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A remark on the Alperin-Mckay conjecture

By

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Introduction

Let G be a finite group and p a prime. For an irreducible character χ of G, let $\operatorname{ht}\chi$ be the height of χ (defined in terms of the p-block of G to which χ belongs). Let N be a normal subgroup of G. We define $\operatorname{ht}_N(\chi)$, the N-*height* of χ, by

$$\operatorname{ht}_N(\chi) = \operatorname{ht}\chi - \operatorname{ht}\xi,$$

where ξ is an irreducible constituent of χ_N. Clearly $\operatorname{ht}_N(\chi)$ does not depend on the choice of ξ. In [M2] we have shown that it always holds that $\operatorname{ht}_N(\chi) \geqq 0$. In the present paper we are concerned with the number of irreducible characters χ with $\operatorname{ht}_N(\chi) = 0$.

Let B be a block of G with defect group D. Let $k_0(B)$ be the number of irreducible characters in B of height 0. We recall the Alperin-McKay conjecture:

(AM) If $\widetilde{B}$ is the Brauer correspondent of B in $N_G(D)$, then $k_0(B) = k_0(\widetilde{B})$.

Let

$$k_0(B, N) = |\{\chi \in \operatorname{Irr}(B); \operatorname{ht}_N(\chi) = 0\}|.$$

We consider the following analogue of (AM).

$(*)$ If $\widetilde{B}$ is the Brauer correspondent of B in $N_G(D)N$, then $k_0(B, N) = k_0(\widetilde{B}, N)$.

For an integer $h \geqq 0$, let

$$\begin{aligned}&\operatorname{Irr}_0(B, N, h) = \{\chi \in \operatorname{Irr}(B); \operatorname{ht}_N(\chi) = 0, \operatorname{ht}\chi = h\} \ \text{ and}\\ &k_0(B, N, h) = |\operatorname{Irr}_0(B, N, h)|.\end{aligned}$$

In Section 1 we prove the following theorem (Theorem 7).

Theorem. *Let D be a defect group of B and let $\widetilde{B}$ be the Brauer correspondent of B in $N_G(D)N$. We assume that (AM) is true for any block of any*

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central extension of any subgroup of G/N. Then $k_0(B, N, h) = k_0(\widetilde{B}, N, h)$ for any integer $h \geqq 0$. In particular, $k_0(B, N) = k_0(\widetilde{B}, N)$.

Since $k_0(B, \{1\}) = k_0(B)$, if $(*)$ holds for all G and N, then (AM) holds for all G. Theorem shows that the converse also holds. Thus (AM) and $(*)$ are equivalent statements. Moreover, we can apply Theorem to reduce (to some extent) the conjecture (AM) itself. In particular, we obtain an alternative proof of part of Okuyama-Wajima's proof of (AM) for p-solvable groups, see Section 2. Recently Isaacs and Navarro [IN] proposed refinements of the McKay conjecture. The methods and results of the present paper could be adapted without much difficulty to one of their refinements, cf. Conjecture B of [IN]. The author is grateful to the referee for valuable suggestions.

1. Proof of Theorem

Let N be a normal subgroup of a group G. Let B be a block of G with defect group D. Let b be a block of N which is covered by B. Let $T_G(b)$ be the inertial group of b in G. We say that D is an *inertial defect group* of B over b if D is a defect group of the Fong-Reynolds correspondent of B over b in $T_G(b)$. For a subgroup H of G containing $N_G(D)$, there is a unique block $\widetilde{B}$ of H with defect group D such that $\widetilde{B}^G = B$. We call $\widetilde{B}$ the Brauer correspondent of B in H.

Lemma 1. *Let $\widetilde{B}$ be the Brauer correspondent of B in $N_G(D)N$. The following are equivalent.*

(i) *$\widetilde{B}$ covers b.*

(ii) *D is an inertial defect group of B over b.*

Proof. Put $T = T_G(b)$ and $H = N_G(D)N$.

(i) $\Rightarrow$ (ii): Let β' be the Fong-Reynolds correspondent of $\widetilde{B}$ over b in $H \cap T$. We claim that D is a defect group of β'. Indeed, let Q be a defect group of β'. We have $Q = D^x$ for some $x \in H$. If we write $x = yz$ with $y \in N_G(D)$ and $z \in N$, then $Q = D^z$. So the claim follows.

Since $N_T(D) \leqq H \cap T$, $\beta := \beta'^T$ is defined and has defect group D. Clearly β covers b, and $\beta^G = (\beta'^T)^G = (\beta'^H)^G = \widetilde{B}^G = B$. So β is the Fong-Reynolds correspondent of B over b in T. Thus the result follows.

(ii) $\Rightarrow$ (i): Let β be the Fong-Reynolds correspondent of B over b in T. So D is a defect group of β. Let $\widetilde{\beta}$ be the Brauer correspondent of β in $H \cap T$. Clearly $\widetilde{\beta}$ covers b. So $\widetilde{\beta}$ is the Fong-Reynolds correspondent of a block B' of H over b in $H \cap T$. Then $B'^G = (\widetilde{\beta}^H)^G = (\widetilde{\beta}^T)^G = \beta^G = B$. Since B' has defect group D, we get $B' = \widetilde{B}$. So $\widetilde{B}$ covers b. This completes the proof. $\square$

Lemma 2. (i) *If D is an inertial defect group of B over b, then D is a defect group of a unique block of DN covering b.*

(ii) *If b is G-invariant, then $N_G(DN) = N_G(D)N$.*

Proof. (i) This is [M1, Lemma 2.2].

(ii) If b' is the block of DN as in (i), b' is $N_G(DN)$-invariant. So the result follows from (i) and the Frattini argument. $\square$

For an irreducible character ξ in b, put

$$\begin{aligned}
&\mathrm{Irr}(B|\xi) = \{\chi \in \mathrm{Irr}(B); (\chi_N, \xi)_N \neq 0\},\\
&\mathrm{Irr}_0(B|\xi) = \{\chi \in \mathrm{Irr}(B|\xi); \mathrm{ht}_N(\chi) = 0\}, \quad \text{and}\\
&k_0(B|\xi) = |\mathrm{Irr}_0(B|\xi)|.
\end{aligned}$$

Let $T_G(\xi)$ be the inertial group of ξ in G.

Lemma 3. *Let ξ be an irreducible character in b. Let D be a defect group of B with $D \leqq T_G(\xi)$. The following are equivalent.*
(i) *D is an inertial defect group of B over b.*
(ii) *There is a block β of $T_G(\xi)$ with the following properties:*
β has defect group D, β covers b, and $\beta^G = B$.

Proof. (i) $\Rightarrow$ (ii): Let $H = N_G(D)N$ and $T = T_G(\xi)$. Let $\widetilde{B}$ be the Brauer correspondent of B in H. By Lemma 1, $\widetilde{B}$ covers b. Take $\zeta \in \mathrm{Irr}(\widetilde{B}|\xi)$ and let η be the Clifford correspondent of ζ over ξ in $H \cap T$. Let β' be the block of $H \cap T$ containing η. Then, since $\eta^H = \zeta$, we have $\beta'^H = \widetilde{B}$. Since β' covers a unique block of DN covering b, there is a defect group Q of β' with $Q \geqq D$ by Lemma 2. Since $\beta'^G = (\beta'^H)^G = \widetilde{B}^G = B$, we have $Q \leqq_G D$. Thus $Q = D$. Then, since $N_T(D) \leqq H \cap T$, $\beta := \beta'^T$ is defined and has defect group D. Clearly β covers b. Since $\beta^G = \beta'^G = B$, the result follows.

(ii) $\Rightarrow$ (i): We see $\beta^{T_G(b)}$ is defined and $(\beta^{T_G(b)})^G = B$. So D is a defect group of $\beta^{T_G(b)}$, which is the Fong-Reynolds correspondent of B over b in $T_G(b)$. This completes the proof. $\square$

Lemma 4. *Let ξ be an irreducible character in b. The following are equivalent.*
(i) *There is $\chi \in \mathrm{Irr}(B|\xi)$ such that $\mathrm{ht}\,\chi = \mathrm{ht}\,\xi$.*
(ii) *For an inertial defect group $D^x (x \in G)$ of B over b, ξ extends to $D^x N$.*

Proof. This follows from [M2, Theorem 2.3]. $\square$

We need the following.

Lemma 5. *Let Z be a central subgroup of G. Let λ be a linear character of Z lying in a block of Z covered by B. Let $\widetilde{B}$ be the Brauer correspondent of B in $N_G(D)$. Assume (AM) is true for any block of G/W with $1 \leqq W \leqq Z$. Then $k_0(B|\lambda) = k_0(\widetilde{B}|\lambda)$.*

Proof. We argue by induction on $|Z|$. If Z is a p'-group, then, $k_0(B|\lambda) = k_0(B)$ and $k_0(\widetilde{B}|\lambda) = k_0(\widetilde{B})$. So the result follows. Thus we may assume $|Z|$ is divisible by p. Assume $W := \mathrm{Ker}(\lambda) \neq 1$. Let β and $\widetilde{\beta}$ be the blocks of G/W and $N_G(D)/W$ dominated by B and $\widetilde{B}$, respectively. Clearly DW/W

is a common defect group of β and $\widetilde{\beta}$, and $\widetilde{\beta}$ is the Brauer correspondent of β in $N_{G/W}(DW/W)$. We see that $k_0(B|\lambda) = k_0(\beta|\lambda)$ and $k_0(\widetilde{B}|\lambda) = k_0(\widetilde{\beta}|\lambda)$. Hence the result follows by induction. So we may assume that λ is faithful. Put $\lambda = \mu \times \nu$, where μ and ν are the p-part and p'-part of λ, respectively. Put $W = \mathrm{Ker}(\mu^p \times \nu)$. Then $|W| = p$. Let β and $\widetilde{\beta}$ be the blocks of G/W and $N_G(D)/W$ dominated by B and $\widetilde{B}$, respectively. For any integer i, $\mu^{pi} \times \nu$ is regarded as a character of Z/W. Then, for the same reason as above, we have

$$k_0(\beta|\mu^{pi} \times \nu) = k_0(\widetilde{\beta}|\mu^{pi} \times \nu).$$

Clearly

$$k_0(B|\mu^{pi} \times \nu) = k_0(\beta|\mu^{pi} \times \nu)$$

and

$$k_0(\widetilde{B}|\mu^{pi} \times \nu) = k_0(\widetilde{\beta}|\mu^{pi} \times \nu).$$

So

$$k_0(B|\mu^{pi} \times \nu) = k_0(\widetilde{B}|\mu^{pi} \times \nu)$$

for any integer i. Let $p^n (n \geqq 1)$ be the order of a p-Sylow subgroup of Z. Then we have

$$k_0(B) = \sum_i k_0(B|\mu^{pi} \times \nu) + \sum_j k_0(B|\mu^j \times \nu),$$

where i runs through integers modulo p^{n-1} and j runs through p'-integers modulo p^n. Since all $\mu^j \times \nu$ are p-conjugate to λ, $k_0(B|\mu^j \times \nu) = k_0(B|\lambda)$ (for the definition of "p-conjugate", see [NT, p. 335]). Thus

$$k_0(B) = \sum_i k_0(B|\mu^{pi} \times \nu) + (p^n - p^{n-1})k_0(B|\lambda).$$

Similarly

$$k_0(\widetilde{B}) = \sum_i k_0(\widetilde{B}|\mu^{pi} \times \nu) + (p^n - p^{n-1})k_0(\widetilde{B}|\lambda).$$

Since $k_0(B) = k_0(\widetilde{B})$, we then get $k_0(B|\lambda) = k_0(\widetilde{B}|\lambda)$.

This completes the proof. □

Let ξ be an irreducible character in b. We choose a set of representatives $\{D^{x_i}; 1 \leqq i \leqq m\} (x_i \in G)$ of $T_G(\xi)$-conjugacy classes of inertial defect groups $D^x (x \in G)$ of B over b with $D^x \leqq T_G(\xi)$ (possibly $m = 0$).

Theorem 6. *Let $\widetilde{B}$ be the Brauer correspondent of B in $N_G(D)N$. We assume that (AM) is true for any block of any central extension of $T_G(\xi)/N$. Then*

$$k_0(B|\xi) = \sum_{i=1}^{m} k_0(\widetilde{B}|\xi^{x_i^{-1}}).$$

Proof. The right hand side makes sense. Indeed, since the block $\widetilde{B}^{x_i}$ of $N_G(D^{x_i})N$ covers b by Lemma 1, $\widetilde{B}$ covers $b^{{x_i}^{-1}}$.

We first consider the case where $m = 0$. We must show $k_0(B|\xi) = 0$. If $k_0(B|\xi) > 0$, then, for some inertial defect group $D^x (x \in G)$ of B over b, ξ extends to D^xN by Lemma 4. So $D^x \leqq T_G(\xi)$, a contradiction. Hence $k_0(B|\xi) = 0$. In the following we assume $m > 0$.

We divide the proof into two cases :

Case I. ξ is G-invariant; Case II. The general case.

Case I. ξ is G-invariant.

In this case $m = 1$. We must show $k_0(B|\xi) = k_0(\widetilde{B}|\xi)$. There exists a (finite) central extension of G,

$$1 \longrightarrow Z \longrightarrow \widehat{G} \xrightarrow{f} G \longrightarrow 1$$

with the following properties: $f^{-1}(N) = Z \times N_1$, for a normal subgroup N_1 of $\widehat{G}$; ξ extends to $\widehat{G}$. (Here we identify N with N_1 via f.)

We fix an extension $\widehat{\xi}$ of ξ to $\widehat{G}$. Let $\widehat{B}$ be the inflation of B to $\widehat{G}$. Since Z is central in $\widehat{G}$, we may choose a defect group $\widehat{D}$ of $\widehat{B}$ so that $\widehat{D}Z/Z = D$. Let λ be an irreducible constituent of $\widehat{\xi}_Z$. We regard λ as a character of ZN/N in a natural way. For any $\chi \in \mathrm{Irr}_0(B|\xi)$, let $\widehat{\chi}$ be the inflation of χ to $\widehat{G}$. Let $\{B_j ; 1 \leqq j \leqq s\}$ be the set of blocks of $\widehat{G}/N$ which are $\widehat{\xi}$-dominated by $\widehat{B}$ and have defect group $\widehat{D}N/N$, cf. [M2] for "$\widehat{\xi}$-domination".

For each $\chi \in \mathrm{Irr}_0(B|\xi)$, there is a unique irreducible character $\theta(\chi)$ of $\widehat{G}/N$ such that $\widehat{\chi} = \widehat{\xi} \otimes \theta(\chi)$. Then if B' is the block of $\widehat{G}/N$ containing $\theta(\chi)$, we have

$$\mathrm{ht}\,\chi = \mathrm{ht}\,\xi + \mathrm{ht}\,\theta(\chi) + d(\widehat{B}) - d(b) - d(B').$$

Since a defect group of B' is contained in $\widehat{D}N/N$ ([M2, Corollary 1.5]), we see that $\mathrm{ht}\,\theta(\chi) = 0$ and that $\widehat{D}N/N$ is a defect group of B'. So $B' = B_j$ for some j. Further, λ^{-1} is a constituent of $\theta(\chi)_{ZN/N}$. Thus $\theta(\chi) \in \mathrm{Irr}_0(B_j|\lambda^{-1})$.

Conversely, let b_1 be the block of $Z = ZN/N$ containing λ^{-1}. Then B_j covers b_1 for any j. Indeed, let $\theta \in \mathrm{Irr}(B_j)$ and let μ be an irreducible constituent of θ_Z. Then, since $\widehat{\xi} \otimes \theta \in \mathrm{Irr}(\widehat{B})$ and $\widehat{B}$ covers the the principal block of Z, a p-complement of Z is contained in $\mathrm{Ker}(\lambda\mu)$. This shows B_j covers b_1. For any j and any $\theta \in \mathrm{Irr}_0(B_j|\lambda^{-1})$ if we set $\zeta = \widehat{\xi} \otimes \theta$, then $\zeta = \widehat{\chi}$ for some $\chi \in \mathrm{Irr}_0(B|\xi)$ and then $\theta = \theta(\chi)$. Thus we get

(1) $\qquad k_0(B|\xi) = \sum_j k_0(B_j|\lambda^{-1})$.

Let $\widehat{H} = f^{-1}(N_G(D)N) \leqq \widehat{G}$. Let β be the inflation of $\widetilde{B}$ to $\widehat{H}$. We show

(2) $\qquad \widehat{H} = N_{\widehat{G}}(\widehat{D})N$.

(3) $\quad \beta$ has defect group $\widehat{D}$ and is the Brauer correspondent of $\widehat{B}$ in $\widehat{H}$.

To prove (2), since $f^{-1}(N_G(D)N) = f^{-1}(N_G(D))N$, it suffices to show $f^{-1}(N_G(D)) = N_{\widehat{G}}(\widehat{D})$. Clearly $f(N_{\widehat{G}}(\widehat{D})) \leqq N_G(D)$. If $f(x) \in N_G(D)$ for

$x \in \widehat{G}$, then $(\widehat{D}Z)^x = \widehat{D}Z$. Considering p-Sylow subgroups of both sides, we get $x \in N_{\widehat{G}}(\widehat{D})$ and the result follows.

We may choose a defect group Q of β such that $QZ/Z = D$. So $QZ = \widehat{D}Z$. Considering p-Sylow subgroups of both sides, we get $Q = \widehat{D}$. Since $\widetilde{B}^G = B$, we get $\beta^{\widehat{G}} = \widehat{B}$. So (3) follows.

Let $\{\beta_k; 1 \leqq k \leqq t\}$ be the set of blocks of $\widehat{H}/N$ which are $\widehat{\xi}_{\widehat{H}}$-dominated by β. Then the same argument as in the above shows

(1)′ $\qquad k_0(\widetilde{B}|\xi) = \sum_k k_0(\beta_k|\lambda^{-1})$.

Now, from the proof of Corollary 2. 5 of [M3], (2) and (3) yield that $s = t$ and, after renumbering, β_j is the Brauer correspondent of B_j in $\widehat{H}/N$ for each $j\,(1 \leqq j \leqq s)$.

We have $N_{\widehat{G}/N}(\widehat{D}N/N) = \widehat{H}/N$ by (2) and Lemma 2. Since $(\widehat{G}/N)/(ZN/N) \cong G/N$, by assumption and Lemma 5, we get $k_0(B_j|\lambda^{-1}) = k_0(\beta_j|\lambda^{-1})$ for each $j\,(1 \leqq j \leqq s)$. Therefore, (1) and (1)′ yield $k_0(B|\xi) = k_0(\widetilde{B}|\xi)$. This completes the proof of Case I.

Case II. The general case.

Put $T = T_G(\xi)$. For each $i\,(1 \leqq i \leqq m)$, let $\mathcal{B}_i$ be the set of blocks β of T with the following properties:

$$\beta \text{ has defect group } D^{x_i}, \beta \text{ covers } b, \text{ and } \beta^G = B,$$

cf. Lemma 3. Let $\chi \in \mathrm{Irr}(B|\xi)$ and let ζ be the Clifford correspondent of χ over ξ in T. We have

$$\mathrm{ht}\,\chi = \mathrm{ht}\,\zeta + d(B) - d(B'),$$

where B' is the block of T containing ζ. Since $\mathrm{ht}\,\zeta \geqq \mathrm{ht}\,\xi$ by [M2, Lemma 2. 2] and $B'^G = B$, we see that $\mathrm{ht}\,\chi = \mathrm{ht}\,\xi$ if and only if $\mathrm{ht}\,\zeta = \mathrm{ht}\,\xi$ and B' has a defect group of the form $D^x (x \in G)$. We note that D^x is then an inertial defect group of B over b by Lemma 3. Thus we get

(4) $$k_0(B|\xi) = \sum_i \sum_{\beta \in \mathcal{B}_i} k_0(\beta|\xi).$$

Fix $i\,(1 \leqq i \leqq m)$. Put $D_i = D^{x_i}$ and $H_i = N_G(D_i)N$. Let $\widetilde{B}_i$ be the Brauer correspondent of B in H_i. By Lemma 1, $\widetilde{B}_i$ covers b. Further, from the structure of H_i, we see that D_i is an inertial defect group of $\widetilde{B}_i$ over b. Clearly $D_i \leqq H_i \cap T$. Let $\widetilde{\mathcal{B}}_i$ be the set of blocks β' of $H_i \cap T$ with the following properties:

$$\beta' \text{ has defect group } D_i,\ \beta' \text{ covers } b, \text{ and } \beta'^{H_i} = \widetilde{B}_i.$$

We see that any H_i-conjugate of D_i is $H_i \cap T$-conjugate to D_i. So by the argument above, we get

(4)′ $$k_0(\widetilde{B}_i|\xi) = \sum_{\beta' \in \widetilde{\mathcal{B}}_i} k_0(\beta'|\xi).$$

For each $\beta \in \mathcal{B}_i$, let $\widetilde{\beta}$ be the Brauer correspondent of β in $N_T(D_i)N =$

$H_i \cap T$. We claim that the map sending β to $\widetilde{\beta}$ is a bijection from $\mathcal{B}_i$ to $\widetilde{\mathcal{B}}_i$.

Let $\beta \in \mathcal{B}_i$. Since $\widetilde{\beta}$ covers b, $\widetilde{\beta}^{H_i}$ is defined. We have $(\widetilde{\beta}^{H_i})^G = (\widetilde{\beta}^T)^G = \beta^G = B$. This shows that $\widetilde{\beta}^{H_i}$ has defect group D_i, and we see $\widetilde{\beta}^{H_i} = \widetilde{B}_i$. So $\widetilde{\beta} \in \widetilde{\mathcal{B}}_i$.

Conversely, let $\beta' \in \widetilde{\mathcal{B}}_i$. Then β' is the Brauer correspondent of a block β of T in $H_i \cap T$. Clearly β covers b. So β^G is defined and $\beta^G = (\beta'^T)^G = \widetilde{B}_i{}^G = B$. Thus we get $\beta \in \mathcal{B}_i$ and $\beta' = \widetilde{\beta}$. The claim follows.

By Case I, we have $k_0(\beta|\xi) = k_0(\widetilde{\beta}|\xi)$ for any i and any $\beta \in \mathcal{B}_i$. Thus we get

$$\begin{aligned} k_0(B|\xi) &= \sum_i \sum_{\beta \in \mathcal{B}_i} k_0(\beta|\xi) \quad \text{(by (4))} \\ &= \sum_i \sum_{\widetilde{\beta} \in \widetilde{\mathcal{B}}_i} k_0(\widetilde{\beta}|\xi) \\ &= \sum_i k_0(\widetilde{B}_i|\xi) \quad \text{(by (4)}'\text{)} \\ &= \sum_i k_0(\widetilde{B}|\xi^{{x_i}^{-1}}) \quad (\text{since } \widetilde{B}_i = \widetilde{B}^{x_i} \text{ for each } i). \end{aligned}$$

This completes the proof. □

Theorem 7. *Let $\widetilde{B}$ be the Brauer correspondent of B in $N_G(D)N$. We assume that (AM) is true for any block of any central extension of any subgroup of G/N. Then $k_0(B,N,h) = k_0(\widetilde{B},N,h)$ for any integer $h \geqq 0$. In particular, $k_0(B,N) = k_0(\widetilde{B},N)$.*

Proof. Put

$$\mathrm{Irr}_0(N,B,h) = \{\xi \in \mathrm{Irr}(N); \mathrm{ht}_N(\chi) = 0 \ \text{ for some } \ \chi \in \mathrm{Irr}(B|\xi), \mathrm{ht}\,\xi = h\}.$$

Let S be a set of of representatives of G-conjugacy classes of $\mathrm{Irr}_0(N,B,h)$. So

$$(1) \qquad k_0(B,N,h) = \sum_{\xi \in S} k_0(B|\xi)$$

by Clifford's theorem.

We claim $\mathrm{Irr}_0(N,\widetilde{B},h) \subseteq \mathrm{Irr}_0(N,B,h)$. Indeed, let $\xi \in \mathrm{Irr}_0(N,\widetilde{B},h)$. Let b be the block of N containing ξ. There is $\zeta \in \mathrm{Irr}(\widetilde{B}|\xi)$ such that $\mathrm{ht}\,\zeta = \mathrm{ht}\,\xi = h$. Then, since any defect group of $\widetilde{B}$ is of the form $D^n (n \in N)$, ξ extends to DN by Lemma 4. Since $\widetilde{B}$ covers b, D is an inertial defect group of B over b by Lemma 1. So, by Lemma 4 again, there is $\chi \in \mathrm{Irr}(B|\xi)$ such that $\mathrm{ht}\,\chi = \mathrm{ht}\,\xi$. So $\xi \in \mathrm{Irr}_0(N,B,h)$, and the claim is proved.

Let $\mathcal{O}_G(\xi)$ be the set of G-conjugates of ξ. Since

$$\mathrm{Irr}_0(N,B,h) = \bigcup_{\xi \in S} \mathcal{O}_G(\xi) \quad \text{(disjoint)},$$

we get

$$\mathrm{Irr}_0(N, \widetilde{B}, h) = \bigcup_{\xi \in S} \mathrm{Irr}_0(N, \widetilde{B}, h) \cap \mathcal{O}_G(\xi) \quad \text{(disjoint)}. \tag{2}$$

Fix $\xi \in S$. Let b be the block of N containing ξ. We choose $\{D^{x_i}; 1 \leqq i \leqq m\}$ as in Theorem 6. We show

$$\mathrm{Irr}_0(N, \widetilde{B}, h) \cap \mathcal{O}_G(\xi) = \bigcup \mathcal{O}_{N_G(D)N}(\xi^{{x_i}^{-1}}) \quad \text{(disjoint)}, \tag{3}$$

where i runs through those indices such that $\xi^{{x_i}^{-1}} \in \mathrm{Irr}_0(N, \widetilde{B}, h)$.

Assume that $\xi^{{x_j}^{-1}} = (\xi^{{x_i}^{-1}})^y$ for $y \in N_G(D)N$. Then $yx_j = x_i t$ for some $t \in T_G(\xi)$. Write $y = zn, z \in N_G(D), n \in N$. Then $D^{x_i t} = D^{n x_j} = D^{x_j (x_j^{-1} n x_j)}$. Since $x_j^{-1} n x_j \in N \leqq T_G(\xi)$, we get $i = j$.

Next let $\eta \in \mathrm{Irr}_0(N, \widetilde{B}, h) \cap \mathcal{O}_G(\xi)$. Put $\eta = \xi^x, x \in G$. Since $\eta \in \mathrm{Irr}_0(N, \widetilde{B}, h)$, η is D-invariant by Lemma 4. So $D^{x^{-1}} \leqq T_G(\xi)$. Further, $\widetilde{B}$ covers b^x, so the block $\widetilde{B}^{x^{-1}}$ of $N_G(D^{x^{-1}})N$ covers b. Thus $D^{x^{-1}}$ is an inertial defect group of B over b by Lemma 1. Thus for some i, $D^{x^{-1}} = D^{x_i t}$, where $t \in T_G(\xi)$. Then $y := x_i t x \in N_G(D)$ and $\eta = \xi^{tx} = (\xi^{{x_i}^{-1}})^y$. Thus (3) is proved.

By Clifford's theorem and (3), the number of those $\zeta \in \mathrm{Irr}_0(\widetilde{B}, N, h)$ which lie over some G-conjugate of ξ equals $\sum_i k_0(\widetilde{B}|\xi^{{x_i}^{-1}})$. Since this equals $k_0(B|\xi)$ by Theorem 6, we obtain

$$\begin{aligned} k_0(\widetilde{B}, N, h) &= \sum_{\xi \in S} k_0(B|\xi) \qquad &&\text{(by (2))} \\ &= k_0(B, N, h) &&\text{(by (1))}. \end{aligned}$$

This completes the proof. □

Since (AM) is true for p-solvable groups ([D], [OW]), we obtain the following.

Corollary 8. *Let $\widetilde{B}$ be the Brauer correspondent of B in $N_G(D)N$. We assume G/N is p-solvable. Then $k_0(B, N, h) = k_0(\widetilde{B}, N, h)$ for any integer $h \geqq 0$. In particular, $k_0(B, N) = k_0(\widetilde{B}, N)$.*

2. Reduction of the conjecture (AM)

In this section we reduce the conjecture (AM) by using Theorem 7.

Let $F^*(G)$ be the generalized Fitting subgroup of G ([S, 6. 6. 10]); namely $F^*(G) = F(G)E(G)$, where $F(G)$ is the Fitting subgroup of G and $E(G)$ is the maximal semisimple normal subgroup of G.

Proposition 9. *Let B be a block of G with defect group D. Suppose that (AM) is true for any block of any group H with $|H : Z(H)| < |G : Z(G)|$, but is not true for B. Then the following holds.*

(i) *For any non-central normal subgroup K of G, $G = N_G(D)K$.*

(ii) *For any normal subgroup K of G, B covers a G-invariant block of K.*

(iii) $G = N_G(D)F^*(G)$.

(*We remark that if G is p-solvable, then it suffices to assume that (AM) is true only for any block of any p-solvable group H with $|H : Z(H)| < |G : Z(G)|$.*)

Proof. Let $\widetilde{B}$ be the Brauer correspondent of B in $N_G(D)$. If G is abelian, (AM) is trivially true. So we assume G is non-abelian.

(i) Let K be a non-central normal subgroup of G. Put $N = Z(G)K$. Then, by assumption, (AM) is true for any block of any central extension of any subgroup of G/N. Thus by Theorem 7, $k_0(B, N, 0) = k_0(B_1, N, 0)$, where B_1 is the Brauer correspondent of B in $N_G(D)N$. We note that $k_0(B, N, 0) = k_0(B)$ and $k_0(B_1, N, 0) = k_0(B_1)$. Thus $k_0(B) = k_0(B_1)$. If $N_G(D)N < G$, then by assumption, $k_0(B_1) = k_0(\widetilde{B})$. So $k_0(B) = k_0(\widetilde{B})$, a contradiction. Thus we have $G = N_G(D)N = N_G(D)K$.

(ii) Let b be a block of K covered by B. The result is clear if $K \leqq Z(G)$. So we assume K is non-central. Let $T = T_G(b)$. Let β be the Fong-Reynolds correspondent of B over b in T. We may assume D is a defect group of β. Since $G = N_G(D)K$ by (i), we have $DK \lhd G$. Let b_1 be a unique block of DK covering b. Then D is a defect group of b_1 by Lemma 2. Let $\widetilde{b}_1$ be the Brauer correspondent of b_1 in $N_{DK}(D)$. Since β covers b_1, there is a block $\widetilde{\beta}$ of $N_T(D)$ such that $\widetilde{\beta}$ covers $\widetilde{b}_1$ and that $\widetilde{\beta}^T = \beta$ by [HK]. Then $\widetilde{\beta}$ has defect group D. So $\widetilde{\beta}$ is the Brauer correspondent of β in $N_T(D)$.

Since $T = T_G(b_1)$, we have $N_T(D) = N_G(D) \cap T_G(\widetilde{b}_1)$. So $\widetilde{\beta}^{N_G(D)}$ is defined. Then $(\widetilde{\beta}^{N_G(D)})^G = (\widetilde{\beta}^T)^G = \beta^G = B$, and hence $\widetilde{\beta}^{N_G(D)} = \widetilde{B}$. Thus $\widetilde{\beta}$ is the Fong-Reynolds correspondent of $\widetilde{B}$ over $\widetilde{b}_1$ in $N_T(D)$. Thus $k_0(\widetilde{\beta}) = k_0(\widetilde{B})$. Also, $k_0(\beta) = k_0(B)$. If $T < G$, then, since $T \geqq Z(G)$, we have $k_0(\beta) = k_0(\widetilde{\beta})$ by assumption. So $k_0(B) = k_0(\widetilde{B})$, a contradiction. Thus we have $T = G$. So b is G-invariant.

(iii) Since $C_G(F^*(G)) \leqq F^*(G)$ ([S, 6. 6. 11]) and G is non-abelian, $F^*(G)$ is non-central. Thus $G = N_G(D)F^*(G)$ by (i). □

Okuyama-Wajima's proof ([OW]) of the conjecture (AM) for p-solvable groups can be divided into two steps.

Step 1. To reduce the conjecture to the case of groups of p-length 1.

Step 2. To prove the conjecture for groups of p-length 1 by [OW, Theorem 2].

We give an alternative proof of Step 1 by using Proposition 9 (within the class of p-solvable groups).

Let G be a p-solvable group and let B be a p-block of G with defect group D. We argue by induction on $|G : Z(G)|$. Since G is p-solvable, $E(G)$ is a p'-group. Thus by Proposition 9 (iii), we may assume $G = N_G(D)O_{p'}(G)$, since $O_p(G) \leqq D$.

By Proposition 9 (ii) we may assume a block of $O_{p'}(G)$ covered by B is G-invariant. So D is a p-Sylow subgroup of G by Fong's theorem. Hence G has p-length 1.

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