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A note on automorphisms of prime and semiprime rings

By

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Abstract

Let f be a centralizing automorphism of a semiprime ring R. Then for all $x \in R$ $(f(x) - x) \in Z(R)$; that is the mapping $f - 1$ maps R into its centre.

A ring R is said to be prime if $aRb = 0$ implies that either $a = 0$ or $b = 0$; and semiprime if $aRa = 0$ implies that $a = 0$. A prime ring is obviously semiprime. We write $[x, y] = xy - yx$. Then $[xy, z] = x[y, z] + [x, z]y$ and $[x, yz] = y[x, z] + [x, y]z$. A mapping $f : R \to R$ is said to be commuting if $[f(x), x] = 0$ and is centralizing if $[f(x), x] \in Z(R)$ (where $Z(R)$ denotes the centre of R) for all $x \in R$. If $f : R \to R$ is additive and commuting, then replacing x by $x + y$ in $[f(x), x] = 0$ we get $[f(x), y] = [x, f(y)]$ for all $x, y \in R$.

The centralising automorphisms on prime and semiprime rings have been investigated by a number of known researchers like Mayne [6], Luh [4] and Bell [1]. In this paper we have proved a number of useful identities for semiprime rings which then provide alternate proofs to the well known results of Mayne [6] and Luh [4].

We first prove the following theorem, which proves certain identities of independent interest and can be used further to investigate centralizing automorphisms.

Theorem 0.1. *Let f be a centralizing automorphism of a semiprime ring R. Then for all $x, y, z \in R$,*

(i) $f(x)[y, z] = x[y, z]$

(ii) $[y, z]f(x) = [y, z]x$

(iii) $(f(x) - x) \in Z(R)$ *for all x; that is the mapping $f - 1$ maps R into its centre.*

Proof.

(i) Since f is centralizing, f is commuting by Lemma 2 of [1]. Further

$$[f(xy), x] = [xy, f(x)] = x[y, f(x)] + [x, f(x)]y = x[y, f(x)] = x[f(y), x]$$

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and therefore

$$(1) \qquad [f(xy), x] = x[f(y), x].$$

Also

$$[f(xy), x] = [f(x)f(y), x] = f(x)[f(y), x] + [f(x), x]f(y) = f(x)[f(y), x]$$

and so by (1)

$$f(x)[f(y), x] = x[f(y), x].$$

Since f is onto, we get $f(x)[y, x] = x[y, x]$. This implies that for all $x, y \in R$,

$$(2) \qquad (f(x) - x)[y, x] = 0.$$

For an arbitrary element $u \in R$, replacing y by uy in (2), we get

$$(f(x) - x)[uy, x] = (f(x) - x)u[y, x] + (f(x) - x)[u, x]y = 0$$

and thus by (2)

$$(3) \qquad (f(x) - x)u[y, x] = 0.$$

Replacing x by $x + z$ in (2), we get

$$(f(x + z) - (x + z))[y, x + z] = 0$$

or

$$(4) \quad (f(x) - x)[y, x] + (f(x) - x)[y, z] + (f(z) - z)[y, x] + (f(z) - z)[y, z] = 0.$$

Using (2) (4), we get

$$(f(x) - x)[y, z] + (f(z) - z)[y, x] = 0$$

and hence

$$(f(x) - x)[y, z] = -(f(z) - z)[y, x].$$

Now for arbitrary $v \in R$, we have

$$(f(x) - x)[y, z]v(f(x) - x)[y, z] = (f(x) - x)[y, z]v(f(z) - z)[y, x] = 0,$$

by using equation (3). Since R is semiprime, we get

$$(f(x) - x)[y, z] = 0$$

for all $x, y, z \in R$.

(ii) Replacing y by uy in (i), we get

$$(5) \qquad (f(x) - x)u[y, z] = 0.$$

This implies that

$$[y,z](f(x)-x)u[y,z](f(x)-x)=0$$

and since R is semiprime, we get

$$[y,z](f(x)-x)=0.$$

(iii) From (i) and (ii) we get

$$(f(x)-x)[y,z]=[y,z](f(x)-x)$$

for all $x,y,z\in R$. Hence by Lemma 1.1.8 of Herstein [3], we have for all $x\in R$, $(f(x)-x)\in Z(R)$. $\square$

The well known result of Mayne [6] and Luh [4] follows as a corollary of the above Theorem.

Corollary 0.2. *Let R be a prime ring and f be a non-trivial centralizing automorphism on R. Then R is a commutative.*

Proof. Since f is non-trivial, there exists an $a\in R$ such that $f(a)\neq a$. From (i) of Theorem 0.1 we have $(f(a)-a)[y,z]=0$. Replacing y by uy, we again get (5) and hence

$$(f(a)-a)R[y,z]=0$$

for all $y,z\in R$. Since R is prime and $(f(a)-a)\neq 0$, we have $[y,z]=0$ and therefore R is commutative. $\square$

Laradji and Thaheem [5] introduced the notion of a dependent element of an automorphism as an algebraic generalization of the concept given by Choda, Kasanara and Nakamoto [2] for C^*- algebras and showed that any dependent element of a centralizing mapping on a semiprime ring is central. The proof of Laradji and Thaheem is quite complicated. We provide a short proof of this result as a simple corollary of Theorem 0.1. But first we need the following definition and lemma.

Definition 0.3. If f is a self map of a ring R and if $a\in R$ is such that $f(x)a=ax$ for all x then a is said to be a dependent element of f. We denote by $D(f)$ the set of all dependent elements of f.

Following is a short proof of the result given by Laradji and Thaheem in [5].

Theorem 0.4. *Let R be a semiprime ring and f be a centralizing automorphism of R. Then $D(f)\subseteq Z(R)$.*

Proof. Since f is centralizing, we have that f is commuting by [1] . Now if $a\in D(f)$ then

$$(f(x)-x)a=f(x)a-xa=ax-xa=[a,x].$$

By Theorem 0.1,

$$a(f(x)-x)[y,z] = [y,z](f(x)-x)a = 0.$$

Since $f(x)-x \in Z(R)$, we have

$$(f(x)-x)a[y,z] = [y,z](f(x)-x)a = 0$$

or

$$[a,x][y,z] = [y,z][a,x] = 0,$$

for all $y,z \in R$. Thus $[a,x] \in Z(R)$ by Lemma 1.1.8 of Herstein [3]. Hence $[a,x][a,x] = ([a,x])^2 = 0$. Since 0 is the only nilpotent central element of R, we get $[a,x] = 0$ for all x. This implies that $a \in Z(R)$ and hence

$$D(f) \subseteq Z(R).$$

□

Corollary 0.5. *If f is a centralizing inner automorphism of a semiprime ring R, then f is the identity map.*

Proof. Let a be an invertible element of R. Assume that $f(x) = axa^{-1}$, for all $x \in R$. Then $f(x)a = ax$. Therefore a is a dependent element of f and hence by Theorem 0.4 $a \in Z(R)$. It follows that $f(x) = x$ for all $x \in R$. □

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