} & \text{if } \lambda < \kappa-1,\\ +\infty & \text{if } \lambda \ge \kappa-1; \end{cases}$$

$$d_\eta = \begin{cases} \dfrac{3\kappa-\eta}{2\kappa-\eta-1} & \text{if } 0 \le \eta < \kappa-2,\\ \dfrac{\kappa+2\ell-\eta}{\kappa+\ell-\eta-1} & \text{if } \kappa-2 \le \eta < \kappa+2\,\dfrac{\ell-\kappa}{\kappa+1},\\ \dfrac{2\kappa}{\kappa-1} & \text{if } \kappa+2\,\dfrac{\ell-\kappa}{\kappa+1} \le \eta < \ell,\\ \dfrac{2\kappa+\ell-\eta}{\kappa+\ell-\eta-1} & \text{if } \ell \le \eta < \kappa+\ell-1,\\ +\infty & \text{if } \eta \ge \kappa+\ell-1; \end{cases}$$

$$d_\mu = \begin{cases} \dfrac{3\kappa-\mu}{\kappa-\mu-2} & \text{if } \mu < \kappa-2,\\ +\infty & \text{if } \mu \ge \kappa-2; \end{cases}$$

and

$$\widetilde{d} = \min\bigl(d_\lambda, d_\eta, d_\mu\bigr)\,.$$

Note that d_λ, d_η, d_μ are increasing with respect to λ, η, μ.

We claim that the Cauchy problem (2.1) is γ^d well-posed for $d < \widetilde{d}$, and $\mathcal{C}^\infty$ well-posed if $\widetilde{d} = +\infty$.

It's easy to see that if (1.3) hold true, (2.2) implies

$$|t|^{\Lambda_2}\,|M_2(t,\tau_j,i\xi)| \lesssim |\tau_j-\tau_k|\,|\tau_j-\tau_l|\ , \tag{2.3}$$

for $j,k,l \in \{1,2,3\}$, $j \ne k$, $j \ne l$, $k \ne l$, with $\Lambda_2 = \max\Bigl([\kappa-\lambda]^+, [\kappa+\ell-\eta]^+\Bigr)$ and

$$|t|^{\Lambda_1}\,|M_1(t,\tau_j,i\xi)| \lesssim |\tau_j-\tau_k|\ , \tag{2.4}$$

for $j, k \in \{1,2\}$, $j \neq k$, with $\Lambda_1 = [\kappa - \mu]^+$.

Without loss of generality, we can assume

$$\lambda \le \kappa - 1\,, \qquad \eta \le \kappa + \ell - 1\,, \qquad \mu \le \kappa - 2\,.$$

According to (1.10) and (1.12), we set:

$$d_{2,0} = \begin{cases} \dfrac{2\kappa - \ell + \Lambda_2}{\kappa - \ell - 1 + \Lambda_2} & \text{if } \ell + 1 < \Lambda_2 \le \kappa + \ell\,, \\ \dfrac{\Lambda_2 + \ell}{\Lambda_2 - 1} & \text{if } 1 < \Lambda_2 \le \ell + 1\,, \\ +\infty\,, & \text{if } \Lambda_2 \le 1\,, \end{cases}$$

$$d_{2,1} = \begin{cases} \dfrac{2\kappa - [\kappa - \Lambda_2]^+}{\kappa - 1 - [\kappa - \Lambda_2]^+} & \text{if } \min(\kappa, \Lambda_2) > 1\,, \\ +\infty\,, & \text{if } \min(\kappa, \Lambda_2) \le 1\,, \end{cases}$$

$$d_{1,0} = \begin{cases} \dfrac{2\kappa + \Lambda_1}{\Lambda_1 - 2} & \text{if } \min(\kappa, \Lambda_1) > 2\,, \\ +\infty\,, & \text{if } \min(\kappa, \Lambda_1) \le 2\,, \end{cases}$$

and $d_0 = \min(d_{2,0}, d_{2,1}, d_{1,0})$.

In order to prove that $d_0 = \widetilde{d}$, let $d'(\Lambda_2) = \min(d_{2,0}, d_{2,1})$; we have:

$$d'(\Lambda_2) = \begin{cases} +\infty\,, & \text{if } \Lambda_2 \le 1, \\ \dfrac{\Lambda_2 + \ell}{\Lambda_2 - 1} & \text{if } 1 < \Lambda_2 \le \ell + 1, \end{cases}$$

if $\kappa = 1$ and

$$d'(\Lambda_2) = \begin{cases} +\infty\,, & \text{if } \Lambda_2 \le 1, \\ \dfrac{\Lambda_2 + \kappa}{\Lambda_2 - 1}\,, & \text{if } 1 < \Lambda_2 \le \kappa, \\ \dfrac{2\kappa}{\kappa - 1}\,, & \text{if } \kappa \le \Lambda_2 \le \ell - 2\dfrac{\ell - \kappa}{\kappa + 1}, \\ \dfrac{\Lambda_2 + \ell}{\Lambda_2 - 1} & \text{if } \ell - 2\dfrac{\ell - \kappa}{\kappa + 1} \le \Lambda_2 \le \ell + 2, \\ \dfrac{2\kappa - \ell + \Lambda_2}{\kappa - \ell - 1 + \Lambda_2} & \text{if } \ell + 2 \le \Lambda_2 \le \ell + \kappa, \end{cases}$$

if $\kappa > 1$.

Since $d'(\Lambda_2)$ is decreasing in Λ_2, we have:

$$d'(\Lambda_2) = \min\Big\{ d'\big([\kappa - \lambda]^+\big)\,, d'(\kappa + \ell - \eta) \Big\}\,,$$

and one easily check that $d'\big([\kappa - \lambda]^+\big) = d_\lambda$ and $d'(\kappa + \ell - \eta) = d_\eta$.

Finally, note that, since $\Lambda_1 = \kappa - \mu$, we have $d_{1,0} = d_\mu$, and we get $d_0 = \widetilde{d}$.

Applying the Fourier transform with respect to the space variable to the equation, we obtain the following ordinary differential equation in t, depending on the parameter ξ:

$$L(t, \partial_t, i\xi)\, v = M_2(t, \partial_t, i\xi)\, v + M_1(t, \partial_t, i\xi)\, v + M_0(t)\, v\,,$$

where the symbol $\widehat{\ }$ denotes the Fourier transform with respect to x and $v = \widehat{u}$.

2.1. Estimate in the pseudodifferential zone

Let $t_1(\xi) = |\xi|^{-(1-1/d_0)}$ (we note $1/d_0 = 0$, if $d_0 = +\infty$), for $|t| \le t_1(\xi)$, $|\xi| \ge 1$, we define the energy:

$$E_1(t, \xi) = |v''|^2 + |\xi|^2\, |v'|^2 + |\xi|^4\, |v|^2\,.$$

We have:

$$\begin{aligned} E_1'(t, \xi) &= 2\,\mathrm{Re}(v''', v'') + 2\,|\xi|^2\,\mathrm{Re}(v'', v') + 2\,|\xi|^4\,\mathrm{Re}(v', v) \\ &\le 2|v'''|\,|v''| + 2\,|\xi|^2\; |v''|\,|v'| + 2\,|\xi|^4\; |v'|\,|v|\,. \end{aligned}$$

Now,

$$|v''| \le \sqrt{E_1(t,\xi)}\,, \qquad |\xi|\,|v'| \le \sqrt{E_1(t,\xi)}\,, \qquad |\xi|^2\; |v| \le \sqrt{E_1(t,\xi)}\,,$$

and

$$\begin{aligned} |v'''| &\le \sum_{k=0}^{2} \big|a_1 v'' + a_2 v' + a_3 v\big| + \sum_{j=0}^{2} \big|M_j(t, \partial_t, i\xi) v\big| \\ &\lesssim |\xi|\, \sqrt{E_1(t,\xi)} + \sqrt{E_1(t,\xi)} \lesssim |\xi| \sqrt{E_1(t,\xi)}\,. \end{aligned}$$

We get:

$$E_1'(t,\xi) \lesssim |\xi|\, E_1(t,\xi)\,,$$

hence, by Gronwall Lemma:

$$E_1(\theta_2, \xi) \le \exp\big(C\,|\xi|\;(\theta_2 - \theta_1)\big)\, E_1(\theta_1) \le \exp\big(C\,|\xi|^{1/d_0}\big)\, E_1(\theta_1)\,, \tag{2.5}$$

for any $-t_1(\xi) \le \theta_1 \le \theta_2 \le t_1(\xi)$.

2.2. Estimate in the hyperbolic zone

Let (see [3], [6], [5], [9])

$$\begin{aligned} [v]_0^2 &= |v|^2\,, \\ [v]_1^2 &= |\,\mathrm{L}_1\, v|^2 + |\,\mathrm{L}_2\, v|^2 + |\,\mathrm{L}_3\, v|^2\,, \\ [v]_2^2 &= |\,\mathrm{L}_{12}\, v|^2 + |\,\mathrm{L}_{23}\, v|^2 + |\,\mathrm{L}_{31}\, v|^2\,, \end{aligned}$$

where

$$\begin{aligned} &\mathrm{L}_j(t,\partial_t,i\xi)v = v' - i\tau_j v && j=1,2,3\\ &\mathrm{L}_{jk}(t,\partial_t,i\xi)v = v'' - i(\tau_j+\tau_k)v' - \tau_j\tau_k v && j,k\in\{1,2,3\}\,,\ j\neq k\,. \end{aligned}$$

For $t_1(\xi) \le |t| \le T$, $|\xi| \ge 1$, we define the energy:

$$E_2(t,\xi) = [v]_2^2 + \frac{t_1^2(\xi)\,|\xi|^2}{t^2}\,[v]_1^2 + \frac{t_1^4(\xi)\,|\xi|^4}{t^4}\,[v]_0^2\,.$$

The energy E_2 is equivalent to E_1 for $t_1(\xi) \le |t| \le T$. In fact, it's easy to see that:

$$E_2(t,\xi) \lesssim E_1(t,\xi)\,.$$

To prove the reversed inequality, we remark that:

$$\mathrm{L}_j\,v - \mathrm{L}_k\,v = -i(\tau_j-\tau_k)v\,,$$

for $j,k\in\{1,2,3\}$, $j\neq k$, and

$$\mathrm{L}_{jl}\,v - \mathrm{L}_{jk}\,v = -i(\tau_l-\tau_k)\,\mathrm{L}_j\,v\,,$$

for $j,k,l\in\{1,2,3\}$, $j\neq k$, $k\neq l$, $l\neq j$, hence

$$\begin{aligned} |\xi|^2\,|v| &\le \frac{|\,\mathrm{L}_1\,v| + |\,\mathrm{L}_2\,v|}{|\tau_1-\tau_2|}\,|\xi|^2 \le \frac{|\,\mathrm{L}_{12}\,v| + |\,\mathrm{L}_{13}\,v|}{|\tau_1-\tau_2|\,|\tau_2-\tau_3|}\,|\xi|^2 + \frac{|\,\mathrm{L}_{21}\,v| + |\,\mathrm{L}_{23}\,v|}{|\tau_1-\tau_2|\,|\tau_1-\tau_3|}\,|\xi|^2\\ &\lesssim \frac{|\,\mathrm{L}_{12}\,v| + |\,\mathrm{L}_{13}\,v| + |\,\mathrm{L}_{23}\,v|}{|t|^{\kappa+\ell}} \lesssim |\xi|^{(\kappa+\ell)(1-1/d_0)}\,\sqrt{E_2(t,\xi)}\,, \end{aligned}$$

since $\tau_2-\tau_3 \lesssim \tau_1-\tau_3$ and $t\ge t_1$. Similarly, since $v' = \mathrm{L}_j\,v + i\tau_j v$, we have:

$$\begin{aligned} |\xi|\,|v'| &\le |\xi|\,|\,\mathrm{L}_1\,v| + |\xi|\,|\tau_1 v| \lesssim \frac{|\,\mathrm{L}_{12}\,v| + |\,\mathrm{L}_{13}\,v|}{|\tau_2-\tau_3|}\,|\xi| + |\xi|^2\,|v|\\ &\lesssim |\xi|^{(\kappa+\ell)(1-1/d_0)}\,\sqrt{E_2(t,\xi)}\,. \end{aligned}$$

Finally, since:

$$v'' = \mathrm{L}_{jk}\,v + i\tau_j\,\mathrm{L}_k\,v + i\tau_k\,\mathrm{L}_j\,v + \tau_j\tau_k v\,,$$

we have:

$$\begin{aligned} |v''| &\le |\,\mathrm{L}_{jk}\,v| + |\tau_j|\,|\,\mathrm{L}_k\,v| + |\tau_k|\,|\,\mathrm{L}_j\,v| + |\tau_j|\,|\tau_k|\,|v|\\ &\lesssim |\xi|^{(\kappa+\ell)(1-1/d_0)}\,\sqrt{E_2(t,\xi)}\,. \end{aligned}$$

Now we derive the energy estimate for E_2. We have:

$$\begin{aligned} E_2'(t) &= \partial_t\,[v]_2^2 - 2\frac{t_1^2(\xi)\,|\xi|^2}{t^3}\,[v]_1^2 + \frac{t_1^2(\xi)\,|\xi|^2}{t^2}\,\partial_t\,[v]_1^2\\ &\quad - 4\frac{t_1^4(\xi)\,|\xi|^4}{t^5}\,[v]_0^2 + \frac{t_1^4(\xi)\,|\xi|^4}{t^4}\,\partial_t\,[v]_0^2\\ &\lesssim \partial_t\,[v]_2^2 + \frac{t_1^2(\xi)\,|\xi|^2}{t^2}\,\partial_t\,[v]_1^2 + \frac{t_1^4(\xi)\,|\xi|^4}{t^4}\,\partial_t\,[v]_0^2 + \frac{1}{|t|}\,E_2(t)\,. \end{aligned} \tag{2.6}$$

Since $v' = \mathrm{L}_1\, v + i\tau_1 v$, and τ_1 is real, we have:

$$\begin{aligned}\partial_t |v|^2 &= 2\,\mathrm{Re}(v', v) = 2\,\mathrm{Re}(\mathrm{L}_1\, v + i\tau_1 v, v)\\ &= 2\,\mathrm{Re}(\mathrm{L}_1\, v, v) + 2\,\mathrm{Re}(i\tau_1 v, v) \le 2|\,\mathrm{L}_1\, v|\,|v| \lesssim [v]_1\;[v]_0\;.\end{aligned} \tag{2.7}$$

We have

$$\begin{aligned}\partial_t\,\mathrm{L}_1\, v &= \partial_t\big(v' - i\tau_1 v\big) = v'' - i\tau_1 v' - i\tau_1' v\\ &= \mathrm{L}_{12}\, v + i\tau_2 v' + \tau_1\tau_2 v - i\tau_1' v = \mathrm{L}_{12}\, v + i\tau_2\,\mathrm{L}_1\, v - i\tau_1' v\,,\end{aligned}$$

hence, since τ_2 is real, we have:

$$\begin{aligned}\partial_t |\,\mathrm{L}_1\, v|^2 &= 2\,\mathrm{Re}(\partial_t\,\mathrm{L}_1\, v, \mathrm{L}_1\, v) = 2\,\mathrm{Re}(\mathrm{L}_{12}\, v + i\tau_2\,\mathrm{L}_1\, v - i\tau_1' v, \mathrm{L}_1\, v)\\ &= 2\,\mathrm{Re}(\mathrm{L}_{12}\, v, \mathrm{L}_1\, v) - 2\,\mathrm{Re}(i\tau_1' v, \mathrm{L}_1\, v) \lesssim [v]_2\,[v]_1 + 2|\tau_1' v|\;[v]_1\;.\end{aligned}$$

Now, since $v = i\dfrac{\mathrm{L}_1\, v - \mathrm{L}_2\, v}{\tau_1 - \tau_2}$, using (1.4), we have:

$$|\tau_1' v| = \Big|\frac{\tau_1'}{\tau_1 - \tau_2}\Big|\,|\,\mathrm{L}_1\, v - \mathrm{L}_2\, v| \lesssim \frac{1}{|t|}\,[v]_1\;,$$

which gives:

$$\partial_t |\,\mathrm{L}_1\, v|^2 \lesssim [v]_2\,[v]_1 + \frac{1}{|t|}\;[v]_1^2\;.$$

We get similar estimations for $\partial_t|\,\mathrm{L}_2\, v|^2$ and $\partial_t|\,\mathrm{L}_3\, v|^2$, so that:

$$\partial_t\,[v]_1 \lesssim [v]_2\,[v]_1 + \frac{1}{|t|}\;[v]_1^2\;. \tag{2.8}$$

Similarly, since

$$\begin{aligned}\partial_t\,\mathrm{L}_{12}\, v &= \partial_t\big(v'' - i(\tau_1+\tau_2)v' - \tau_1\tau_2 v\big)\\ &= v''' - i(\tau_1+\tau_2)v'' - \tau_1\tau_2 v' - i(\tau_1'+\tau_2')v' - (\tau_1'\tau_2 + \tau_1\tau_2')v\\ &= Lv + i\tau_3 v'' + \tau_3(\tau_1+\tau_2)v' - i\tau_1\tau_2\tau_3 v - i(\tau_1'+\tau_2')v' - (\tau_1'\tau_2 + \tau_1\tau_2')v\\ &= Lv + i\tau_3\,\mathrm{L}_{12}\, v - i\tau_1'\,\mathrm{L}_2\, v - i\tau_2'\,\mathrm{L}_1\, v\,,\end{aligned}$$

we have:

$$\begin{aligned}&2\,\mathrm{Re}(\partial_t\,\mathrm{L}_{12}\, v, \mathrm{L}_{12}\, v)\\ &\qquad= 2\,\mathrm{Re}(Lv + i\tau_3\,\mathrm{L}_{12}\, v - i\tau_1'\,\mathrm{L}_2\, v - i\tau_2'\,\mathrm{L}_1\, v, \mathrm{L}_{12}\, v)\\ &\qquad= 2\,\mathrm{Re}(Lv, \mathrm{L}_{12}\, v) - 2\,\mathrm{Re}(i\tau_1'\,\mathrm{L}_2\, v, \mathrm{L}_{12}\, v) - 2\,\mathrm{Re}(i\tau_2'\,\mathrm{L}_1\, v, \mathrm{L}_{12}\, v)\\ &\qquad\le 2\,|Lv|\;|\,\mathrm{L}_{12}\, v| + 2|\tau_1'|\,|\,\mathrm{L}_2\, v|\,|\,\mathrm{L}_{12}\, v| + 2|\tau_2'|\,|\,\mathrm{L}_1\, v|\,|\,\mathrm{L}_{12}\, v|\,.\end{aligned}$$

Now: $\mathrm{L}_2\, v = i\dfrac{\mathrm{L}_{12}\, v - \mathrm{L}_{23}\, v}{\tau_1 - \tau_3}$, so that:

$$|\tau_1'\,\mathrm{L}_2\, v| \le \Big|\frac{\tau_1'}{\tau_1 - \tau_3}\Big|\,|\,\mathrm{L}_{12}\, v - \mathrm{L}_{23}\, v| \lesssim \frac{1}{|t|}\;[v]_2\;.$$

Similarly we have $|\tau_1' \operatorname{L}_2 v| \lesssim \frac{1}{|t|}\ [v]_2$, and we get:

$$\partial_t |\operatorname{L}_{12} v|^2 \lesssim |Lv|\, [v]_2 + \frac{1}{|t|}\ [v]_2^2 \,.$$

We have similar estimations for $\partial_t |\operatorname{L}_{23} v|^2$ and $\partial_t |\operatorname{L}_{31} v|^2$, so that:

$$\partial_t\, [v]_2 \lesssim |Lv|\, [v]_2 + \frac{1}{|t|}\ [v]_2^2 \,. \tag{2.9}$$

Inserting (2.7), (2.8) and (2.9) in (2.6):

$$\begin{aligned} E_2'(t) &\le |Lv|\, [v]_2 + \frac{1}{|t|}\ [v]_2^2 + \frac{t_1^2(\xi)\, |\xi|^2}{t^2}\ [v]_2\, [v]_1 + \frac{t_1^2(\xi)\, |\xi|^2}{t^2}\ \frac{1}{|t|}\ [v]_1^2 \\ &\quad + \frac{t_1^4(\xi)\, |\xi|^4}{t^4}\ [v]_1\, [v]_0 + \frac{1}{|t|} E_2(t) \\ &\lesssim |Lv|\, [v]_2 + \big(1 + t_1(\xi)\, |\xi|\big) \frac{1}{|t|}\ E_2(t) \\ &\lesssim \big(|M_2 v| + |M_1 v| + |M_0 v|\big)\, [v]_2 + |\xi|^{1/d_0}\ \frac{1}{|t|}\ E_2(t) \,. \end{aligned}$$

We claim that

$$|M_j v| \lesssim |\xi|^{1/d_0}\ \frac{1}{|t|}\ E_2(t) \,, \quad \text{for } t \ge t_1(\xi) \tag{2.10}$$

for $j = 1, 2$. Indeed, assuming (2.10) we have:

$$E_2'(t) \lesssim |\xi|^{1/d_0}\ \frac{1}{|t|}\ E_2(t) \,,$$

and by Gronwall Lemma:

$$E_2(\theta_2, \xi) \le \exp\Big[C\, |\xi|^{1/d_0}\ \log \frac{\theta_2}{\theta_1}\Big]\ E_2(\theta_1) \lesssim |\xi|^{C_1} \exp\big(C_2\, |\xi|^{1/d_0}\big)\ E_2(\theta_1) \,, \tag{2.11}$$

for any θ_1, θ_2 such that $-T \le \theta_1 \le \theta_2 \le -t_1(\xi)$ or $t_1(\xi) \le \theta_1 \le \theta_2 \le T$, for some constant C_1 and C_2. Combining (2.5) and (2.11), we can conclude the proof of Theorem 1.1 by standard arguments.

Now we prove (2.10) for $j = 2$. A simple calculation shows that:

$$\begin{aligned} &b_0 v'' + i b_1 \xi v' - b_2 \xi^2 v \\ &\quad = b_0 \big(\operatorname{L}_{23} v + i(\tau_2 + \tau_3) v' + \tau_2 \tau_3 v\big) + i b_1 \xi v' - b_2 \xi^2 v \\ &\quad = b_0 \operatorname{L}_{23} v + i\big(b_0(\tau_2 + \tau_3) + b_1\big) v' + (b_0 \tau_2 \tau_3 - b_2) v \\ &\quad = b_0 \operatorname{L}_{23} v + i\big(b_0(\tau_2 + \tau_3) + b_1\big)(\operatorname{L}_3 v + i \tau_3 v) + (b_0 \tau_2 \tau_3 - b_2) v \\ &\quad = b_0 \operatorname{L}_{23} v + i\big(b_0(\tau_2 + \tau_3) + b_1\big) \operatorname{L}_3 v \\ &\qquad - \big(b_0(\tau_2 + \tau_3) + b_1\big) \tau_3 v + (b_0 \tau_2 \tau_3 - b_2) v \\ &\quad = b_0 \operatorname{L}_{23} v + i\big(b_0(\tau_2 + \tau_3) + b_1\big) \operatorname{L}_3 v - \big(b_0 \tau_3^2 + b_1 \tau_3 + b_2\big) v \,, \end{aligned} \tag{2.12}$$

then, noting that

$$b_0(\tau_2+\tau_3)+b_1\xi=\frac{M_2(\tau_3)-M_2(\tau_2)}{\tau_3-\tau_2}\,,$$

where $M_2(\tau_j)=M_2\big(t,\tau_j(t,\xi),\xi\big)$, we get:
(2.13)

$$\big|b_0v''+ib_1v'-b_2v\big|\lesssim|b_0|\;|\,\mathrm{L}_{23}\,v|+\Big|\frac{M_2(\tau_3)-M_2(\tau_2)}{\tau_3-\tau_2}\Big|\,|\mathrm{L}_3\,v|+\big|M_2(\tau_3)\big|\,|v|\,.$$

Estimation of $\big|M_2(\tau_3)\big|\,|v|$.

We have three different estimates of $\big|M_2(\tau_3)\big|\,|v|$.

First estimate. We have:

$$\big|M_2(\tau_3)\big|=|\tau_3-\tau_2|\,|\tau_3-\tau_1|\;\frac{\big|M_2(\tau_3)\big|}{|\tau_3-\tau_2|\,|\tau_3-\tau_1|}\lesssim|t|^{\kappa+\ell-\Lambda_2}\,|\xi|^2\,,$$

and by the definition of E_2:

$$\big|M_2(\tau_3)\big|\,|v|\lesssim\frac{|t|^{\kappa+\ell-\Lambda_2+3}}{t_1^2\;|\xi|}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\,. \tag{2.14}$$

Second estimate. Since: $|v|\lesssim\dfrac{|\,\mathrm{L}_1\,v-\mathrm{L}_3\,v|}{|\tau_1-\tau_3|}$, we have:

$$\begin{aligned}\big|M_2(\tau_3)\big|\,|v|&\lesssim\big|M_2(\tau_3)\big|\,\frac{[v]_1}{|\tau_1-\tau_3|}\lesssim|\tau_3-\tau_2|\;\frac{\big|M_2(\tau_3)\big|}{|\tau_1-\tau_3|\,|\tau_2-\tau_3|}\;[v]_1\\&\lesssim\frac{|t|^{\ell-\Lambda_2+2}}{t_1}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\,.\end{aligned} \tag{2.15}$$

Third estimate. We have:

$$\begin{aligned}|v|&\lesssim\frac{|\,\mathrm{L}_2\,v-\mathrm{L}_3\,v|}{|\tau_2-\tau_3|}\lesssim\frac{|\,\mathrm{L}_{12}\,v-\mathrm{L}_{23}\,v|}{|\tau_2-\tau_3|\,|\tau_1-\tau_3|}+\frac{|\,\mathrm{L}_{13}\,v-\mathrm{L}_{23}\,v|}{|\tau_2-\tau_3|\,|\tau_1-\tau_2|}\\&\lesssim\frac{[v]_2}{|\tau_3-\tau_1|\,|\tau_3-\tau_2|}\,,\end{aligned}$$

since $|\tau_1-\tau_3|\approx|\tau_1-\tau_2|$, hence:

$$\big|M_2(\tau_3)\big|\,|v|\lesssim\frac{\big|M_2(\tau_3)\big|}{|\tau_3-\tau_1|\,|\tau_3-\tau_2|}\;[v]_2\lesssim|t|^{1-\Lambda_2}\;\frac{1}{|t|}\,\sqrt{E_2(t)}\,. \tag{2.16}$$

Now we combine the three estimates to estimate $\big|M_2(\tau_3)\big|\,|v|$. If $\Lambda_2\le1$, (2.16) for $t_1(\xi)\le|t|\le T$ gives:

$$\big|M_2(\tau_3)\big|\,|v|\lesssim\frac{1}{|t|}\,\sqrt{E_2(t)}\,.$$

If $1 < \Lambda_2 \le \ell + 2$, we consider (2.15) for $t_1(\xi) \le |t| \le t_2(\xi)$ and (2.16) for $t_2(\xi) \le |t| \le T$, with $t_2(\xi) = |\xi|^{-(1-1/d_0)/(\ell+1)}$; we have:

$$\begin{aligned}\big|M_2(\tau_3)\big|\,|v| &\lesssim \frac{|t|^{\ell-\Lambda_2+2}}{t_1}\,\frac{1}{|t|}\,\sqrt{E_2(t)} \lesssim \frac{t_2^{\ell-\Lambda_2+2}}{t_1}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\\ &\lesssim |\xi|^{(1-1/d_0)[1-(\ell-\Lambda_2+2)/(\ell+1)]}\,\frac{1}{|t|}\,\sqrt{E_2(t)} \lesssim |\xi|^{1/d_0}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\end{aligned}$$

for $t_1(\xi) \le |t| \le t_2(\xi)$, and:

$$\begin{aligned}\big|M_2(\tau_3)\big|\,|v| &\lesssim |t|^{1-\Lambda_2}\,\frac{1}{|t|}\,\sqrt{E_2(t)} \lesssim t_2^{1-\Lambda_2}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\\ &\lesssim |\xi|^{(1-1/d_0)(\Lambda_2-1)/(\ell+1)}\,\frac{1}{|t|}\,\sqrt{E_2(t)} \lesssim |\xi|^{1/d_0}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\end{aligned}$$

for $t_2(\xi) \le |t| \le T$, if $d_0 \le \dfrac{\Lambda_2+\ell}{\Lambda_2-1}$.

Finally, if $\ell+2 \le \Lambda_2 \le \kappa+\ell$, we consider (2.14) for $t_1(\xi) \le |t| \le t_2(\xi)$ and (2.15) for $t_2(\xi) \le |t| \le T$, with $t_2(\xi) = |\xi|^{-(1-1/d_0)/(\kappa+1)}$:

$$\begin{aligned}\big|M_2(\tau_3)\big|\,|v| &\lesssim \frac{|t|^{\kappa+\ell-\Lambda_2+3}}{t_1^2}\,\frac{1}{|t|}\,\sqrt{E_2(t)} \le \frac{t_2^{\kappa+\ell-\Lambda_2+3}}{t_1^2}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\\ &\le |\xi|^{(1-1/d_0)[2-(\kappa+\ell-\Lambda_2+3)/(\kappa+1)]}\,\frac{1}{|t|}\,\sqrt{E_2(t)} \le |\xi|^{1/d_0}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\end{aligned}$$

for $t_1(\xi) \le |t| \le t_2(\xi)$, and

$$\begin{aligned}\big|M_2(\tau_3)\big|\,|v| &\lesssim \frac{|t|^{\ell-\Lambda_2+2}}{t_1}\,\frac{1}{|t|}\,\sqrt{E_2(t)} \le \frac{t_2^{\ell-\Lambda_2+2}}{t_1}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\\ &\le |\xi|^{(1-1/d_0)[1-(\ell-\Lambda_2+2)/(\kappa+1)]}\,\frac{1}{|t|}\,\sqrt{E_2(t)} \le |\xi|^{1/d_0}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\,,\end{aligned}$$

for $t_2(\xi) \le |t| \le T$, if $d_0 \le \dfrac{2\kappa-\ell+\Lambda_2}{\kappa-\ell-1+\Lambda_2}$.

Estimation of $\Big|\dfrac{M_2(\tau_3)-M_2(\tau_2)}{\tau_3-\tau_2}\Big|\,|\mathrm{L}_3\,v|$**.** We have two different estimates of $\Big|\dfrac{M_2(\tau_3)-M_2(\tau_2)}{\tau_3-\tau_2}\Big|$.

First estimate. We have:

$$\begin{aligned}\Big|\frac{M_2(\tau_3)}{\tau_3-\tau_2}\Big| &= |\tau_3-\tau_1|\,\frac{\big|M_2(\tau_3)\big|}{|\tau_3-\tau_2|\,|\tau_3-\tau_1|} \lesssim |t|^{\kappa-\Lambda_2}\,|\xi|\\ \Big|\frac{M_2(\tau_2)}{\tau_3-\tau_2}\Big| &= |\tau_2-\tau_1|\,\frac{\big|M_2(\tau_2)\big|}{|\tau_2-\tau_3|\,|\tau_2-\tau_1|} \lesssim |t|^{\kappa-\Lambda_2}\,|\xi|\,,\end{aligned}$$

which gives:

$$\Big|\frac{M_2(\tau_2)-M_2(\tau_3)}{\tau_3-\tau_2}\Big| \lesssim |t|^{[\kappa-\Lambda_2]^+}\,|\xi|\ ,$$

and using the definition of E_2:

$$\begin{aligned}\Big|\frac{M_2(\tau_3)-M_2(\tau_2)}{\tau_3-\tau_2}\Big|\,|\mathrm{L}_3\,v| &\lesssim \Big|\frac{M_2(\tau_3)-M_2(\tau_2)}{\tau_3-\tau_2}\Big|\,\frac{|t|}{t_1\,|\xi|}\,\sqrt{E_2(t)}\\ &\lesssim \frac{|t|^{[\kappa-\Lambda_2]^++2}}{t_1}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\ .\end{aligned} \tag{2.17}$$

Second estimate. Since:

$$|\,\mathrm{L}_3\,v| \lesssim \frac{|\,\mathrm{L}_{13}\,v-\mathrm{L}_{23}\,v|}{|\tau_1-\tau_2|} \lesssim \frac{[v]_2}{|\tau_1-\tau_2|}\ ,$$

we have:

$$\begin{aligned}\Big|\frac{M_2(\tau_3)-M_2(\tau_2)}{\tau_3-\tau_2}\Big|\,|\mathrm{L}_3\,v| &\lesssim \Big|\frac{M_2(\tau_3)-M_2(\tau_2)}{\tau_3-\tau_2}\Big|\,\frac{[v]_2}{|\tau_1-\tau_2|}\\ &\lesssim |t|^{[\kappa-\Lambda_2]^+-\kappa+1}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\ .\end{aligned} \tag{2.18}$$

Now we combine the three estimates to estimate $\Big|\dfrac{M_2(\tau_3)-M_2(\tau_2)}{\tau_3-\tau_2}\Big|$.
If $\Lambda_2\le 1$, (2.18) gives

$$\Big|\frac{M_2(\tau_3)-M_2(\tau_2)}{\tau_3-\tau_2}\Big|\,|\mathrm{L}_3\,v| \lesssim \frac{1}{|t|}\,\sqrt{E_2(t)}\ .$$

If $\Lambda_2>1$, we consider (2.17) for $t_1(\xi)\le|t|\le t_2(\xi)$ and (2.18) for $t_2(\xi)\le|t|\le T$, with $t_2=|\xi|^{-(1-1/d_0)/(\kappa+1)}$:

$$\begin{aligned}\Big|\frac{M_2(\tau_3)-M_2(\tau_2)}{\tau_3-\tau_2}\Big|\,|\mathrm{L}_3\,v| &\lesssim \frac{|t|^{[\kappa-\Lambda_2]^++2}}{t_1}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\\ &\le \frac{t_2^{[\kappa-\Lambda_2]^++2}}{t_1}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\\ &\le |\xi|^{(1-1/d_0)[1-([\kappa-\Lambda_2]^++2)/(\kappa+1)]}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\\ &\le |\xi|^{1/d_0}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\end{aligned}$$

for $t_1(\xi)\le|t|\le t_2(\xi)$, and

$$\Big|\frac{M_2(\tau_3)-M_2(\tau_2)}{\tau_3-\tau_2}\Big|\,|\mathrm{L}_3\,v| \lesssim |t|^{[\kappa-\Lambda_2]^+-\kappa+1}\,\frac{1}{|t|}\,\sqrt{E_2(t)}$$

$$\begin{aligned}
&\le t_2^{[\kappa-\Lambda_2]^+-\kappa+1}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\\
&\le |\xi|^{-(1-1/d_0)([\kappa-\Lambda_2]^+-\kappa+1)/(\kappa+1)}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\\
&\le |\xi|^{1/d_0}\,\frac{1}{|t|}\,\sqrt{E_2(t)}
\end{aligned}$$

for $t_2(\xi) \le |t| \le T$, if $d_0 \le \dfrac{2\kappa - [\kappa-\Lambda_2]^+}{\kappa - 1 - [\kappa-\Lambda_2]^+}$.

Now we prove (2.10) for $j = 1$. We have

$$\begin{aligned}
M_1(t,\partial_t,i\xi)v &= c_0 v' + ic_1\xi v = c_0(\mathrm{L}_2\,v + i\tau_2 v) + ic_1\xi v\\
&= c_0\,\mathrm{L}_2\,v + (ic_0\tau_2 + ic_1\xi)v\\
&= c_0\,\mathrm{L}_2\,v + M_1\big(t,\tau_2(t,i\xi),i\xi\big)\,v\,,
\end{aligned} \tag{2.19}$$

hence:

$$\big|M_1(t,\partial_t,i\xi)v\big| \le |c_0|\,|\mathrm{L}_3\,v| + \big|M_1(\tau_2)\big|\,|v|\,,$$

where $M_1(\tau_2) = M_1\big(t,\tau_2(t,i\xi),i\xi\big)$.

We have two different estimates of $\big|M_1(\tau_2)\big|\,|v|$.

First estimate. Using (2.4), we have:

$$\big|M_1(\tau_2)\big| = |\tau_1-\tau_2|\,\frac{\big|M_1(\tau_2)\big|}{|\tau_1-\tau_2|} \lesssim |t|^{\kappa-\Lambda_1}\,|\xi|\;,$$

so that:

$$\big|M_1(\tau_2)\big|\,|v| \lesssim |t|^{\kappa-\Lambda_1}\,|\xi|\;|v| \lesssim \frac{|t|^{\kappa-\Lambda_1+3}}{t_1^2\,|\xi|}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\,. \tag{2.20}$$

Second estimate. Since $|v| \lesssim \dfrac{|\,\mathrm{L}_1\,v - \mathrm{L}_2\,v|}{|\tau_1-\tau_2|}$, we have:

$$\big|M_1(\tau_2)\big|\,|v| \lesssim |t|^{-\Lambda_1}\;[v]_1 \lesssim \frac{|t|^{2-\Lambda_1}}{t_1\,|\xi|}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\,. \tag{2.21}$$

We combine (2.20) and (2.21) to prove (2.10) for $j = 1$.

If $\Lambda_1 \le 2$, (2.21) gives

$$\big|M_1(\tau_2)\big|\,|v| \lesssim \frac{1}{|t|}\,\sqrt{E_2(t)}\,.$$

If $\Lambda_1 > 2$, we consider (2.20) for $t_1(\xi) \le |t| \le t_2(\xi)$ and (2.21) for $t_2(\xi) \le |t| \le T$, with $t_2 = |\xi|^{-(1-1/d_0)/(\kappa+1)}$, we have:

$$\big|M_1(\tau_2)\big|\,|v| \lesssim \frac{|t|^{\kappa-\Lambda_1+3}}{t_1^2\,|\xi|}\,\frac{1}{|t|}\,\sqrt{E_2(t)} \le \frac{t_2^{\kappa-\Lambda_1+3}}{t_1^2\,|\xi|}\,\frac{1}{|t|}\,\sqrt{E_2(t)}$$

$$\le |\xi|^{(1-1/d_0)[2-(\kappa-\Lambda_1+3)/(\kappa+1)]-1}\,\frac{1}{|t|}\,\sqrt{E_2(t)} \le |\xi|^{1/d_0}\,\frac{1}{|t|}\,\sqrt{E_2(t)}$$

for $t_1(\xi) \le |t| \le t_2(\xi)$, and

$$\begin{aligned}\big|M_1(\tau_2)\big|\,|v| &\lesssim \frac{|t|^{2-\Lambda_2}}{t_1\,|\xi|}\,\frac{1}{|t|}\,\sqrt{E_2(t)} \le \frac{t_2^{2-\Lambda_2}}{t_1\,|\xi|}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\\ &\le |\xi|^{(1-1/d_0)[1-(2-\Lambda_2)/(\kappa+1)]-1}\,\frac{1}{|t|}\,\sqrt{E_2(t)} \le |\xi|^{1/d_0}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\end{aligned}$$

for $t_2(\xi) \le |t| \le T$, if $d_0 \le \dfrac{2\kappa+\Lambda_1}{\Lambda_1-2}$.

This completes the proof of Theorem 1.1, for third order operators.

3. Some preliminary lemmas

Let

$$\begin{aligned}
[v]_0^2 &= |v|^2\,,\\
[v]_1^2 &= |\,\mathrm{L}_1\,v|^2 + |\,\mathrm{L}_2\,v|^2 + |\,\mathrm{L}_3\,v|^2\,,\\
[v]_2^2 &= |\,\mathrm{L}_{12}\,v|^2 + |\,\mathrm{L}_{23}\,v|^2 + |\,\mathrm{L}_{31}\,v|^2\\
&\ \vdots\\
[v]_j^2 &= \sum_{1\le k_1<k_2<\cdots<k_j\le m} \big|\mathrm{L}_{k_1,\dots,k_j}(t,\partial_t,i\xi)v\big|^2 \qquad (1\le j\le m)
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{L}_k\,v &= v' - i\tau_k v & & k=1,2,\dots,m,\\
\mathrm{L}_{jk}\,v &= v'' - i(\tau_j+\tau_k)v' - \tau_j\tau_k v\,, & & j,k\in\{1,2,\dots,m\}\,,\ j\ne k\,,\\
&\ \vdots
\end{aligned}$$

and, in general, $\mathrm{L}_{k_1,\dots,k_j}(t,\partial_t,i\xi)$ is the operator with symbol:

$$\mathrm{L}_{k_1,\dots,k_j}(t,\tau,\xi) = \prod_{l=1,\dots,j}\big(\tau - i\tau_{k_l}(t,\xi)\big)\,.$$

Let $\sigma_j(\tau_1,\dots,\tau_m)$ be the elementary symmetric function of order j:

$$\sigma_j(\tau_1,\dots,\tau_m) = \sum_{1\le k_1<k_2\cdots<k_j\le m} \tau_{k_1}\tau_{k_2}\cdots\tau_{k_j}\,,\quad j\ge 1$$

and $\sigma_0(\tau_1,\dots,\tau_m)=1$; we note also:

$$\sigma_j(\tau_1,\dots,\widehat{\tau}_\ell,\dots,\tau_m) = \sum_{\substack{1\le k_1<k_2\cdots<k_j\le m\\ \ell\notin\{k_1,k_2,\dots,k_j\}}} \tau_{k_1}\tau_{k_2}\cdots\tau_{k_j}\,.$$

The following identities are clear:

(3.1)
$$\sigma_j(\tau_1,\dots,\tau_m)=\begin{cases}\sigma_j(\tau_1,\dots,\widehat{\tau_\ell},\dots,\tau_m)+\tau_\ell\sigma_{j-1}(\tau_1,\dots,\widehat{\tau_\ell},\dots,\tau_m)\,, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } 1\le j\le m-1\,,\\ \tau_\ell\sigma_{m-1}(\tau_1,\dots,\widehat{\tau_\ell},\dots,\tau_m)\,, \qquad\qquad \text{if } j=m\,,\end{cases}$$

(3.2)
$$\partial_t\sigma_j(\tau_1,\dots,\tau_m)=\sum_{\ell=1}^{m}\tau'_\ell\sigma_{j-1}(\tau_1,\dots,\widehat{\tau_\ell},\dots,\tau_m)\,,\quad \text{if } 1\le j\le m\,.$$

Given $k_1,k_2,\dots,k_j\in\{1,2,\dots,m\}$, with $j\le m$ and $h\le j$, we define in a obvious way $\sigma_h(\tau_{k_1},\dots,\tau_{k_j})$ and $\sigma_h(\tau_{k_1},\dots,\widehat{\tau_{k_l}},\dots,\tau_{k_j})$. We have identities similar to (3.1) and (3.2).

Remark that

(3.3)
$$\mathrm{L}_{k_1,\dots,k_j}(t,\partial_t,i\xi)=\sum_{h=0}^{j}(-i)^h\sigma_h(\tau_{k_1},\dots,\tau_{k_j})\partial_t^{j-h}\,.$$

We will consider also the operators $\mathrm{L}_{k_1,\dots,\widehat{k}_h,\dots,k_j}$ whose symbol is given by:

$$\mathrm{L}_{k_1,\dots,\widehat{k}_h,\dots,k_j}(t,\tau,\xi)=\prod_{\substack{l=1,\dots,j\\ l\ne h}}\big(\tau-i\tau_{k_l}(t,\xi)\big)\,,$$

hence:

$$\mathrm{L}_{k_1,\dots,\widehat{k}_h,\dots,k_j}(t,\partial_t,i\xi)v=\sum_{h=0}^{j-1}(-i)^h\sigma_h(\tau_{k_1},\dots,\widehat{\tau_{k_l}}\dots,\tau_{k_j})\partial_t^{j-1-h}v\,.$$

We have:

$$\mathrm{L}_1\,v-\mathrm{L}_2\,v=-i(\tau_1-\tau_2)v\,,$$

and consequently:

(3.4)
$$|v|\le\frac{|\,\mathrm{L}_1\,v-\mathrm{L}_2\,v|}{|\tau_1-\tau_2|}\le\frac{|\,\mathrm{L}_1\,v|+|\,\mathrm{L}_2\,v|}{|\tau_1-\tau_2|}\lesssim\frac{[v]_1}{|\tau_1-\tau_2|}\,.$$

Similarly, we have:

$$\mathrm{L}_{12}\,v-\mathrm{L}_{13}\,v=-i(\tau_2-\tau_3)\,\mathrm{L}_1\,v\,,$$

and consequently:

(3.5)
$$|\,\mathrm{L}_1\,v|\le\frac{|\,\mathrm{L}_{12}\,v-\mathrm{L}_{13}\,v|}{|\tau_2-\tau_3|}\le\frac{|\,\mathrm{L}_{12}\,v|+|\,\mathrm{L}_{13}\,v|}{|\tau_2-\tau_3|}\lesssim\frac{[v]_2}{|\tau_2-\tau_3|}\,.$$

We have an analogous estimation for $\mathrm{L}_2\,v$:

(3.6)
$$|\,\mathrm{L}_2\,v|\le\frac{|\,\mathrm{L}_{12}\,v-\mathrm{L}_{23}\,v|}{|\tau_1-\tau_3|}\le\frac{|\,\mathrm{L}_{12}\,v|+|\,\mathrm{L}_{23}\,v|}{|\tau_1-\tau_3|}\lesssim\frac{[v]_2}{|\tau_1-\tau_3|}\,.$$

Combining (3.4), (3.5) and (3.6), we get:

$$|v| \leq \frac{[v]_2}{|\tau_1 - \tau_2|\,|\tau_2 - \tau_3|} + \frac{[v]_2}{|\tau_1 - \tau_2|\,|\tau_1 - \tau_3|} \lesssim \frac{[v]_2}{|\tau_1 - \tau_2|\,|\tau_2 - \tau_3|}\,, \tag{3.7}$$

since $|\tau_2 - \tau_3| \lesssim |\tau_1 - \tau_3|$.

More generally, we have:

Lemma 3.1. *Let $l_1, l_2, \ldots, l_p \in \{1, 2, \ldots, m\}$, $p \leq m-2$, with $l_i \neq l_j$ if $i \neq j$. Let $k_1, k_2, \ldots, k_q \in \{1, 2, \ldots, m\} \setminus \{l_1, l_2, \ldots, l_p\}$ with $k_1 < k_2 < \cdots < k_q$, $q \geq 2$ and $p + q \leq m$.*

Then we have:

$$|\,\mathrm{L}_{l_1,\ldots,l_p}\, v| \lesssim \frac{[v]_{p+q-1}}{|\tau_{k_1} - \tau_{k_2}|\,|\tau_{k_2} - \tau_{k_3}| \cdots |\tau_{k_{q-1}} - \tau_{k_q}|}\,. \tag{3.8}$$

Proof. In order to simplify the notations, in this proof we will omit the $l_1, \ldots, l_p$ from the indices noting $\widetilde{\mathrm{L}}$ the operator $\mathrm{L}_{l_1,\ldots,l_p}$ and noting $\widetilde{\mathrm{L}}_{k_1,\ldots,k_q}$ instead of $\mathrm{L}_{k_1,\ldots,k_q,l_1,\ldots,l_p}$.

To prove (3.8), it will be enough to prove:

$$|\widetilde{\mathrm{L}}v| \lesssim \frac{\sum_{l=1}^{q} \left|\widetilde{\mathrm{L}}_{k_1,\ldots,\widehat{k_l},\ldots,k_q} v\right|}{|\tau_{k_1} - \tau_{k_2}|\,|\tau_{k_2} - \tau_{k_3}| \cdots |\tau_{k_{q-1}} - \tau_{k_q}|}\,, \tag{3.9}$$

where $\widetilde{\mathrm{L}}_{k_1,\ldots,\widehat{k_l},\ldots,k_q}$ is the operator with symbol

$$\widetilde{\mathrm{L}}_{k_1,\ldots,\widehat{k_l},\ldots,k_q}(t,\tau,\xi) = \prod_{h=1,\ldots,p} \big(\tau - i\tau_{l_h}(t,\xi)\big) \prod_{\substack{h=1,\ldots,q \\ h \neq l}} \big(\tau - i\tau_{k_h}(t,\xi)\big)\,,$$

that is

$$\widetilde{\mathrm{L}}_{1,\ldots,\widehat{l},\ldots,q}(t,\partial_t,i\xi) = \sum_{h=0}^{p+q-1} (-i)^h \sigma_h(\tau_{l_1},\ldots,\tau_{l_p},\tau_{k_1},\ldots,\widehat{\tau_{k_l}}\ldots,\tau_{k_q})\partial_t^{p+q-1-h}\,.$$

To prove (3.9), we proceed by induction on q. We consider the case $q = 2$.

Let $k', k'' \in \{1, 2, \ldots, m\} \setminus \{l_1, l_2, \ldots, l_p\}$ with $k' \neq k''$, using (3.1) and (3.3), we have:

$$\begin{aligned}
\widetilde{\mathrm{L}}_{k'}v - \widetilde{\mathrm{L}}_{k''}v &= \sum_{h=0}^{j+1} (-i)^h \sigma_h(\tau_{l_1},\ldots,\tau_{l_p},\tau_{k'})\partial_t^{j+1-h} v \\
&\qquad - \sum_{h=0}^{j+1} (-i)^h \sigma_h(\tau_{l_1},\ldots,\tau_{l_p},\tau_{k''})\partial_t^{j+1-h} v \\
&= \sum_{h=1}^{j+1} (-i)^h (\tau_{k'} - \tau_{k''})\sigma_{h-1}(\tau_{l_1},\ldots,\tau_{l_p})\partial_t^{j+1-h} v
\end{aligned}$$

$$= -i(\tau_{k'} - \tau_{k''}) \sum_{h=0}^{j} (-i)^h \sigma_h(\tau_{l_1}, \dots, \tau_{l_p}) \partial_t^{j-h} v$$
$$= -i(\tau_{k'} - \tau_{k''}) \widetilde{\mathrm{L}} v\,,$$

which gives:

$$|\widetilde{\mathrm{L}} v| \le \frac{|\widetilde{\mathrm{L}}_{k'} v| + |\widetilde{\mathrm{L}}_{k''} v|}{|\tau_{k'} - \tau_{k''}|}\,. \tag{3.10}$$

Assuming (3.9) for $q \ge 2$, let us prove (3.9) for $q+1$. Applying (3.10) to $\widetilde{\mathrm{L}}_{k_1,\dots,\widehat{k_l},\dots,k_q}$ instead of $\widetilde{\mathrm{L}}$, and $k' = k_l$, $k'' = k_{q+1}$, we have:

$$|\widetilde{\mathrm{L}}_{k_1,\dots,\widehat{k_l},\dots,k_q} v| \le \frac{|\widetilde{\mathrm{L}}_{k_1,\dots,k_q} v| + |\widetilde{\mathrm{L}}_{k_1,\dots,\widehat{k_l},\dots,k_q,k_{q+1}} v|}{|\tau_{k_l} - \tau_{k_{q+1}}|}$$
$$\le \frac{|\tau_{k_q} - \tau_{k_{q+1}}|}{|\tau_{k_l} - \tau_{k_{q+1}}|} \frac{|\widetilde{\mathrm{L}}_{k_1,\dots,k_q} v| + |\widetilde{\mathrm{L}}_{k_1,\dots,\widehat{k_l},\dots,k_{q+1}} v|}{|\tau_{k_q} - \tau_{k_{q+1}}|}\,,$$

which gives (3.9) for $q+1$, since $|\tau_{k_q} - \tau_{k_{q+1}}| \lesssim |\tau_{k_l} - \tau_{k_{q+1}}|$. □

Let $\mathcal{L}_0 = 1$, $\mathcal{L}_1 = \mathrm{L}_m$, $\mathcal{L}_2 = \mathrm{L}_{m-1,m}$, $\dots$, $\mathcal{L}_j = \mathrm{L}_{m-j+1,m-j+2,\dots,m}$, $\dots$, $\mathcal{L}_m = \mathrm{L}_{m,m-1,\dots,2,1}$. Using (3.8), we have:

$$|\mathcal{L}_j v| \lesssim \frac{[v]_{j+k}}{|\tau_1 - \tau_2||\tau_2 - \tau_3| \cdots |\tau_k - \tau_{k+1}|} \lesssim \frac{[v]_{j+k}}{|t|^{s_k} |\xi|^k}\,, \tag{3.11}$$

for any j, k such that $j + k \le m - 1$.

Let $M(X)$ be a polynomial of degree j in one variable, we define:

$$\Delta_0[M](X_0) = M(X_0)$$
$$\Delta_1[M](X_0,X_1) = \frac{\Delta_0[M](X_0) - \Delta_0[M](X_1)}{X_0 - X_1} = \frac{M(X_0) - M(X_1)}{X_0 - X_1}$$
$$\vdots$$
$$\Delta_k[M](X_0, \dots, X_{k-2}, X_{k-1}, X_k)$$
$$= \frac{\Delta_{k-1}[M](X_0, \dots, X_{k-2}, X_{k-1}) - \Delta_{k-1}[M](X_0, \dots, X_{k-2}, X_k)}{X_{k-1} - X_k}\,.$$

We will note simply Δ_k instead of $\Delta_k[M]$ if there is no risk of confusion.

Proposition 3.1. *If $M(X)$ is a polynomial of degree j, then $\Delta_k[M](X_0, X_1, \dots, X_k)$ is a symmetric polynomial of degree $j-k$ in $X_0, X_1, \dots, X_k$.*

Proof. Since Δ_k is a linear operator, it's enough to prove the proposition for $M(X) = X^j$. We claim that:

$$\Delta_k[X^j](X_0, X_1, \dots, X_k) = \sum_{j_0+j_1+\cdots+j_k=j-k} X_0^{j_0} X_1^{j_1} \cdots X_k^{j_k}\,, \quad \text{if } k \le j\,. \tag{3.12}$$

We proceed by induction. For $k = 0$, (3.12) is obvious, hence assuming it for $k-1$ we prove it for k. We have (indicating Δ_k for $\Delta_k[X^m]$):

$$\begin{aligned}\Delta_k(X_0,\dots,X_k) &= \frac{\Delta_{k-1}(X_0,\dots,X_{k-2},X_{k-1}) - \Delta_{k-1}(X_0,\dots,X_{k-2},X_k)}{X_{k-1}-X_k} \\ &= \sum_{j_0+\cdots+j_{k-1}=j-k+1} X_0^{j_0}\cdots X_{k-2}^{j_{k-2}} \frac{X_{k-1}^{j_{k-1}} - X_k^{j_{k-1}}}{X_{k-1}-X_k} \\ &= \sum_{j_0+\cdots+j_{k-1}=j-k+1} X_0^{j_0}\cdots X_{k-2}^{j_{k-2}} \sum_{i_1+i_2=j_{k-1}-1} X_{k-1}^{i_1}X_k^{i_2} \\ &= \sum_{j_0+\cdots+j_{k-2}+i_1+i_2=j-k} X_0^{j_0}\cdots X_{k-2}^{j_{k-2}} X_{k-1}^{i_1}X_k^{i_2}\,,\end{aligned}$$

and we get the (3.12). □

Note that if $M(X) = \alpha_j X^j + \alpha_{j-1}X^{j-1} + \cdots + \alpha_1 X + \alpha_0$, then:

$$\Delta_k[M](0,\dots,0) = \begin{cases}\alpha_k & \text{if } k \le j\,,\\ 0 & \text{if } k > j\,.\end{cases}$$

Let $M(t,\partial_t,i\xi)$ be a differential operator, considering its symbol $M(t,\tau,i\xi)$ as a polynomial in τ with coefficients depending on t,ξ, we can construct the functions $\Delta_j[\sigma(M)]$ as before. For sake of brevity, we denote them simply by $\Delta_j[M]$ instead of $\Delta_j[\sigma(M)]$.

If M_j verifies Assumption 3 then $\Delta_k[M_j]$ verifies similar estimations.

For example, let $m = 3$: Assumption 3 implies that $|t|^{\Lambda_2}\,\big|M_2(\tau_j)\big|$ is bounded by $|\tau_j-\tau_k|\,|\tau_j-\tau_l|$, for $j\neq k$, $k\neq l$, $l\neq j$. We have:

$$\begin{aligned}\frac{\big|\Delta_1[M_2](\tau_2,\tau_3)\big|}{|\tau_2-\tau_1|} &= \frac{\big|M_2(\tau_2)-M_2(\tau_3)\big|}{|\tau_2-\tau_3|\,|\tau_2-\tau_1|} \\ &\le \frac{\big|M_2(\tau_2)\big|}{|\tau_2-\tau_1|\,|\tau_2-\tau_3|} + \frac{|\tau_3-\tau_1|}{|\tau_2-\tau_1|}\,\frac{\big|M_2(\tau_3)\big|}{|\tau_3-\tau_2|\,|\tau_3-\tau_1|}\,,\end{aligned}$$

hence $|t|^{\Lambda_2}\,\big|\Delta_1[M_2](\tau_2,\tau_3)\big|$ is bounded by $|\tau_2-\tau_1|$.

In general, we have:

Lemma 3.2. *If M_j verifies Assumption* 3 *then for any $j_0 < j_1 < \cdots < j_k$, we have*:

(3.13)
$$|t|^{\Lambda_j}\,\Big|\Delta_k[M_j](\tau_{j_0},\dots,\tau_{j_k})\Big| \lesssim \prod_{l\in\{1,\dots,j+1\}\setminus\{j_0,j_1,\dots,j_k\}} |\tau_{j_0}-\tau_l|\,,\quad 0\le k\le j\,.$$

In particular:

$$\Big|\Delta_k[M_j](\tau_{j+1-k},\dots,\tau_{j+1})\Big| \lesssim |t|^{[s_{j-k}-\Lambda_j]^+}\,|\xi|^{j-k}\,. \tag{3.14}$$

Proof. We prove (3.13) by induction. For $k = 0$ there is nothing to prove. Assuming (3.13) for $k-1$, we prove it for k. Using the symmetry of Δ_k with respect to its arguments and (1.8):

$$
\begin{aligned}
&\Big|\frac{\Delta_k(\tau_{j_0},\tau_{j_1},\dots,\tau_{j_k})}{\prod_{l\notin\{j_0,j_1,\dots,j_k\}}(\tau_{j_0}-\tau_l)}\Big| \\
&\quad= \Big|\frac{\Delta_{k-1}(\tau_{j_0},\tau_{j_2},\dots,\tau_{j_k})-\Delta_{k-1}(\tau_{j_1},\tau_{j_2},\dots,\tau_{j_k})}{(\tau_{j_0}-\tau_{j_1})\prod_{l\notin\{j_0,j_1,\dots,j_k\}}(\tau_{j_0}-\tau_l)}\Big| \\
&\quad\le \Big|\frac{\Delta_{k-1}(\tau_{j_0},\tau_{j_2},\dots,\tau_{j_k})}{(\tau_{j_0}-\tau_{j_1})\prod_{l\notin\{j_0,j_1,\dots,j_k\}}(\tau_{j_0}-\tau_l)}\Big| \\
&\qquad+\Big|\frac{\Delta_{k-1}(\tau_{j_1},\tau_{j_2},\dots,\tau_{j_k})}{(\tau_{j_0}-\tau_{j_1})\prod_{l\notin\{j_0,j_1,\dots,j_k\}}(\tau_{j_0}-\tau_l)}\Big| \\
&\quad\le \Big|\frac{\Delta_{k-1}(\tau_{j_0},\tau_{j_2},\dots,\tau_{j_k})}{\prod_{l\notin\{j_0,j_2,\dots,j_k\}}(\tau_{j_0}-\tau_l)}\Big| \\
&\qquad+\Big|\frac{\prod_{l\notin\{j_0,j_1,\dots,j_k\}}(\tau_{j_1}-\tau_l)}{\prod_{l\notin\{j_0,j_1,\dots,j_k\}}(\tau_{j_0}-\tau_l)}\Big|\Big|\frac{\Delta_{k-1}(\tau_{j_1},\tau_{j_2},\dots,\tau_{j_k})}{(\tau_{j_0}-\tau_{j_1})\prod_{l\notin\{j_0,j_1,\dots,j_k\}}(\tau_{j_1}-\tau_l)}\Big| \\
&\quad\le \Big|\frac{\Delta_{k-1}(\tau_{j_0},\tau_{j_2},\dots,\tau_{j_k})}{\prod_{l\notin\{j_0,j_2,\dots,j_k\}}(\tau_{j_0}-\tau_l)}\Big| \\
&\qquad+\Big|\frac{\prod_{l\notin\{j_0,j_1,\dots,j_k\}}(\tau_{j_1}-\tau_l)}{\prod_{l\notin\{j_0,j_1,\dots,j_k\}}(\tau_{j_0}-\tau_l)}\Big|\Big|\frac{\Delta_{k-1}(\tau_{j_1},\tau_{j_2},\dots,\tau_{j_k})}{\prod_{l\notin\{j_1,\dots,j_k\}}(\tau_{j_1}-\tau_l)}\Big|,
\end{aligned}
$$

and we get (3.13) by induction hypothesis. □

We use the operators $\mathcal{L}_k$ and the $\Delta_k[M_j]$ to have a nice decomposition of the lower order terms $M_j(t,\partial_t,i\xi)$. For example, if $j=1$ we have (see (2.19)):

$$b_0v' + ib_1v = b_0\mathcal{L}_1v + i(b_0\tau_m + b_1)v\,,$$

and, if $j=2$ we have (see (2.12)):

$$
\begin{aligned}
b_0v'' + ib_1v' - b_2v = b_0\mathcal{L}_2v &+ i\big(b_0(\tau_{m-1}+\tau_m)+b_1\big)\mathcal{L}_1v \\
&- \big(b_0\tau_m^2 + b_1\tau_m + b_2\big)v\,.
\end{aligned}
$$

Similarly, for $j=3$:

$$
\begin{aligned}
b_0v''' + ib_1v'' - b_2v' - ib_3v = {}& b_0\mathcal{L}_3v + i\big(b_0(\tau_m+\tau_{m-1}+\tau_{m-2})+b_1\big)\mathcal{L}_2v \\
&-\big(b_0(\tau_m^2+\tau_m\tau_{m-1}+\tau_{m-1}^2)+b_1(\tau_m+\tau_{m-1})+b_2\big)\mathcal{L}_1v \\
&-i\big(b_0\tau_m^3+b_1\tau_m^2+b_2\tau_m+b_3\big)v\,.
\end{aligned}
$$

Proposition 3.2. *Let $M_j(t,\partial_t,i\xi)$ be a differential operator of order j, then we have the following decomposition*:

$$M_j = \sum_{k=0}^{j} \Delta_k[M](i\tau_{j-k},\dots,i\tau_j)\mathcal{L}_k\,. \tag{3.15}$$

Proof. We proceed by induction on j. For $j = 0$, (3.15) is obvious. Assuming (3.15) for $j-1$, we prove it for j. Since $\Delta_j[M_j]$ is the leading term of M, $M - \Delta_j[M]\mathcal{L}_j$ is of order $\le j-1$, hence, using induction hypothesis, we have:

$$M - \Delta_j[M]\mathcal{L}_j = \sum_{k=0}^{j-1} \Delta_k\big[M(t,\partial_t) - \Delta_j[M]\mathcal{L}_j\big](i\tau_{j-k},\dots,i\tau_j)\mathcal{L}_k$$

noting that Δ_k are linear and $\Delta_k[\mathcal{L}_j] = 0$ if $k < j$, we get:

$$M - \Delta_j[M]\mathcal{L}_j = \sum_{k=0}^{j-1} \Delta_k[M](i\tau_{j-k},\dots,i\tau_j)\mathcal{L}_k\,,$$

and hence the desired assertion. ☐

Finally, we need an estimation of the time derivatives of the $\mathrm{L}_{k_1,\dots,k_j}$. As seen in §2, we have:

$$\partial_t|\,\mathrm{L}_1\,v|^2 \le 2|\,\mathrm{L}_{12}\,v|\,|\,\mathrm{L}_1\,v| + 2\,|\tau_1'|\;|v|\,|\,\mathrm{L}_1\,v|\,,$$

and

$$\partial_t|\,\mathrm{L}_{12}\,v|^2 \le 2|\,\mathrm{L}_{123}\,v|\,|\,\mathrm{L}_{12}\,v| + 2\,|\tau_1'|\;|\,\mathrm{L}_2\,v|\,|\,\mathrm{L}_{12}\,v| + 2\,|\tau_2'|\;|\,\mathrm{L}_1\,v|\,|\,\mathrm{L}_{12}\,v|\,.$$

In general, we have:

Lemma 3.3. *Let $k_1,k_2,\dots,k_j \in \{1,2,\dots,m\}$, $j \le m-1$. For any $k' \in \{1,2,\dots,m\}\setminus\{k_1,k_2,\dots,k_j\}$ we have*

$$(3.16)\quad \partial_t|\,\mathrm{L}_{k_1,\dots,k_j}\,v|^2 \lesssim |\,\mathrm{L}_{k_1,k_2,\dots,k_j,k'}\,v|\,|\,\mathrm{L}_{k_1,\dots,k_j}\,v| + \sum_{l=1}^{j}|\tau_{k_l}'|\,|\,\mathrm{L}_{k_1,\dots,\widehat{k_l},\dots,k_j}\,v|\,|\,\mathrm{L}_{k_1,\dots,k_j}\,v|\,.$$

Proof. We prove at first:

$$(3.17)\quad \partial_t\,\mathrm{L}_{k_1,\dots,k_j}\,v = \mathrm{L}_{k_1,k_2,\dots,k_j,k'}\,v + i\tau_{k'}\,\mathrm{L}_{k_1,\dots,k_j}\,v - \sum_{l=1}^{j} i\tau_{k_l}'\,\mathrm{L}_{k_1,\dots,\widehat{k_l},\dots,k_j}\,v\,.$$

Using (3.1) and (3.2), we have:

$$\partial_t\,\mathrm{L}_{k_1,\dots,k_j}\,v = \sum_{h=0}^{j}(-i)^h\sigma_h(\tau_{k_1},\dots,\tau_{k_j})\partial_t^{j+1-h}v + \sum_{h=0}^{j}(-i)^h\partial_t\sigma_h(\tau_{k_1},\dots,\tau_{k_j})\partial_t^{j-h}v$$

$$
\begin{aligned}
&= \sum_{h=0}^{j+1} (-i)^h \sigma_h(\tau_{k_1}, \dots, \tau_{k_j}, \tau_{k'}) \partial_t^{j+1-h} v \\
&\quad - \sum_{h=1}^{j+1} (-i)^h \tau_{k'} \sigma_{h-1}(\tau_{k_1}, \dots, \tau_{k_j}) \partial_t^{j+1-h} v \\
&\quad + \sum_{h=1}^{j} \sum_{l=1}^{j} (-i)^h \tau'_{k_l} \sigma_{h-1}(\tau_{k_1}, \dots, \widehat{\tau_{k_l}}, \dots, \tau_{k_j}) \partial_t^{j-h} v \\
&= \mathrm{L}_{k_1,k_2,\dots,k_j,k'}\, v + i\tau_{k'}\, \mathrm{L}_{k_1,\dots,k_j}\, v - i \sum_{l=1}^{j} \tau'_{k_l}\, \mathrm{L}_{k_1,\dots,\widehat{k_l},\dots,k_j}\, v\,.
\end{aligned}
$$

Since $\tau_{k'}$ is real, we get (3.16) form (3.17):

$$
\begin{aligned}
\partial_t \big|\mathrm{L}_{k_1,\dots,k_j}\, v\big|^2 &= 2\,\mathrm{Re}\,\big\langle \mathrm{L}_{k_1,k_2,\dots,k_j,k'}\, v, \mathrm{L}_{k_1,\dots,k_j}\, v\big\rangle_{L^2} \\
&\qquad - 2\,\mathrm{Re} \left\langle i \sum_{l=1}^{j} \tau'_{k_l}\, \mathrm{L}_{k_1,\dots,\widehat{k_l},\dots,k_j}\, v, \mathrm{L}_{k_1,\dots,k_j}\, v \right\rangle_{L^2} \\
&\le 2\big|\mathrm{L}_{k_1,k_2,\dots,k_j,k'}\, v\big|\big|\mathrm{L}_{k_1,\dots,k_j}\, v\big| \\
&\qquad + 2 \sum_{l=1}^{j} |\tau'_{k_l}| \big|\mathrm{L}_{k_1,\dots,\widehat{k_l},\dots,k_j}\, v\big|\big|\mathrm{L}_{k_1,\dots,k_j}\, v\big|\,.
\end{aligned}
$$

□

4. Energy estimates

By Fourier transform, we get the equation:

$$
L(t, \partial_t, i\xi)\, v = \sum_{j=0}^{m-1} M_j(t, \partial_t, i\xi)\, v\,,
$$

where $v = \widehat{u}$.

4.1. Estimate in the pseudodifferential zone

For $|t| \le t_1(\xi)$, $t_1(\xi) = |\xi|^{-(1-1/d_0)}$,
$|\xi| \ge 1$, we define the energy:

$$
E_1(t, \xi) = \sum_{j=0}^{m-1} |\xi|^{2(m-j-1)}\, |v^{(j)}|^2\,.
$$

We have:

$$
E'_1(t, \xi) = 2 \sum_{j=0}^{m-1} |\xi|^{2(m-j-1)}\, \mathrm{Re}\big(v^{(j+1)}, v^{(j)}\big) \le 2 \sum_{j=0}^{m-1} |\xi|^{2(m-j-1)}\, |v^{(j+1)}|\, |v^{(j)}|
$$

$$\leq |v^{(m)}|\,|v^{(m-1)}| + 2(m-2)\,|\xi|\; E_1(t,\xi) \lesssim |\xi|\, E_1(t,\xi)\,,$$

since:

$$\begin{aligned}|v^{(m)}| &\leq \sum_{k=0}^{m-1} |a_k|\; |\xi|^k\; |v^{(m-k)}| + \sum_{j=0}^{m-1} \big|M_j(t,\partial_t,i\xi)v\big| \\ &\lesssim |\xi|\,\sqrt{E_1(t,\xi)} + \sqrt{E_1(t,\xi)} \lesssim |\xi|\sqrt{E_1(t,\xi)}\,.\end{aligned}$$

By Gronwall Lemma:

$$E_1(\theta_2,\xi) \leq \exp\big(C\,|\xi|\;(\theta_2-\theta_1)\big)\,E_1(\theta_1) \leq \exp\big(C\,|\xi|^{1/d_0}\big)\,E_1(\theta_1)\,, \tag{4.1}$$

for any $-t_1(\xi) \leq \theta_1 \leq \theta_2 \leq t_1(\xi)$.

4.2. Estimate in the hyperbolic zone

For $t_1(\xi) \leq |t| \leq T$, $|\xi| \geq 1$, we define the energy:

$$E_2(t,\xi) = \sum_{j=0}^{m-1} \Big(\frac{t_1(\xi)\,|\xi|}{t}\Big)^{2(m-1-j)}\; [v]_j^2\;.$$

The energy E_2 is equivalent to E_1 for $t_1(\xi) \leq |t| \leq T$, that is:

$$E_1(t,\xi) \lesssim |\xi|^{M_1}\; E_2(t,\xi)\,, \qquad \text{and} \qquad E_2(t,\xi) \lesssim |\xi|^{M_2}\, E_1(t,\xi)\,,$$

for some $M_1, M_2 \geq 0$. In fact, it's easy to see that: $E_2(t,\xi) \lesssim E_1(t,\xi)$. To prove the reversed inequality, thanks to Lemma 3.1, we see that:

$$\begin{aligned}|\xi|^{m-1}\;|v| &\lesssim \frac{[v]_{m-1}}{|t|^{s_{m-1}}} \lesssim \frac{[v]_{m-1}}{t_1^{s_{m-1}}} = |\xi|^{(1-1/d_0)s_{m-1}}\;[v]_{m-1} \\ &\leq |\xi|^{(1-1/d_0)s_{m-1}}\;\sqrt{E_2(t,\xi)}\,.\end{aligned}$$

To estimate $|\xi|^{m-2}\;|v'|$, we remark that:

$$|\xi|^{m-2}\;|v'| = |\xi|^{m-2}\;|\,\mathrm{L}_m\, v + i\tau_m v| \lesssim |\xi|^{m-2}\;|\,\mathrm{L}_m\, v| + |\xi|^{m-1}\;|v|$$

hence it's enough to estimate $|\xi|^{m-2}\;|\,\mathrm{L}_m\, v|$. Using again Lemma 3.1, we have:

$$\begin{aligned}|\xi|^{m-2}\;|\,\mathrm{L}_m\, v| &\lesssim \frac{[v]_{m-1}}{|t|^{s_{m-2}}} \lesssim \frac{[v]_{m-1}}{t_1^{s_{m-2}}} = |\xi|^{(1-1/d_0)s_{m-2}}\;[v]_{m-1} \\ &\leq |\xi|^{(1-1/d_0)s_{m-2}}\;\sqrt{E_2(t,\xi)}\,.\end{aligned}$$

In general, using Proposition 3.2, we have

$$|v^{(j)}| \leq \sum_{k=0}^{j} \Big|\Delta_k[\tau^j](\tau_{m-k},\dots,\tau_m)\Big|\,|\mathcal{L}_k v|\,,$$

where, thanks to (3.12), we have:

$$\Big|\Delta_k[\tau^j](\tau_{m-k},\dots,\tau_m)\Big| \le \sum_{j_0+j_1+\cdots+j_k=j-k} |\tau_0^{j_0}|\,|\tau_1^{j_1}|\cdots|\tau_k^{j_k}| \lesssim |\xi|^{j-k} \ ,$$

hence:

$$|\xi|^{m-1-j}\ |v^{(j)}| \lesssim \sum_{k=0}^{j} |\xi|^{m-1-k}\ |\mathcal{L}_k v| \,.$$

Using (3.11), we have:

$$\begin{aligned} |\xi|^{m-1-k}\ |\mathcal{L}_k v| &\lesssim \frac{[v]_{m-1}}{|t|^{s_{m-k-1}}} \lesssim \frac{[v]_{m-1}}{t_1^{s_{m-k-1}}} = |\xi|^{(1-1/d_0)s_{m-k-1}}\ [v]_{m-1} \\ &\le |\xi|^{(1-1/d_0)s_{m-k-1}}\ \sqrt{E_2(t,\xi)}\,. \end{aligned}$$

Now we derive the energy estimate for E_2. We have:

$$\begin{aligned} E_2'(t) &= 2\sum_{j=0}^{m-1}\Big(\frac{t_1(\xi)\,|\xi|}{t}\Big)^{2(m-1-j)}\ \partial_t\,[v]_j^2 \\ &\quad - \frac{2}{t}\sum_{j=0}^{m-1}(m-1-j)\Big(\frac{t_1(\xi)\,|\xi|}{t}\Big)^{2(m-1-j)}\ [v]_j^2 \\ &\lesssim 2\sum_{j=0}^{m-1}\Big(\frac{t_1(\xi)\,|\xi|}{t}\Big)^{2(m-1-j)}\ \partial_t\,[v]_j^2 + \frac{1}{|t|}\,E_2(t)\,. \end{aligned}$$

Now, recalling (3.16), we get:

$$\begin{aligned} \sum_{j=0}^{m-1}\Big(\frac{t_1(\xi)\,|\xi|}{t}\Big)^{2(m-1-j)}\ \partial_t\,[v]_j^2 &\le \sum_{j=0}^{m-1}\Big(\frac{t_1(\xi)\,|\xi|}{t}\Big)^{2(m-1-j)}\ [v]_{j+1}\,[v]_j \\ &+ \sum_{j=0}^{m-1}\Big(\frac{t_1(\xi)\,|\xi|}{t}\Big)^{2(m-1-j)} \sum_{1\le k_1<\cdots<k_j\le m}\ \sum_{l=1}^{j}|\tau_{k_l}'|\big|\mathrm{L}_{k_1,\dots,\widehat{k_l},\dots,k_j}\,v\big|\,[v]_j \ , \end{aligned} \tag{4.2}$$

for $j = 0, 1\dots, m-1$.

We claim that all the terms in the right hand side of (4.2) are estimated by $|\xi|^{1/d_0}\ \dfrac{1}{|t|}\,E_2(t)$, so that we have:

$$E_2'(t) \lesssim |\xi|^{1/d_0}\ \frac{1}{|t|}\,E_2(t)\,,$$

and by Gronwall Lemma:

$$E_2(\theta_2,\xi) \le \exp\Big[C\,|\xi|^{1/d_0}\ \log\frac{\theta_2}{\theta_1}\Big]\,E_2(\theta_1) \lesssim |\xi|^{C_1}\exp\big(C_2\,|\xi|^{1/d_0}\big)\,E_2(\theta_1)\,, \tag{4.3}$$

for any θ_1, θ_2 such that $-T \le \theta_1 \le \theta_2 \le -t_1(\xi)$ or $t_1(\xi) \le \theta_1 \le \theta_2 \le T$, for some constant C_1 and C_2. Combining (4.1) and (4.3), we can conclude the proof by standard arguments.

We prove that all the terms in the first sum of the right hand side of (4.2) are estimated by $|\xi|^{1/d_0} \, \dfrac{1}{|t|} \, E_2(t)$. We have:

$$\Big(\frac{t_1(\xi)\,|\xi|}{t}\Big)^{2(m-1-j)}\,[v]_{j+1}\,[v]_j \le \frac{t_1(\xi)\,|\xi|}{|t|}\,E_2(t) \le \frac{|\xi|^{1/d_0}}{|t|}\,E_2(t)\,,$$

for $j = 0, 1, \ldots, m-2$, and

$$\Big(\frac{t_1(\xi)\,|\xi|}{t}\Big)^{2(m-1-j)}\,[v]_{j+1}\,[v]_j \le [v]_m\,\sqrt{E_2(t)}\,,$$

for $j = m-1$. Using (3.15), we have:

$$[v]_m = |Lv| = |Mv| \le \sum_{j=0}^{m-1} |M_j v| \le \sum_{j=0}^{m-1}\sum_{k=0}^{j} \big|\Delta_k[M_j](\tau_{m-k},\ldots,\tau_m)\big|\,|\mathcal{L}_k v|\,.$$

We claim that Assumptions 1 and 3 imply:

(4.4)
$$\big|\Delta_k[M_j](\tau_{m-k},\ldots,\tau_m)\big|\,|\mathcal{L}_k v| \lesssim |\xi|^{1/d_0}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\,, \quad \text{for } t_1(\xi) \le |t| \le T\,,$$

for $0 \le k < j \le m$.

By (3.11) and the definition of E_2, we get:

$$|\mathcal{L}_k v(t)| \lesssim \frac{[v]_{k+h}}{|t|^{s_h}\,|\xi|^h} \lesssim \frac{|t|^{m-k-h-s_h}}{t_1^{m-1-k-h}\,|\xi|^{m-1-k}}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\,,$$

for $h = 1, \ldots, m-k-1$, hence, by (3.14):

$$\big|\Delta_k[M_j](\tau_{m-k},\ldots,\tau_m)\big|\,|\mathcal{L}_k v| \lesssim \frac{|t|^{[s_{j-k}-\Lambda_j]^+ - s_h + m-k-h}}{t_1^{m-1-k-h}\,|\xi|^{m-1-j}}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\,, \tag{4.5}$$

for $h = 1, \ldots, j-k$.

If $\min(s_{j-k}, \Lambda_j) \le m-j$, (4.5) with $h = j-k$, gives:

$$\begin{aligned}\big|\Delta_k[M_j](\tau_{m-k},\ldots,\tau_m)\big|\,|\mathcal{L}_k v| &\lesssim \frac{|t|^{[s_{j-k}-\Lambda_j]^+ - s_{j-k} + m-j}}{t_1^{m-1-j}\,|\xi|^{m-1-j}}\,\frac{1}{|t|}\,\sqrt{E_2(t)} \\ &\lesssim \frac{1}{|t|}\,\sqrt{E_2(t)}\,,\end{aligned}$$

since $[s_{j-k}-\Lambda_j]^+ - s_{j-k} + m-j \ge 0$ and $t_1 \ge |\xi|^{-\frac{1}{d_0}} \ge |\xi|^{-1}$. If $\min(s_{j-k}, \Lambda_j) > m-j$, let $h = h(j,k)$ be as in (1.12), we use (4.5) with $h(j,k)-1$ for $t_1(\xi) \le |t| \le t_2(\xi)$ and (4.5) with $h(j,k)$ for $t_2(\xi) \le |t| \le T$, where

$$t_2(\xi) = |\xi|^{-\frac{1}{\kappa_h+1}(1-1/d_0)} \qquad \big(h = h(j,k)\big)\,,$$

so that (noting simply h for $h(j,k)$):

$$\big|\Delta_k[M_j](\tau_{m-k},\ldots,\tau_m)\big||\mathcal{L}_k v| \lesssim \frac{t_2^{[s_{j-k}-\Lambda_j]^+-s_{h-1}+m-k-h+1}}{t_1^{m-k-h}\,|\xi|^{m-1-j}}\,\frac{1}{|t|}\,\sqrt{E_2(t)}$$
$$\lesssim |\xi|^{1/d_0}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\,,$$

for $t_1(\xi) \le |t| \le t_2(\xi)$, and

$$\big|\Delta_k[M_j](\tau_{m-k},\ldots,\tau_m)\big||\mathcal{L}_k v| \lesssim \frac{t_2^{[s_{j-k}-\Lambda_j]^+-s_h+m-k-h}}{t_1^{m-1-k-h}\,|\xi|^{m-1-j}}\,\frac{1}{|t|}\,\sqrt{E_2(t)}$$
$$\lesssim |\xi|^{1/d_0}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\,,$$

for $t_2(\xi) \le |t| \le T$, by the definition of d_0.

Now we prove that the terms in the second sum in (4.2) are estimated by $|\xi|^{1/d_0}\,\dfrac{1}{|t|}\,E_2(t)$, too. Here also we have three different estimates.

First estimate. Using Assumption 2 we have:

$$|\tau'_{k_l}| = \frac{|\tau'_{k_l}|}{|\tau_{k_l}-\tau_{k^*}|}\,|\tau_{k_l}-\tau_{k^*}| \lesssim |t|^{\kappa_1-\Lambda_m-1}\,|\xi|\,,$$

hence:

(4.6)
$$\Big(\frac{t_1(\xi)\,|\xi|}{t}\Big)^{m-1-j}|\tau'_{k_l}|\big|\mathrm{L}_{k_1,\ldots,\widehat{k_l},\ldots,k_j}\,v\big| \lesssim \Big(\frac{t_1(\xi)\,|\xi|}{t}\Big)^{m-1-j}\,|t|^{\kappa_1-\Lambda_m-1}\,|\xi|\,\,[v]_{j-1}$$
$$\lesssim \frac{|t|^{\kappa_1+1-\Lambda_m}}{t_1}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\,.$$

Second estimate. Let $k^* \in \{1,\ldots,m\}\setminus\{k_1,\ldots,\ldots,k_j\}$, using (3.10) and Assumption 2 we have:

(4.7)
$$\Big(\frac{t_1(\xi)\,|\xi|}{t}\Big)^{m-1-j}|\tau'_{k_l}|\big|\mathrm{L}_{k_1,\ldots,\widehat{k_l},\ldots,k_j}\,v\big| \lesssim \Big(\frac{t_1(\xi)\,|\xi|}{t}\Big)^{m-1-j}|\tau'_{k_l}|\,\frac{[v]_j}{|\tau_{k_l}-\tau_{k^*}|}$$
$$\lesssim |t|^{-\Lambda_m}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\,.$$

Third estimate. Using Bronšteĭn Lemma to estimate $|\tau'_{k_l}|$, we get:

$$\Big(\frac{t_1(\xi)\,|\xi|}{t}\Big)^{m-1-j}|\tau'_{k_l}|\big|\mathrm{L}_{k_1,\ldots,\widehat{k_l},\ldots,k_j}\,v\big| \lesssim \Big(\frac{t_1(\xi)\,|\xi|}{t}\Big)^{m-1-j}\,|\xi|\,\,[v]_{j-1}$$
$$\lesssim \frac{t^2}{t_1}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\,. \tag{4.8}$$

If $\Lambda_m = 0$, using (4.7) for $t_1(\xi) \le |t| \le T$, we get:

$$\Big(\frac{t_1(\xi)\,|\xi|}{t}\Big)^{m-1-j} |\tau'_{k_l}| |\mathrm{L}_{k_1,\dots,\widehat{k_l},\dots,k_j}\, v| \lesssim \frac{1}{|t|}\sqrt{E_2(t)}\,.$$

If $0 < \Lambda_m \le \kappa_1 - 1$, using (4.6) for $t_1(\xi) \le |t| \le t_2(\xi)$ and (4.7) for $t_2(\xi) \le |t| \le T$, with $t_2 = |\xi|^{-1/(\kappa_1+\Lambda_m+1)}$, we get:

$$\Big(\frac{t_1(\xi)\,|\xi|}{t}\Big)^{m-1-j} |\tau'_{k_l}| |\mathrm{L}_{k_1,\dots,\widehat{k_l},\dots,k_j}\, v| \lesssim \frac{|\xi|^{1/d_0}}{|t|}\sqrt{E_2(t)}\,. \tag{4.9}$$

If $\Lambda_m > \kappa_1 - 1$, we use (4.8) instead of (4.6) for $t_1(\xi) \le |t| \le t_2(\xi)$ and (4.7) for $t_2(\xi) \le |t| \le T$, with $t_2 = |\xi|^{-1/2(\Lambda_m+1)}$, and we get again (4.9).

5. Proof of Theorem 1.2

As remarked in the introduction, Assumption 1 implies Assumption 2 with $\Lambda_m = \kappa_{m-1} - 1$; moreover if no condition on the lower order terms is assumed, then Assumption 3 is satisfied with $\Lambda_j = s_j$. Hence we can apply Theorem 1.1 and it's enough to prove that $d^* = d_0$.

We have:

$$d_{j,k} = \Big(1 - \frac{(m-j)(\kappa_{h(j,k)}+1)}{s_{h(j,k)} + \big(m-k-h(j,k)\big)\kappa_{h(j,k)}}\Big)^{-1},$$

where $h(j,k)$ defined as in (1.12).

$$h(j,k) = \min\Big\{h \in \mathbb{N}\setminus\{0\} \;\Big|\; s_h + h \ge m-k\Big\}. \tag{5.1}$$

Since $h(j,k)$ does not depend on j, we have $d_{j,k} \ge d_{m-1,k}$, for any k, and we will note $h(k)$ instead of $h(j,k)$.

Note that $h(j,k) \le \Big\lceil \dfrac{m-k}{2} \Big\rceil$.

We claim that $d_{m-1,k} \ge d_{m-1,0}$, for any k. Note that $h(k)$ is decreasing with respect to k, but this does not imply immediately that $d_{m-1,k}$ is decreasing with respect to k, so we have to proceed as follows.

If $h(k-1) = h(k)$, then clearly $d_{m-1,k-1} \le d_{m-1,k}$. On the other side, if $h(k-1) > h(k)$, we have:

$$s_{h(k)} + h(k) = m-k\,, \tag{5.2}$$

$$h(k-1) = h(k) + 1\,. \tag{5.3}$$

In fact, if (5.2) doesn't hold, then $s_{h(k)} + h(k) \ge m-(k-1)$, and, by the definition (5.1) we have $h(k-1) \le h(k)$, which contradicts the assumption $h(k-1) > h(k)$.

Using (5.2), we have:

$$s_{h(k)+1} + h(k) + 1 = s_{h(k)} + \kappa_{h(k)+1} + h(k) + 1 = m - k + \kappa_{h(k)+1} + 1 > m-(k-1)\,,$$

which gives $h(k-1) \le h(k)+1$ and hence (5.3).

Now

$$s_{h(k-1)} = s_{h(k)+1} = s_{h(k)} + \kappa_{h(k)+1} = s_{h(k)} + \kappa_{h(k-1)}\,,$$

hence we have:

$$\begin{aligned}\frac{s_{h(k-1)} + \big(m-k-h(k-1)\big)\kappa_{h(k-1)}}{\kappa_{h(k-1)}+1} \\ &= \frac{s_{h(k)} + \kappa_{h(k)+1} + \big(m-k-h(k)-1\big)\kappa_{h(k-1)}}{\kappa_{h(k-1)}+1} \\ &= m-k-h(k)\,,\end{aligned}$$

which gives $d_{m-1,k-1} = d_{m-1,k}$, since, using (5.2) we have:

$$d_{j,k} = \Big(1 - \frac{1}{m-k-h(k)}\Big)^{-1}.$$

Hence we have:

$$\min_{j,k} d_{j,k} = d_{m-1,0} = d^*\,.$$

To prove that $d^* \le d_m$, it will be enough to prove that

$$d^* \le 2 + \frac{2}{\kappa_{m-1}-1}\,. \tag{5.4}$$

It's easy to check (5.4) for $m=2,3$ (see Examples 1.2 and 1.3), so we can assume $m \ge 4$. Since $h = h(0) \le \left\lceil \frac{m}{2} \right\rceil$ we have $m-h \ge 2$.

If $s_h \ge 2$ and $m-h \ge 2$ then clearly

$$2\kappa_h + 2 \le s_h + (m-h)\kappa_h\,, \tag{5.5}$$

which gives $d^* \le 2$, and (5.4) is obviously satisfied. On the other side, if $s_h = 1$, then $h=1$ and $\kappa_1 = 1$. In this case (5.5) still holds true, unless $m \le 3$.

Hence $d^* = d_0$, and applying Theorem 1.1 we complete the proof of Theorem 1.2. □

6. An example of operator not verifying Assumption 1

Before to give the proof of Theorem 1.3, we give a sufficient condition for the operator L to verify Assumption 1 or 1′.

Let

$$\begin{aligned}\Delta_2 &= (\tau_1-\tau_2)^2 + (\tau_1-\tau_3)^2 + (\tau_1-\tau_4)^2 + (\tau_2-\tau_3)^2 + (\tau_2-\tau_4)^2 + (\tau_3-\tau_4)^2 \\ \Delta_3 &= (\tau_1-\tau_2)^2(\tau_2-\tau_3)^2(\tau_3-\tau_1)^2 + (\tau_2-\tau_3)^2(\tau_3-\tau_4)^2(\tau_4-\tau_2)^2 \\ &\quad + (\tau_3-\tau_4)^2(\tau_4-\tau_1)^2(\tau_1-\tau_3)^2 + (\tau_4-\tau_1)^2(\tau_1-\tau_2)^2(\tau_2-\tau_4)^2\end{aligned}$$

$$\Delta_4(=\Delta) = \prod_{j<k}(\tau_j-\tau_k)^2$$
$$\widetilde{\Delta} = (\tau_1-\tau_2)^2(\tau_3-\tau_4)^2 + (\tau_1-\tau_3)^2(\tau_2-\tau_4)^2 + (\tau_1-\tau_4)^2(\tau_2-\tau_3)^2$$

The functions Δ_2, Δ_3, Δ_4 and $\widetilde{\Delta}$ are symmetric polynomials of the characteristic roots, hence, by Newton's Theorem, they can be expressed as functions of the coefficients of L.

We note that if L verifies Assumption 1, that is:

$$\tau_1-\tau_2 \approx t^{\kappa_1}\,, \quad \tau_2-\tau_3 \approx t^{\kappa_2}\,, \quad \tau_3-\tau_4 \approx t^{\kappa_3}\,,$$

with $\kappa_1 \le \kappa_2 \le \kappa_3$, then:

$$\Delta_2 \approx t^{2\kappa_1}\,, \quad \Delta_3 \approx t^{4\kappa_1+2\kappa_2}\,, \quad \Delta_4 \approx t^{6\kappa_1+4\kappa_2+2\kappa_3} \quad \widetilde{\Delta} \approx t^{2\kappa_1+2\kappa_2}\,,$$

whereas if L verifies Assumption $1'$, we have:

$$\Delta_2 \approx t^{2\kappa}\,, \quad \Delta_3 \approx t^{4\kappa+2\ell_1}\,, \quad \Delta_4 \approx t^{8\kappa+2\ell_1+2\ell_2}\,, \quad \widetilde{\Delta} \approx t^{4\kappa}\,.$$

Hence, assuming that:

$$\Delta_2 \approx t^{2\delta_2}\,, \quad \Delta_3 \approx t^{2\delta_3}\,, \quad \Delta_4 \approx t^{2\delta_4}\,, \quad \widetilde{\Delta} \approx t^{2\widetilde{\delta}}\,,$$

we can easily see that if $\widetilde{\delta} > 2\delta_1$, then L verifies Assumption 1 with

$$\kappa_1 = \delta_1\,, \quad \kappa_2 = \delta_3 - 2\kappa_1\,, \quad \kappa_3 = \delta_4 - 3\kappa_1 - 2\kappa_2\,,$$

whereas if $\widetilde{\delta} = 2\delta_1$, and $\delta_3 > 3\delta_1$, then L verifies Assumption $1'$ with:

$$\kappa = \delta_1\,, \quad \ell_1 = \delta_3 - 2\kappa\,, \quad \ell_2 = \delta_4 - 4\kappa - \ell_1\,;$$

finally, if $\widetilde{\delta} = 2\delta_1$ and $\delta_3 = 3\delta_1$ then L verifies both Assumptions 1 (with $\kappa_1 = \kappa_2 = \delta_1$, and $\kappa_3 = \delta_4 - 5\delta_1$) and $1'$ (with $\kappa = \ell_1 = \delta_1$ and $\ell_2 = \delta_4 - 5\delta_1$).

The proof of Theorem 1.3 is similar to that of Theorem 1.1, and it will be enough to prove (4.4).

We have:

$$\widetilde{d}_{3,0} = \begin{cases} +\infty & \text{if } \min(2\kappa+\ell_2,\Lambda_3) \le 1 \\ \dfrac{\min(2\kappa+\ell_2,\Lambda_3)+\ell_2}{\min(2\kappa+\ell_2,\Lambda_3)-1} & \text{if } 1 < \min(2\kappa+\ell_2,\Lambda_3) \le \ell_2+2 \\ \dfrac{2\kappa-\ell_2+\min(2\kappa+\ell_2,\Lambda_3)}{\kappa-\ell_2+\min(2\kappa+\ell_2,\Lambda_3)-1} & \text{if } \min(2\kappa+\ell_2,\Lambda_3) > \ell_2+2 \end{cases}$$

$$\widetilde{d}_{3,1} = \begin{cases} +\infty & \text{if } \min(\kappa+\ell_1,\Lambda_3) \le 1 \\ \dfrac{\ell_1+\min(\kappa+\ell_1,\Lambda_3)}{\min(\kappa+\ell_1,\Lambda_3)-1} & \text{if } 1 < \min(\kappa+\ell_1,\Lambda_3) \le 2+\ell_1 \\ \dfrac{2\kappa-\ell_1+\min(\kappa+\ell_1,\Lambda_3)}{\kappa-\ell_1+\min(\kappa+\ell_1,\Lambda_3)-1} & \text{if } \min(\kappa+\ell_1,\Lambda_3) \ge 2+\ell_1 \end{cases}$$

$$\widetilde{d}_{3,2} = \begin{cases} +\infty & \text{if } \min(\kappa, \Lambda_3) \le 1\,, \\ \dfrac{\min(\kappa, \Lambda_3) + \kappa}{\min(\kappa, \Lambda_3) - 1} & \text{if } \min(\kappa, \Lambda_3) > 1\,, \end{cases}$$

$$\widetilde{d}_{2,0} = \begin{cases} +\infty & \text{if } \min(2\kappa, \Lambda_2) \le 2\,, \\ \dfrac{2\kappa + \min(2\kappa, \Lambda_2)}{\min(2\kappa, \Lambda_2) - 2} & \text{if } \min(2\kappa, \Lambda_2) > 2\,, \end{cases},$$

$$\widetilde{d}_{2,1} = \begin{cases} +\infty & \text{if } \min(\kappa, \Lambda_2) \le 2\,, \\ \dfrac{2\kappa + \min(\kappa, \Lambda_2)}{\min(\kappa, \Lambda_2) - 2} & \text{if } \min(\kappa, \Lambda_2) > 2\,, \end{cases}$$

$$\widetilde{d}_{1,0} = \begin{cases} +\infty & \text{if } \min(\kappa, \Lambda_1) \le 3\,, \\ \dfrac{3\kappa + \min(\kappa, \Lambda_1)}{\min(\kappa, \Lambda_1) - 3} & \text{if } \min(\kappa, \Lambda_1) > 3\,. \end{cases}$$

Estimation of $|M_3(\tau_4)|\,|v|$. Since:

$$|v| \lesssim \frac{|t|^3}{t_1^3\,|\xi|^3}\,\sqrt{E_2(t)} \tag{6.1}$$

$$|v| \lesssim \frac{|\,\mathrm{L}_4\,v - \mathrm{L}_1\,v|}{|\tau_4 - \tau_1|} \lesssim \frac{1}{|t|^\kappa\,|\xi|}\,[v]_1 \lesssim \frac{|t|^{2-\kappa}}{t_1^2\,|\xi|^3}\,\sqrt{E_2(t)} \tag{6.2}$$

$$\begin{aligned} |v| &\lesssim \frac{|\mathrm{L}_4\,v - \mathrm{L}_1\,v|}{|\tau_4 - \tau_1|} \lesssim \frac{|\mathrm{L}_{24}\,v - \mathrm{L}_{34}\,v|}{|\tau_4 - \tau_1|\,|\tau_2 - \tau_3|} + \frac{|\mathrm{L}_{12}\,v - \mathrm{L}_{13}\,v|}{|\tau_4 - \tau_1|\,|\tau_2 - \tau_3|} \\ &\lesssim \frac{[v]_2}{t^{2\kappa}\,|\xi|^2} \lesssim \frac{|t|^{1-2\kappa}}{t_1\,|\xi|^3}\,\sqrt{E_2(t)}\,, \end{aligned} \tag{6.3}$$

moreover:

$$\frac{|\mathrm{L}_{24}\,v|}{|\tau_4 - \tau_1|\,|\tau_2 - \tau_3|} \lesssim \frac{|\mathrm{L}_{124}\,v - \mathrm{L}_{234}\,v|}{|\tau_4 - \tau_1|\,|\tau_2 - \tau_3|\,|\tau_1 - \tau_3|} \lesssim \frac{1}{t^{2\kappa+\ell_1}\,|\xi|^3}\,[v]_3$$

$$\frac{|\mathrm{L}_{34}\,v|}{|\tau_4 - \tau_1|\,|\tau_2 - \tau_3|} \lesssim \frac{|\mathrm{L}_{134}\,v - \mathrm{L}_{234}\,v|}{|\tau_4 - \tau_1|\,|\tau_2 - \tau_3|\,|\tau_1 - \tau_2|} \lesssim \frac{1}{t^{3\kappa}\,|\xi|^3}\,[v]_3$$

$$\frac{|\mathrm{L}_{12}\,v|}{|\tau_4 - \tau_1|\,|\tau_2 - \tau_3|} \lesssim \frac{|\mathrm{L}_{123}\,v - \mathrm{L}_{124}\,v|}{|\tau_4 - \tau_1|\,|\tau_2 - \tau_3|\,|\tau_3 - \tau_4|} \lesssim \frac{1}{t^{3\kappa}\,|\xi|^3}\,[v]_3$$

$$\frac{|\mathrm{L}_{13}\,v|}{|\tau_4 - \tau_1|\,|\tau_2 - \tau_3|} \lesssim \frac{|\mathrm{L}_{123}\,v - \mathrm{L}_{134}\,v|}{|\tau_4 - \tau_1|\,|\tau_2 - \tau_3|\,|\tau_2 - \tau_4|} \lesssim \frac{1}{t^{2\kappa+\ell_2}\,|\xi|^3}\,[v]_3$$

hence:

$$|v| \lesssim \frac{1}{t^{2\kappa+\ell_2}\,|\xi|^3}\,[v]_3 \lesssim \frac{1}{t^{2\kappa+\ell_2}\,|\xi|^3}\,\sqrt{E_2(t)}\,. \tag{6.4}$$

On the other side, by Assumption 3, we have:

$$\begin{aligned} |M_3(\tau_4)| &= |\tau_4 - \tau_1|\,|\tau_4 - \tau_2|\,|\tau_4 - \tau_3|\;\frac{|M_3(\tau_4)|}{|\tau_4 - \tau_1|\,|\tau_4 - \tau_2|\,|\tau_4 - \tau_3|} \\ &\lesssim |t|^{[2\kappa+\ell_2-\Lambda_3]^+}\,|\xi|^3\,. \end{aligned} \tag{6.5}$$

Combining (6.1), (6.2) (6.3) and (6.4) with (6.5), we get four different estimates of $|M_3(\tau_4)|\,|v|$:

$$\begin{aligned} |M_3(\tau_4)|\,|v| &\lesssim \frac{|t|^{[2\kappa+\ell_2-\Lambda_3]^+ +4}}{t_1^3}\,\frac{1}{|t|}\,\sqrt{E_2(t)} \\ &= \frac{|t|^{2\kappa+\ell_2+4-\min(2\kappa+\ell_2,\Lambda_3)}}{t_1^3}\,\frac{1}{|t|}\,\sqrt{E_2(t)} \end{aligned} \tag{6.6}$$

$$\begin{aligned} |M_3(\tau_4)|\,|v| &\lesssim \frac{|t|^{[2\kappa+\ell_2-\Lambda_3]^+ -\kappa+3}}{t_1^2}\,\frac{1}{|t|}\,\sqrt{E_2(t)} \\ &= \frac{|t|^{\kappa+\ell_2+3-\min(2\kappa+\ell_2,\Lambda_3)}}{t_1^2}\,\frac{1}{|t|}\,\sqrt{E_2(t)} \end{aligned} \tag{6.7}$$

$$\begin{aligned} |M_3(\tau_4)|\,|v| &\lesssim \frac{|t|^{[2\kappa+\ell_2-\Lambda_3]^+ -2\kappa+2}}{t_1}\,\frac{1}{|t|}\,\sqrt{E_2(t)} \\ &= \frac{|t|^{\ell_2+2-\min(2\kappa+\ell_2,\Lambda_3)}}{t_1}\,\frac{1}{|t|}\,\sqrt{E_2(t)} \end{aligned} \tag{6.8}$$

$$\begin{aligned} |M_3(\tau_4)|\,|v| &\lesssim |t|^{[2\kappa+\ell_2-\Lambda_3]^+ -2\kappa-\ell_2+1}\,[v]_3 \\ &= |t|^{1-\min(2\kappa+\ell_2,\Lambda_3)}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\,. \end{aligned} \tag{6.9}$$

Note that if we use the estimation

$$|v| \lesssim \frac{|\mathrm{L}_1\,v-\mathrm{L}_2\,v|}{|\tau_1-\tau_2|} \lesssim \frac{|\mathrm{L}_{12}\,v-\mathrm{L}_{13}\,v|}{|\tau_1-\tau_2|\,|\tau_1-\tau_3|} + \frac{|\mathrm{L}_{12}\,v-\mathrm{L}_{23}\,v|}{|\tau_1-\tau_2|\,|\tau_2-\tau_3|} \lesssim \frac{[v]_2}{t^{\kappa+\ell_1}\,|\xi|^2}$$

instead of (6.3), we get:

$$|M_3(\tau_4)|\,|v| \lesssim \frac{|t|^{\kappa-\ell_1+\ell_2+2-\min(2\kappa+\ell_2,\Lambda_3)}}{t_1}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\,,$$

which is weaker than (6.8) if $\ell_1 > \kappa$.

Now we combine these estimates to estimate $|M_3(\tau_4)|\,|v|$.

If $\min(2\kappa+\ell_2,\Lambda_3) \le 1$ from (6.9), we get:

$$|M_3(\tau_4)|\,|v| \lesssim \frac{1}{|t|}\,\sqrt{E_2(t)}\,.$$

If $1 < \min(2\kappa+\ell_2,\Lambda_3) \le \ell_2+2$, we use (6.8) for $t_1(\xi) \le |t| \le t_2(\xi)$ and (6.9) for $t_2(\xi) \le |t| \le T$, with $t_2(\xi) = |\xi|^{-(1-1/\widetilde{d}_0)/(\ell_2+1)}$; we have:

$$\begin{aligned} \big|M_3(\tau_4)\big|\,|v| &\lesssim \frac{|t|^{\ell_2+2-\min(2\kappa+\ell_2,\Lambda_3)}}{t_1}\,\frac{1}{|t|}\,\sqrt{E_2(t)} \\ &\lesssim \frac{t_2^{\ell_2+2-\min(2\kappa+\ell_2,\Lambda_3)}}{t_1}\,\frac{1}{|t|}\,\sqrt{E_2(t)} \end{aligned}$$

$$\lesssim |\xi|^{(1-1/\widetilde{d}_0)[1-(\ell_2-\min(2\kappa+\ell_2,\Lambda_3)+2)/(\ell_2+1)]}\,\frac{1}{|t|}\,\sqrt{E_2(t)}$$
$$\lesssim |\xi|^{1/\widetilde{d}_0}\,\frac{1}{|t|}\,\sqrt{E_2(t)}$$

for $t_1(\xi) \le |t| \le t_2(\xi)$, and:

$$\begin{aligned}\big|M_3(\tau_4)\big|\,|v| &\lesssim |t|^{1-\min(2\kappa+\ell_2,\Lambda_3)}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\\ &\lesssim t_2^{1-\min(2\kappa+\ell_2,\Lambda_3)}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\\ &\lesssim |\xi|^{(1-1/\widetilde{d}_0)(\min(2\kappa+\ell_2,\Lambda_3)-1)/(\ell_2+1)}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\\ &\lesssim |\xi|^{1/\widetilde{d}_0}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\end{aligned}$$

for $t_2(\xi) \le |t| \le T$, if $\widetilde{d}_0 \le \dfrac{\min(2\kappa+\ell_2,\Lambda_3)+\ell_2}{\min(2\kappa+\ell_2,\Lambda_3)-1}$.

If $\ell_2+2 < \min(2\kappa+\ell_2,\Lambda_3) \le \kappa+\ell_2+3$ we consider (6.7) for $t_1(\xi) \le |t| \le t_2(\xi)$ and (6.6) for $t_2(\xi) \le |t| \le T$, with $t_2(\xi) = |\xi|^{-(1-1/\widetilde{d}_0)/(\kappa+1)}$; we have:

$$\begin{aligned}\big|M_3(\tau_4)\big|\,|v| &\lesssim \frac{|t|^{\kappa+\ell_2+3-\min(2\kappa+\ell_2,\Lambda_3)}}{t_1^2}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\\ &\lesssim \frac{t_2^{\kappa+\ell_2+3-\min(2\kappa+\ell_2,\Lambda_3)}}{t_1^2}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\\ &\lesssim |\xi|^{(1-1/\widetilde{d}_0)[2-(\kappa+\ell_2-\min(2\kappa+\ell_2,\Lambda_3)+3)/(\kappa+1)]}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\\ &\lesssim |\xi|^{1/\widetilde{d}_0}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\,,\end{aligned}$$

for $t_1(\xi) \le |t| \le t_2(\xi)$, and:

$$\begin{aligned}\big|M_3(\tau_4)\big|\,|v| &\lesssim \frac{|t|^{\ell_2+2-\min(2\kappa+\ell_2,\Lambda_3)}}{t_1}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\\ &\lesssim \frac{t_2^{\ell_2+2-\min(2\kappa+\ell_2,\Lambda_3)}}{t_1}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\\ &\lesssim |\xi|^{(1-1/\widetilde{d}_0)[1-(\ell_2-\min(2\kappa+\ell_2,\Lambda_3)+2)/(\kappa+1)]}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\\ &\lesssim |\xi|^{1/\widetilde{d}_0}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\,,\end{aligned}$$

for $t_2(\xi) \le |t| \le T$, if $\widetilde{d}_0 \le \dfrac{2\kappa-\ell_2+\min(2\kappa+\ell_2,\Lambda_3)}{\kappa-\ell_2+\min(2\kappa+\ell_2,\Lambda_3)-1}$.

If $\min(2\kappa+\ell_2,\Lambda_3) > \kappa+\ell_2+3$, we consider (6.6) for $t_1(\xi) \le |t| \le t_2(\xi)$ and (6.7) for $t_2(\xi) \le |t| \le T$, with $t_2(\xi) = |\xi|^{-(1-1/\widetilde{d}_0)/(\kappa+1)}$; we have:

$$\begin{aligned}
\big|M_3(\tau_4)\big|\,|v| &\lesssim \frac{|t|^{2\kappa+\ell_2-\min(2\kappa+\ell_2,\Lambda_3)+4}}{t_1^3}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\\
&\lesssim \frac{t_2^{2\kappa+\ell_2-\min(2\kappa+\ell_2,\Lambda_3)+4}}{t_1^3}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\\
&\lesssim |\xi|^{(1-1/\widetilde{d}_0)[3-(2\kappa+\ell_2-\min(2\kappa+\ell_2,\Lambda_3)+4)/(\kappa+1)]}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\\
&\lesssim |\xi|^{1/\widetilde{d}_0}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\,,
\end{aligned}$$

for $t_1(\xi) \le |t| \le t_2(\xi)$, and:

$$\begin{aligned}
\big|M_3(\tau_4)\big|\,|v| &\lesssim \frac{|t|^{\kappa+\ell_2+3-\min(2\kappa+\ell_2,\Lambda_3)}}{t_1^2}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\\
&\lesssim \frac{t_2^{\kappa+\ell_2+3-\min(2\kappa+\ell_2,\Lambda_3)}}{t_1^2}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\\
&\lesssim |\xi|^{(1-1/\widetilde{d}_0)[2-(\kappa+\ell_2-\min(2\kappa+\ell_2,\Lambda_3)+3)/(\kappa+1)]}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\\
&\lesssim |\xi|^{1/\widetilde{d}_0}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\,,
\end{aligned}$$

for $t_2(\xi) \le |t| \le T$, if $\widetilde{d}_0 \le \dfrac{2\kappa-\ell_2+\min(2\kappa+\ell_2,\Lambda_3)}{\kappa-\ell_2+\min(2\kappa+\ell_2,\Lambda_3)-1}$.

Estimation of $|\Delta_1[M_3](\tau_3,\tau_4)|\,|\mathrm{L}_4\,v|$**.** We have:

$$|\mathrm{L}_4\,v| \lesssim \frac{|t|^2}{t_1^2\,|\xi|^2}\,\sqrt{E_2(t)} \tag{6.10}$$

$$|\mathrm{L}_4\,v| \lesssim \frac{|\,\mathrm{L}_{14}\,v-\mathrm{L}_{24}\,v|}{|\tau_1-\tau_2|} \lesssim \frac{[v]_2}{|\tau_1-\tau_2|} \lesssim \frac{|t|^{1-\kappa}}{t_1\,|\xi|^2}\,\sqrt{E_2(t)} \tag{6.11}$$

$$|\mathrm{L}_4\,v| \lesssim \frac{|\,\mathrm{L}_{14}\,v-\mathrm{L}_{24}\,v|}{|\tau_1-\tau_2|} \lesssim \frac{|\,\mathrm{L}_{124}\,v-\mathrm{L}_{134}\,v|}{|\tau_1-\tau_2|\,|\tau_2-\tau_3|} + \frac{|\,\mathrm{L}_{124}\,v-\mathrm{L}_{234}\,v|}{|\tau_1-\tau_2|\,|\tau_1-\tau_3|} \tag{6.12}$$

$$\lesssim \frac{|t|^{-\kappa-\ell_1}}{|\xi|^2}\,\sqrt{E_2(t)}\,. \tag{6.13}$$

On the other side, by Assumption 3, we have:

$$\begin{aligned}
\Big|\frac{M_3(\tau_3)}{\tau_4-\tau_3}\Big| &= |\tau_3-\tau_2|\,|\tau_3-\tau_1|\,\frac{|M_3(\tau_3)|}{|\tau_3-\tau_1|\,|\tau_3-\tau_2|\,|\tau_3-\tau_4|} \lesssim |t|^{\kappa+\ell_1-\Lambda_3}\,|\xi|^2\\
\Big|\frac{M_3(\tau_4)}{\tau_4-\tau_3}\Big| &= |\tau_4-\tau_1|\,|\tau_4-\tau_2|\,\frac{|M_3(\tau_4)|}{|\tau_4-\tau_1|\,|\tau_4-\tau_2|\,|\tau_4-\tau_3|} \lesssim |t|^{\kappa+\ell_2-\Lambda_3}\,|\xi|^2\,,
\end{aligned}$$

we get:

$$|\Delta_1[M_3](\tau_3,\tau_4)| \lesssim |t|^{[\kappa+\ell_1-\Lambda_3]^+}\,|\xi|^2\,. \tag{6.14}$$

Combinig (6.10), (6.11), (6.13) with (6.14), we get three different estimates of $|\Delta_1[M_3](\tau_3,\tau_4)|\,|\mathrm{L}_4\,v|$:

$$|\Delta_1[M_3](\tau_3,\tau_4)|\,|\mathrm{L}_4\,v| \lesssim \frac{|t|^{\kappa+\ell_1+3-\min(\kappa+\ell_1,\Lambda_3)}}{t_1^2}\,\frac{1}{|t|}\,\sqrt{E_2(t)} \tag{6.15}$$

$$|\Delta_1[M_3](\tau_3,\tau_4)|\,|\mathrm{L}_4\,v| \lesssim \frac{|t|^{\ell_1+2-\min(\kappa+\ell_1,\Lambda_3)}}{t_1}\,\frac{1}{|t|}\,\sqrt{E_2(t)} \tag{6.16}$$

$$|\Delta_1[M_3](\tau_3,\tau_4)|\,|\mathrm{L}_4\,v| \lesssim \frac{|t|^{1-\min(\kappa+\ell_1,\Lambda_3)}}{t_1}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\,. \tag{6.17}$$

Now we combine these three estimates to estimate $|\Delta_1[M_3](\tau_3,\tau_4)|$.

If $\min(\kappa+\ell_1,\Lambda_3)\le 1$, then (6.17) gives

$$|\Delta_1[M_3](\tau_3,\tau_4)|\,|\mathrm{L}_4\,v| \lesssim \frac{1}{|t|}\,\sqrt{E_2(t)}\,.$$

If $1<\min(\kappa+\ell_1,\Lambda_3)\le \ell_1+2$, we consider (6.16) for $t_1(\xi)\le|t|\le t_2(\xi)$ and (6.17) for $t_2(\xi)\le|t|\le T$, with $t_2(\xi)=|\xi|^{-(1-1/\widetilde{d}_0)/(\ell_1+1)}$; we have:

$$\begin{aligned}|\Delta_1[M_3](\tau_3,\tau_4)|\,|\mathrm{L}_4\,v| &\lesssim \frac{|t|^{\ell_1+2-\min(\kappa+\ell_1,\Lambda_3)}}{t_1}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\\ &\lesssim \frac{t_2^{\ell_1+2-\min(\kappa+\ell_1,\Lambda_3)}}{t_1}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\\ &\lesssim |\xi|^{(1-1/\widetilde{d}_0)[1-(\ell_1+2-\min(\kappa+\ell_1,\Lambda_3))/(\ell_1+1)]}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\\ &\lesssim |\xi|^{1/\widetilde{d}_0}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\end{aligned}$$

for $t_1(\xi)\le|t|\le t_2(\xi)$, and:

$$\begin{aligned}|\Delta_1[M_3](\tau_3,\tau_4)|\,|\mathrm{L}_4\,v| &\lesssim |t|^{1-\min(\kappa+\ell_1,\Lambda_3)}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\\ &\lesssim t_2^{1-\min(\kappa+\ell_1,\Lambda_3)}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\\ &\lesssim |\xi|^{-(1-1/\widetilde{d}_0)(1-\min(\kappa+\ell_1,\Lambda_3))/(\ell_1+1)}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\\ &\lesssim |\xi|^{1/\widetilde{d}_0}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\end{aligned}$$

for $t_2(\xi)\le|t|\le T$, if $\widetilde{d}_0 \le \dfrac{\min(\kappa+\ell_1,\Lambda_3)+\ell_1}{\min(\kappa+\ell_1,\Lambda_3)-1}$.

If $\min(\kappa+\ell_1,\Lambda_3) \geq \ell_1+2$, we consider (6.15) for $t_1(\xi) \leq |t| \leq t_2(\xi)$ and (6.16) for $t_2(\xi) \leq |t| \leq T$, with $t_2(\xi) = |\xi|^{-(1-1/\widetilde{d}_0)/(\ell_1+1)}$; we have:

$$\begin{aligned}|\Delta_1[M_3](\tau_3,\tau_4)|\,|\mathrm{L}_4\,v| &\lesssim \frac{|t|^{\kappa+\ell_1+3-\min(\kappa+\ell_1,\Lambda_3)}}{t_1^2}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\\ &\lesssim \frac{t_2^{\kappa+\ell_1+3-\min(\kappa+\ell_1,\Lambda_3)}}{t_1^2}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\\ &\lesssim |\xi|^{(1-1/\widetilde{d}_0)[2-(\kappa+\ell_1+3-\min(\kappa+\ell_1,\Lambda_3))/(\ell_1+1)]}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\\ &\lesssim |\xi|^{1/\widetilde{d}_0}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\,,\end{aligned}$$

for $t_1(\xi) \leq |t| \leq t_2(\xi)$, and:

$$\begin{aligned}|\Delta_1[M_3](\tau_3,\tau_4)|\,|\mathrm{L}_4\,v| &\lesssim \frac{|t|^{\ell_1+2-\min(\kappa+\ell_1,\Lambda_3)}}{t_1}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\\ &\lesssim \frac{t_2^{\ell_1+2-\min(\kappa+\ell_1,\Lambda_3)}}{t_1}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\\ &\lesssim |\xi|^{(1-1/\widetilde{d}_0)[1-(\ell_1+2-\min(\kappa+\ell_1,\Lambda_3))/(\ell_1+1)]}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\\ &\lesssim |\xi|^{1/\widetilde{d}_0}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\,,\end{aligned}$$

for $t_2(\xi) \leq |t| \leq T$, if $\widetilde{d}_0 \leq \dfrac{2\kappa-\ell_1+\min(\kappa+\ell_1,\Lambda_3)}{\kappa-\ell_1+\min(\kappa+\ell_1,\Lambda_3)-1}$.

Estimation of $|\Delta_2[M_3](\tau_2,\tau_3,\tau_4)|\,|\mathrm{L}_{34}\,v|$. Since:

$$\begin{aligned}|\mathrm{L}_{34}\,v| &\lesssim \frac{|t|}{t_1\,|\xi|}\,\sqrt{E_2(t)}\\ |\mathrm{L}_{34}\,v| &\lesssim \frac{|\mathrm{L}_{134}\,v-\mathrm{L}_{234}\,v|}{|\tau_1-\tau_2|} \lesssim |t|^{-\kappa}\,\sqrt{E_2(t)}\end{aligned}$$

and, thanks to Assumption 3:

$$\begin{aligned}|\Delta_2[M_3](\tau_2,\tau_3,\tau_4)| \lesssim \left|\frac{M_3(\tau_2)}{(\tau_2-\tau_3)(\tau_2-\tau_4)}\right| + \left|\frac{M_3(\tau_3)}{(\tau_3-\tau_2)(\tau_3-\tau_4)}\right|\\ + \left|\frac{M_3(\tau_4)}{(\tau_4-\tau_2)(\tau_4-\tau_3)}\right| \lesssim |t|^{[\kappa-\Lambda_3]^+}\,|\xi|\,,\end{aligned}$$

we get two different estimates of $|\Delta_2[M_3](\tau_2,\tau_3,\tau_4)|\,|\mathrm{L}_{34}\,v|$

$$|\Delta_2[M_3](\tau_2,\tau_3,\tau_4)|\,|\mathrm{L}_{34}\,v| \lesssim \frac{|t|^{\kappa+2-\min(\kappa,\Lambda_3)}}{t_1}\,\frac{1}{|t|}\,\sqrt{E_2(t)} \tag{6.18}$$

$$|\Delta_2[M_3](\tau_2,\tau_3,\tau_4)|\,|\mathrm{L}_{34}\,v| \lesssim |t|^{1-\min(\kappa,\Lambda_3)}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\,. \tag{6.19}$$

If $\min(\kappa,\Lambda_3)\le 1$ from (6.19), we get:

$$|\Delta_2[M_3](\tau_2,\tau_3,\tau_4)|\,|\mathrm{L}_{34}\,v| \lesssim \frac{1}{|t|}\,\sqrt{E_2(t)}\,.$$

If $1<\min(\kappa,\Lambda_3)\le\kappa+2$, we consider (6.18) for $t_1(\xi)\le|t|\le t_2(\xi)$ and (6.19) for $t_2(\xi)\le|t|\le T$, with $t_2(\xi)=|\xi|^{-(1-1/\widetilde{d}_0)/(\kappa+1)}$; we have:

$$\begin{aligned}
|\Delta_2[M_3](\tau_2,\tau_3,\tau_4)|\,|\mathrm{L}_{34}\,v| &\lesssim \frac{|t|^{\kappa+2-\min(\kappa,\Lambda_3)}}{t_1}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\\
&\lesssim \frac{t_2^{\kappa+2-\min(\kappa,\Lambda_3)}}{t_1}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\\
&\lesssim |\xi|^{(1-1/\widetilde{d}_0)[1-(\kappa+2-\min(\kappa,\Lambda_3))/(\kappa+1)]}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\\
&\lesssim |\xi|^{1/\widetilde{d}_0}\,\frac{1}{|t|}\,\sqrt{E_2(t)}
\end{aligned}$$

for $t_1(\xi)\le|t|\le t_2(\xi)$, and:

$$\begin{aligned}
|\Delta_2[M_3](\tau_2,\tau_3,\tau_4)|\,|\mathrm{L}_{34}\,v| &\lesssim |t|^{1-\min(\kappa,\Lambda_3)}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\\
&\lesssim t_2^{1-\min(\kappa,\Lambda_3)}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\\
&\lesssim |\xi|^{(1-1/\widetilde{d}_0)(1-\min(\kappa,\Lambda_3))/(\kappa+1)}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\\
&\lesssim |\xi|^{1/\widetilde{d}_0}\,\frac{1}{|t|}\,\sqrt{E_2(t)}
\end{aligned}$$

for $t_2(\xi)\le|t|\le T$, if $\widetilde{d}_0\le\dfrac{\min(\kappa,\Lambda_3)+\kappa}{\min(\kappa,\Lambda_3)-1}$.

Estimation of $|M_2(\tau_4)|\,|v|$. Since:

$$|M_2(\tau_4)|=|\tau_4-\tau_1|\,|\tau_4-\tau_3|\,\frac{|M_2(\tau_4)|}{|\tau_4-\tau_1|\,|\tau_4-\tau_3|}\lesssim |t|^{[2\kappa-\Lambda_2]^+}\,|\xi|^2\,,$$

using (6.1)-(6.3), we have three different estimates of $|M_2(\tau_4)|\,|v|$:

$$|M_2(\tau_4)|\,|v|\lesssim\frac{|t|^{2\kappa+4-\min(2\kappa,\Lambda_2)}}{t_1^3\,|\xi|}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\,,\tag{6.20}$$

$$|M_2(\tau_4)|\,|v|\lesssim|M_2(\tau_4)|\,\frac{[v]_1}{|\tau_4-\tau_1|}\lesssim\frac{|t|^{\kappa+3-\min(2\kappa,\Lambda_2)}}{t_1^2\,|\xi|}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\,,\tag{6.21}$$

$$|M_2(\tau_4)|\,|v|\lesssim\frac{|t|^{2-\min(2\kappa,\Lambda_2)}}{t_1\,|\xi|}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\,.\tag{6.22}$$

Now we combine these estimates to estimate $|M_2(\tau_4)|\,|v|$.

If $\min(2\kappa, \Lambda_2) \le 2$, (6.22) for $t_1(\xi) \le |t| \le T$ gives:

$$|M_2(\tau_4)|\,|v| \lesssim \frac{1}{|t|}\sqrt{E_2(t)}\,.$$

If $2 < \min(2\kappa, \Lambda_2) \le \kappa + 3$, we consider (6.21) for $t_1(\xi) \le |t| \le t_2(\xi)$ and (6.22) for $t_2(\xi) \le |t| \le T$, with $t_2(\xi) = |\xi|^{-(1-1/\widetilde{d}_0)/(\kappa+1)}$; we have:

$$\begin{aligned}|M_2(\tau_4)|\,|v| &\lesssim \frac{|t|^{\kappa+3-\min(2\kappa,\Lambda_2)}}{t_1^2\,|\xi|}\frac{1}{|t|}\sqrt{E_2(t)} \lesssim \frac{t_2^{\kappa+3-\min(2\kappa,\Lambda_2)}}{t_1^2\,|\xi|}\frac{1}{|t|}\sqrt{E_2(t)}\\ &\lesssim |\xi|^{(1-1/\widetilde{d}_0)[2-(\kappa+3-\min(2\kappa,\Lambda_2))/(\kappa+1)]-1}\frac{1}{|t|}\sqrt{E_2(t)}\\ &\lesssim |\xi|^{1/\widetilde{d}_0}\frac{1}{|t|}\sqrt{E_2(t)}\end{aligned}$$

for $t_1(\xi) \le |t| \le t_2(\xi)$, and:

$$\begin{aligned}|M_2(\tau_4)|\,|v| &\lesssim \frac{|t|^{2-\min(2\kappa,\Lambda_2)}}{t_1\,|\xi|}\frac{1}{|t|}\sqrt{E_2(t)} \lesssim \frac{t_2^{2-\min(2\kappa,\Lambda_2)}}{t_1\,|\xi|}\frac{1}{|t|}\sqrt{E_2(t)}\\ &\lesssim |\xi|^{(1-1/\widetilde{d}_0)[1-(2-\min(2\kappa,\Lambda_2))/(\kappa+1)]-1}\frac{1}{|t|}\sqrt{E_2(t)}\\ &\lesssim |\xi|^{1/\widetilde{d}_0}\frac{1}{|t|}\sqrt{E_2(t)}\end{aligned}$$

for $t_2(\xi) \le |t| \le T$, if $\widetilde{d}_0 \le \dfrac{\kappa + \min(2\kappa,\Lambda_2)}{\min(2\kappa,\Lambda_2) - 1}$.

If $\min(2\kappa, \Lambda_2) > \kappa + 3$, we consider (6.20) for $t_1(\xi) \le |t| \le t_2(\xi)$ and (6.21) for $t_2(\xi) \le |t| \le T$, with $t_2(\xi) = |\xi|^{-(1-1/\widetilde{d}_0)/(\kappa+1)}$; we have:

$$\begin{aligned}|M_2(\tau_4)|\,|v| &\lesssim \frac{|t|^{2\kappa+4-\min(2\kappa,\Lambda_2)}}{t_1^3\,|\xi|}\frac{1}{|t|}\sqrt{E_2(t)} \lesssim \frac{t_2^{2\kappa+4-\min(2\kappa,\Lambda_2)}}{t_1^3\,|\xi|}\frac{1}{|t|}\sqrt{E_2(t)}\\ &\lesssim |\xi|^{(1-1/\widetilde{d}_0)[3-(2\kappa-\min(2\kappa,\Lambda_2)+4)/(\kappa+1)]-1}\frac{1}{|t|}\sqrt{E_2(t)}\\ &\lesssim |\xi|^{1/\widetilde{d}_0}\frac{1}{|t|}\sqrt{E_2(t)}\end{aligned}$$

for $t_1(\xi) \le |t| \le t_2(\xi)$, and:

$$\begin{aligned}|M_2(\tau_4)|\,|v| &\lesssim \frac{|t|^{\kappa+3-\min(2\kappa,\Lambda_2)}}{t_1^2\,|\xi|}\frac{1}{|t|}\sqrt{E_2(t)} \lesssim \frac{t_2^{\kappa+3-\min(2\kappa,\Lambda_2)}}{t_1^2\,|\xi|}\frac{1}{|t|}\sqrt{E_2(t)}\\ &\lesssim |\xi|^{(1-1/\widetilde{d}_0)[2-(\kappa+3-\min(2\kappa,\Lambda_2))/(\kappa+1)]-1}\frac{1}{|t|}\sqrt{E_2(t)}\\ &\lesssim |\xi|^{1/\widetilde{d}_0}\frac{1}{|t|}\sqrt{E_2(t)}\end{aligned}$$

for $t_2(\xi) \le |t| \le T$, if $\widetilde{d}_0 \le \dfrac{\kappa + \min(2\kappa,\Lambda_2)}{\min(2\kappa,\Lambda_2) - 1}$.

Estimation of $|\Delta_1[M_2](\tau_3,\tau_4)|\,|\mathrm{L}_4\, v|$. Since

$$\left|\frac{M_2(\tau_3)}{\tau_4-\tau_3}\right| = |\tau_3-\tau_2|\ \frac{|M_2(\tau_3)|}{|\tau_3-\tau_2|\,|\tau_3-\tau_4|} \lesssim |t|^{\kappa-\Lambda_2}\ |\xi|$$
$$\left|\frac{M_2(\tau_4)}{\tau_4-\tau_3}\right| = |\tau_4-\tau_1|\ \frac{|M_2(\tau_4)|}{|\tau_4-\tau_1|\,|\tau_4-\tau_3|} \lesssim |t|^{\kappa-\Lambda_2}\ |\xi|\,,$$

we have:

$$|\Delta_1[M_2](\tau_3,\tau_4)| \lesssim |t|^{[\kappa-\Lambda_2]^+}\ |\xi|\,,$$

hence, using (6.10) and (6.11), we get two different estimates of $|\Delta_1[M_2](\tau_3,\tau_4)|$:

$$|\Delta_1[M_2](\tau_3,\tau_4)|\,|\mathrm{L}_4\, v| \lesssim \frac{|t|^{\kappa+3-\min(\kappa,\Lambda_2)}}{t_1^2\,|\xi|}\ \frac{1}{|t|}\ \sqrt{E_2(t)}\ , \tag{6.23}$$

$$|\Delta_1[M_2](\tau_3,\tau_4)|\,|\mathrm{L}_4\, v| \lesssim \frac{|t|^{2-\min(\kappa,\Lambda_2)}}{t_1^2\,|\xi|}\ \frac{1}{|t|}\ \sqrt{E_2(t)}\ . \tag{6.24}$$

Now we combine these estimates to estimate $|M_2(\tau_4)|\,|v|$.

If $\min(\kappa,\Lambda_2)\le 2$, (6.24) for $t_1(\xi)\le|t|\le T$ gives:

$$|M_2(\tau_4)|\,|v| \lesssim \frac{1}{|t|}\ \sqrt{E_2(t)}\ .$$

If $\min(\kappa,\Lambda_2)>2$, we consider (6.23) for $t_1(\xi)\le|t|\le t_2(\xi)$ and (6.24) for $t_2(\xi)\le|t|\le T$, with $t_2(\xi)=|\xi|^{-(1-1/\widetilde{d}_0)/(\kappa+1)}$; we have:

$$\begin{aligned}|M_2(\tau_4)|\,|v| &\lesssim \frac{|t|^{\kappa+3-\min(\kappa,\Lambda_2)}}{t_1^2\,|\xi|}\ \frac{1}{|t|}\ \sqrt{E_2(t)}\\ &\lesssim \frac{t_2^{\kappa+3-\min(\kappa,\Lambda_2)}}{t_1^2\,|\xi|}\ \frac{1}{|t|}\ \sqrt{E_2(t)}\\ &\lesssim |\xi|^{(1-1/\widetilde{d}_0)[2-(\kappa+3-\min(\kappa,\Lambda_2))/(\kappa+1)]-1}\ \frac{1}{|t|}\ \sqrt{E_2(t)}\\ &\lesssim |\xi|^{1/\widetilde{d}_0}\ \frac{1}{|t|}\ \sqrt{E_2(t)}\end{aligned}$$

for $t_1(\xi)\le|t|\le t_2(\xi)$, and:

$$\begin{aligned}|M_2(\tau_4)|\,|v| &\lesssim \frac{|t|^{2-\min(\kappa,\Lambda_2)}}{t_1\,|\xi|}\ \frac{1}{|t|}\ \sqrt{E_2(t)}\\ &\lesssim \frac{t_2^{2-\min(\kappa,\Lambda_2)}}{t_1\,|\xi|}\ \frac{1}{|t|}\ \sqrt{E_2(t)}\\ &\lesssim |\xi|^{(1-1/\widetilde{d}_0)[1-(2-\min(\kappa,\Lambda_2))/(\kappa+1)]-1}\ \frac{1}{|t|}\ \sqrt{E_2(t)}\\ &\lesssim |\xi|^{1/\widetilde{d}_0}\ \frac{1}{|t|}\ \sqrt{E_2(t)}\end{aligned}$$

for $t_2(\xi)\le|t|\le T$, if $\widetilde{d}_0 \le \dfrac{\kappa+\min(\kappa,\Lambda_2)}{\min(\kappa,\Lambda_2)-1}$.

Estimation of $|M_1(\tau_4)|\,|v|$. Since

$$|M_1(\tau_4)| \lesssim |t|^{[\kappa,\Lambda_1]^+}\,|\xi|\ ,$$

using (6.1) and (6.2), we have two different estimates of $|M_1(\tau_4)|\,|v|$:

$$|M_1(\tau_4)|\,|v| \lesssim \frac{|t|^{\kappa+4-\min(\kappa,\Lambda_1)}}{t_1^3\,|\xi|^2}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\ , \tag{6.25}$$

$$|M_1(\tau_4)|\,|v| \lesssim \frac{|t|^{3-\min(\kappa,\Lambda_1)}}{t_1^2\,|\xi|^2}\,\frac{1}{|t|}\,\sqrt{E_2(t)}\ . \tag{6.26}$$

Now we combine these estimates to estimate $|M_1(\tau_4)|\,|v|$.

If $\min(\kappa,\Lambda_1) \le 3$, (6.26) for $t_1(\xi) \le |t| \le T$ gives:

$$|M_1(\tau_4)|\,|v| \lesssim \frac{1}{|t|}\,\sqrt{E_2(t)}\ .$$

If $\min(\kappa,\Lambda_1) > 3$, we consider (6.25) for $t_1(\xi) \le |t| \le t_2(\xi)$ and (6.26) for $t_2(\xi) \le |t| \le T$, with $t_2(\xi) = |\xi|^{-(1-1/\widetilde{d}_0)/(\kappa+1)}$; we have:

$$\begin{aligned}
|M_1(\tau_4)|\,|v| &\lesssim \frac{|t|^{\kappa+4-\min(\kappa,\Lambda_1)}}{t_1^3\,|\xi|^2}\,\frac{1}{|t|}\,\sqrt{E_2(t)} \\
&\lesssim \frac{t_2^{\kappa+4-\min(\kappa,\Lambda_1)}}{t_1^3\,|\xi|^2}\,\frac{1}{|t|}\,\sqrt{E_2(t)} \\
&\lesssim |\xi|^{(1-1/\widetilde{d}_0)[2-(\kappa+4-\min(\kappa,\Lambda_1))/(\kappa+1)]-2}\,\frac{1}{|t|}\,\sqrt{E_2(t)} \\
&\lesssim |\xi|^{1/\widetilde{d}_0}\,\frac{1}{|t|}\,\sqrt{E_2(t)}
\end{aligned}$$

for $t_1(\xi) \le |t| \le t_2(\xi)$, and:

$$\begin{aligned}
|M_1(\tau_4)|\,|v| &\lesssim \frac{|t|^{2-\min(\kappa,\Lambda_1)}}{t_1^2\,|\xi|^2}\,\frac{1}{|t|}\,\sqrt{E_2(t)} \\
&\lesssim \frac{t_2^{2-\min(\kappa,\Lambda_1)}}{t_1^2\,|\xi|^2}\,\frac{1}{|t|}\,\sqrt{E_2(t)} \\
&\lesssim |\xi|^{(1-1/\widetilde{d}_0)[1-(2-\min(\kappa,\Lambda_1))/(\kappa+1)]-2}\,\frac{1}{|t|}\,\sqrt{E_2(t)} \\
&\lesssim |\xi|^{1/\widetilde{d}_0}\,\frac{1}{|t|}\,\sqrt{E_2(t)}
\end{aligned}$$

for $t_2(\xi) \le |t| \le T$, if $\widetilde{d}_0 \le \dfrac{3\kappa+\min(\kappa,\Lambda_1)}{\min(\kappa,\Lambda_1)-3}$.

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