J. Math. Kyoto Univ. (JMKYAZ)
43-3 (2003), 567–583

A conditional limit theorem for generalized diffusion processes

By

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Abstract

Let $\mathbf{X} = \{X(t) : t \geq 0\}$ be a one-dimensional generalized diffusion process with initial state $X(0) > 0$, hitting time $\tau_{\mathbf{X}}(0)$ at the origin and speed measure m which is regularly varying at infinity with exponent $1/\alpha - 1 > 0$. It is proved that, for a suitable function $u(c)$, the probability law of $\{u(c)^{-1}X(ct) : 0 < t \leq 1\}$ conditioned by $\{\tau_{\mathbf{X}}(0) > c\}$ converges as $c \to \infty$ to the conditioned $2(1-\alpha)$-dimensional Bessel excursion on natural scale and that the latter is equivalent to the $2(1-\alpha)$-dimensional Bessel meander up to a scale transformation. In particular, the distribution of $u(c)^{-1}X(c)$ converges to the Weibull distribution. From the conditional limit theorem we also derive a limit theorem for some of regenerative process associated with $\mathbf{X}$.

1. Introduction

A number of limit theorems of generalized diffusion processes and their functionals have been obtained in the literature; see e.g. Kasahara (1975), Minami et al. (1985), Ogura (1989), Stone (1963), Watanabe (1995), and Yamazato (1990) among others. Let $\mathbf{X} = \{X(t) : t \geq 0\}$ be a non-negative generalized diffusion process with speed measure $m(dx)$ which is regularly varying at infinity with exponent $1/\alpha - 1 > 0$. Stone (1963) proved that, for a suitable scale function $u(c)$, the distribution of $\{u(c)^{-1}X(ct) : t \geq 0\}$ converges in distribution as $c \to \infty$ to a $2(1-\alpha)$-dimensional Bessel diffusion process on natural scale, and Kasahara (1975) showed that essentially only Bessel processes can arise in this kind of limits. See Lamperti (1962, 1972) for discussions of scale limits leading to more general classes of processes. On the other hand, a number of conditional limit theorems for Brownian motion and random walks have been proved which lead to Brownian meander and Brownian excursion

1991 *Mathematics Subject Classification(s).* 60J60, 60J25, 60F05
Received March 4, 2002
Revised November 28, 2002
*Supported by the National Natural Science Foundation 10131040, Supported in part by Grant-in-Aid for Scientific Research 11640103, Supported in part by Grant-in-Aid for Scientific Research 11304003.

processes; see e.g. Belkin (1972), Bolthausen (1976), Durrett et al. (1977), Iglehart (1974), and Shimura (1983). Similar conditional limit theorems for branching processes have also been studied; see e.g. Athreya and Ney (1972), Lamperti and Ney (1968), Li (2000) and the references therein.

In this paper, we prove a conditional limit theorem of the generalized diffusion process $\mathbf{X} = \{X(t) : t \geq 0\}$. Suppose that $X(0) > 0$ and let $\tau_{\mathbf{X}}(0) = \inf\{t \geq 0 : X(t) = 0\}$. We prove that the distribution of $\{u(c)^{-1}X(ct) : 0 < t \leq 1\}$ under $\boldsymbol{P}\{\cdot\,|\tau_{\mathbf{X}}(0) > c\}$ converges in distribution as $c \to \infty$ and we characterize the limit in terms of Bessel excursion and Bessel meander. In particular, the conditional distribution of $u(c)^{-1}X(c)$ converges to the Weibull distribution. From the conditional limit theorem we also derive a limit theorem for some of regenerative processes associated with $\mathbf{X}$. Bessel meanders and their generalizations have been studied by Yor (1992a,b, 1997).

2. Preliminaries

Let $\mathcal{M}$ be the totality of left continuous, non-decreasing functions $m : [0,\infty) \to [0,\infty]$ with $m(0) = 0$. For any $m \in \mathcal{M}$, set $l_m = \sup\{x \geq 0 : m(x) < \infty\}$. We shall identify $m \in \mathcal{M}$ with the measure m on $[0,\infty)$ determined by $m([0,x)) = m(x)$. Note that $m(\{l_m\}) = \infty$ is possible. We sometimes think $m \in \mathcal{M}$ as a measure on $(-\infty,\infty)$. Given $m \in \mathcal{M}$, let E_m denote its closed support and let $m^{-1} \in \mathcal{M}$ denote its inverse function, that is, $m^{-1}(0) = 0$ and $m^{-1}(x) = \sup\{y \geq 0 : m(y) < x\}$ for $x > 0$. Let $\mathcal{M}_0$ be the set of elements $m \in \mathcal{M}$ such that $0 \in E_m$. For any given one-dimensional process $\mathbf{X} = \{X(t) : t \geq 0\}$ and a point x in its state space we define the hitting time $\tau_{\mathbf{X}}(x) = \inf\{t \geq 0 : X(t) = x\}$.

Let $m \in \mathcal{M}$ and let $\mathbf{B} = \{B(t) : t \geq 0\}$ be a one-dimensional Brownian motion with initial state $B(0) = 0$ and generator d^2/dx^2. Let $l(t,x)$ denote the local time of $\mathbf{B}$. Of course, $\mathbf{B}$ and $l(t,x)$ are defined on some probability space $(\Omega, \mathcal{F}, \boldsymbol{P})$. Let

$$S(x,t) = \int_{[0,\infty)} l(t, y-x)m(dy), \qquad t \geq 0,\ x \in \mathbb{R}, \tag{2.1}$$

and

$$S^{-1}(x,t) = \sup\{r : S(x,r) \leq t\}, \qquad t \geq 0,\ x \in \mathbb{R}. \tag{2.2}$$

For any $x \geq 0$ we define

$$X(x,t) = x + B(S^{-1}(x,t)), \qquad t \geq 0. \tag{2.3}$$

Then $\mathbf{X}(x) := \{X(x,t) : t \geq 0, \boldsymbol{P}\}$ is a cádlág strong Markov process in E_m, whose life time is the hitting time at l_m. This process is the so-called *generalized diffusion process* with speed measure $m(dx)$; see e.g. Kasahara (1975) and Stone (1963). The process has formal generator $d^2/dm(x)dx$. Its transition function can be characterized in terms of Krein's correspondence as follows. Consider the integral equations

$$\phi(x,\lambda) = 1 + \lambda \int_0^x dy \int_{[0,y)} \phi(z,\lambda) m(dz) \tag{2.4}$$

and

$$\psi(x,\lambda) = x + \lambda \int_0^x dy \int_{[0,y)} \psi(z,\lambda) m(dz). \tag{2.5}$$

For each $\lambda > 0$, the equations have unique solutions $\phi(x,\lambda)$ and $\psi(x,\lambda)$ respectively. Furthermore, for each $x \geq 0$, both $\phi(x,\lambda)$ and $\psi(x,\lambda)$ can be extended to entire functions of λ. The *characteristic function* $h(\lambda)$ of $m(x)$ is defined as

$$h(\lambda) = \int_0^\infty \frac{dx}{\phi(x,\lambda)^2} = \lim_{x\to\infty} \frac{\psi(x,\lambda)}{\phi(x,\lambda)} \tag{2.6}$$

(under the conventions $1/\infty = 0$ and $1/0 = \infty$). The function $h(\lambda)$ has representation

$$h(\lambda) = a + \int_{[0,\infty)} \frac{\sigma(ds)}{\lambda + s}, \qquad \lambda > 0, \tag{2.7}$$

where $a = \inf\{x \geq 0 : m(x) > 0\}$ and $\sigma(ds)$ is a Radon measure on $[0,\infty)$ satisfying

$$\int_{[0,\infty)} \frac{\sigma(ds)}{1+s} < \infty.$$

The correspondence between m and (a,σ) is one-to-one and known as *Krein's correspondence*. Let $p(t,x,y)$ denote the density with respect to $m(dy)$ of the transition function of $\mathbf{X}(x)$. Then we have

$$p(t,x,y) = \int_{[0,\infty)} \mathrm{e}^{-\lambda t} \phi(x,-\lambda)\phi(y,-\lambda)\sigma(d\lambda), \qquad x \geq 0,\ y \geq 0. \tag{2.8}$$

We refer the reader to Itô and McKean (1965), Kac and Krein (1974), Kasahara (1975), Kotani and Watanabe (1982), and Yamazato (1990) for more detailed explanations of those results.

For $m \in \mathcal{M}_0$ and for $X(x,t)$ defined by (2.3), let $X^0(x,t) := X(x, t \wedge \tau_{\mathbf{X}(x)}(0))$. Then $\mathbf{X}^0(x) := \{X^0(x,t) : t \geq 0, \boldsymbol{P}\}$ is the *absorbing barrier generalized diffusion process*. It is known that $m^{-1} \in \mathcal{M}_0$ has characteristic function $1/\lambda h(\lambda)$, which may be represented as

$$\frac{1}{\lambda h(\lambda)} = m(0^+) + \frac{1}{\lambda l_m} + \int_{(0,\infty)} \frac{\sigma^0(ds)}{s(\lambda+s)}, \qquad \lambda > 0, \tag{2.9}$$

where $\sigma^0(ds)$ is a Radon measure on $(0,\infty)$ satisfying

$$\int_{(0,\infty)} \frac{\sigma^0(ds)}{s(1+s)} < \infty;$$

see Minami et al. (1985; Lemma 3). The transition density of the absorbing barrier process $\mathbf{X}^0(x)$ is given by

$$p^0(t,x,y) = \int_{(0,\infty)} \mathrm{e}^{-\lambda t}\psi(x,-\lambda)\psi(y,-\lambda)\sigma^0(d\lambda), \qquad x>0,\ y>0; \tag{2.10}$$

see Minami et al. (1985; (3.17)). Now we have the following

Lemma 2.1. *For $x \in E_m$, we have $\tau_{\mathbf{X}(x)}(0) = S(x,\tau_{\mathbf{B}}(-x))$ and $\tau_{\mathbf{B}}(-x) = S^{-1}(x,\tau_{\mathbf{X}(x)}(0))$ (a.s.).*

Proof. For $t > \tau_{\mathbf{B}}(-x)$ we have $l(t,-x) > l(\tau_{\mathbf{B}}(-x),-x)$. Since $0 \in E_m$, by (2.1) and the continuity of $l(\cdot,\cdot)$ we get $S(x,t) > \sigma := S(x,\tau_{\mathbf{B}}(-x))$. That is, $\tau_{\mathbf{B}}(-x)$ is an increasing point of $S(x,\cdot)$. Therefore, $S^{-1}(x,\sigma) = \tau_{\mathbf{B}}(-x)$. This implies that $X(x,\sigma) = B(S^{-1}(x,\sigma)) + x = 0$, and hence $\tau_{\mathbf{X}(x)}(0) \le \sigma$. On the other hand, since $S(x,\cdot)$ is continuous, $S^{-1}(x,\cdot)$ is strictly increasing. Then for any $t < \sigma$, we have $S^{-1}(x,t) < S^{-1}(x,\sigma) = \tau_{\mathbf{B}}(-x)$. From this it follows that $X(x,t) = B(S^{-1}(x,t)) + x > 0$, yielding $\tau_{\mathbf{X}(x)}(0) \ge \sigma$. The second equality is immediate. □

In the sequel, we consider a sequence $m_n \in \mathcal{M}_0$ for $n = 0,1,\ldots$. Let $\mathbf{X}_n(x)$ and $\mathbf{X}_n^0(x)$ denote the corresponding generalized diffusion processes, and let $p_n(t,x,y)$ and $p_n^0(t,x,y)$ denote their transition densities. We also write ψ_n, σ_n^0 and so on for the corresponding quantities associated with m_n. An earlier version of the following result was proved by Stone (1963). We here present it in a form which is more convenient for our application in the sequel.

Theorem 2.1. *Suppose that $E_{m_0} = [0,\infty)$. If $E_{m_n} \ni x_n \to x_0$ and $m_n(x) \to m_0(x)$ for all continuity points $x \ge 0$ of m_0, then $X_n(x_n,t) \to X_0(x_0,t)$ and $X_n^0(x_n,t) \to X_0^0(x_0,t)$ uniformly in $t \in [0,\tau_{\mathbf{X}_0^0(x_0)}(0)]$ (a.s.).*

Proof. Let $S_n(x,t)$ be defined by (2.1) with m replaced by m_n. Under the assumption, $S_0(x_0,t)$ is strictly increasing in $t \in [0,\tau_{\mathbf{B}}(-x_0)]$ and $\tau_{\mathbf{B}}(-x_0)$ is an increasing point of $S(x_0,\cdot)$, so $S_0^{-1}(x_0,t)$ is continuous in $t \in [0,\tau_{\mathbf{X}_0(x_0)}(0)]$ by virtue of Lemma 2.1. By (2.1) we have $S_n(x_n,t) \to S_0(x_0,t)$ for all $t \ge 0$, and hence $S_n^{-1}(x_n,t) \to S_0^{-1}(x_0,t)$ for all $t \in [0,\tau_{\mathbf{X}_0(x_0)}(0)]$. Note that the functions are non-decreasing, $S_0(x_0,t)$ is continuous in $t \in [0,\infty)$ and $S_0^{-1}(x_0,t)$ is continuous in $t \in [0,\tau_{\mathbf{X}_0(x_0)}(0)]$. Consequently $S_n(x_n,t) \to S_0(x_0,t)$ uniformly in $t \in [0,\tau_{\mathbf{B}}(-x_0)]$ and $S_n^{-1}(x_n,t) \to S_0^{-1}(x_0,t)$ uniformly in $t \in [0,\tau_{\mathbf{X}_0(x_0)}(0)]$. Then the assertions follow from the relations $\tau_{\mathbf{X}_0(x_0)}(0) = \tau_{\mathbf{X}_0^0(x_0)}(0)$, $X_n(x_n,t) = B(S_n^{-1}(x_n,t)) + x_n$ and

$$\begin{aligned} X_n^0(x_n,t) &= B(S_n^{-1}(x_n, t\wedge\tau_{\mathbf{X}_n(x_n)}(0))) + x_n \\ &= B(S_n^{-1}(x_n,t)\wedge\tau_{\mathbf{B}}(-x_n))) + x_n, \end{aligned}$$

where we have used Lemma 2.1 for the last equality. □

Theorem 2.2. *Suppose that $E_{m_0} = [0,\infty)$ and $m_n(x) \to m_0(x)$ for all continuity points $x \ge 0$ of m_0. Then for any $t > 0$ and $a > 0$ we have*

$$\lim_{n\to\infty} \sup_{0<x,y<a} \frac{1}{xy} |p_n^0(t,x,y) - p_0^0(t,x,y)| = 0. \tag{2.11}$$

Proof. By (2.5), for any $\lambda \in I\!R$ and $x_2 > x_1 > 0$ we have

$$\begin{aligned}
\left|\frac{\psi(x_2,\lambda)}{x_2} - \frac{\psi(x_1,\lambda)}{x_1}\right| &= \frac{|\lambda|}{x_1 x_2}\left| x_1 \int_{x_1}^{x_2} dy \int_{[0,y)} \psi(z,\lambda) m(dz) \right. \\
&\qquad \left. - (x_2 - x_1) \int_0^{x_1} dy \int_{[0,y)} \psi(z,\lambda) m(dz) \right| \\
&\le \frac{|\lambda|}{x_1 x_2}\left[x_1 \int_{x_1}^{x_2} dy \int_{[0,x_2)} |\psi(z,\lambda)| m(dz) \right. \\
&\qquad \left. + (x_2 - x_1) \int_0^{x_1} dy \int_{[0,x_2)} |\psi(z,\lambda)| m(dz) \right] \\
&\le 2|\lambda|(x_2 - x_1) \int_{[0,x_2)} \frac{|\psi(z,\lambda)|}{z} m(dz) \\
&\le 2|\lambda|(x_2 - x_1) m(x_2) \cosh\left\{\sqrt{2\,|\lambda|\, x_2 m(x_2)}\right\},
\end{aligned}$$

where we have used the inequality

$$|\psi(x,\lambda)| \le x \cosh\left\{\sqrt{2|\lambda| x m(x)}\right\}, \qquad x \ge 0,\ \lambda \in I\!R; \tag{2.12}$$

see Ogura (1989; (5.4)). Using these and (2.10) we get

$$\begin{aligned}
&\left|\frac{p_n^0(t,x_2,y)}{x_2 y} - \frac{p_n^0(t,x_1,y)}{x_1 y}\right| \\
&\quad \le \int_{(0,\infty)} \left|\frac{\psi_n(x_2,\lambda)}{x_2} - \frac{\psi_n(x_1,\lambda)}{x_1}\right| \left|\frac{\psi_n(y,\lambda)}{y}\right| \mathrm{e}^{-\lambda t} \sigma_n^0(d\lambda) \\
&\quad \le 2|\lambda|(x_2 - x_1) m_n(x_2) \\
&\qquad \cdot \int_{(0,\infty)} \cosh\left\{\sqrt{2|\lambda| x_2 m_n(x_2)}\right\} \cosh\left\{\sqrt{2|\lambda| y m_n(y)}\right\} \mathrm{e}^{-\lambda t} \sigma_n^0(d\lambda).
\end{aligned}$$

By Ogura (1989; Lemma 5.3) and the symmetry of $p_n^0(t,\cdot,\cdot)$, for every $t > 0$ and $a > 0$ the sequence $\{(xy)^{-1} p_n^0(t,x,y)\}$ is equi-continuous and uniformly bounded in $(x,y) \in (0,a] \times (0,a]$. Thus there is a subsequence $\{n_k\} \subset \{n\}$ such that $(xy)^{-1} p_{n_k}^0(t,x,y)$ converges to some function $q(t,x,y)$. Since $p_n^0(t,x,y) \to p_0^0(t,x,y)$ by Ogura (1989; Theorem 2.1), we must have $q(t,x,y) = (xy)^{-1} p_0^0(t,x,y)$, yielding the desired result. □

3. Conditional limit theorem

Given an interval $I \subset I\!R$, let $D(I)$ denote the set of cadlag paths from I to $I\!R$. We topologize $D(I)$ by the convention that $w_n \to w_0$ in $D(I)$ if and only if

$w_n(t) \to w_0(t)$ uniformly in $t \in [a,b]$ for each bounded closed interval $[a,b] \subset I$. Let $C(I)$ denote the subspace of $D(I)$ comprising of continuous paths. Fix a function $m \in \mathcal{M}_0$ that is regularly varying at infinity and assume that

$$m(x) \sim x^{1/\alpha-1}K(x), \qquad x \to \infty, \tag{3.1}$$

for a constant $0 < \alpha < 1$ and a slowly varying function $K(x)$. Let $u(\cdot)$ denote the inverse function $x^{1/\alpha}K(x)$. It is easy to check that $u(x) \sim x^{\alpha}L(x)$ for some slowly varying function $L(x)$. If we define $X(x,t)$ by (2.3), then $\mathbf{X}_c := \{X_c(t) \equiv u(c)^{-1}X(x,ct) : t \geq 0, \boldsymbol{P}\}$ is a generalized diffusion process with initial state $u(c)^{-1}x$ and speed measure $m_c(x) := c^{-1}u(c)m(u(c)x)$. Observe that

$$\lim_{c\to\infty} m_c(x) = \lim_{u\to\infty}[u^{1/\alpha}K(u)]^{-1}u^{1/\alpha}x^{1/\alpha-1}K(ux) = x^{1/\alpha-1}, \quad x \geq 0. \tag{3.2}$$

The generalized diffusion process $\mathbf{X}_0 = \{X_0(t) : t \geq 0, \boldsymbol{P}\}$ with speed measure $m_0(dx) := (1/\alpha - 1)x^{1/\alpha-2}dx$ is called a *reflecting* $2(1-\alpha)$*-dimensional Bessel diffusion process on natural scale.* Indeed, $\mathbf{X}_0$ has generator $d^2/dm_0dx = \alpha(1-\alpha)^{-1}x^{2-1/\alpha}d^2/dx^2$ so that $\{\sqrt{2\alpha(1-\alpha)}X_0^{1/2\alpha}(t) : t \geq 0, \boldsymbol{P}\}$ is a standard $2(1-\alpha)$-dimensional Bessel diffusion with reflecting barrier 0. We shall also need to consider the generalized diffusion processes with absorbing boundary condition at the origin. For any $c \geq 0$, let $P_c(t)$ denote semigroup of $\mathbf{X}_c$ and let $P_c^0(t)$ denote semigroup of the corresponding absorbing barrier processes.

Lemma 3.1. *The transition function $P_0^0(t,x,dy)$ has density*

$$p_0^0(t,x,y) = \frac{\alpha\sqrt{xy}}{t}\exp\left\{-\frac{\alpha(1-\alpha)(x^{1/\alpha}+y^{1/\alpha})}{t}\right\} I_\alpha\left(\frac{2\alpha(1-\alpha)(xy)^{1/2\alpha}}{t}\right) \tag{3.3}$$

with respect to $m_0(dy)$, where

$$I_\alpha(z) = \sum_{n=0}^{\infty}\frac{(z/2)^{2n+\alpha}}{n!\Gamma(\alpha+n+1)}. \tag{3.4}$$

Proof. It is not hard to check that $Y_0(t) := 2\alpha(1-\alpha)X_0(t)^{1/\alpha}$ is the squared Bessel process generated by $2xd^2/dx^2 + 2(1-\alpha)d/dx$ with absorbing boundary condition at zero. By Borodin and Salminen (1996; p. 117), $Y_0(t)$ has transition function

$$Q_0^0(t,x,dy) = \frac{1}{2t}\exp\left\{-\frac{x+y}{2t}\right\} I_\alpha\left(\frac{\sqrt{xy}}{t}\right)\left(\frac{x}{y}\right)^{\alpha/2} dy, \qquad t,x,y > 0.$$

Then (3.3) follows by a simple transformation. □

Let us describe a σ-finite Markovian measure associated with $P_0^0(t)$ which plays an important role in the study of our conditional limit theorems. For $t > 0$ and $y > 0$, let $\kappa_t(dy) = \kappa_t(y)m_0(dy)$, where

$$\kappa_t(y) = \frac{\alpha^{\alpha+1}(1-\alpha)^\alpha y}{\Gamma(1+\alpha)t^{\alpha+1}}\exp\left\{-\frac{\alpha(1-\alpha)}{t}y^{1/\alpha}\right\}. \tag{3.5}$$

It is simple to check that

$$\kappa_t(y) = \lim_{x\downarrow 0} x^{-1} p_0^0(t,x,y), \qquad t>0,\ y>0, \tag{3.6}$$

and $(\kappa_t)_{t>0}$ form an entrance law for $P_0^0(t)$, that is $\kappa_{s+t} = \kappa_s P_0^0(t)$ for all $s>0$ and $t>0$. By the general theory of Markov processes, there is a σ-finite measure $\boldsymbol{Q}$ on $C(0,\infty)$ such that

$$\begin{aligned}&\boldsymbol{Q}\{w(t_1)\in dx_1, w(t_2)\in dx_2,\dots,w(t_n)\in dx_n, t_n<\tau_w(0)\}\\ &\qquad = \kappa_{t_1}(dx_1)P_0^0(t_2-t_1,x_1,dx_2)\cdots P_0^0(t_n-t_{n-1},x_{n-1},dx_n)\end{aligned}$$

for $0<t_1<t_2<\cdots$ and $x_1,x_2,\cdots>0$. Indeed, for $\boldsymbol{Q}$-almost all paths $w\in C(0,\infty)$ we have $w(0^+)=0$, $\tau_w(0)<\infty$ and $w(t)\equiv w(t\wedge\tau_w(0))$, where $\tau_w(0)=\inf\{t>0: w(t)=0\}$. In the theory of diffusion processes, $\boldsymbol{Q}$ is known as the excursion law of the α-Bessel diffusion; see e.g. Biane and Yor (1987) and Pitman and Yor (1992, 1998) for some properties of the excursion law.

Lemma 3.2. *Suppose $x>0$ and let $x_c = u(c)^{-1}x$. Then for each $t>0$, we have $x_c^{-1}P_c^0(t,x_c,\cdot)\to\kappa_t(\cdot)$ weakly as $c\to\infty$.*

Proof. Let $p_c^0(t,x,y)$ denote the density of $P_c^0(t,x,dy)$ with respect to $m_c(dy)$. For $b>a>0$ we may appeal Theorem 2.2 and (3.6) to see that

$$\begin{aligned}\lim_{c\to\infty} x_c^{-1}P_c^0(t,x_c,[a,b]) &= \lim_{c\to\infty}\int_a^b x_c^{-1}p_c^0(t,x_c,y)m_c(dy)\\ &= \int_a^b \kappa_t(y)m_0(dy) = \kappa_t([a,b]).\end{aligned}$$

By Yamazato (1990; Theorem 4), we have

$$\begin{aligned}P_c^0(t,x_c,(0,\infty)) &\sim \frac{[\alpha(1-\alpha)]^\alpha x}{\Gamma(1+\alpha)(ct)^\alpha L(ct)}\\ &\sim \frac{[\alpha(1-\alpha)]^\alpha x_c}{\Gamma(1+\alpha)t^\alpha} = \kappa_t(0,\infty)x_c, \qquad c\to\infty.\end{aligned} \tag{3.7}$$

Then $x_c^{-1}P_c^0(t,x_c,\cdot)\to\kappa_t(\cdot)$ weakly as $c\to\infty$. □

Lemma 3.3. *Suppose $\{f_c : c\geq 0\}$ is a bounded family of Borel functions on $[0,\infty)$ such that $f_c(x_c)\to f_0(x)$ whenever $c\to\infty$ and $x_c\to x$. If $\{p_c : c\geq 0\}$ is a family of Borel probabilities on $[0,\infty)$ such that $p_c\to p_0$ weakly as $c\to\infty$, then*

$$\lim_{c\to\infty}\int_{[0,\infty)} f_c(x)p_c(dx) = \int_{[0,\infty)} f_0(x)p_0(dx). \tag{3.8}$$

Proof. By Skorokhod's result, we can construct a family of random variables $\{\xi_c : c\geq 0\}$ such that ξ_c has distribution p_c and $\xi_c\to\xi_0$ as $c\to\infty$ (a.s.). Then the assumption implies that $f_c(\xi_c)\to f_0(\xi_0)$ as $c\to\infty$ (a.s.), and (3.8) follows by bounded convergence theorem. □

Theorem 3.1. *For each $x > 0$, the distribution on $D(0,1]$ of the rescaled processes $\{u(c)^{-1}X(x,ct) : 0 < t \le 1\}$ under $\boldsymbol{P}\{\cdot|\tau_{\mathbf{X}(x)}(0) > c\}$ converges as $c \to \infty$ to $\boldsymbol{Q}_0 := \boldsymbol{Q}\{\cdot|\tau_w(0) > 1\}$.*

Proof. Let $\boldsymbol{P}_x^c$ denote the distribution on $D[0,\infty)$ of the generalized diffusion process with initial state $x \ge 0$ and speed measure $m_c(dy)$. Suppose that $0 < r < 1$ and $F(\cdot)$ is a continuous function on $D[r,1]$. By (3.7) and the Markov property,

$$\begin{aligned}
&\lim_{c\to\infty} \boldsymbol{E}\{F((X_c(t))_{r\le t\le 1})|\tau_{\mathbf{X}(x)}(0) > c\} \\
&\quad = \lim_{c\to\infty} \boldsymbol{P}\{\tau_{\mathbf{X}(x)}(0) > c\}^{-1}\boldsymbol{E}\{F((X_c(t))_{r\le t\le 1}); \tau_{\mathbf{X}(x)}(0) > c\} \\
&\quad = \lim_{c\to\infty} \frac{\Gamma(1+\alpha)u(c)}{[\alpha(1-\alpha)]^\alpha x} \int_0^\infty P_c^0(r, u(c)^{-1}x, dy) \\
&\qquad \cdot \boldsymbol{P}_y^c\{F((w(t-r))_{r\le t\le 1}); \tau_w(0) > 1-r\} \\
&\quad = \frac{1}{\kappa_1(0,\infty)} \int_0^\infty \kappa_r(dy)\boldsymbol{P}_y^0\{F((w(t-r))_{r\le t\le 1}); \tau_w(0) > 1-r\} \\
&\quad = \frac{1}{\kappa_1(0,\infty)} \boldsymbol{Q}\{F((w(t))_{r\le t\le 1}); \tau_w(0) > 1\},
\end{aligned}$$

where we also used Theorem 2.1 and Lemmas 3.2 and 3.3 for the third equality. □

Corollary 3.1. *For each $x > 0$, the distribution of $u(c)^{-1}X(x,c)$ under $\boldsymbol{P}\{\cdot|\tau_{\mathbf{X}(x)}(0) > c\}$ converges as $c \to \infty$ to the Weibull distribution*

$$\kappa_1(0,\infty)^{-1}\kappa_1(dy) = (1-\alpha)y^{1/\alpha-1}\exp\{-\alpha(1-\alpha)y^{1/\alpha}\}dy, \qquad y > 0. \tag{3.9}$$

From Corollary 3.1 and a theorem of Mitov et al. (1996) we can deduce a limit theorem for some kind of regenerative processes associated with generalized diffusions. Let G be a probability measure on $(0,\infty)$ such that

$$g := \int_{(0,\infty)} yG(dy) < \infty. \tag{3.10}$$

Let $\mathbf{X}_i = \{X_i(t) : t \ge 0\}$, $i = 1, 2, \dots$, be a sequence of i.i.d. generalized diffusions with initial distribution G and speed measure $m(dy)$, and let ξ_i, $i = 1, 2, \dots$, be a sequence of i.i.d. non-negative random variables with $\boldsymbol{E}\xi_i < \infty$. Assume that the two families are defined on the same probability space and are independent of each other. Let $\sigma_0 = 0$ and let

$$\sigma_n = \sum_{i=1}^n (\tau_i + \xi_i), \qquad n = 1, 2, \dots,$$

where $\tau_i = \tau_{\mathbf{X}_i}(0)$. Then $\sigma_n \to \infty$ as $n \to \infty$ (a.s.). The regenerative process $\{Y(t) : t \ge 0\}$ is defined by

$$Y(t) = X_n((t-\sigma_{n-1}) \wedge \tau_n), \qquad \sigma_{n-1} \le t < \sigma_n. \tag{3.11}$$

Corollary 3.2. *For each $x \geq 0$,*

$$\lim_{t\to\infty} \boldsymbol{P}\{u(t)^{-1}Y(t) \leq x\}$$
$$= \pi^{-1}\sin(\pi\alpha)\int_0^1 y^{-\alpha}(1-y)^{\alpha-1}(1-\exp\{-\alpha(1-\alpha)x^{1/\alpha}y^{-1}\})dy.$$

Proof. With Corollary 3.1 in hand, the only thing we need to do is to check the tail of the hitting time $\tau_{\mathbf{X}(x)}(0)$ is of the right order and to appeal the general result of Mitov et al. (1996, 2001). Observe that $u(x,\lambda) := \boldsymbol{E}\exp\{-\lambda\tau_{\mathbf{X}(x)}(0)\}$ satisfies

$$\frac{d}{dm(x)}\frac{d^+}{dx}u(x,\lambda) = \lambda u(x,\lambda), \qquad u(0,\lambda)=1, \quad x\geq 0,\ \lambda>0, \tag{3.12}$$

where d^+/dx denotes the right derivative. But, it is well-known that

$$u(x,\lambda) = \phi(x,\lambda) - h(\lambda)^{-1}\psi(x,\lambda), \qquad x\geq 0,\ \lambda>0; \tag{3.13}$$

see e.g. Itô and McKean (1965; p. 129). By (2.4), (2.5), and (3.13) we get

$$\frac{d^+}{dx}u(0,\lambda) = \lambda m(0^+) - h(\lambda)^{-1}, \qquad \lambda > 0.$$

Combining this with (3.12) we get

$$u(x,\lambda) = 1 - h(\lambda)^{-1}x + \lambda\int_0^x dy\int_{[0,y)} u(z,\lambda)m(dz), \qquad x\geq 0,\ \lambda>0.$$

Then it is easy to see that $0 \leq h(\lambda)[1-u(x,\lambda)] \leq x$, and $h(\lambda)[1-u(x,\lambda)] \to x$ as $\lambda\to 0$. Now the dominated convergence theorem yields

$$h(\lambda)[1-\boldsymbol{E}\exp\{-\lambda\tau_i\}] = \int_{(0,\infty)} h(\lambda)[1-u(x,\lambda)]G(dx) \to g, \qquad \lambda\to 0.$$

Using Kasahara (1975; Theorem 2) we get

$$1-\boldsymbol{E}\exp\{-\lambda\tau_i\} \sim \frac{g}{h(\lambda)} \sim \frac{[\alpha(1-\alpha)]^\alpha\Gamma(1-\alpha)g\lambda^\alpha}{\Gamma(1+\alpha)L(1/\lambda)}, \qquad \lambda\to 0.$$

By Tauberian theorem,

$$\boldsymbol{P}\{\tau_i > c\} \sim \frac{[\alpha(1-\alpha)]^\alpha g}{\Gamma(1+\alpha)c^\alpha L(c)}, \qquad c\to\infty;$$

see e.g. Feller (1971; p. 447). Then the result follows immediately from Corollary 3.1 and Mitov et al. (1996; Theorem 1) (see also Mitov et al. (2001; Theorem 2.1)). □

Note that the three limit distributions obtained above are universal, in the sense that they only depend on the constant $0 < \alpha < 1$ rather than the explicit form of the speed measure. In particular, the Weibull distribution came from the conditional excursion law of the Bessel diffusion. These are rather similar to the results for critical branching processes; see e.g. Athreya and Ney (1972), Lamperti and Ney (1968), and Li (2000). It would be interesting if one could establish conditional limit theorems for generalized diffusions under the assumption (3.1) with $\alpha = 1$, which would correspond to the theorems for sub-critical branching processes and lead to limit laws depending on the speed measure explicitly; see e.g. Li (2000; Theorems 4.1 and 4.3).

4. Conditioned Bessel excursion and Bessel meander

In this section, we give a characterization for the limit law $\boldsymbol{Q}_0$ in Theorem 3.1 in terms of stochastic differential equation. From this characterization we get that $\boldsymbol{Q}_0$ is in fact the law of the $2(1-\alpha)$-dimensional Bessel meander on natural scale. Let us define a conservative inhomogeneous transition semigroup $(Q_{s,t})_{0\leq s\leq t\leq 1}$ on the state space $(0,\infty)$ by

$$(4.1)\qquad Q_{s,t}(x,dy) = \left(P_0^0(1-s)1(x)\right)^{-1} P_0^0(1-t)1(y)P_0^0(t-s,x,dy).$$

In view of (3.6) and (4.1), we can extend $(Q_{s,t})_{0\leq s\leq t\leq 1}$ to a transition semigroup on $[0,\infty)$ by continuity. It is easy to check that $\boldsymbol{Q}_0$ coincides with the distribution on $C(0,1]$ of a Markov process with semigroup $(Q_{s,t})_{0\leq s\leq t\leq 1}$ and initial state zero. The following result is already known; see Yor (1992a, Section 2) for a detailed discussion of the result and its variations. We here include a short proof of the result based on Lemma 3.1 for completeness.

Lemma 4.1. *For $t>0$ and $x\geq 0$, we have $P_0^0(t)1(x) = \Gamma(\alpha)^{-1}F(t,x)$, where*

$$(4.2)\qquad F(t,x) = \frac{1}{(2t)^\alpha}\int_0^{2\alpha(1-\alpha)x^{1/\alpha}} u^{\alpha-1}\mathrm{e}^{-u/2t}du.$$

Proof. By Lemma 3.1 we have

$$\begin{aligned}
P_0^0(t)1(x) &= \frac{(1-\alpha)}{t}\sqrt{x}\exp\left\{-\frac{\alpha(1-\alpha)}{t}x^{1/\alpha}\right\}\sum_{n=0}^\infty \frac{[\alpha(1-\alpha)x^{1/2\alpha}]^{2n+\alpha}}{n!\Gamma(n+\alpha+1)t^{2n+\alpha}} \\
&\quad\cdot\int_0^\infty y^{(n+1)/\alpha-1}\exp\left\{-\frac{\alpha(1-\alpha)}{t}y^{1/\alpha}\right\}dy \\
&= \frac{(1-\alpha)}{t}\sqrt{x}\exp\left\{-\frac{\alpha(1-\alpha)}{t}x^{1/\alpha}\right\} \\
&\quad\cdot\sum_{n=0}^\infty \frac{[\alpha(1-\alpha)x^{1/2\alpha}]^{2n+\alpha}}{n!\Gamma(n+\alpha+1)t^{2n+\alpha}}\cdot\frac{n!\alpha t^{n+1}}{[\alpha(1-\alpha)]^{n+1}} \\
&= \exp\left\{-\frac{\alpha(1-\alpha)}{t}x^{1/\alpha}\right\}\sum_{n=0}^\infty\frac{[\alpha(1-\alpha)]^{n+\alpha}x^{n/\alpha+1}}{\Gamma(n+\alpha+1)t^{n+\alpha}} \\
&= \frac{1}{\Gamma(\alpha)}\int_0^{\alpha(1-\alpha)x^{1/\alpha}/t} z^{\alpha-1}\mathrm{e}^{-z}dz,
\end{aligned}$$

where we have used the equality

$$\mathrm{e}^{-y}\sum_{n=0}^{\infty}\frac{y^{n+\alpha}}{\Gamma(n+\alpha+1)}=\frac{1}{\Gamma(\alpha)}\int_0^y z^{\alpha-1}\mathrm{e}^{-z}dz,$$

which can be checked by differentiating both sides in $y \geq 0$. □

Theorem 4.1. *Let $\{B(t) : 0 \leq t \leq 1\}$ be a standard Brownian motion (with generator $2^{-1}d^2/dx^2$) and let $\{Z(t) : 0 \leq t < 1\}$ be the solution to*

$$\begin{aligned} dZ(t) = &\frac{(2\alpha)^{1/2}Z(t)^{1-1/2\alpha}}{(1-\alpha)^{1/2}}dB(t) \\ &+\frac{2Z(t)^{1-1/\alpha}}{(1-\alpha)H(t,Z(t))}\exp\left\{-\frac{\alpha(1-\alpha)Z(t)^{1/\alpha}}{1-t}\right\}dt \end{aligned} \tag{4.3}$$

with $Z(0) = 0$, where

$$H(t,x)=\frac{1}{[2\alpha(1-\alpha)]^{\alpha}x}\int_0^{2\alpha(1-\alpha)x^{1/\alpha}}u^{\alpha-1}\mathrm{e}^{-u/2(1-t)}du, \quad 0\leq t<1,\ x>0. \tag{4.4}$$

Then the distribution of $\{Z(t) : 0 \leq t < 1\}$ on $C([0, 1))$ coincides with $\boldsymbol{Q}_0$.

Proof. Fix $r > 0$ and $x > 0$ and let $X_r^0(x,t) = X(x,(t-r)\wedge\tau_{\mathbf{X}(x)}(0))$ for $t \geq r$, where $\mathbf{X}(x) = \{X(x,t) : t \geq 0, \boldsymbol{P}\}$ is the generalized diffusion process defined by (2.3) with speed measure $m_0(dy) := (1/\alpha - 1)y^{1/\alpha-2}dy$. Let

$$\rho(t) = F(1,x)^{-1}F(1-t, X_r^0(x,t)), \qquad r \leq t < 1. \tag{4.5}$$

Since $\lim_{t\downarrow 0} F(t,x) = \Gamma(\alpha)$ if $x > 0$, $= 0$ if $x = 0$, we have

$$\begin{aligned}\rho(1) :&= \lim_{t\uparrow 1}\rho(t) \\ &= F(1,x)^{-1}\Gamma(\alpha)1_{\{X_r^0(x,1)>0\}} \\ &= F(1,x)^{-1}\Gamma(\alpha)1_{\{\tau_{\mathbf{X}(x)}(0)>1-r\}} \quad \text{a.s.}\end{aligned}$$

We set $\boldsymbol{Q}_{r,x}(d\omega) = \rho(1,\omega)\boldsymbol{P}(d\omega)$. Then $\{X_r^0(x,t) : r \leq t < 1\}$ under the probability measure $\boldsymbol{Q}_{r,x}$ is a Markov process with semigroup $(Q_{s,t})_{r\leq s\leq t<1}$. Recall Lemma 2.1 and observe that

$$S(x,t)=\int_0^{\infty} l(t,y-x)m_0(dy)=\frac{1-\alpha}{\alpha}\int_0^t (B(s)+x)^{1/\alpha-2}ds$$

for $t \leq \tau_{\mathbf{B}}(-x)$ and

$$S^{-1}(x,t)=\frac{\alpha}{1-\alpha}\int_0^t (B(S^{-1}(x,u))+x)^{2-1/\alpha}du=\frac{\alpha}{1-\alpha}\int_0^t X(x,u)^{2-1/\alpha}du$$

for $t \le \tau_{\mathbf{X}(x)}(0)$. Using Lemma 2.1 again one sees that

$$\begin{aligned} X_r^0(x,t) &= x + B(S^{-1}(x,(t-r)\wedge \tau_{\mathbf{X}(x)}(0))) \\ &= x + B(S^{-1}(x,(t-r))\wedge \tau_{\mathbf{B}}(-x)). \end{aligned}$$

Therefore $\{X_r^0(x,t) : r \le t < 1\}$ is a continuous martingale with quadratic variation process

$$\begin{aligned} \langle X_r^0(x,\cdot)\rangle(t) &= \frac{2\alpha}{1-\alpha}\int_0^{(t-r)\wedge\tau_{\mathbf{X}(x)}(0)} X(x,s)^{2-1/\alpha}ds \\ &= \frac{2\alpha}{1-\alpha}\int_r^{t\wedge\tau_{\mathbf{X}_r^0(x)}(0)} X_r^0(x,s)^{2-1/\alpha}ds. \end{aligned}$$

By Itô's formula

$$d\rho(t) = F(1,x)^{-1}F'_x(1-t, X_r^0(x,t))dX_r^0(x,t), \qquad r \le t < 1, \tag{4.6}$$

where

$$F'_x(1-t,x) = \frac{1}{\alpha}\left[\frac{\alpha(1-\alpha)}{1-t}\right]^\alpha \exp\left\{-\frac{\alpha(1-\alpha)}{1-t}x^{1/\alpha}\right\}, \qquad r \le t < 1.$$

It follows that

$$\begin{aligned} \rho(t)^{-1}d\langle\rho, X_r^0(x,\cdot)\rangle(t) &= F(1-t,X_r^0(x,t))^{-1}F'_x(1-t,X_r^0(x,t))d\langle X_r^0(x)\rangle(t) \\ &= \frac{2}{1-\alpha}\left[\frac{\alpha(1-\alpha)}{1-t}\right]^\alpha X_r^0(x,t)^{2-1/\alpha}F(1-t,X_r^0(x,t))^{-1} \\ &\quad \cdot \exp\left\{-\frac{\alpha(1-\alpha)}{1-t}X_r^0(x,t)^{1/\alpha}\right\}1_{\{t<\tau_{\mathbf{X}_r^0(x)}(0)\}}dt. \end{aligned}$$

By the relation

$$F(1-t,x) = \frac{[\alpha(1-\alpha)]^\alpha x}{(1-t)^\alpha}H(t,x), \tag{4.7}$$

we have

$$\begin{aligned} \rho(t)^{-1}d\langle\rho, X_r^0(x,\cdot)\rangle(t) &= \frac{2}{1-\alpha}X_r^0(x,t)^{1-1/\alpha}H(t,X_r^0(x,t))^{-1} \\ &\quad \cdot \exp\left\{-\frac{\alpha(1-\alpha)}{1-t}X_r^0(x,t)^{1/\alpha}\right\}1_{\{t<\tau_{\mathbf{X}_r^0(x)}(0)\}}dt. \end{aligned}$$

We note that $\{\rho(t) : r \le t \le 1\}$ is continuous martingale. Then we may appeal Girsanov's formula to see that

$$\begin{aligned} dX_r^0(x,t) &= dM(t) + \frac{2}{1-\alpha}X_r^0(x,t)^{1-1/\alpha}H(t,X_r^0(x,t))^{-1} \\ &\quad \cdot \exp\left\{-\frac{\alpha(1-\alpha)}{1-t}X_r^0(x,t)^{1/\alpha}\right\}1_{\{t<\tau_{\mathbf{X}_r^0(x)}(0)\}}dt, \end{aligned}$$

where $\{M(t) : r \le t < 1\}$ under $\boldsymbol{Q}_{r,x}$ is a continuous martingale with quadratic variation process

$$\langle M\rangle(t) = \frac{2\alpha}{1-\alpha}\int_r^{t\wedge\tau_{\mathbf{X}^0(x)}(0)} X_r^0(x,s)^{2-1/\alpha}ds;$$

see e.g. Chung and Williams (1990; Theorem 9.8). But, $\boldsymbol{Q}_{r,x}\{\tau_{\mathbf{X}_r^0(x)} > 1\} = 1$, so $\{X_r^0(x,t) : r \le t < 1\}$ under $\boldsymbol{Q}_{r,x}$ satisfies equation (4.3) on the time interval $[r,1)$. Now the desired result is immediate since $\boldsymbol{Q}_0\{w(r) > 0\} = 1$ for each $0 < r < 1$. □

Recall that the $2(1-\alpha)$-dimensional Bessel meander $\{M_\alpha(t) : 0 \le t \le 1\}$ is defined by

$$M_\alpha(t) = \frac{1}{\sqrt{1-g_\alpha}}R_{-\alpha}(g_\alpha + t(1-g_\alpha)), \qquad 0 \le t \le 1, \tag{4.8}$$

where $\{R_{-\alpha}(t) : t \ge 0\}$ is the $2(1-\alpha)$-dimensional Bessel process starting at zero and $g_\alpha = \sup\{0 \le t \le 1 : R_{-\alpha}(t) = 0\}$; see Yor (1992b, p. 42). Let $\{X_\alpha(t) : 0 \le t \le 1\}$ be the squared $2(1+\alpha)$-dimensional Bessel process starting at zero, which is governed by the equation

$$dX(t) = 2\sqrt{X(t)}dB(t) + 2(1+\alpha)dt, \qquad X(0) = 0, \tag{4.9}$$

where $\{B(t) : 0 \le t \le 1\}$ is a standard Brownian motion.

It is known that the processes $\{M_\alpha(t) : 0 \le t \le 1\}$ and $\{X_\alpha(t) : 0 \le t \le 1\}$ are related in the following way: For any bounded measurable function F on $C([0,1))$,

$$\boldsymbol{E}\{F(M_\alpha^2(t) : 0 \le t \le 1)\} = c(\alpha)\boldsymbol{E}\{F(X_\alpha(t) : 0 \le t \le 1)X_\alpha(1)^{-\alpha}\}, \tag{4.10}$$

where $c(\alpha) > 0$ is a constant; see Yor (1992b, p. 42).

Using those and Theorem 4.1 we now prove the following

Theorem 4.2. *Let $\{Z(t) : 0 \le t < 1\}$ be defined by* (4.3)*. Then the probability law of $\{\sqrt{2\alpha(1-\alpha)}Z(t)^{1/2\alpha} : 0 \le t \le 1\}$ coincides with the $2(1-\alpha)$-dimensional Bessel meander $\{M_\alpha(t) : 0 \le t \le 1\}$.*

Proof. Let $Y(t) = 2\alpha(1-\alpha)Z(t)^{1/\alpha}$. By Itô's formula, we have

$$dY(t) = 2\sqrt{Y(t)}dB(t) + 2(1-\alpha)dt + \frac{4Y(t)V_x'(t,Y(t))}{V(t,Y(t))}dt, \tag{4.11}$$

where

$$V(t,x) = \int_0^x u^{\alpha-1}\mathrm{e}^{-u/2(1-t)}du, \qquad 0 \le t < 1,\ x \ge 0. \tag{4.12}$$

Let

$$N(t) = \exp\left\{\int_0^t b(s,X(s))dB(s) - \frac{1}{2}\int_0^t b(s,X(s))^2ds\right\}, \qquad 0 \le t \le 1,$$

where

$$b(t,x) = 2\sqrt{x}\left[\frac{V_x'(t,x)}{V(t,x)} - \frac{\alpha}{x}\right], \qquad 0 \le t < 1,\ x > 0.$$

By Girsanov's formula,

$$\boldsymbol{E}\{F(Y(t): 0 \le t \le 1)\} = \boldsymbol{E}\{F(X(t): 0 \le t \le 1)N(1)\}. \tag{4.13}$$

Setting

$$U(t,x) = \log V(t,x) - \alpha \log x, \qquad 0 \le t < 1,\ x > 0,$$

one may check that

$$U_x' = \frac{V_x'}{V} - \frac{\alpha}{x}, \qquad U_t' = \frac{x}{1-t}\frac{V_x'}{V} - \frac{\alpha}{1-t},$$

and

$$U_{xx}'' = -\left(\frac{V_x'}{V}\right)^2 + \frac{V_{xx}''}{V} + \frac{\alpha}{x^2} = -\left(\frac{V_x'}{V}\right)^2 + \left(\frac{\alpha-1}{x} - \frac{1}{2(1-t)}\right)\frac{V_x'}{V} + \frac{\alpha}{x^2}.$$

Then we have

$$\begin{aligned}
&U_t' + 2(1+\alpha)U_x' + 2xU_{xx}'' + \frac{\alpha}{1-t} \\
&\quad = \frac{x}{1-t}\frac{V_x'}{V} + 2(1+\alpha)\left(\frac{V_x'}{V} - \frac{\alpha}{x}\right) \\
&\qquad + 2x\left[-\left(\frac{V_x'}{V}\right)^2 + \left(\frac{\alpha-1}{x} - \frac{1}{2(1-t)}\right)\frac{V_x'}{V} + \frac{\alpha}{x^2}\right] \\
&\quad = \frac{2(1+\alpha)V_x'}{V} - \frac{2\alpha(1+\alpha)}{x} + 2x\left[-\left(\frac{V_x'}{V}\right)^2 + \frac{\alpha-1}{x}\frac{V_x'}{V} + \frac{\alpha}{x^2}\right] \\
&\quad = 2x\left[-\left(\frac{V_x'}{V}\right)^2 + \frac{2\alpha}{x}\frac{V_x'}{V} - \frac{\alpha^2}{x^2}\right] \\
&\quad = -\frac{1}{2}b(t,x)^2.
\end{aligned}$$

By Itô's formula,

$$U(t,X(t)) - U(0,x) = \int_0^t b(s,X(s))dB(s) - \frac{1}{2}\int_0^t b(s,X(s))^2 ds + \alpha\log(1-t).$$

Then we have

$$N(1) = \lim_{t\to 1}\exp\{U(t,X(t)) - U(0,x) - \alpha\log(1-t)\} = c(\alpha)X(1)^{-\alpha},$$

and the desired result follows from (4.10) and (4.13). □

By the above result, our Theorem 3.1 can be regarded as an extension of the first conditional limit theorem of Durrett et al. (1977), where convergence to Brownian meander was considered.

Acknowledgement. The authors would like to thank the referee for the valuable comments.

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