J. Math. Kyoto Univ. (JMKYAZ)
43-3 (2003), 451–464

Analytic Jacobi Eisenstein series and the Shimura method

By

Bernhard E. Heim

Abstract

In this paper it is proven that analytic Jacobi Eisenstein series always admit a meromorphic continuation on the whole complex plane and statements about the location of possible poles are given. Moreover a new interpretation of Shimura's approach to the standard L-function of a Siegel modular form is presented.

The purpose of this paper is twofold. First and mainly we prove the analytic continuation (which has not known before) and study the location of pole of analytic Jacobi Eisenstein series. Applications of this results have been presented in [He98a], [He98b]. Secondly we present a new interpretation of Shimura's approach to automorphic L-functions and indicate several improvements. It is maybe important to note, that the analytic continuation of Jacobi Eisenstein series does not follow from the work of Langlands [La76]. So this fact seems to be interesting in its own way.

In [Sh75a] Goro Shimura introduced a method to study in a precise way the poles of the Rankin L-function $D_f(s)$ attached to an elliptic cusp form f by integrating the cusp form against an Eisenstein series of half-integral weight multiplied with a theta series. Later on this method was generalized to standard L-functions of Siegel modular forms ([An79], [A-K79], [Sh94]).

In this paper we show that Shimura's integral can be interpreted as the Petersson scalar product of an analytic Jacobi Eisenstein series (introduced by Arakawa [Ar90], [Ar92], [Ar94]) restricted to the Siegel variable against the modular form.

The reason for this is the following insight: an analytic Jacobi Eisenstein series can be interpreted as the product of a Siegel Eisenstein series (of integral or half integral weight) with a theta series, up to some trace. As a by-product we get the meromorphic continuation and a precise description of possible poles of analytic Jacobi Eisenstein series of every degree and index. To get the functional equation one may have to proceed in a different way. This has been done in the elliptic case by Tsuneo Arakawa in [Ar90] and [Ar92], using methods

Received March 21, 2000
Revised May 28, 2001

of Roelcke and Selberg. He worked with vector-valued half-integral Eisenstein series.

It should be mentioned that the Fourier-Jacobi expansion of an analytic Siegel Eisenstein series is not a Jacobi Eisenstein series in the sense of Arakawa. This is in contrast to the holomorphic situation.

Acknowledgements. This work was inspired by Tsuneo Arakawa's papers [Ar90] and [Ar94]. The author would like to thank him, Siegfried Böcherer and Paul Garrett for useful discussions and encouragement. But without the hospitality of the Max-Planck Institut für Mathematik in Bonn and a Max-Planck fellowship, this work would not have been possible.

Notation. For an associative ring R with identity element, we denote by $R^\times$ the group of all invertible elements of R and by $R^{m,n}$ the module of all $m \times n$ matrices with entries in R. If M is a matrix, M^t, $\det(M)$, and $\mathrm{tr}(M)$ stand for its transpose, determinant, and trace. We put $M_n(R) = R^{n,n}$, $Gl_n(R) = M_n(R)^\times$, $Sl_n(R) = \{M \in Gl_n(R) | \det(M) = 1\}$ (when R is commutative). The identity and zero elements of $M_n(R)$ are denoted by 1_n and 0_n respectively (when n needs to be stressed). Let $J_n = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ then $Sp_n(R) = \{M \in Gl_{2n}(R) | M^t J_n M = J_n\}$.

$$P_{n,r}(R) = \left\{ \begin{pmatrix} * & * \\ 0_{n+r,n-r} & * \end{pmatrix} \in Sp_n(R) \right\},$$

$$P^J_{n,r}(R) = \left\{ \begin{pmatrix} 1_{n-r} & * \\ 0_{n+r,n-r} & * \end{pmatrix} \in Sp_n(R) \right\}.$$

For convenience let $\Gamma_n = Sp_n(\mathbb{Z})$, $P_{n,r} = P_{n,r}(\mathbb{Z})$, $\Gamma = \Gamma_1$ and $\Gamma_\infty = P_{1,0}$. For real symmetric matrices A and B, we write $A > B$ ($A \geq B$) to indicate that $A - B$ is positive definite (positive semi-definite) and $A[B] = B^t AB$ if A, B are suitable. If $A_1, A_2, \ldots, A_n$ are square matrices, $[A_1, A_2, \ldots, A_n]$ denotes the matrix with $A_1, A_2, \ldots, A_n$ in the diagonal blocks and 0 in all other blocks. Let A_n^+ denote the set of all half-integral positive definite matrices of size n. Let $Z \in \mathbb{C}^{n,n}$, then we put $e\{Z\} = e^{2\pi i\,(\mathrm{tr}\,Z)}$ and $\mathrm{Re}(Z), \mathrm{Im}(Z)$ for the real and imaginary part of Z and for $M \in \mathbb{C}^{m,n}$ we put occasionally $x_M = \mathrm{Re}(M)$ and $y_M = \mathrm{Im}(M)$. Further $\delta(Z) = \det(\mathrm{Im}\,(Z))$. By $\overline{a}$ we denote the complex conjugation of an element $a \in \mathbb{C}$.

1. Modular forms

1.1. Siegel modular forms

The group $Sp_n(\mathbb{R})$ acts on the Siegel upper half space $\mathbb{H}_n$ of degree n by

$$(M, Z) \mapsto M(Z) = (AZ + B)(CZ + D)^{-1},$$

where $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ with $A, B, C, D \in \mathbb{R}^{n,n}$. We denote by M_n^k be the space of Siegel modular forms and S_n^k the subspace of cusp forms of degree n and weight k for the Siegel modular group Γ_n. Let $J(M,Z) = (CZ+D)$ and $j(M,Z) = \det(CZ+D)$. Then every $F \in M_n^k$ satisfies the transformation law

$$F(Z) = F(M(Z))\, j(M,Z)^{-k} \qquad \text{for all} \quad M \in \Gamma_n,\ Z \in \mathbb{H}_n. \tag{1}$$

The right side of the equation (1) is usually abbreviated by $(F|_k M)(Z)$. For arbitrary complex valued functions F and G on $\mathbb{H}_n$, which satisfy (1) for a congruence subgroup Γ' of Γ_n, we define the Petersson integral, convergence assumed, by

$$\langle F, G\rangle_{\Gamma'} = \int_{\Gamma' \backslash \mathbb{H}_n} F(Z)\, \overline{G(Z)}\, \delta(Z)^k d^*Z. \tag{2}$$

Here $Z = X + iY$ and $d^*Z = \det(Y)^{-(n+1)} dXdY$ denotes the symplectic volume element. Every $F \in S_n^k$ has a Fourier expansion of the form $F(Z) = \sum_{T \in A_n^+} a^F(T) e\{TZ\}$. If σ is an automorphism of $\mathbb{C}$, then

$$F^\sigma(Z) = \sum_{T \in A_n^+} a^F(T)^\sigma e\{TZ\} \in S_n^k \tag{3}$$

([Sh75b]). We denote by # the complex-conjugation in $\mathbb{C}$. If we assume that $F \in S_n^k$ is a Hecke eigenform with Satake-parameters $(\alpha_{0,p}; \alpha_{1,p}; \ldots; \alpha_{n,p})$, then the standard zeta function $D_F^n(s)$ of F is given by

$$D_F^n(s) = \prod_p \left\{ (1-p^{-s}) \prod_{j=1}^n (1-\alpha_{j,p}p^{-s})\,(1-\alpha_{j,p}^{-1}p^{-s}) \right\}^{-1}. \tag{4}$$

For more details the reader is referred to Andrianov [An87] or Klingen [Kl90].

1.2. Jacobi Eisenstein series

Every element $g \in P_{n+l,n}^J(\mathbb{R})$ can be parameterized by

$$((\lambda, \mu; \rho), M) \in (\mathbb{R}^{l,n} \times \mathbb{R}^{l,n} \times \mathbb{R}^{l,l}) \times Sp_n(\mathbb{R}) \tag{5}$$

determined by

$$g = \begin{pmatrix} 1_l & 0 & \rho & \mu \\ 0 & 1_n & \mu^t & 0 \\ 0 & 0 & 1_l & 0 \\ 0 & 0 & 0 & 1_n \end{pmatrix} \begin{pmatrix} 1_l & \lambda & 0 & 0 \\ 0 & 1_n & 0 & 0 \\ 0 & 0 & 1_l & 0 \\ 0 & 0 & -\lambda^t & 1_n \end{pmatrix} (1_{2l} \times M). \tag{6}$$

An important normal subgroup is given by the Heisenberg group $H_{n,l}(\mathbb{R})$, consisting of all elements h of the form $h = (\lambda, \mu; \rho) = ((\lambda, \mu; \rho), 1_{2n})$. The group $P_{n+l,n}^J(\mathbb{R})$ acts on $D_{n,l} = \mathbb{H}_n \times \mathbb{C}^{l,n}$ by

$$\begin{gathered} P_{n+l,n}^J(\mathbb{R}) \times D_{n,l} \longrightarrow D_{n,l} \\ g(\tau, z) = (M(\tau), zJ(M,\tau)^{-1} + \lambda M(\tau) + \mu), \end{gathered} \tag{7}$$

where $g = ((\lambda, \mu; \rho), M)$ and $(\tau, z) \in D_{n,l}$. It is sometimes convenient to regard $Sp_n(\mathbb{R})$ as a subgroup of $P_{n+l,n}^J(\mathbb{R})$.

Definition 1.1. Let $k, l, n \in \mathbb{N}$ and $S \in A_l^+$. The factor of automorphy $j_{k,S}$ of index S and weight k for $P_{n+l,n}^J(\mathbb{R})$ is the map

$$(8) \qquad j_{k,S} : \begin{cases} P_{n+l,n}^J(\mathbb{R}) \times D_{n,l} & \longrightarrow \quad D_{n,l} \\ (g, (\tau, z)) & \mapsto \quad j_{k,S}(g, (\tau, z)) \end{cases},$$

where

$$(9) \quad j_{k,S}(g, (\tau, z)) = j(M, \tau)^k e\{-S\rho\} \\ e\{-S[\lambda]\, M(\tau) - 2\lambda^t SzJ(M, \tau)^{-1} + S[z]J(M, \tau)^{-1}c\}$$

for $g = ((\lambda, \mu; \rho), M)$ and $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Let $F : D_{n,l} \longrightarrow \mathbb{C}$, then the generalized Petersson slash operator $|_{J,k}$ is given by

$$(10) \qquad (F|_{J,k}g)\,(\tau, z) := j_{S,k}(g, (\tau, z))^{-1} F(g(\tau, z)).$$

Let $\mathcal{J}_{k,S}^n$ and $\mathcal{J}_{k,S}^{n,\mathrm{cusp}}$ denote the space of Jacobi forms and Jacobi cusp forms, respectively, of weight k, index S and degree (n, l) for $P_{n+l,n}^J(\mathbb{Z})$. We abbreviate $P_{n+l,n}^J(\mathbb{Z})$ by $\Gamma_{n,l}^J$. For more details see Arakawa [Ar94, Sections 1.1 and 1.2]. Similarly as Eisenstein series on the Siegel upper half plane, one defines Jacobi Eisenstein series. For each r with $0 \leq r \leq n$, let

$$(11) \quad \Gamma_{n,l,r}^J = \{((0\lambda_2, \mu; \rho), M) \,|\, \lambda_2 \in \mathbb{Z}^{l,r}, \mu \in \mathbb{Z}^{l,n}, \rho \in \mathbb{Z}^{l,l} \text{ and } M \in P_{n,r}\}$$

and

$$(12) \qquad *_J : \begin{cases} D_{n,l} & \longrightarrow \quad D_{r,l} \\ (\tau, z) & \mapsto \quad \left(\tau\left[\binom{0}{1_r}\right], z\binom{0}{1_r}\right) \end{cases}.$$

Definition 1.2. Let $k, l, n \in \mathbb{N}$ with k even, $0 \leq r \leq n$ and $S \in A_l^+$. To $\phi \in \mathcal{J}_{k,S}^{r,\mathrm{cusp}}$ we attach an analytic Jacobi Eisenstein series of Klingen type on $D_{n,l}$ defined by:

(13)
$$E_{n,r}^{S,k}((\tau, z), \phi); s) = \sum_{\gamma \in \Gamma_{n,l,r}^J \backslash \Gamma_{n,l}^J} \phi(\gamma(\tau, z)_{*_J})\; j_{k,S}(\gamma, (\tau, z))^{-1} \left(\frac{\delta(M(\tau))}{\delta(M(\tau)_*)}\right)^s,$$

here $\gamma = (h, M)$, $s \in \mathbb{C}$ with $\mathrm{Re}(s)$ sufficiently large and $\left(\begin{smallmatrix} \tilde{\tau}_1 & \tilde{\tau}_2 \\ * & \tilde{\tau}_4 \end{smallmatrix}\right)_* = \tilde{\tau}_4 \in \mathbb{H}_r$. If $r = 0$, we put $E_n^{S,k}((\tau, z); s) = E_{n,0}^{S,k}((\tau, z), 1; s)$.

Let $\phi \in \mathcal{J}_{k,S}^n$, then $(\mathcal{D}_0\phi)(\tau) := \phi(\tau, 0) \in M_n^k$.

Remark 1.3.
a) The Eisenstein series is absolutely convergent for $k + 2\,\mathrm{Re}(s) > n + r + l + 1$. If $k > n + r + l + 1$, then $E_{n,r}^{S,k}(\phi, (\tau, z); 0) \in \mathcal{J}_{k,S}^n$.
b) Let us put

$$(14) \qquad \chi_\mu^{k,s;J}(g, (\tau, z)) = j_{k,S}(g, (\tau, z))^{-1} |j(M, \tau)|^{-2s},$$

then we have the cocycle relation

$$\chi_\mu^{k,s;J}(gh,(\tau,z)) = \chi_\mu^{k,s;J}(g,h(\tau,z))\ \chi_\mu^{k,s;J}(h,(\tau,z)) \tag{15}$$

for $g,h \in P^J_{n+l,n}(\mathbb{R})$ and

$$E_n^{S,k}((\tau,z),s) = \delta(\tau)^s \sum_{\gamma\in\Gamma^J_{(n,l),0}\backslash\Gamma^J_{n,l}} \chi_\mu^{k,s;J}(\gamma,(\tau,z)).$$

2. The Main fomula

The Jacobi Eisenstein series introduced in the last section, can be used to study analytic and arithmetic properties of the standard L-function attached to Siegel modular forms. In this section we present a formula, which can be used as a starting point in this direction.

Let $S \in A_l^+$. Then we attach to S the theta series

$$\Theta_S^o(\tau) = \sum_{\lambda\in\mathbb{Z}^{l,n}} e\{S[\lambda]\tau\}, \qquad (\tau \in \mathbb{H}_n). \tag{16}$$

It is obvious, that one can choose

$$R_{l,n} = \{((\lambda,0;0),M) \mid \lambda \in \mathbb{Z}^{l,n} \text{ and } M \in P_{n,0}\backslash\Gamma\} \tag{17}$$

for a representative system of $\Gamma^J_{n,l,0}\setminus\Gamma^J_n$. Thus a simple pullback formula of the Jacobi Eisenstein series via the natural embedding

$$\mathbb{H}_n \longrightarrow \mathbb{H}_n \times \mathbb{C}^{l,n}, \qquad \tau \mapsto (\tau,0)$$

is given by

$$E_n^{S,k}\left((\tau,0),s\right) = \delta(\tau)^s \sum_{M\in P_{n,0}\backslash\Gamma_n} \chi_\mu^{k,s}(M,\tau)\ \Theta_S^o(M(\tau)), \tag{18}$$

where

$$\chi_\mu^{k,s}(M,\tau) = j(M,\tau)^{-k}\ \ |\, j(M,\tau)\,|^{-2s}\ .$$

Because the generalized automorphic factor $\chi_\mu^{k,s;J}$ splits into

$$\chi_\mu^{k,s}(M,\tau)\ e\{S[\lambda]M(\tau)\}. \tag{19}$$

For further simplifications we put: $\Gamma_n(s) = \prod_{\nu=1}^n \Gamma(s-(\nu-1)/2)$ and

$$\mathcal{D}_0 E_n^{S,k}(s) := E_n^{S,k}((\tau,0),s).$$

Theorem 2.1 (Main formula)**.** *Let $k,l,n\in\mathbb{N}$ and k even. Let $S \in A_l^+$ and $F \in S_n^k$. Let $s\in\mathbb{C}$ with $2\,\mathrm{Re}(s)+k > n+l+1$. Then*

$$\begin{aligned}\langle \mathcal{D}_0 E_n^{S,k}(s)\,;\,F\rangle_{\Gamma_n} &= (4\pi)^{-n(k+s-\frac{n+1}{2})}\pi^{n(n-1)/4}\,\Gamma_n\left(s+k-\frac{n+1}{2}\right)\\ &\quad\times D_S^{n,l}\left(F^\#, k+s-\frac{n+1}{2}\right).\end{aligned} \tag{20}$$

Here

$$D_S^{n,l}(G,s) := \sum_{\lambda \in \mathbb{Z}^{l,n} \,/\, Gl_n(\mathbb{Z})} a^G(S[\lambda])\ (\det(S[\lambda]))^{-s}$$

for $G \in S_n^k$.

Proof. After all our previous observations, this can be done in a straightforward manner. Let $P_n = \{Y \in \mathbb{R}^{n,n} \,|\, iY \in I\!H_n\}$ and R_n a fundamental domain of the action of $Gl_n(\mathbb{Z})$ on P_n, via $Y \mapsto Y[U^t]$.

$$\langle \mathcal{D}_0 E_n^{S,k}(s)\,;\,F\rangle_{\Gamma_n} \tag{21}$$

$$= \boxed{\boxed{\int_{\Gamma_{n,0}\backslash I\!H_n} \Theta_S^o(\tau)\overline{F(\tau)}\delta(\tau)^{k+s}\, d^*\tau}}$$

$$= \int_{X\in(\mathbb{R}/\mathbb{Z})^{n,n}} \int_{R_n} \Theta^o(X+iY)\,\overline{F(X+iY)}\delta(Y)^{k+s-n-1}\, dX\, dY \tag{22}$$

$$= \sum_{\lambda\in\mathbb{Z}^{l,n}\,/Gl_n(\mathbb{Z})} \overline{a(S[\lambda])} \int_{y>0} \det(y)^{k+s-n-1}\, e^{-4\pi \mathrm{tr}(S[\lambda]y)}\, dy. \tag{23}$$

Obviously, we have only to care for $S[\lambda] > 0$. But if this is the case, then $\int_y$ can be calculated by applying Lemma 2 given in [Kl90] (Chapter 6), which proves the theorem. □

Thus $\left\langle \mathcal{D}_0 E_n^{S,k}(s)\,;\,F\right\rangle_{\Gamma_n} \equiv 0$, if $l < n$. Let now F be a Hecke eigenform. It had been shown by Andrianov [An79], Böcherer [Boe86] and Shimura [Sh94], that $D_S^{n,n}(F,s)$ is essentially an Euler product obtained from the eigenvalues, i.e. the standard L-function of F.

Moreover, Andrianov and Shimura used a similar expression as given in the box, given in (21), to study analytic properties of the L-function. Actually, this approach had been introduced by Shimura [Sh75a] to give a precise description of the poles of the Rankin L-function related to $D_1^{1,1}(F,s)$. Hence the **Shimura method** can be interpreted as the Petersson scalar product of a cusp form with an analytic Jacobi Eisenstein serieson restricted on the Siegel variable. This view point avoids somehow the theory of modular forms of half-integral weight. But nevertheless, in the next section we will go somehow in the opposite direction to study analytic properties of the Jacobi Eisenstein series.

3. Jacobi Eisenstein series and theta series

Shimura established an isomorphism between $\mathcal{J}_{k,S}^n$ and a certain space of vector-valued Siegel modular forms [Zi89, Theorem 3.3]. Main ingredients of the isomorphism are theta series with characteristic (a,b) on $D_{n,l}$ of the following type:

$$\Theta_{S,a,b}(\tau,z) := \sum_{\lambda\in\mathbb{Z}^{l,n}} e\{S[\lambda+a]\tau + 2(\lambda+a)^t S(z+b)\} \tag{24}$$

with $(a,b) \in \mathbb{Q}^{l,n} \times \mathbb{Q}^{l,n}$. For $n = l = 1$ see [E-Z85] and for simplification of the notation let $\Theta_S = \Theta_{S,0,0}$.
In a non-holomorphic setting (e.g. Jacobi Eisenstein series $E^{S,k}_{n,r}(\,*\,,s)$) the situation becomes much more complicated. A first approach to connect non-holomorphic Jacobi Eisenstein series with Siegel Eisenstein series has been established by Arakawa [Ar90] (Proposition 3.1) and [Ar92] (Theorem 6.2), at least in the case $n = 1$. He related the Jacobi Eisenstein series to Eisenstein series with theta multiplier system on $\mathbb{H}_1$. By making use of a general theory for Eisenstein series of $\mathrm{SL}_2(\mathbb{Z})$ due to Selberg and Roelcke, he was able to establish the meromorphic continuation and the functional equation of the Jacobi Eisenstein series. Here we would like to state a slightly different approach, which reduces the problem of meromorphic continuation to that of certain Siegel Eisenstein series (of half-integral weight).

Remark 3.1. Let $g = ((\lambda, 0; 0), M) \in \Gamma^J_{n,l}$. Then we have

$$J_{k,S}(g,(\tau,z)) = j(M,\tau)^k\, e\{S[z]J(M,\tau)^{-1}\,c\} \\ \times e\{-S[\lambda]M(\tau) - 2\lambda^t SzJ(M,\tau)^{-1}\}. \tag{25}$$

This gives us the following description of analytic Jacobi Eisenstein series.

$$\begin{aligned}\boxed{E^{S,k}_n((\tau,z),s)} &= \delta(\tau)^s \sum_{M=\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\in\Gamma_{n,0}\backslash\Gamma_n} \chi^{k,s}_\mu(M,\tau)\; e\{S[z]J(M,\tau)^{-1}\,c\} \\ &\quad\times \sum_{\lambda\in\mathbb{Z}^{l,n}} e\{S[\lambda]M(\tau) + 2\lambda^t SzJ(M,\tau)^{-1}\} \\ &= \boxed{\delta(\tau)^s \sum_{M=\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\in\Gamma_{n,0}\backslash\Gamma_n} \chi^{k,s}_\mu(M,\tau)\;\; e\{-S[z]J(M,\tau)^{-1}\,c\}\;\; \Theta_S\left(M(\tau,z)\right).}\end{aligned} \tag{26}$$

To employ the transformation law of the theta series Θ_S we decompose $\Gamma_{n,0}\backslash\Gamma_n$ into

$$(P_{n,0}\setminus\Gamma^n_0(q)) \times (\Gamma^n_0(q)\setminus\Gamma_n).$$

Here q denotes the level of the quadratic form $2S$, and

$$\Gamma^n_0(q) = \left\{\gamma = \begin{pmatrix} A & B \\ C & D\end{pmatrix} \in \Gamma_n \,\middle|\, C \equiv 0\ (q)\right\}.$$

By $\chi^{(n)}_S(M)$ we denote the multiplier, an eighth root of unity, attached to $S \in A^+_l$ and $\Gamma^n_0(q)$ ([A-M75]). For example in the case $l = n = 1$ and $S = 1$ we have

$$\chi^{(1)}_1(M) = \left(\frac{c}{d}\right)\epsilon_d^{-1} \qquad\qquad \left(M = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma_0(4)\right), \tag{27}$$

here $\epsilon_d = 1$ if $d \equiv 1(4)$ and i if $d \equiv 3(4)$ and $(\frac{c}{d})$ is a generalized quadratic residue symbol (Jacobi symbol, for more details [Sh73]). Let $\tau \in \mathbb{C}^\times$. For the square root $\tau^{1/2}$ we always choose the branch with $-\pi/2 < \arg(\tau^{1/2}) \leq \pi/2$ and put $\tau^{r/2} = (\tau^{1/2})^r$ for all $r \in \mathbb{Z}$.

$$\left(\left(\frac{c}{d}\right) \epsilon_d^{-1}(c\tau + d)^{1/2}\right)^2 = \left(\frac{-1}{d}\right) j(M,\tau). \tag{28}$$

More general (e.g. [Boe83, Proposition 9]) we have

Lemma 3.2. *Let $S \in A_l^+$ and let q be the level of S. Then for $M \in \Gamma_0^n(q)$ we have*

$$\Theta_S(M(\tau,z)) = e\{S[z]\, J(M,\tau)^{-1}c\}\; h_S(M,\tau)\, \Theta_S(\tau,z), \tag{29}$$

with $h_S(M,\tau) = j(M,\tau)^{l/2}\chi_S^{(n)}(M)$ and $(\tau,z) \in D_{n,l}$.

Definition 3.3. Let $k,l,n \in \mathbb{N}$ and k even and let $S \in A_l^+$ with level q. Let $s \in \mathbb{C}$ with $2\,\mathrm{Re}(s) + k > n + l + 1$. An (Siegel-)Eisenstein series of general type is defined by

$$\begin{aligned} E_n^{S,k-l/2,q}(\tau,s) &= \delta(\tau)^s \sum_{\gamma \in P_{n,0}\backslash\Gamma_0^n(q)} \chi_\mu^{k,s}(\gamma,\tau)\, h_S(\gamma,\tau) \\ &= \delta(\tau)^s \sum_{\gamma \in P_{n,0}\backslash\Gamma_0^n(q)} \chi_S^{(n)}(\gamma)\; \chi_\mu^{k-l/2,s}(\gamma,\tau). \end{aligned} \tag{30}$$

Let us recall some well known results on multiplier systems coming from theta series ([An87], Chapter 1). Let $S \in A_l^+$ with level q.

- Let $l = 2g$ with $g \in \mathbb{N}$. Then $\chi_S^{(n)}(M) = \psi_S \det(D))$, where ψ_S is a certain real Dirichlet character modulo q.
- Let $l = 2g + 1$ with $g \in \mathbb{N}$. Then $\chi_S^{(n)}(M) = \chi_{1_n}^{(n)}(M)\; \psi_S'(\det(D))$, where ψ_S' is again a certain real Dirichlet character modulo q. Further let

$$(\chi_{1_n}^{(n)}(M))^2 = \mathrm{sgn}(\det D)\, \left(\frac{-4}{|\det(D)|}\right) \qquad \text{and} \qquad (\chi_{1_n}^{(n)}(M))^4 = 1,$$

here $\mathrm{sgn}(x) = x/|x|$ and $\left(\frac{\cdot}{\cdot}\right)$ denotes the Jacobi symbol.

Remark 3.4. Let $l = 2g$ then the function defined in (30) is an ordinary Eisenstein series (of integer weight) with real Dirichlet character ψ_S. If $l = 2g + 1$, then it is an Eisenstein series of half-integral weight with Dirichlet character ψ_S'. Both Eisenstein series possess a meromorphic continuation on the whole complex plane (Langlands [La76], see also Shimura [Sh94], the remark given in the last paragraph on p. 562).

Thus the $E_n^{S,k-l/2,q}(\tau,s)$ is absolute convergent and behaves like an automorphic form with respect to $\Gamma_0^n(q)$. Further it admits a meromorphic continuation on the whole complex plane.

Definition 3.5. Let F be a complex-valued function on $D_{n,l}$, which transforms like a Jacobi form (of weight k and index S) with respect to a congruence subgroup Γ' of Γ_n. Then the n^{th}-trace of F is defined by

$$(\mathbf{Tr}_{\Gamma'}^{\Gamma_n} F)(\tau, z) = \sum_{M \in \Gamma' \backslash \Gamma_n} (F|_{J,k} M)(\tau, z). \tag{31}$$

This function transforms like a Jacobi form with respect to Γ_n. Of course, we can also apply this construction to functions which are restricted to the symplectic variable τ (i.e. $z = 0$).

Theorem 3.6. *Let $k, l, n \in \mathbb{N}$ and $S \in A_l^+$. Let k be even and q be the level of S and $s \in \mathbb{C}$ with $\mathrm{Re}(s) + k > n + l + 1$. Then we have for $(\tau, z) \in D_{n,l}$:*

$$E_n^{S,k}((\tau, z), s) = \mathbf{Tr}_{\Gamma_0^n(q)}^{\Gamma_n}(E_n^{S,k-l/2,q}(\tau, s)\, \Theta_S(\tau, z)). \tag{32}$$

Theorem 3.6 gives us a tool to study analytic properties of the Jacobi Eisenstein series, by reducing everything to the theory of (Siegel-)Eisenstein series of integral or half-integral weight and properties of the trace map $\mathbf{Tr}$.

Let Γ' be a congruence subgroup of Γ_n and let us assume that F behaves like automorphic forms with respect to Γ' and G with respect to Γ. Then we have, if the scalar products exist,

$$\langle \mathbf{Tr}_{\Gamma'}^{\Gamma_n} F, G \rangle_{\Gamma_n} = \langle F, G \rangle_{\Gamma'}. \tag{33}$$

This principle works of course in much more general situations. A first application is the following. If we make the same assumptions as in Theorem 2.1, then we have

$$I(F, s) := \langle \mathcal{D}_0 E_n^{S,k}(s)\,;\, F \rangle_{\Gamma_n} = \langle E_n^{S,k,q}(*, s)(\mathcal{D}_0(\Theta_S))\,;\, F) \rangle_{\Gamma_0^n(q)}. \tag{34}$$

Thus the meromorphic continuation (and functional equation) of $\mathcal{D}_0 E_n^{S,k}(s)$ and $E_n^{S,k,q}(\tau, s)$ transfer directly to $I(F, s)$. Moreover one gets a sharp picture about the possible poles of $I(F, s)$. It is of course also interesting to compare directly the properties of the corresponding Eisenstein series of Jacobi and Siegel type.

4. Applications

4.1. Analytic properties of Jacobi Eisenstein series

Let $U \subseteq \mathbb{C}$ be an open subset and $F(\tau, z, s)$ a function $F : D_{n,l} \times U \longrightarrow \mathbb{C}$ holomorphic in s. Further, let Γ' a congruence subgroup of Γ and $F|_{J,k}\gamma = F$ for $S \in A_l^+$ and $\gamma \in \Gamma'$. Then $\mathbf{Tr}_{\Gamma'}^{\Gamma} F$ remains holomorphic. Hence we have the following

Theorem 4.1. *Let $l, n \in \mathbb{N}$ and $k \in 2\mathbb{Z}$ and $S \in A_l^+$. Then $E_n^{S,k}((\tau, z),$ $s)$ has a meromorphic continuation to the whole complex plane with possible poles coming from Eisenstein series $E_n^{S,k-l/2,q}((\tau, z), s)$ of integral or half-integral weight* (*the latter can be studied by* [Sh75a], [Sh94] (*Theorem* 7.3)).

Let us be more precise, at least in the simplest (non-trivial) case $n = l = 1$. Let $K \in \mathbb{N}$ be odd, $N \in 4\mathbb{N}$ and ω be a Dirichlet character modulo N which satisfies $\omega(-1) = 1$. Further, let $L(s,\omega) = \sum \omega(n)\, n^{-s}$, $\tilde{j}(\gamma, z) = \Theta_1^o(\gamma(z)) \,/\, \Theta_1^o(z)$ for $\gamma \in \Gamma_0^1(4)$ and

$$E(z,s,K,\omega) = \delta(z)^{s/2} \sum_{\gamma \in P_{1,0} \backslash \Gamma_0^1(N)} \omega(d)\, \tilde{j}(\gamma,z)^{-K}\, |\tilde{j}(\gamma,z)|^{-2s}. \tag{35}$$

We put $\lambda_0 = 0$ or 1 according $\lambda = (1-K)/2$ is even or odd. Then it is well known [Sh75a] (Proposition 3), that

$$F(z,s,K,\omega) = \Gamma\left(\frac{s+K}{2}\right) \Gamma\left(\frac{s-\lambda+\lambda_0}{2}\right) L(2s-2\lambda,\omega^2) E(z,s,K,\omega), \tag{36}$$

multiplied with $(s-\lambda-1)$ can be continued to a holomorphic function in the whole complex-plane. The factor $(s-\lambda-1)$ is unnecessary either if $(|K|+1)/2$ is even or if ω^2 is non-trivial. Hence we get:

Corollary 4.2. *Let $k \in 2\mathbb{N}$ and $\chi_0(N)$ be the principal Dirichlet character modulo N, then*

$$\Gamma(s+k-1/2)\Gamma(s+k/2)\; L(4s+2k-2,\chi_0(4))\;\; E_1^{1,k}((\tau,z),s) \tag{37}$$

can be continued to a holomorphic function in the whole complex-plane. In particular $E_1^{1,k}((\tau,z),(3-2k)/4) \equiv 0$.

Next we would like to compare theses results with [Ar90]. For simplicity let $\kappa = k/2 - 1/4$ and $E^A(s) = E_1^{1,k}((\tau,z), s-\kappa)$. From the work of Arakawa one can deduce the following

Proposition 4.3. *The Eisenstein series $E^A(s)$ admits a meromorphic continuation on the whole complex plane with functional equation*

$$E^A(1-s) = \Phi(1-s)E^A(s), \tag{38}$$

here

$$\Phi(s) = 2^{3/2-2s} e^{-\pi i k/2} \frac{\Gamma(2s-1)}{\Gamma(s+\kappa)\Gamma(s-\kappa)} \frac{\zeta(4s-2)}{\zeta(4s-1)}. \tag{39}$$

Poles of the Eisenstein series can only occur in the strip $1/4 < \mathrm{Re}(s) < 1/2$, when $\zeta(2-4s)$ has (non-trivial) zeroes.

The result about possible poles is equivalent to the statement that the function

$$\tilde{E}^A(s) = \Gamma\left(2s+\frac{1}{2}\right)\, \zeta(4s-1)\, \left(s-\frac{1}{2}\right)\, E^A(s)$$

is holomorphic. Which is again the same as to say that

$$\Gamma(2s+k)\, \zeta(4s+2k-2)\, (2s+k-3/2)\;\, E_1^{1,k}((\tau,z),s) \tag{40}$$

is holomorphic.

Remark 4.4. Corollary 4.2 and Proposition 4.3 give the same candidates for possible poles of the analytic Jacobi Eisenstein series. But of course, Arakawa's approach has the advantage of giving a functional equation. Supplementary the corollary determines a value of s where the Eisenstein series vanishes, which is not obvious from the Proposition 4.3. Hence (40) can be improved to the statement that

$$\Gamma(2s+k)\,\zeta(4s+2k-2)\;E_1^{1,k}((\tau,z),s) \tag{41}$$

is holomorphic.

4.2. Properties of the standard L-function

As already stated, Theorem 2.1 can be used to study properties of the standard L-function $D_F^n(s)$ attached to a Hecke eigenform $F \in S_n^k$. For simplicity and because Arakawa's result is at the moment only available for Eisenstein series on $I\!H_1 \times \mathbb{C}^l$ we demonstrate this in the case of elliptic cusp forms.

Let us recall, that $D_1^{1,1}(F,s)/2 = \sum_{\lambda\in\mathbb{N}} a^F(\lambda^2)\;\lambda^{-2s}$, which is equal to

$$\zeta(4s-2k+2)^{-1}D_F^1(2s),$$

here $D_F(s)$ is the Euler product of degree 3 attached to a normalized Hecke eigenform $F \in S_1^k$. The *Main formula*, given in Theorem 2.1, specialized to this case leads to

$$\begin{aligned}\langle \mathcal{D}_0 E_1^{1,k}(s)\,,\,F\rangle_\Gamma &= (4\pi)^{-(k+s-1)}\;\Gamma(s+k-1)\,D_1^{1,1}(F,k+s-1)\\ &= (4\pi)^{-(k+s-1)}\Gamma(s+k-1)\zeta(4s+2k-2)^{-1}\\ &\quad\times D_F^1(2s+2k-2).\end{aligned}$$

This integral representation of $D_F^1(s)$ and the holomorphy of the expression given in (41) leads to the holomorphy of

$$\Gamma(s-k+2)\;\Gamma(s/2-k)\;D_F^1(s). \tag{42}$$

Thus $D_F^1(s)$ is a holomorphic function. Proposition 4.3 allows the standard zeta function to have only one possible pole (at $s = k-1/2$), but from Corollary 4.2, we know that 'there' the Eisenstein series has a zero, which gives the desired result. Hence the holomorphy of the L-function can be deduced directly from the analytic behavior of an (Jacobi) Eisenstein series, which can be seen as a slight refinement of the Shimura method (e.g. [Sh75a]).

Remark 4.5. These observations suggest that generalizing Proposition 4.3 to Jacobi Eisenstein series of degree n should lead to a better understanding of the analytic behavior of the standard zeta function attached to a Siegel Hecke eigenform of arbitrary degree ([Mi91]).

Max-Planck Institut für Mathematik
Gottfried-Claren Strasse 26
D-53225 Bonn
e-mail: Bernhard_Heim@wes.de

Kriegstraße 62
D-60326 Frankfurt
e-mail: Bernhard_Heim@wes.de

References

[An79] A. N. Andrianov, *The multiplicative arithmetic of Siegel modular forms*, Uspekhi Mat. Nauk **34** (1979), 67–135; Engl. transl. Russian Math. Surveys **34** (1979), 75–148.

[An87] ———, *Quadratic Forms and Hecke Operators*, Grundlehren der math., Wissenschaften 286, Berlin, Heidelberg, New York, Springer, 1987.

[A-K79] A. N. Andrianov and V. L. Kalinin, *On the analytic properties of standard zeta functions of Siegel modular forms*, Trudy Mat. Inst. Steklov. **142** (1976), 22–45; English Transl. Math. USSR Sb. **35** (1979), 1–17.

[A-M75] A. N. Andrianov and G. N. Maloletkin, *Behaviour of theta series of degree N under modular substitutions*, Math. USSR-Izv. **9** (1975), 227–241.

[Ar90] T. Arakawa, *Real Analytic Eisenstein Series for the Jacobi Group*, Abh. Math. Sem. Univ. Hamburg **60** (1990), 131–148.

[Ar92] ———, *Selberg Zeta Functions and Jacobi Forms*, Adv. Stud. Pure Math. **21** (1992), 181–218.

[Ar94] ———, *Jacobi Eisenstein series and a basis problem for Jacobi forms*, Comm. Mathematici Universitatis Sancti Pauli **43**-2 (1994), 181–216.

[Boe83] S. Böcherer, *Über die Fourier-Jacobi-Entwicklung der Siegelschen Eisensteinreihen*, Math. Z. **183** (1983), 21–46.

[Boe85b] ———, *Über die Funktionalgleichung automorpher L-Funktionen zur Siegelschen Modulgruppe*, J. Reine Angew. Math. **362** (1985), 146–168.

[Boe86] ———, *Ein Rationalitätssatz für formale Heckereihen zur Siegelschen Modulgruppe*, Abh. Math. Sem. Univ. Hamburg **56** (1986), 35–47.

[Du97] J. Dulinski, *L-Functions for Jacobi Forms on $\mathbb{H} \times \mathbb{C}$*, Results Math. **31** (1997), 75–94.

[E-Z85] M. Eichler and D. Zagier, *The theory of Jacobi forms*, Progr. Math. **55**, Birkhäuser Verlag, 1985.

[Ga84] P. Garrett, *Pullbacks of Eisenstein series; Applications*, In: Automorphic forms of several variables. Taniguchi symposium 1983, Birkhäuser, 1984.

[He99] B. Heim, *Pullbacks of Eisenstein series*, Hecke-Jacobi theory and automorphic L-functions, In: Automorphic Forms, Automorphic Representations and Arithmetic. Proceedings of Symposia of Pure Mathematics 66, part 2.

[He98a] ———, *L-functions for Jacobi forms and the basis problem*, MPI preprint 1998 (99), submitted.

[He98b] ———, *Quotients of L-functions*, Nagoya Math. J. **160** (2000), 143–159.

[Ib92] T. Ibukiyama, *On Jacobi Forms and Siegel Modular Forms of Half Integral Weights*, Com. Mathematici Universitatis Sancti Pauli, 1992, 109–124.

[Kl90] H. Klingen, *Introductory lectures on Siegel modular forms*, Cambridge University Press, Cambridge, 1990.

[La76] R. Langlands, *On the functional equations satisfied by Eisenstein series*, Lecture Notes in Math. **544**, Springer Verlag, Berlin-Heidelberg-New York, 1976.

[Mi91] S. Mizumoto, *Poles and residues of standard L-functions attached to Siegel modular forms*, Math. Ann. **289** (1991), 589–612.

[Mu89] A. Murase, *L-functions attached to Jacobi forms of degree n*, Part I. The basic identity, J. Reine Angew. Math. **401** (1989), 122–156.

[Sh73] G. Shimura, *On modular forms of half integral weight*, Ann. of Math. **97** (1973), 440–448.

[Sh75a] ———, *On the holomorphy of certain Dirichlet series*, Proc. London Math. Soc., 3rd ser. **31** (1975), 79–98.

[Sh75b] ———, *On the Fourier coefficients of modular forms of several variables*, Nachr. Akad. Wiss. Göttingen Math.- Phys. Kl. II **17** (1975), 261–268.

[Sh85] ———, *On Eisenstein series of half-integral weight*, Duke Math. J. **52** (1985), 281–324.

[Sh94] ———, *Euler products and Fourier coefficients of automorphic forms on symplectic groups*, Invent. Math. **116** (1994), 531–576.

[Za77] D. Zagier, *Modular forms whose Fourier coefficients involve zeta functions of quadratic fields*, in Modular Functions of One Variable VI, Lecture Notes in Math. **627**, Springer Verlag, Berlin-Heidelberg-New York, 1977, pp. 105–169.

[Zi89] C. Ziegler, *Jacobi forms of higher degree*, Abh. Math. Sem. Univ. Hamburg **59** (1989), 191–224.