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Rational homology cobordisms of Seifert fibred rational homology three spheres

By

Takayuki Mukawa*

1. Introduction

The relation between a 3-manifold and the 4-manifolds which it bounds is important in many aspects of low-dimensional topology. For example, under the situation that Σ is a $\mathbb{Q}$-homology 3-sphere and $\Sigma \to S^3$ is a cyclic branched covering of degree p^n (p: prime) branched over a knot K in S^3, if K is slice, then Σ bounds a $\mathbb{Z}_p$-homology 4-ball.

The application of gauge theory to homology cobordisms of homology 3-spheres was initiated by Fintushel and Stern [FS1], and has been developed into several directions by many authors ([FS2], [F], [M], [R]). In [FS1], Fintushel and Stern defined a numerical invariant for Seifert fibred $\mathbb{Z}$-homology 3-spheres, and showed that if the invariant is positive, then the Seifrt fibred $\mathbb{Z}$-homology 3-sphere can not bound a positive definite 4-manifold whose 1st homology contains no 2-torsion.

In this paper, we treat the case where the 1st homology contains 2-torsion, in other words, $\mathbb{Q}$-homology cobordisms of Seifert fibred $\mathbb{Q}$-homology 3-spheres by applying the fundamental works of Fintushel-Stern [FS1] and Donaldson [D2], [D3].

Let M^3 be a Seifert fibred $\mathbb{Q}$-homology 3-sphere with Seifert invariant $\{0; (1, -b), (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\}$, where $\alpha_1, \ldots, \alpha_n$ are pairwise relatively prime integers ≥ 2. (For Seifert invariants we use the definition in [NR].) We assign to M^3 the orientation as the link of an algebraic singularity. Then, by plumbing, M^3 bounds a simply connected, negative definite 4-manifold. Define $c = \alpha_1 \cdots \alpha_n(-b + \sum_{i=1}^n (\beta_i/\alpha_i))$, and

$$R(M) = \frac{2c}{\alpha} - 3 + n + \sum_{i=1}^{n} \frac{2}{\alpha_i} \sum_{k=1}^{\alpha_i - 1} \cot \frac{\pi \alpha c_i^* k}{\alpha_i^2} \cot \frac{\pi k}{\alpha_i} \sin^2 \frac{\pi k}{\alpha_i},$$

where $\alpha = \alpha_1 \cdots \alpha_n$, and $c_i c_i^* \equiv 1 \pmod{\alpha_i}$ for $i = 1, \ldots, n$. (Note that $c > 0$ and $H_1(M^3; \mathbb{Z}) = \mathbb{Z}_c$.)

Then following is our main result.

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*Deceased

Theorem 1.1. *If $R(M) > 0$ and $c/\alpha < 4/\alpha_i$ $(i = 1, \dots, n)$, then M^3 cannot bound an oriented, compact, smooth, positive definite 4-manifold V^4. In particular, any connected sum of M^3 cannot bound a $\mathbb{Q}$-homology 4-ball.*

Note that when M^3 is a $\mathbb{Z}$-homology 3-sphere and $H_1(V^4; \mathbb{Z})$ contains no 2-torsion, our theorem coincides with Fintushel-Stern's result ([FS1, Theorem 1.1]).

On the other hand, Yu ([Y]) showed that when M^3 is a $\mathbb{Z}_2$-homology 3-sphere and $H_1(V^4; \mathbb{Z})$ contains no 2-torsion Fintushel-Stern's result still holds, if we add the condition $c/\alpha < 4/\alpha_i$ $(i = 1, \dots, n)$ (, which corresponds to compactness of moduli space).

Theorem 1.1 can be considered as a corollary of the following theorem which concerns the non-existence of certain orbifolds:

Let X^4 be a pseudofree orbifold in the sense of [FS1]. This means that there is a smooth, fixed point free S^1-action on a smooth 5-manifold Q^5, and its orbit space Q^5/S^1 is X^4. Let $E_1, \dots, E_n$ be the exceptional orbits of the S^1-action on Q^5, and let $\mathbb{Z}_{a_i}$ (resp. $(a_i; r_i, s_i)$) be the isotropy (resp. the slice type) of E_i. (Here, we assume that $a_1, \dots, a_n$ are pairwise relatively prime.) E_i corresponds with a singular point of X^4, whose neighbourhood is a cone on the lens space $L(a_i, b_i)$. (Here, $r_i s_i^{-1} \equiv b_i \pmod{a_i}$.) The quotient map $Q^5 \to Q^5/S^1 = X^4$ becomes a principal S^1-bundle when it is restricted to $D(X^4) = X^4 \setminus \bigcup_{i=1}^n \operatorname{int}(cL(a_i, b_i))$. Its Euler class $e \in H^2(D(X); \mathbb{Z})$ is called a pseudofree Euler class of X^4. Define

$$\begin{aligned}\mu(e) := \sharp[\{\pi(f); f \in H^2(D(X); \mathbb{Z}), f^2 = e^2, f \equiv e (\text{mod } 2),\\ i_j^*(f) = \pm i_j^*(e) \forall j\}/\{v \sim -v\}]\end{aligned}$$

(Here, $\pi : H^2(D(X); \mathbb{Z}) \to \operatorname{Fr} H^2(D(X); \mathbb{Z})$ is the projection onto the free part, and $i_j : L(a_i, b_j) \to D(X)$ is the inclusion $(j = 1, \dots, n)$.) and

$$R(X, e) = -2e^2 - 3 + n + \sum_{i=1}^{n} \frac{2}{a_i} \sum_{k-1}^{a_i - 1} \cot \frac{\pi k r_i}{a_i} \cot \frac{\pi k s_i}{a_i} \sin^2 \frac{\pi k}{a_i}.$$

Then, we have

Theorem 1.2. *There does not exist a pseudofree orbifold which satisfies the following conditions*:

(i) *the intersection form on X^4 is negative definite,*
(ii) $H_1(D(X); \mathbb{R}) = 0$,
(iii) $\iota^*(\operatorname{Tor} H^2(D(X); \mathbb{Z})) = 0$ *where $\iota : \partial D(X) \to D(X)$ is the inclusion,*
(iv) $-e^2 < 4/a_i$ $(i = 1, \dots, n)$,
(v) $\mu(e) = 1$,
(vi) $R(X, e) > 0$.

Since both theorems are proved similarly, we will prove Theorem 1.1 only.

We briefly explain the difference between Fintushel-Stern's result and ours. In Fintushel-Stern's case (also Yu's case), the number of reducible self-dual connections is *a priori* odd by Hodge theory. On the other hand, reducible connections correspond to singular points of moduli space. By using $\mathbb{Z}_2$-cohomology class, Fintushel-Stern concluded that the number of singular points must be even (, hence obtained a contradiction). In our case, since we allow $H_1(M^3;\mathbb{Z})$ and $H_1(V^4;\mathbb{Z})$ to contain 2-torsion, two difficulties arise. One is that other types of reducible connections than S^1 may exist. The other is that even if reducible connections are all S^1, the number of them is always even. For the former, we will show that the anti-self-duality equation is transverse enough to exclude such reducible connections in generic situation. In order to deal with the latter, we use integral cohomology class of moduli space (, which was introduced by Donaldson [D2]). To carry out this we explicitly calculate orientation of moduli space by extending Donaldson's method ([D3]) to orbifold case.

The organization of this paper is as follows. In Section 2, we assembled basic facts about orbifolds and gauge theory. There it is proved that the number of S^1-reducible connections is always even, by computing the homology of the orbifold. This fact is not used directly, but important for our approach as mentioned above. In Section 3, it is proved that other types of reducible connections than S^1 are removed away by a generic perturbation. In Sections 4 and 5 we calculate orientation of the moduli space. Finally, in Section 6 we prove Theorem 1.1.

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2. Generalities

From now on, we suppose that a Seifert fibred $\mathbb{Q}$-homology 3-sphere M^3 with $R(M) > 0$ and $c/\alpha < 4/\alpha_i$ $(i = 1, \ldots, n)$ bounds an oriented, compact, smooth, positive definite 4-manifold V^4. By surgering out the free part of $H_1(V^4;\mathbb{Z})$ we can assume that $H_1(V^4;\mathbb{R}) = 0$.

(a) Pseudofree orbifolds

Let C denote the mapping cylinder of the Seifert fibration $M \to S^2$. Then C is a 4-manifold with n singular points which have neighbourhood which are cones on the lens spaces $L(\alpha_i, -(\alpha c_i^*)/\alpha_i)$ and the boundary of C is M^3. The rational intersection form of C as a rational homology manifold is definite. We choose an orientation of C so that its intersection form is negative definite. This orientation is compatible with that of the boundary M^3. Define the pseudofree S^1-action on $M^3 \times D^2$ to be the diagonal action of the S^1-action on M^3 by the action of the Seifert fibration and the S^1-action on D^2 as multiplication of complex numbers. The orbit space $(M^3 \times D^2)/S^1$ is identified with the mapping cylinder C. This S^1-action on $M^3 \times D^2$ is compatible with the obvious S^1-action on $V^4 \times S^1$ via the S^1-equivariant diffeomorphism $\varphi : \partial(M^3 \times D^2) = M^3 \times S^1 \to M^3 \times S^1 = \partial(V^4 \times S^1)$ $((x, z) \mapsto (z^{-1} \cdot x, z))$, so these actions glue together to give the pseudofree S^1-action on $Q^5 := (M^3 \times D^2) \cup_\varphi (V^4 \times S^1)$, and the orbit space $Q^5/S^1 = C \cup_{M^3} (-V^4) =: X^4$ can be naturally considered as a negative definite V-manifold. (Note that this X^4 corresponds to the one in Theorem 1.2.)

We compute the slice type of the S^1-action on Q^5. Let $(\alpha_i;\nu_i)$ denote the slice type of the S^1-action on M^3 at the $\mathbb{Z}_{\alpha_i}$-orbit, where $\beta_i\nu_i \equiv 1 \pmod{\alpha_i}$. By definition, $c = \alpha(-b + \sum_{j=1}^n \beta_j/\alpha_j)$ and $cc_i^* \equiv 1 \pmod{\alpha_i}$, so $1 \equiv cc_i^* \equiv \alpha(\beta_i/\alpha_i)c_i^* \pmod{\alpha_i}$. Hence, $\nu_i \equiv (\alpha c_i^*)/\alpha_i \pmod{\alpha_i}$. The exceptional orbits of the S^1-action on Q^5 are naturally in one to one correspondence with those on $M^3 \times D^2$. Hence, the S^1-action on Q^5 has n exceptional orbits with slice type $(\alpha_i;\nu_i,1) = (\alpha_i;(\alpha_i c_i^*)/\alpha_i, 1)$ $(i = 1,\dots,n)$.

Let us compute the (co-)homology of X^4. Define

$$W^4 := C \setminus \bigcup_{i=1}^n \operatorname{int}\left(cL\left(\alpha_i, -\frac{\alpha c_i^*}{\alpha_i}\right)\right), \quad \text{and}$$

$$D(X^4) := X^4 \setminus \bigcup_{i=1}^n \operatorname{int}\left(cL\left(\alpha_i, -\frac{\alpha c_i^*}{\alpha_i}\right)\right) = W^4 \cup_{M^3} (-V^4).$$

By the exact sequences

$$\begin{array}{cccccc}
\cdots \to H_2\left(\bigcup_{i=1}^n L\left(\alpha_i, -\frac{\alpha c_i^*}{\alpha}\right);\mathbb{Z}\right) & \to H_2(W;\mathbb{Z}) \to & H_2\left(W, \bigcup_{i=1}^n L\left(\alpha_i, -\frac{\alpha c_i^*}{\alpha}\right);\mathbb{Z}\right) & \\
\| & & \| & \\
0 & & \mathbb{Z} & \\
\to H_1\left(\bigcup_{i=1}^n L\left(\alpha_i, -\frac{\alpha c_i^*}{\alpha}\right);\mathbb{Z}\right) & \to H_1(W;\mathbb{Z}) \to & H_1\left(W, \bigcup_{i=1}^n L\left(\alpha_i, -\frac{\alpha c_i^*}{\alpha}\right);\mathbb{Z}\right) & \\
\| & & \| & \\
\bigoplus_{i=1}^n \mathbb{Z}_{\alpha_i} & & \mathbb{Z}^{n-1} & \\
\to \widetilde{H}_0\left(\bigcup_{i=1}^n L\left(\alpha_i, -\frac{\alpha c_i^*}{\alpha}\right);\mathbb{Z}\right) & \to \widetilde{H}_0(W;\mathbb{Z}) \to & H_0\left(W, \bigcup_{i=1}^n L\left(\alpha_i, -\frac{\alpha c_i^*}{\alpha}\right);\mathbb{Z}\right) & \to 0, \\
\| & \| & \| & \\
\mathbb{Z}^{n-1} & 0 & 0 &
\end{array}$$

and

$$\begin{array}{ccccccc}
\cdots & \to & H_2(M;\mathbb{Z}) & \to & H_2(W;\mathbb{Z}) & \to & H_2(W,M;\mathbb{Z}) & \\
& & \| & & & & \| & \\
& & 0 & & & & \mathbb{Z} & \\
& \to & H_1(M;\mathbb{Z}) & \to & H_1(W;\mathbb{Z}) & \to & H_1(W,M;\mathbb{Z}) & \\
& & \| & & & & \| & \\
& & \mathbb{Z}_c & & & & 0 & \\
& \to & \widetilde{H}_0(M;\mathbb{Z}) & \to & \widetilde{H}_0(W;\mathbb{Z}) & \to & H_0(W,M;\mathbb{Z}) & \to 0, \\
& & \| & & \| & & \| & \\
& & 0 & & 0 & & 0 &
\end{array}$$

we see that $H_1(W^4;\mathbb{Z}) = 0$ and $H_2(W^4;\mathbb{Z}) = \mathbb{Z}$. Hence $H_1(D(X^4);\mathbb{R}) = 0$ and $i^*(\operatorname{Tor} H^2(D(X^4);\mathbb{Z})) = 0$, where $i : \partial D(X^4) \hookrightarrow D(X^4)$ is the inclusion.

By definition, the pseudofree Euler class $e \in H^2(D(X^4);\mathbb{Z})$ of X^4 is the Euler class of the principal S^1-bundle $Q^5|_{D(X^4)} \to D(X^4)$. This lies in the image of the inclusion

$$H^2(W,M;\mathbb{Z}) = H^2(D(X^4),V^4;\mathbb{Z}) \hookrightarrow H^2(D(X^4);\mathbb{Z}).$$

Define the cross section s of the complex line bundle $(Q^5|_{W^4}) \times_{S^1} \mathbb{C} \to W^4$ associated to the principal S^1-bundle $Q^5|_{W^4} \to W^4$ as follows:

$$s : W^4 \left(\subset C = \frac{M^3 \times D^2}{S^1} \right) \to (Q^5|_{W^4}) \times_{S^1} \mathbb{C} \subset (M^3 \times D^2) \times_{S^1} \mathbb{C}.$$
$$[(x,z)] \mapsto [((x,z),z)]$$

Then we have

$$\begin{array}{ccccccccc} H^2(W,M;\mathbb{Z}) & \overset{\cap[W,\partial W]}{\to} & H_2\left(W, L\left(\alpha_i, -\dfrac{\alpha c_i^*}{\alpha}\right); \mathbb{Z}\right) & \cong & H_2(C;\mathbb{Z}) & \cong & \mathbb{Z}. \\ e & \mapsto & [s^{-1}(0)] & \mapsto & \left[\dfrac{M \times 0}{S^1}\right] & \mapsto & 1 \end{array}$$

Thus we have

$$\begin{array}{c} H^2(W,M;\mathbb{Z}) = H^2(D(X),V;\mathbb{Z})(=\mathbb{Z}) \subset H^2(D(X);\mathbb{Z})\ . \\ 1 \longmapsto e \end{array}$$

We compute the rational self-intersection number $e^2 \in \mathbb{Q}$ of e.

By the following diagrams

$$\begin{array}{ccccccccc} H^1(\partial W,M;\mathbb{Z}) & \to & H^2(W,\partial W;\mathbb{Z}) & \overset{\alpha\cdot}{\to} & H^2(W,M;\mathbb{Z}) & \to & H^2(\partial W,M;\mathbb{Z}) & \to & H^3(W,\partial W;\mathbb{Z}) \\ || & & || & & || & & || & & || \\ 0 & & \mathbb{Z} & & \mathbb{Z} & & \oplus_{i=1}^n \mathbb{Z}_{\alpha_i} & & 0 \\ & & & & 1 = e & \mapsto & \oplus_{i=1}^n 1 & & \\ & & 1 = f & \mapsto & \alpha = \alpha e & & & & \end{array}$$

$$\begin{array}{ccccc} & & & & H^1\left(\partial W, \cup_{i=1}^n L\left(\alpha_i, -\dfrac{\alpha c_i^*}{\alpha}\right); \mathbb{Z}\right) \\ & & & & || \\ & & & & 0 \\ \to H^2(W,\partial W;\mathbb{Z}) & \overset{c\cdot}{\to} & H^2\left(W, \cup_{i=1}^n L\left(\alpha_i, -\dfrac{\alpha c_i^*}{\alpha}\right); \mathbb{Z}\right) & \to & H^2\left(\partial W, \cup_{i=1}^n L\left(\alpha_i, -\dfrac{\alpha c_i^*}{\alpha}\right); \mathbb{Z}\right) \\ || & & || & & || \\ \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z}_c \\ & & 1 & \mapsto & 1 \\ 1 = f & \mapsto & c(:= f') & & \\ \to H^3(W,\partial W;\mathbb{Z}) & & & & \\ || & & & & \\ 0 & & & & \end{array}$$

$$\begin{array}{ccc} H^2\left(W, \cup_{i=1}^n L\left(\alpha_i, -\dfrac{\alpha c_i^*}{\alpha}\right); \mathbb{Z}\right) & \overset{\cap[W,\partial W]}{\to} & H_2(W,M;\mathbb{Z}) \\ || & & || \\ \mathbb{Z} & & \mathbb{Z} \\ 1 & \mapsto & 1 \\ c = f' & \mapsto & c = f' \cap [W,\partial W] \end{array}$$

$$\begin{array}{ccc} H^2(W,M;\mathbb{Z}) & \to & \mathrm{Hom}(H^2(W,M;\mathbb{Z}),\mathbb{Z}), \\ \| & & \| \\ \mathbb{Z} & & \mathbb{Z} \\ 1=e & \mapsto & (\mathbb{Z}\ni 1\mapsto 1\in\mathbb{Z}) \end{array}$$

we have up to sign

$$\begin{aligned} e^2 &= \frac{1}{\alpha}\langle e\cup f,[W,\partial W]\rangle \\ &= \frac{1}{\alpha}\langle e\cup f',[W,\partial W]\rangle \\ &= \frac{1}{\alpha}\langle e, f'\cap[W,\partial W]\rangle \\ &= \frac{c}{\alpha}. \end{aligned}$$

Since X^4 is negative definite, it follows that $e^2<0$.
Hence we have

Proposition 2.1. $e^2=-c/\alpha$.

(b) The setting for gauge theory

The quotient map $Q^5\to Q^5/S^1=X^4$ of the pseudofree S^1-action on Q^5 can be naturally considered as a principal S^1-V-bundle. In fact, over each $L(\alpha_i,-(\alpha c_i^*)/\alpha)$

$$\begin{array}{ccc} Q^5|_{L(\alpha_i,-\frac{\alpha c_i^*}{\alpha})} & = & S^3\times_{\mathbb{Z}_{\alpha_i}} S^1 \\ \downarrow & & \downarrow \\ L\left(\alpha_i,-\dfrac{\alpha c_i^*}{\alpha}\right) & = & S^3/\mathbb{Z}_{\alpha_i}, \end{array}$$

and this extends naturally to the S^1-V-bundle over $cL(\alpha_i,-(\alpha c_i^*)/\alpha)$

$$\begin{array}{l} D^4\times_{\mathbb{Z}_{\alpha_i}} S^1 \\ \quad\downarrow \\ D^4/\mathbb{Z}_{\alpha_i}=cL\left(\alpha_i,-\dfrac{\alpha c_i^*}{\alpha}\right). \end{array}$$

Let $L\to X^4$ be the complex line V-bundle associated to $Q^5\to X^4$. L stabilizes to an $SO(3)$-V-bundle $E=L\oplus\underline{\mathbb{R}}$. Let adE be the V-bundle associated to E by the adjoint representation (or equivalently the standard representation). Let $\mathrm{Aut}\,E$ be the fibre bundle associated to E by the conjugate action. Choose and fix a Riemannian V-metric on X^4 and a smooth $SO(3)$-V-connection ∇_0 on E. Using ∇_0 and the Levi-Civita connection of X^4, we take Sobolev completion of various function spaces.

Notations:

$$\begin{aligned}
\mathcal{C}_E &:= \{L^2_3\text{-}SO(3)\text{-}V\text{-connections on } E\} = \nabla^0 + L^2_3(\Omega^1(adE)) \\
\mathcal{G}_E &:= \{L^2_4\text{-}V\text{-gauge transformations on } E\} = L^2_4(\Omega^0(\mathrm{Aut}\, E)) \\
\mathcal{C}^*_E &:= \{\nabla \in \mathcal{C}_E; \nabla \text{ is irreducible}\} \\
\mathcal{A}_E &:= \{\nabla \in \mathcal{C}_E; F_\nabla + {}_*F_\nabla = 0 \text{ (i.e. } \nabla \text{ is anti-self-dual)}\} \\
\mathcal{A}^*_E &:= \mathcal{A}_E \cap \mathcal{C}^*_E \\
\mathcal{B}_E &:= \mathcal{C}_E/\mathcal{G}_E \\
\mathcal{B}^*_E &:= \mathcal{C}^*_E/\mathcal{G}_E \\
\mathfrak{M}_E &:= \mathcal{A}_E/\mathcal{G}_E \\
\mathfrak{M}^*_E &:= \mathcal{A}^*_E/\mathcal{G}_E
\end{aligned}$$

Here and hereafter, L^2_3 is the space of sections whose derivatives up to 3rd order is of L^2 class.

The moduli space which we will consider is $\mathfrak{M}_E$.

(c) Compactness of the moduli space

By Uhlenbeck's bubble theorem [U1], [U2], [D1], ends of $\mathfrak{M}_E$ correspond to the "*curvature concentration phenomenon.*" Since $p_1(E) = e^2 = -c/\alpha > -4/\alpha_i$ $(i = 1, \ldots, n)$, the curvature can not concentrate in the neighbourhood of either a smooth point or a branched point [FS1], [FL].

Hence, we have

Proposition 2.2. *$\mathfrak{M}_E$ is compact.*

(d) Virtual dimension of the moduli space

Consider a connection $\nabla \in \mathcal{A}_E$. We then have the Atiyah-Hitchin-Singer complex [AHS]

$$0 \longrightarrow \Omega^0(E) \xrightarrow{\nabla} \Omega^1(E) \xrightarrow{d^\nabla_+} \Omega^2_+(E) \longrightarrow 0$$

which has cohomology groups $\mathbb{H}^0_\nabla$, $\mathbb{H}^1_\nabla$, $\mathbb{H}^2_\nabla$.

Let us compute

$$-\dim \mathbb{H}^0_\nabla + \dim \mathbb{H}^1_\nabla - \dim \mathbb{H}^2_\nabla.$$

For $g = e^{(2\pi\sqrt{-1}k)/\alpha_i} \in \mathbb{Z}_{\alpha_i}$, let $r_i(g)$ and $s_i(g)$ denote the rotation angles of the action of g on D^4. Then $r_i(g) = (2\pi\alpha c^*_i k)/\alpha_i^2$ and $s_i(g) = (2\pi k)/\alpha_i$, because the slice type of the S^1-action on Q^5 at the $\mathbb{Z}_{\alpha_i}$-orbit is $(\alpha_i; (\alpha c^*_i)/\alpha_i, 1)$.

Also, the rotation angle of the action of g on S^1 is $t_i(g) = (2\pi k)/\alpha_i$.

By Kawasaki's index theorem over V-manifolds [K], we have

$$\begin{aligned}
&-\dim \mathbb{H}^0_\nabla + \dim \mathbb{H}^1_\nabla - \dim \mathbb{H}^2_\nabla \\
&\quad = -\left\langle ch(E \otimes_{\mathbb{R}} \mathbb{C}) \left(2 + \frac{p_1(\mathrm{T}\,X)}{3} + \frac{e(\mathrm{T}\,X)}{2}\right)\left(1 - \frac{p_1(\mathrm{T}\,X)}{12}\right), [X]\right\rangle
\end{aligned}$$

$$+\sum_{i=1}^{n}\frac{1}{\alpha_i}\sum_{g\in\mathbb{Z},g\neq 1}\left\{\frac{1}{2}\left(-1-\cot\frac{r_i(g)}{2}\cot\frac{s_i(g)}{2}\right)\right\}\left(3-4\sin^2\frac{t_i(g)}{2}\right).$$

Here, the first term is

$$\begin{aligned}&-2p_1(E)-\frac{3}{2}\left(\frac{p_1(\mathrm{T}\,X)}{3}+e(\mathrm{T}\,X)\right)\\&\quad=-2e^2-\frac{3}{2}\left\{(\sigma(X)+\chi(X))-\sum_{i=1}^{n}\frac{1}{\alpha_i}\sum_{g\in\mathbb{Z},g\neq 1}(\sigma_g(X)+\chi_g(X))\right\}.\end{aligned}$$

(Here, $\sigma(X)$ is the signature of X, $\chi(X)$ is the Euler number of X, $\sigma_g(X) = \cot(r_i(g)/2)\cot(s_i(g)/2)$, $\chi_g(X) = 1$.)

$$=-2e^2-3+\frac{3}{2}\sum_{i=1}^{n}\frac{1}{\alpha_i}\sum_{g\in\mathbb{Z},g\neq 1}\left(\cot\frac{r_i(g)}{2}\cot\frac{s_i(g)}{2}+1\right).$$

Thus, we have

$$\begin{aligned}&-\dim\mathbb{H}^0_\nabla+\dim\mathbb{H}^1_\nabla-\dim\mathbb{H}^2_\nabla\\&\quad=-2e^2-3+\frac{3}{2}\sum_{i=1}^{n}\frac{1}{\alpha_i}\sum_{g\in\mathbb{Z},g\neq 1}\left(\cot\frac{r_i(g)}{2}\cot\frac{s_i(g)}{2}+1\right)\\&\qquad+\frac{1}{2}\sum_{i=1}^{n}\frac{1}{\alpha_i}\sum_{g\in\mathbb{Z},g\neq 1}\left(-1-\cot\frac{r_i(g)}{2}\cot\frac{s_i(g)}{2}\right)\left(3-4\sin^2\frac{t_i(g)}{2}\right)\\&\quad=\frac{2c}{\alpha}-3+n+\sum_{i=1}^{n}\frac{2}{\alpha_i}\sum_{k=1}^{\alpha_i-1}\cot\frac{\pi\alpha c_i^*k}{\alpha_i^2}\cot\frac{\pi k}{\alpha_i}\sin^2\frac{\pi k}{\alpha_i}\\&\quad=R(M).\end{aligned}$$

As in [FS1, Corollary 6.3], $R(M)$ turns out to be an odd integer ≥ -1.

(e) S^1-reductions

Definition 2.1. We call $\nabla\in\mathcal{C}_E$ an S^1-reducible connection when the following equivalent conditions are satisfied. (See [FU] for the proof of equivalence.)

(1) (E,∇) admits a reduction of the structure group to $S^1\subset SO(3)$.

(2) $\Gamma_\nabla=\{g\in\mathcal{G}_E; g(\nabla)=\nabla\}$ is equal to S^1.

(3) $\nabla=\nabla^{L_1}\oplus d$, where ∇^{L_1} is an $SO(2)$-V-connection on an $SO(2)$-V-bundle L_1 which satisfies $L_1\oplus\underline{\mathbb{R}}=E$, and d is the exterior differentiation on the trivial one-dimensional bundle $\underline{\mathbb{R}}$.

First we have a classification of topological S^1-reductions of E.

Lemma 2.1. *$L_1 \oplus \underline{\mathbb{R}}$ is equivalent (as an $SO(3)$-V-bundle) to $L \oplus \underline{\mathbb{R}}$ if and only if*

(i) $c_1(L_1)^2 = c_1(L)^2$,

(ii) $c_1(L_1) \equiv c_1(L) (\mathrm{mod}\ 2)$, *and*

(iii) $i_j^* c_1(L_1) = \pm i_j^* c_1(L) \in H^2(L(\alpha_j, -(\alpha c_j^*/\alpha_j); \mathbb{Z})$ *for each* $j = 1, \dots, n$.

(Here $i_j : L(\alpha_j, -(\alpha c_j^*)/\alpha_j) \hookrightarrow D(X)$ *is the inclusion.)*

Define

$$\mu(e) := \sharp[\{\pi(f); f \in H^2(D(X); \mathbb{Z}), f^2 = e^2, f \equiv e (\mathrm{mod}\ 2), \\ i_j^*(f) = \pm i_j^*(e) \forall j\} / \{v \sim -v\}],$$

where $\pi : H^2(D(X); \mathbb{Z}) \to \mathrm{Fr}\, H^2(D(X); \mathbb{Z})$ is the projection.

By Lemma 2.1, up to orientation, the number of topological S^1-reductions of E is just $\mu(e) \cdot |H_1(D(X); \mathbb{Z})|$. As we remarked in Section 1, this number is even when $H_1(V; \mathbb{Z})$ has 2-torsion.

We compute $\mu(e)$. By the following diagrams

$$\begin{array}{ccccccccc}
H^1(M;\mathbb{Z}) & \to & H^2(W,M;\mathbb{Z}) & \to & H^2(W;\mathbb{Z}) & \to & H^2(M;\mathbb{Z}) & \to & H^3(W,M;\mathbb{Z}) \\
|| & & || & & || & & || & & || \\
0 & & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z}_c & & \mathbb{Z}^{n-1} \\
 & & & & 1 & \mapsto & 1 & & \\
 & & e|_{(W,M)} = 1 & \mapsto & c = e|_W & & & &
\end{array}$$

and

$$\begin{array}{ccc}
\mathbb{Z} = H^2(W,M;\mathbb{Z}) = H^2(D(X),V;\mathbb{Z}) & \overset{\text{inj.}}{\to} & H^2(D(X);\mathbb{Z}) \\
\downarrow c\cdot & & \downarrow \text{inj.} \\
\mathbb{Z} = H^2(W;\mathbb{Z}) & \subset & H^2(W;\mathbb{Z}) \oplus H^2(V;\mathbb{Z}),
\end{array}$$

we have

$$\begin{array}{ccccccc}
H^1(M;\mathbb{Z}) & \to & H^2(D(X);\mathbb{Z}) & \to & H^2(W;\mathbb{Z}) \oplus H^2(V;\mathbb{Z}) & \overset{\text{surj.}}{\to} & H^2(M;\mathbb{Z}) \\
|| & & & & & & || \\
0 & & e & \mapsto & (c,0) & & \mathbb{Z}_c
\end{array}.$$

From this, it is easy to see that $\mu(e) = 1$. Geometrically, this implies that the free part of the 1st Chern classes (in $H^2(D(X); \mathbb{Z})$) of topological S^1-reductions of E is unique up to orientation.

(f) Canonical isomorphisms between determinant lines associated to addition of instantons

First, we define determinant line bundles. For a V-bundle $\xi \to X$ with structure group $SO(3)$ or $U(N)$, let $\mathcal{C}_\xi$ be the set of L_3^2-V-connections on ξ, $\mathcal{G}_\xi$ be the set of L_4^2-V-gauge transformations ob ξ, and $\mathcal{B}_\xi$ be the orbit space $\mathcal{C}_\xi / \mathcal{G}_\xi$.

By assigning to each $A \in \mathcal{C}_\xi$ the determinant line

$$\Lambda(\mathfrak{D}_A) := \det(\operatorname{Ker}\mathfrak{D}_A) \otimes_{\mathbb{R}} \det(\operatorname{Coker}\mathfrak{D}_A)^*$$

of the 1st order real elliptic operator

$$\mathfrak{D}_A := -d_A^* \oplus d_A^+ : \Omega^1(ad\xi) \to (\Omega^0 \oplus \Omega_+^2)(ad\xi),$$

we obtain a real line bundle $\coprod_{A\in\mathcal{C}_\xi} \Lambda(\mathfrak{D}_A) \to \mathcal{C}_\xi$. The action of $\mathcal{G}_\xi$ on $\mathcal{C}_\xi$ lifts to the determinant lines. We remark that, for any $A \in \mathcal{C}_\xi$, the action of the isotropy group Γ_A on $\Lambda(\mathfrak{D}_A)$ is trivial. Hence, the bundle $\coprod_{A\in\mathcal{C}_\xi} \Lambda(\mathfrak{D}_A) \to \mathcal{C}_\xi$ descends to form a bundle $\Lambda_\xi \to \mathcal{B}_\xi$, which we call the determinant line bundle of $\mathcal{G}_\xi$.

Let $\mathbb{E}$ be a $U(2)$-V-bundle over X^4, x be a smooth point in the V-manifold X^4, λ be a small positive number, and

$$\rho : (ad\mathbb{E})_x \to \Lambda_+^2 \, \mathrm{T}_x^* \, X$$

be an isomorphism of $SO(3)$-spaces. For any $U(2)$-V-connection A on $\mathbb{E}$, we denote by $\widetilde{A} = A' \sharp_\rho J_\lambda$ a connection formed by flattening A over the annulus:

$$\Omega = \{y \in X; MN^{-1}\sqrt{\lambda} < d(x,y) < MN\sqrt{\lambda}\}$$

and attaching a "*flattened instanton*" J_λ of scale λ according to the glueing parameter ρ ([D2]). Here N and M are fixed large numbers.

$\widetilde{A}$ is carried by a $U(2)$-V-bundle $\widetilde{\mathbb{E}} \to X$ with

$$\begin{aligned}
&c_1(\widetilde{\mathbb{E}}) = c_1(\mathbb{E}) \in H^2(D(X);\mathbb{Z}),\\
&c_2(\widetilde{\mathbb{E}}) = c_2(\mathbb{E}) \in \mathbb{Q},\\
&\widetilde{\mathbb{E}}|_{cL(\alpha_i, -\frac{\alpha c_i^*}{\alpha_i})} \cong \mathbb{E}|_{cL(\alpha_i, -\frac{\alpha c_i^*}{\alpha_i})} \qquad \text{(as V-bundles)} \quad (i = 1, \dots, n).
\end{aligned}$$

The following theorem is due to Donaldson [D3, Proposition (3.20)].

Theorem 2.1. *There is a "canonical" isomorphism*

$$j_x : \Lambda_{\mathbb{E}}|_{[A]} \to \Lambda_{\widetilde{\mathbb{E}}}|_{[\widetilde{A}]} \ ,$$

which is well-defined up to positive constants.

3. Non-existence of $O(2)$-reducible connections

When there is no 2-torsion in the 1st homology of the base space, the structure group of an $SO(3)$-bundle may reduce to S^1 only. So, it suffices to consider S^1-reducible A.S.D. connections for the singularity of the moduli space. However, in our case, since $H^1(D(X);\mathbb{Z})$ may contain 2-torsion, other types of reductions may happen.

The types to be considered are the following:

homology group of ∇	1	finite cyclic subgroup of S^1	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	S^1	$O(2)$	$SO(3)$
isotropy group of ∇	$SO(3)$	$O(2)$ or S^1	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	S^1	$\mathbb{Z}_2$	1

Here,

$$\mathbb{Z}_2 = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix} \right\},$$

$$\mathbb{Z}_2 \oplus \mathbb{Z}_2 = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix} \right\},$$

$$S^1 = \left\{ \begin{pmatrix} P & \\ & 1 \end{pmatrix}; P \in SO(2) \right\},$$

$$O(2) = \left\{ \begin{pmatrix} P & \\ & 1 \end{pmatrix}; P \in SO(2) \right\} \cup \left\{ \begin{pmatrix} Q & \\ & 1 \end{pmatrix}; Q \in O(2), \det Q = -1 \right\}.$$

Since $p_1(E) = c_1(L)^2 = e^2 < 0$, E is not flat. Hence, the reductions whose holonomy group is a finite group can not exist.

The reducible connection ∇ with holonomy group $O(2)$ can be decomposed as $\nabla = \nabla^\eta \oplus \nabla^\epsilon$ corresponding to the splitting $E \cong \eta \otimes \epsilon$ of the bundle, where η is a non-orientable $O(2)$-V-bundle and ϵ is a non-orientable real line V-bundle. We call such a reducible connection to be $O(2)$-reducible. The isotropy group of $O(2)$-reducible connection is $\mathbb{Z}_2$. So cobordism argument would fail if $\mathfrak{M}_E$ had such a singularity. However, as we will show in Theorem 3.1, we can find a generic perturbation such that no A.S.D. connection is $O(2)$-reducible.

Theorem 3.1. *For a generic V-metric on X^4, $E = L \oplus \underline{\mathbb{R}}$ has no $O(2)$-reducible A.S.D. connections.*

To prove Theorem 3.1, it suffices to prove Theorem 3.2.

Theorem 3.2. *Let η be a non-orientable, non-flat $O(2)$-V-bundle over X^4. Then, for a generic V-metric on X^4, η has no irreducible A.S.D. V-connections.*

The proof of Theorem 3.2 occupies the rest of this section.

(a) Index computations

Suppose that $\eta \to X$ has an A.S.D. connection.

First, by the following diagram

$$\begin{array}{ccccc} H^1(D(X);\mathbb{Z}_2) & \xrightarrow{\beta} & \operatorname{Tor} H^2(D(X);\mathbb{Z}) & \subset & H^2(D(X);\mathbb{Z}) \\ i^* \downarrow & & i^* \downarrow & & i^* \downarrow \\ H^1(\partial D(X);\mathbb{Z}_2) & \xrightarrow{\beta} & \operatorname{Tor} H^2(\partial D(X);\mathbb{Z}) & \subset & H^2(\partial D(X);\mathbb{Z}), \end{array}$$

it is easy to see that $i^*H^1(D(X);\mathbb{Z}_2)=0$. (Here, $i:\partial D(X)\to D(X)$ is the inclusion.) Hence, $ad\eta$ is trivial when restricted to $\partial D(X)$.

Using this, the index of the Atiyah-Hitchin-Singer complex

$$0 \longrightarrow \Omega^0(ad\eta) \xrightarrow{d^D} \Omega^1(ad\eta) \xrightarrow{d^D_+} \Omega^2_+(ad\eta) \longrightarrow 0$$

is:

$$\begin{aligned}
&-\dim \mathbb{H}^0_D + \dim \mathbb{H}^1_D - \dim \mathbb{H}^2_D \\
&\quad = -\left\langle ch((ad\eta)\otimes_{\mathbb{R}}\mathbb{C})\left(2+\frac{p_1(TX)}{3}+\frac{e(TX)}{2}\right)\left(1-\frac{p_1(TX)}{12}\right),[X]\right\rangle \\
&\qquad + \sum_{i=1}^{n}\frac{1}{\alpha_i}\sum_{g\in\mathbb{Z},g\neq 1}\left\{\frac{1}{2}\left(-1-\cot\frac{r_i(g)}{2}\cot\frac{s_i(g)}{2}\right)\right\}\cdot 1 \\
&\quad = -\frac{1}{2}\left\{(\sigma(X)+\chi(X))-\sum_{i=1}^{n}\frac{1}{\alpha_i}\sum_{k=1}^{\alpha_i-1}\left(\cot\frac{r_i(g)}{2}\cot\frac{s_i(g)}{2}+1\right)\right\} \\
&\qquad + \sum_{i=1}^{n}\frac{1}{\alpha_i}\sum_{g\in\mathbb{Z},g\neq 1}\left\{\frac{1}{2}\left(-1-\cot\frac{r_i(g)}{2}\cot\frac{s_i(g)}{2}\right)\right\} \\
&\quad = -(1-b_1(X)+b_2^+(X)) \\
&\quad = -1.
\end{aligned}$$

(b) Transversality

Notations:

X^4 : as in Section 2

η : a non-orientable, non-flat $O(2)$-V-bundle over X^4

$ad\eta$: the V-bundle associated to η by the adjoint representation

$\mathcal{C}^*_\eta$: the set of irreducible L^2_3-V-connections on η

$\mathcal{G}_\eta$: the set of L^2_4-V-gauge transformations on η

$\mathcal{B}^*_\eta$: the orbit space $\mathcal{C}^*_\eta/\mathcal{G}_\eta$

$Met_X := C^k(\mathrm{GL}^+(\mathrm{T}X))$
Here + means orientation-preserving. $k \gg 4$

$met_X := C^k(\mathrm{End}(\mathrm{T}X))$

g : a base V-metric on X^4

$\Omega^i(ad\eta)_{4-i}$: the L^2_{4-i}-completion of the space of i-forms with values in $ad\eta$ $(i=0,1)$

$\Omega^2_+(ad\eta)_2$: the L^2_2-completion of the space of self-dual 2-forms (with respect to g) with values in $ad\eta$

$\mathcal{ASD}^*_3 := \{(D,\varphi)\in\mathcal{C}^*_\eta\times Met_X; F_D$ is anti-self-dual with respect to $\varphi^*g\}$

Proposition 3.1 (cf. [FU]). *Define a map $\mathcal{P}$ by*

$$\begin{aligned}\mathcal{P} : \mathcal{C}^*_\eta \times Met_X &\to \Omega^2_+(ad\eta)_2\\ (D,\varphi) &\mapsto p_+((\varphi^{-1})^* F_D).\end{aligned}$$

(Here, $p_+ : \Omega^2 \to \Omega^2_+$ is the projection onto the self-dual part.) Then, $\mathcal{P}$ is smooth and $0 (\in \Omega^2_+(ad\eta)_2)$ is its regular value.

Proof. Smoothness is clear.
We will prove that $(\delta\mathcal{P})_{(D,\varphi)}$ is surjective.

$$\begin{array}{ccccc} (\delta\mathcal{P})_{(D,\varphi)} & : & \mathrm{T}_D\,\mathcal{C}^*_\eta \oplus \mathrm{T}_\varphi\, Met_X & \to & \Omega^2_+(ad\eta)_2 \\ \| & & \| & & \\ (\delta_1\mathcal{P})_{(D,\varphi)} \oplus (\delta_2\mathcal{P})_{(D,\varphi)} & : & \Omega^1(ad\eta)_3 \oplus met_X & \to & \Omega^2_+(ad\eta)_2 \end{array}$$

Here,

$$\begin{aligned}(\delta_1\mathcal{P})_{(D,\varphi)} : \Omega^1(ad\eta)_3 &\to \Omega^2_+(ad\eta)_2,\\ A &\mapsto p_+((\varphi^{-1})^* d^D A)\\ (\delta_2\mathcal{P})_{(D,\varphi)} : met_X &\to \Omega^2_+(ad\eta)_2.\\ r &\mapsto p_+((\varphi^{-1})^*(r^* F_D))\end{aligned}$$

It suffices to prove $\mathrm{Coker}(\delta\mathcal{P})_{(D,\varphi)} = 0$.
Choose and fix an element $\Phi \in \mathrm{Coker}(\delta\mathcal{P})_{(D,\varphi)}$.
For an arbitrary $A \in \Omega^1(ad\eta)_3$, we have

$$\begin{aligned}0 &= \int_X (p_+((\varphi^{-1})^* d^D A), \Phi)_g\\ &= \int_X (d^D A, \varphi^*\Phi)_{\varphi^* g}\\ &= \int_X (A, d^{D_*}\widetilde{\Phi})_{\varphi^* g},\end{aligned}$$

where d^{D_*} is the formal adjoint of d^D w.r.t. $\varphi^* g$, and $\varphi^*\Phi = \Phi$.
Hence,

$$d^{D_*}\widetilde{\Phi} = 0 \qquad \text{(point wise)}. \tag{3.1}$$

Similarly, for $r \in met_X$, we have

$$\begin{aligned}0 &= \int_X (p_+((\varphi^{-1})^*(r^* F_D)), \Phi)_g\\ &= \int_X (r^* F_D, \widetilde{\Phi})_{\varphi^* g}.\end{aligned}$$

Hence,

$$(r^* F_D, \widetilde{\Phi})_{\varphi^* g} = 0 \quad \text{(point wise)} \qquad \text{for all } r \in met_X.$$

Hence, by [FU, Lemma 3.7], we have

$$(F_D, \widetilde{\Phi})_{ad\eta} = 0. \tag{3.2}$$

$\mathcal{P}(D, \varphi) = 0$ implies that F_D is anti-self-dual with respect to $\varphi^* g$, so we have

$$(d^D d^{D*} + d^{D*} d^D) F_D = 0.$$

By the elliptic regularity, F_D is continuous.

Since D is not flat, there exists an open set U in X^4 such that

$$F_D \neq 0 \qquad \text{on} \quad U. \tag{3.3}$$

Note that the fibres of $ad\eta$ are 1-dimensional, since the Lie algebra of $O(2)$ is 1-dimensional. Hence, (3.2) and (3.3) imply

$$\widetilde{\Phi} = 0 \qquad \text{on} \quad U. \tag{3.4}$$

On the other hand, the anti-self-duality of $\widetilde{\Phi}$ (with respect to $\varphi^* g$) and (3.1) imply

$$(d^D d^{D*} + d^{D*} d^D) \widetilde{\Phi} = 0. \tag{3.5}$$

By the unique continuation theorem [A], (3.4) and (3.5) imply $\widetilde{\Phi} = 0$. Hence, $\Phi = 0$. Thus, we have proved $\mathrm{Coker}(\delta \mathcal{P})_{(D,\varphi)} = 0$. □

Proposition 3.2. *$\mathcal{ASD}_3^*/\mathcal{G} \subset \mathcal{B}_\eta^* \times Met_X$ is a submanifold.*

Proof. This is proved just as in [FU, Theorem 3.16]. □

Proof of Theorem 3.2.

$$\begin{array}{ccccc}
\mathcal{P}^{-1}(0) = \mathcal{ASD}_3^* & \subset & \mathcal{C}_\eta^* \times Met_X & \xrightarrow{\mathcal{P}} & \Omega_+^2(ad\eta) \\
\downarrow & & \downarrow & & \\
\mathcal{ASD}_3^*/\mathcal{G}_\eta & \subset & \mathcal{B}_\eta^* \times Met_X & & \\
\downarrow \bar{\pi} & & \downarrow \pi & & \\
Met_X & = & Met_X & &
\end{array}$$

Just as in the proof of [FU, Theorem 3.17], $\bar{\pi}$ turns out to be a Fredholm map. Hence, by the Sard-Smale theorem [FU], there exists a Baire subset Met'_X in Met_X such that Met'_Xconsists of regular values of $\bar{\pi}$.

Now, we suppose that $\bar{\pi}^{-1}(\varphi) \neq \phi$ for some $\varphi \in Met'_X$.

Let $[(D, \varphi)]$ be an element of $\bar{\pi}^{-1}(\varphi)$. Then, D is an irreducible A.S.D. V-connection on η w.r.t. $\varphi^* g$ and its Atiyah-Hitchin-Singer complex

$$0 \longrightarrow \Omega^0(ad\eta) \xrightarrow{d^D} \Omega^1(ad\eta) \xrightarrow{d_+^D} \Omega_+^2(ad\eta) \longrightarrow 0$$

has an index equal to -1, since by Section 3 (a) we have

$$-\dim \mathbb{H}^0_D + \dim \mathbb{H}^1_D - \dim \mathbb{H}^2_D = -1. \tag{3.6}$$

Since D is irreducible and the center of $O(2)$ is $\mathbb{Z}_2$, we have

$$\mathbb{H}^0_D = 0. \tag{3.7}$$

Just as in the proof of [FU, Theorem 3.17], we have

$$\dim \operatorname{Coker}(\delta\bar{\pi})_{[(D,\varphi)]} = \dim \mathbb{H}^2_D, \tag{3.8}$$

$$\dim \operatorname{Ker}(\delta\bar{\pi})_{[(D,\varphi)]} = \dim \mathbb{H}^1_D. \tag{3.9}$$

Moreover, since φ is a regular value of $\bar{\pi}$, we have

$$\dim \operatorname{Coker}(\delta\bar{\pi})_{[(D,\varphi)]} = 0. \tag{3.10}$$

(3.6) through (3.8) and (3.10) imply $\dim \mathbb{H}^1_D = -1$. This is a contradiction. Hence, it is concluded that $\bar{\pi}^{-1}(\varphi) = \phi$ for all $\varphi \in Met'_X$.
Thus, we have proved Theorem 3.2. □

From now on, we assume that X^4 is equipped with a generic V-metric which satisfies Theorem 3.1.

4. Orientability of $\mathfrak{M}_E$

Since the maximal exterior power $\Lambda^{\mathrm{top}}\,\mathrm{T}\,\mathfrak{M}^*_E$ of the tangent bundle of $\mathfrak{M}^*_E$ is the restriction of the determinant line bundle Λ_E to $\mathfrak{M}^*_E$, in order to prove that $\mathfrak{M}^*_E$ is orientable it suffices to show that Λ_E is trivial.

The adjoint bundle of the $U(2)$-V-bundle $L \oplus \underline{\mathbb{C}}$ is $E = L \oplus \underline{\mathbb{R}}$, and $\mathcal{B}_E$ is naturally considered as the quotient space of the following action of $H^1(D(X);\mathbb{Z}_2)$ on $\mathcal{B}_{L\oplus\underline{\mathbb{C}}}$:

$$\begin{aligned} H^1(D(X);\mathbb{Z}_2) \times \mathcal{B}_{L\oplus\underline{\mathbb{C}}} &\to \mathcal{B}_{L\oplus\underline{\mathbb{C}}}. \\ (\rho, [\nabla^{L\oplus\underline{\mathbb{C}}}\,]) &\mapsto [\nabla^{L\oplus\underline{\mathbb{C}}} \otimes \nabla^{\rho}] \end{aligned}$$

Here ∇^{ρ} is the flat connection on L_ρ, where L_ρ is the S^1-V-bundle over X^4 whose 1st Chern class $c_1(L_\rho|_{D(X)})$ is the image of ρ by the Bockstein homomorphism:

$$\beta : H^1(D(X);\mathbb{Z}_2) \to H^2(D(X);\mathbb{Z}).$$

This action naturally lifts to the action on $\Lambda_{L\oplus\underline{\mathbb{C}}} \to \mathcal{B}_{L\oplus\underline{\mathbb{C}}}$, and its quotient is $\Lambda_E \to \mathcal{B}_E$. Hence it suffices to prove the following two Lemmas:

Lemma 4.1. *$\Lambda_{L\oplus\underline{\mathbb{C}}} \to \mathcal{B}_{L\oplus\underline{\mathbb{C}}}$ is trivial.*

Lemma 4.2. *$H^1(D(X);\mathbb{Z}_2)$ acts trivially on each fibre of $\Lambda_{L\oplus\underline{\mathbb{C}}} \to \mathcal{B}_{L\oplus\underline{\mathbb{C}}}$.*

(a) The proof of Lemma 4.1

Let $\zeta \to X$ be an $U(n)$-V-bundle and $l \to X$ be an $U(1)$-V-bundle. Choose and fix a $U(1)$-V-connection ∇^l. A natural map $\iota : \mathcal{B}_\zeta \to \mathcal{B}_{\zeta\oplus l}$ ($[\nabla^\zeta] \mapsto [\nabla^\zeta \oplus \nabla^l]$) is then defined.

Corresponding to the bundle splitting

$$ad(\zeta \oplus l) = ad\zeta \oplus \underline{\mathbb{R}} \oplus \zeta \otimes l^*,$$

the operator

$$\mathfrak{D}_{\nabla^\zeta\oplus\nabla^l} = -d^*_{\nabla^\zeta\oplus\nabla^l} \oplus d^+_{\nabla^\zeta\oplus\nabla^l} : \Omega^1(ad(\zeta \oplus l)) \to (\Omega^0 \oplus \Omega^2_+)(ad(\zeta \oplus l))$$

splits as follows:

$$\begin{array}{ccc}
\Omega^1(ad(\zeta \oplus l)) & \xrightarrow{\mathfrak{D}_{\nabla^\zeta\oplus\nabla^l}} & (\Omega^0 \oplus \Omega^2_+)(ad(\zeta \oplus l)) \\
|| & & || \\
\Omega^1(ad\zeta) & \xrightarrow{\mathfrak{D}_{\nabla^\zeta}} & (\Omega^0 \oplus \Omega^2_+)(ad\zeta) \\
\oplus & & \oplus \\
\Omega^1 & \xrightarrow{-d^*\oplus d^+} & \Omega^0 \oplus \Omega^2_+ \\
\oplus & & \oplus \\
\Omega^1(\zeta \otimes l^*) & \xrightarrow{\mathfrak{D}_{\nabla^\zeta\oplus\nabla^{l^*}}} & (\Omega^0 \oplus \Omega^2_+)(\zeta \otimes l^*).
\end{array}$$

It follows that

$$\iota^*\Lambda_{\zeta\oplus l} \cong \Lambda_\zeta \otimes_{\mathbb{R}} \underline{\mathbb{R}} \otimes_{\mathbb{R}} (\overset{\text{top}}{\Lambda}(\text{complex vector bundle})).$$

Hence, for an arbitrary loop $\phi : S^1 \to \mathcal{B}_\zeta$, we have

$$\begin{aligned}
\langle w_1(\Lambda_{\zeta\oplus l}), \iota_*[\phi]\rangle &= \langle w_1(\iota_*\Lambda_{\zeta\oplus l}), [\phi]\rangle \\
&= \langle w_1(\Lambda_\zeta), [\phi]\rangle.
\end{aligned}$$

Hence, in order to prove $\Lambda_\zeta \to \mathcal{B}_\zeta$ is trivial it suffices to show $\Lambda_{\zeta\oplus l} \to \mathcal{B}_{\zeta\oplus l}$ is trivial. Hence, in order to prove Lemma 4.1, it suffices to show $\Lambda_{(L\oplus L^*)^{\oplus m}\oplus\underline{\mathbb{C}}} \to \mathcal{B}_{(L\oplus L^*)^{\oplus m}\oplus\underline{\mathbb{C}}}$ is trivial for a large $m \in \mathbb{Z}$.

Let x_0 be a smooth point in X^4. We define

$$\begin{aligned}
\mathcal{G}_{x_0,(L\oplus L^*)^{\oplus m}\oplus\underline{\mathbb{C}}} &= \{g \in \mathcal{G}_{(L\oplus L^*)^{\oplus m}\oplus\underline{\mathbb{C}}} ; g = \text{id. at } x_0\} \quad \text{and} \\
\widetilde{\mathcal{B}}_{(L\oplus L^*)^{\oplus m}\oplus\underline{\mathbb{C}}} &= \mathcal{C}_{(L\oplus L^*)^{\oplus m}\oplus\underline{\mathbb{C}}}/\mathcal{G}_{x_0,(L\oplus L^*)^{\oplus m}\oplus\underline{\mathbb{C}}}.
\end{aligned}$$

The determinant line bundle is clearly defined over $\widetilde{\mathcal{B}}_{(L\oplus L^*)^{\oplus m}\oplus\underline{\mathbb{C}}}$, too. In order to prove $\Lambda_{(L\oplus L^*)^{\oplus m}\oplus\underline{\mathbb{C}}} \to \mathcal{B}_{(L\oplus L^*)^{\oplus m}\oplus\underline{\mathbb{C}}}$ is trivial, it suffices to show $\Lambda_{(L\oplus L^*)^{\oplus m}\oplus\underline{\mathbb{C}}} \to \widetilde{\mathcal{B}}_{(L\oplus L^*)^{\oplus m}\oplus\underline{\mathbb{C}}}$ is trivial.

As an auxiliary tool we introduce the following V-bundle $F \to X$ and its gauge group:

The bundle $(L \oplus L^*)^{\oplus m} \oplus \underline{\mathbb{C}}|_{D(X)} \to D(X)$ is trivializable. So, choose and fix a trivialization of $(L \oplus L^*)^{\oplus m} \oplus \underline{\mathbb{C}}|_{D(X)} \to D(X)$. We restrict it to

$\partial D(X) = \cup_{i=1}^n L(\alpha_i, -(\alpha c_i^*)/\alpha_i)$ and extend it to $\cup_{i=1}^n cL(\alpha_i, -(\alpha c_i^*)/\alpha_i)$ by the trivialization. Thus we have a V-bundle $F \to X$. Define

$$\begin{aligned}
&\mathcal{G}_{x_0,F} = \{g \in \mathcal{G}_F; g = \text{id. at } x_0\},\\
&\mathcal{G}'_{x_0,(L\oplus L^*)^{\oplus m}\oplus\underline{\mathbb{C}}} = \Big\{g \in \mathcal{G}_{x_0,(L\oplus L^*)^{\oplus m}\oplus\underline{\mathbb{C}}}; g = \text{id. at the cone point}\\
&\qquad\qquad\qquad\text{of } cL\left(\alpha_i, -\frac{\alpha c_i^*}{\alpha_i}\right) \text{ for } 1 \le i \le n\Big\},\\
&\mathcal{G}'_{x_0,F} = \Big\{g \in \mathcal{G}_{x_0,F}; g = \text{id. at the cone point}\\
&\qquad\qquad\qquad\text{of } cL\left(\alpha_i, -\frac{\alpha c_i^*}{\alpha_i}\right) \text{ for } 1 \le i \le n\Big\}.
\end{aligned}$$

Lemma 4.3.
(i) $\mathcal{G}_{x_0,(L\oplus L^*)^{\oplus m}\oplus\underline{\mathbb{C}}}/\mathcal{G}'_{x_0,(L\oplus L^*)^{\oplus m}\oplus\underline{\mathbb{C}}} \cong \prod_{i=1}^n G_{\alpha_i}$,

$$\textit{where } G_{\alpha_i} = \begin{cases} U(m)\times U(m) & \textit{if} \quad \alpha_i \ne 2,\\ U(2m) & \textit{if} \quad \alpha_i = 2.\end{cases}$$

(ii) $\mathcal{G}_{x_0,F}/\mathcal{G}'_{x_0,F} \cong \prod_{i=1}^n SU(2m+1)$,
(iii) $\mathcal{G}'_{x_0,(L\oplus L^*)^{\oplus m}\oplus\underline{\mathbb{C}}}$ *is homotopy equivalent to* $\mathcal{G}'_{x_0,F}$.

Proof. The proof of (ii) and (iii) are easy. So, we prove (i) only.

G_{α_i} is the set of values which a V-gauge transformation of $(L\oplus L^*)^{\oplus m}\oplus \underline{\mathbb{C}} \to X$ can take at the cone point of $cL(\alpha_i, -(\alpha c_i^*)/\alpha_i)$. Hence we have

$$\begin{aligned}
G_{\alpha_i} &= \{P \in SU(2m+1); P\cdot \mathrm{diag}\,(h, h^{-1}, h, h^{-1}, \ldots, h, h^{-1}, 1)\\
&= \mathrm{diag}(h, h^{-1}, h, h^{-1}, \ldots, h, h^{-1}, 1)\cdot P \text{ for } \forall h \in \mathbb{Z}_{\alpha_i}\}\\
&\cong \begin{cases} U(m)\times U(m) & \text{if} \quad \alpha_i \ne 2,\\ U(2m) & \text{if} \quad \alpha_i = 2.\end{cases}
\end{aligned}$$

Here,

$$\mathrm{diag}(h, h^{-1}, h, h^{-1}, \ldots, h, h^{-1}, 1) = \begin{bmatrix} h &&&&&&\\ & h^{-1} &&&&&\\ && h &&&&\\ &&& h^{-1} &&&\\ &&&& \ddots &&\\ &&&&& h &&\\ &&&&&& h^{-1} &\\ &&&&&&& 1 \end{bmatrix}.$$

$\square$

By the homotopy exact sequence of the fibration $\mathcal{C}_{(L\oplus L^*)^{\oplus m}\oplus\underline{\mathbb{C}}} \to \widetilde{\mathcal{B}}_{(L\oplus L^*)^{\oplus m}\oplus\underline{\mathbb{C}}}$, we have

$$\begin{aligned}1 = \pi_1(\mathcal{C}_{(L\oplus L^*)^{\oplus m}\oplus\underline{\mathbb{C}}}) \to \pi_1(\widetilde{\mathcal{B}}_{(L\oplus L^*)^{\oplus m}\oplus\underline{\mathbb{C}}}) \underset{\cong}{\to} \pi_0(\mathcal{G}_{(L\oplus L^*)^{\oplus m}\oplus\underline{\mathbb{C}}}) \\ \to \pi_0(\mathcal{C}_{(L\oplus L^*)^{\oplus m}\oplus\underline{\mathbb{C}}}) = 1.\end{aligned} \tag{4.1}$$

By the homotopy exact sequence of the fibrations

$$\mathcal{G}_{x_0,(L\oplus L^*)^{\oplus m}\oplus\underline{\mathbb{C}}} \to \frac{\mathcal{G}_{x_0,(L\oplus L^*)^{\oplus m}\oplus\underline{\mathbb{C}}}}{\mathcal{G}'_{x_0,(L\oplus L^*)^{\oplus m}\oplus\underline{\mathbb{C}}}} \quad \text{and} \quad \mathcal{G}_{x_0,F} \to \frac{\mathcal{G}_{x_0,F}}{\mathcal{G}'_{x_0,F}},$$

we have

$$\begin{array}{ccccc} \pi_1\left(\dfrac{\mathcal{G}_{x_0,(L\oplus L^*)^{\oplus m}\oplus\underline{\mathbb{C}}}}{\mathcal{G}'_{x_0,(L\oplus L^*)^{\oplus m}\oplus\underline{\mathbb{C}}}}\right) & \to & \pi_0(\mathcal{G}'_{x_0,(L\oplus L^*)^{\oplus m}\oplus\underline{\mathbb{C}}}) & \underset{\text{onto}}{\to} & \pi_0(\mathcal{G}_{x_0,(L\oplus L^*)^{\oplus m}\oplus\underline{\mathbb{C}}}) \\ & & \downarrow\cong & & \\ 1 = \pi_1\left(\dfrac{\mathcal{G}_{x_0,F}}{\mathcal{G}'_{x_0,F}}\right) & \to & \pi_0(\mathcal{G}'_{x_0,F}) & \underset{\cong}{\to} & \pi_0(\mathcal{G}_{x_0,F}) \end{array} \tag{4.2}$$

$$\to \pi_0\left(\frac{\mathcal{G}_{x_0,(L\oplus L^*)^{\oplus m}\oplus\underline{\mathbb{C}}}}{\mathcal{G}'_{x_0,(L\oplus L^*)^{\oplus m}\oplus\underline{\mathbb{C}}}}\right) = 1$$

$$\to \pi_0\left(\frac{\mathcal{G}_{x_0,F}}{\mathcal{G}'_{x_0,F}}\right) = 1.$$

Moreover, we have

$$\begin{aligned}\pi_0(\mathcal{G}_{x_0,F}) &= [X^4, SU(2m+1)]_* \\ &= [X^4, K(\mathbb{Z},3)]_* \\ &\cong H^3(X^4;\mathbb{Z}) \\ &= H^1(D(X^4);\mathbb{Z}).\end{aligned} \tag{4.3}$$

By (4.1) through (4.3), we see that an arbitrary loop in $\widetilde{\mathcal{B}}_{(L\oplus L^*)^{\oplus m}\oplus\underline{\mathbb{C}}}$ can be represented by $\phi_\gamma(t) = [(A^*\sharp_{\rho(t)}J_\lambda) \oplus \theta']$ $(t \in S^1)$ for some $[\gamma] \in H^1(D(X^4);\mathbb{Z})$, where A^* is a V-connection on a $U(2)$-V-bundle over X^4, θ' is a V-connection on a $U(2m-1)$-V-bundle over X^4, and ρ is a lift of γ to $\mathrm{Hom}(ad\mathbb{E}, \Lambda^2_+ \mathrm{T}^* X)$. (See [D3])

On the other hand, $\Lambda_{(L\oplus L^*)^{\oplus m}\oplus\underline{\mathbb{C}}} \to \widetilde{\mathcal{B}}_{(L\oplus L^*)^{\oplus m}\oplus\underline{\mathbb{C}}}$ is trivial over ϕ_γ by Theorem 2.1.

Hence, $\Lambda_{(L\oplus L^*)^{\oplus m}\oplus\underline{\mathbb{C}}} \to \widetilde{\mathcal{B}}_{(L\oplus L^*)^{\oplus m}\oplus\underline{\mathbb{C}}}$ is trivial.

The proof of Lemma 4.1 is complete.

(b) The proof of Lemma 4.2

An orientation α_X of $\det H^1(X)\otimes\det(H^0(X)\oplus H^2_+(X))$ is called a homology orientation of X^4. Note that since $b_2^+(X) = b_1(X) = 0$, α_X is uniquely determined by the orientation of X^4.

Let $(\nabla^L \oplus d) \otimes \nabla^\rho$ (resp. $\nabla^L \oplus d$) be a reducible connection on $L \oplus \underline{\mathbb{C}}$, compatible with a decomposition

$$L \oplus \underline{\mathbb{C}} \cong (L \oplus \underline{\mathbb{C}}) \otimes L_\rho \quad (\text{resp. } L \oplus \underline{\mathbb{C}} \cong (L \oplus \underline{\mathbb{C}}) \otimes \underline{\mathbb{C}}).$$

A homology orientation α_X of X^4 and the complex structure of L naturally determine an orientation $o_{[(\nabla^L \oplus d)\otimes\nabla^\rho]}(L, L_\rho, \alpha_X)$ (resp. $o_{[\nabla^L \oplus d]}(L, \underline{\mathbb{C}}, \alpha_X)$) of the determinant line $\Lambda_{L\oplus\underline{\mathbb{C}}}|_{[(\nabla^L\oplus d)\otimes\nabla^\rho]}$ (resp. $\Lambda_{L\oplus\underline{\mathbb{C}}}|_{[\nabla^L\oplus d]}$) corresponding to $[(\nabla^L \oplus d) \otimes \nabla^\rho]$ (resp. $[\nabla^L \oplus d]$) ([D3]).

In order to prove Lemma 4.2, it suffices to show that $o_{[\nabla^L\oplus d]}(L, \underline{\mathbb{C}}, \alpha_X)$ and $o_{[(\nabla^L\oplus d)\otimes\nabla^\rho]}(L, L_\rho, \alpha_X)$ assign the same orientation to the trivial bundle $\Lambda_{L\oplus\underline{\mathbb{C}}} \to \mathcal{B}_{L\oplus\underline{\mathbb{C}}}$.

By Wu's theorem [W], for a sufficiently large $l \in \mathbb{N}$, $\hat{X}^4 = X^4 \sharp l(S^2 \times S^2)$ becomes an almost complex V-manifold whose almost complex structure is compatible with the orientation of X^4. (Here the connected sum is formed in the smooth part of X^4.)

Two operators

$$-d^* \oplus d^+ : \Omega^1_{\hat{X}} \to \Omega^0_{\hat{X}} \oplus \Omega^2_{+\hat{X}} \quad \text{and} \quad -\bar{\partial}^* \oplus \bar{\partial} : \Omega^{0,1}_{\hat{X}} \to (\Omega^0_{\hat{X}})^{\mathbb{C}} \oplus \Omega^{0,2}_{\hat{X}}$$

have the same symbol under a natural identification, so there exists a canonical isomorphism between $\det \operatorname{ind}(-d^* \oplus d^+)$ and $\det \operatorname{ind}(-\bar{\partial}^* \oplus \bar{\partial})$. (Here $(\Omega^0_{\hat{X}})^{\mathbb{C}} = \Omega^0_{\hat{X}} \oplus \Omega^0_{\hat{X}} \cdot \omega$, and ω is the fundamental 2-form of $\hat{X}^4$). $\Omega^{0,1}_{\hat{X}}$ and $\Omega^{0,2}_{\hat{X}}$ are complex vector spaces by the almost complex structure of $\hat{X}^4$. We make $(\Omega^0_{\hat{X}})^{\mathbb{C}}$ into a complex vector space by:

$$I \cdot \omega = -1, \qquad I \cdot 1 = \omega. \tag{4.4}$$

Then $-\bar{\partial}^* \oplus \bar{\partial}$ is complex linear. We define the homology orientation $\alpha_{\hat{X}}$ of $\hat{X}^4$ to be the orientation of $\det \operatorname{ind}(-d^* \oplus d^+)$ induced by the complex orientation of $\det \operatorname{ind}(-\bar{\partial}^* \oplus \bar{\partial})$.

We assume that $\hat{X}^4 = (X^4 \setminus B^4_X) \cup (l(S^2 \times S^2) \setminus B^4_{l(S^2\times S^2)})$, where B^4_X and $B^4_{l(S^2\times S^2)}$ are open 4-balls in X^4 and $l(S^2 \times S^2)$. Note that $L \to X$ (resp. $L_\rho \to X$) is trivial in $\bar{B}^4_X$. We choose and fix its trivialization, restrict it to $\partial\bar{B}^4_X$, and extend it to $l(S^2 \times S^2) \setminus B^4_{l(S^2\times S^2)}$. Thus we have a line bundle $\hat{L} \to \hat{X}$ (resp. $\hat{L}_\rho \to \hat{X}$).

As in the proof of Lemma 4.1, $\Lambda_{\hat{L}\oplus\underline{\mathbb{C}}} \to \mathcal{B}_{\hat{L}\oplus\underline{\mathbb{C}}}$ turns out to be trivial.

Let $A_{(L\oplus\underline{\mathbb{C}})\otimes L_\rho}$ (resp. $A_{L\oplus\underline{\mathbb{C}}}$) be a reducible connection on $L \oplus \underline{\mathbb{C}}$ which is compatible with a decomposition

$$L \oplus \underline{\mathbb{C}} \cong (L \oplus \underline{\mathbb{C}}) \otimes L_\rho \quad (\text{resp. } L \oplus \underline{\mathbb{C}} \cong (L \oplus \underline{\mathbb{C}}) \otimes \underline{\mathbb{C}}),$$

and is trivial over $\bar{B}^4_X$ with respect to the trivialization fixed above. Extend $A_{(L\oplus\underline{\mathbb{C}})\otimes L_\rho}$ (resp. $A_{L\oplus\underline{\mathbb{C}}}$) to $l(S^2 \times S^2) \setminus B^4_{l(S^2\times S^2)}$ trivially. Then we get a reducible connection $\hat{A}_{(\hat{L}\oplus\underline{\mathbb{C}})\otimes\hat{L}_\rho}$ (resp. $\hat{A}_{\hat{L}\oplus\underline{\mathbb{C}}}$) on $\hat{L} \oplus \underline{\mathbb{C}}$, compatible with a decomposition

$$\hat{L} \oplus \underline{\mathbb{C}} \cong (\hat{L} \oplus \underline{\mathbb{C}}) \otimes \hat{L}_\rho \quad (\text{resp. } (\hat{L} \oplus \underline{\mathbb{C}}) \otimes \underline{\mathbb{C}}).$$

Let $o_{[\hat{A}_{(\hat{L}\oplus\underline{\mathbb{C}})\otimes\hat{L}_\rho}]}(\hat{L},\hat{L}_\rho,\alpha_{\hat{X}})$ (resp. $o_{[\hat{A}_{\hat{L}\oplus\underline{\mathbb{C}}}]}(\hat{L},\underline{\mathbb{C}},\alpha_{\hat{X}})$) be an orientation of the determinant line $\Lambda_{\hat{L}\oplus\underline{\mathbb{C}}}|_{[\hat{A}_{(\hat{L}\oplus\underline{\mathbb{C}})\otimes\hat{L}_\rho}]}$ (resp. $\Lambda_{\hat{L}\oplus\underline{\mathbb{C}}}|_{[\hat{A}_{\hat{L}\oplus\underline{\mathbb{C}}}]}$) which is determined by $\alpha_{\hat{X}}$ and the complex structure of $\hat{L}$.

Then, by the excision argument ([D3, Lemma (3.26)]) we see that

$$\frac{o_{[A_{L\oplus\underline{\mathbb{C}}}]}(L,\underline{\mathbb{C}},\alpha_X)}{o_{[A_{(L\oplus\underline{\mathbb{C}})\otimes L_\rho}]}(L,L_\rho,\alpha_X)}=\frac{o_{[\hat{A}_{\hat{L}\oplus\underline{\mathbb{C}}}]}(\hat{L},\underline{\mathbb{C}},\alpha_{\hat{X}})}{o_{[\hat{A}_{(\hat{L}\oplus\underline{\mathbb{C}})\otimes\hat{L}_\rho}]}(\hat{L},\hat{L}_\rho,\alpha_{\hat{X}})}. \tag{4.5}$$

On the other hand, since α_X and the complex structure of X^4 "*uniformally*" determine orientations of the determinant lines corresponding to reducible connections on $L\oplus\underline{\mathbb{C}}$ which are compatible with a decomposition

$$L\oplus\underline{\mathbb{C}}\cong(L\oplus\underline{\mathbb{C}})\otimes L_\rho\quad(\text{resp. } L\oplus\underline{\mathbb{C}}\cong(L\oplus\underline{\mathbb{C}})\otimes\underline{\mathbb{C}}),$$

it follows that $o_{[(\nabla^L\oplus d)\otimes\nabla^\rho]}(L,L_\rho,\alpha_X)$ (resp. $o_{[\nabla^L\oplus d]}(L,\underline{\mathbb{C}},\alpha_X)$) and $o_{[A_{(L\oplus\underline{\mathbb{C}})\otimes L_\rho}]}(L,L_\rho,\alpha_X)$ (resp. $o_{[A_{L\oplus\underline{\mathbb{C}}}]}(L,\underline{\mathbb{C}},\alpha_X)$) assign the same orientation to $\Lambda_{L\oplus\underline{\mathbb{C}}}\to\mathcal{B}_{L\oplus\underline{\mathbb{C}}}$. Hence we have

$$\frac{o_{[\nabla^L\oplus d]}(L,\underline{\mathbb{C}},\alpha_X)}{o_{[(\nabla^L\oplus d)\otimes\nabla^\rho]}(L,L_\rho,\alpha_X)}=\frac{o_{[A_{L\oplus\underline{\mathbb{C}}}]}(L,\underline{\mathbb{C}},\alpha_X)}{o_{[A_{(L\oplus\underline{\mathbb{C}})\otimes L_\rho}]}(L,L_\rho,\alpha_X)}. \tag{4.6}$$

By (4.5) and (4.6), we have

$$\frac{o_{[\nabla^L\oplus d]}(L,\underline{\mathbb{C}},\alpha_X)}{o_{[(\nabla^L\oplus d)\otimes\nabla^\rho]}(L,L_\rho,\alpha_X)}=\frac{o_{[\hat{A}_{\hat{L}\oplus\underline{\mathbb{C}}}]}(\hat{L},\underline{\mathbb{C}},\alpha_{\hat{X}})}{o_{[\hat{A}_{(\hat{L}\oplus\underline{\mathbb{C}})\otimes\hat{L}_\rho}]}(\hat{L},\hat{L}_\rho,\alpha_{\hat{X}})}. \tag{4.7}$$

We want to prove that the left hand side of (4.7) is equal to 1. We consider the right hand side of (4.7).

Let $I_{\hat{X}}$ be the complex structures on $\Omega^1_{\hat{X}}(\underline{\mathbb{R}}\oplus\hat{L})$ and $(\Omega^0_{\hat{X}}\oplus\Omega^2_{+\hat{X}})(\underline{\mathbb{R}}\oplus\hat{L})$ which are defined by the almost complex structure of $\hat{X}^4$ and (4.4). Define

$$\mathfrak{D}_{\hat{A}_{\hat{L}\oplus\underline{\mathbb{C}}}}=-d^*_{\hat{A}_{\hat{L}\oplus\underline{\mathbb{C}}}}\oplus d^+_{\hat{A}_{\hat{L}\oplus\underline{\mathbb{C}}}}:\Omega^1_{\hat{X}}(\underline{\mathbb{R}}\oplus\hat{L})\to(\Omega^0_{\hat{X}}\oplus\Omega^2_{+\hat{X}})(\underline{\mathbb{R}}\oplus\hat{L})\quad\text{and}$$

$$\mathfrak{D}^t_{\hat{A}_{\hat{L}\oplus\underline{\mathbb{C}}}}=(1-t)\mathfrak{D}_{\hat{A}_{\hat{L}\oplus\underline{\mathbb{C}}}}-tI_{\hat{X}}\mathfrak{D}_{\hat{A}_{\hat{L}\oplus\underline{\mathbb{C}}}}I_{\hat{X}}\quad\left(0\le t\le\frac{1}{2}\right).$$

Since $\mathfrak{D}^{1/2}_{\hat{A}_{\hat{L}\oplus\underline{\mathbb{C}}}}$ commutes with $I_{\hat{X}}$, $\det\operatorname{ind}\mathfrak{D}^{1/2}_{\hat{A}_{\hat{L}\oplus\underline{\mathbb{C}}}}$ inherits an orientation induced by $I_{\hat{X}}$.

In the following decomposition

$$\begin{array}{ccccc}
\mathfrak{D}^{\frac{1}{2}}_{\hat{A}_{\hat{L}\oplus\underline{\mathbb{C}}}} & : & \Omega^1_{\hat{X}}(\underline{\mathbb{R}}\oplus\hat{L}) & \longrightarrow & (\Omega^0_{\hat{X}}\oplus\Omega^2_{+\hat{X}})(\underline{\mathbb{R}}\oplus\hat{L}) \\
\| & & \| & & \| \\
-\bar{\partial}^*\oplus\bar{\partial} & : & \Omega^{0,1}_{\hat{X}} & \longrightarrow & (\Omega^0_{\hat{X}})^{\mathbb{C}}\oplus\Omega^{0,2}_{\hat{X}} \\
\oplus & & \oplus & & \oplus \\
(-\bar{\partial}^*\oplus\bar{\partial})_{\hat{L}} & : & \Omega^{0,1}_{\hat{X}}\otimes_{\mathbb{C}}\hat{L} & \longrightarrow & ((\Omega^0_{\hat{X}})^{\mathbb{C}}\oplus\Omega^{0,2}_{\hat{X}})\otimes_{\mathbb{C}}\hat{L} \\
\oplus & & \oplus & & \oplus \\
(-\bar{\partial}^*\oplus\bar{\partial})_{\hat{L}^{-1}} & : & \Omega^{0,1}_{\hat{X}}\otimes_{\mathbb{C}}\hat{L}^{-1} & \longrightarrow & ((\Omega^0_{\hat{X}})^{\mathbb{C}}\oplus\Omega^{0,2}_{\hat{X}})\otimes_{\mathbb{C}}\hat{L}^{-1},
\end{array}$$

the complex structures defined by $\hat{L}$ and by the base space X^4 agree on the second term and are opposite on the third term. Similarly, for the first term, our homology orientation $\alpha_{\hat{X}}$ uses the same complex structure as that defined by $I_{\hat{X}}$. So the orientation of $\det \operatorname{ind} \mathfrak{D}^{1/2}_{\hat{A}_{(\hat{L}\oplus\underline{\mathbb{C}})\otimes\hat{L}_\rho}}$ defined by $\alpha_{\hat{X}}$ and the complex structure of $\hat{L}$ compares with the orientation of $\det \operatorname{ind} \mathfrak{D}^{1/2}_{\hat{A}_{(\hat{L}\oplus\underline{\mathbb{C}})\otimes\hat{L}_\rho}}$ defined by $I_{\hat{X}}$ with the sign

$$(-1)^{\operatorname{ind}(-\bar{\partial}^*\oplus\bar{\partial})_{\hat{L}^{-1}}}.$$

Similarly, the orientation of $\det \operatorname{ind} \mathfrak{D}^{1/2}_{\hat{A}_{(\hat{L}\oplus\underline{\mathbb{C}})\otimes\hat{L}_\rho}}$ defined by $\alpha_{\hat{X}}$ and the complex structure of $\hat{L}$ compares with the orientation of $\det \operatorname{ind} \mathfrak{D}^{1/2}_{\hat{A}_{(\hat{L}\oplus\underline{\mathbb{C}})\otimes\hat{L}_\rho}}$ defined by $I_{\hat{X}}$ with the sign

$$(-1)^{\operatorname{ind}(-\bar{\partial}^*\oplus\bar{\partial})_{\hat{L}^{-1}}}.$$

On the other hand, we can give $\det \operatorname{ind} \mathfrak{D}^{1/2} \to \mathcal{B}_{\hat{L}\oplus\underline{\mathbb{C}}}$ a trivialization by orientations of fibres defined by $I_{\hat{X}}$.

Hence, by continuity in a parameter $t \in [0, 1/2]$, we see that

$$\frac{o_{[\hat{A}_{\hat{L}\oplus\underline{\mathbb{C}}}]}(\hat{L}, \underline{\mathbb{C}}, \alpha_{\hat{X}})}{o_{[\hat{A}_{(\hat{L}\oplus\underline{\mathbb{C}})\otimes\hat{L}_\rho}]}(\hat{L}, \hat{L}_\rho, \alpha_{\hat{X}})} = 1.$$

Hence, we have proved Lemma 4.2.

Theorem 4.1. *$\mathfrak{M}^*_{\hat{L}\oplus\underline{\mathbb{R}}}$ is orientable.*

5. Comparison of orientations at reducible connections

Let $\nabla^L \oplus d$ be an S^1-reducible A.S.D. V-connection on $E = L \oplus \underline{\mathbb{R}}$, where ∇^L is an A.S.D. V-connection on L and d is the exterior differentiation. Let $\nabla^{L'} \oplus d$ be another reducible A.S.D. V-connection on E, compatible with a decomposition

$$E \cong L' \oplus \underline{\mathbb{R}}.$$

Let $o(L)$ (resp. $o(L')$) be an orientation of $\Lambda_E|_{[\nabla^L\oplus d]}$ (resp. $\Lambda_E|_{[\nabla^{L'}\oplus d]}$) defined by α_X and the complex structure of L (resp. L').

Theorem 5.1. *The orientation which $o(L)$ assigns to the trivial bundle $\Lambda_E \to \mathcal{B}_E$ and the one which $o(L')$ assigns to $\Lambda_E \to \mathcal{B}_E$ compare with the sign*

$$(-1)^{\operatorname{ind}(-\bar{\partial}^*\oplus\bar{\partial})_{L^{-1}} - \operatorname{ind}(-\bar{\partial}^*\oplus\bar{\partial})_{L'^{-1}}}.$$

Proof. This is proved by a similar argument as in Section 4. So for simplicity we assume that X^4 is a Kähler V-manifold.

Moreover, we assume that the homology orientation α_X of X^4 coincides with the orientation of $\det \operatorname{ind}(-\bar{\partial}^* \oplus \bar{\partial}) : \Omega_X^{0,1} \to (\Omega_X^0)^{\mathbb{C}} \oplus \Omega_X^{0,2})$ defined by the complex structure of X^4.

In the following decomposition

$$\begin{array}{ccccc} \mathfrak{D}_{\nabla^L \oplus d} & : & \Omega_X^1(\underline{\mathbb{R}} \oplus L) & \to & (\Omega_X^0 \oplus \Omega_{+X}^2)(\underline{\mathbb{R}} \oplus L) \\ || & & || & & || \\ -\bar{\partial}^* \oplus \bar{\partial} & : & \Omega_X^{0,1} & \to & (\Omega_X^0)^{\mathbb{C}} \oplus \Omega_X^{0,2} \\ \oplus & & \oplus & & \oplus \\ (-\bar{\partial}^* \oplus \bar{\partial})_L & : & \Omega_X^{0,1} \otimes_{\mathbb{C}} L & \to & ((\Omega_X^0)^{\mathbb{C}} \oplus \Omega_X^{0,2}) \otimes_{\underline{\mathbb{C}}} L \\ \oplus & & \oplus & & \oplus \\ (-\bar{\partial}^* \oplus \bar{\partial})_{L^{-1}} & : & \Omega_X^{0,1} \otimes_{\mathbb{C}} L^{-1} & \to & ((\Omega_X^0)^{\mathbb{C}} \oplus \Omega_X^{0,2}) \otimes_{\underline{\mathbb{C}}} L^{-1}, \end{array}$$

the complex structures defined by L and by X^4 agree on the second term and are opposite on the third term. Similarly, for the first term, our homology orientation α_X uses the same complex structure as that defined by $-\bar{\partial}^* \oplus \bar{\partial}$. Hence, the orientation of $\Lambda_E|_{[\nabla^L \oplus d]}$ defined by the complex structure of X^4 compares with $o(L)$ with the sign

$$(-1)^{\operatorname{ind}(-\bar{\partial}^* \oplus \bar{\partial})_{L^{-1}}}.$$

Similarly, the orientation of $\Lambda_E|_{[\nabla^{L'} \oplus d]}$ defined by the complex structure of X^4 compares with $o(L')$ with the sign

$$(-1)^{\operatorname{ind}(-\bar{\partial}^* \oplus \bar{\partial})_{L'^{-1}}}.$$

On the other hand, orientations of fibres of $\Lambda_E \to \mathcal{B}_E$ defined by the complex structure of X^4 determine a trivialization of $\Lambda_E \to \mathcal{B}_E$.

Hence, $o(L)$ and $o(L')$ compare with the sign

$$(-1)^{\operatorname{ind}(-\bar{\partial}^* \oplus \bar{\partial})_{L^{-1}} - \operatorname{ind}(-\bar{\partial}^* \oplus \bar{\partial})_{L'^{-1}}}.$$

□

Now, recall that $\mu(e) = 1$. Hence, if $\{L_i \oplus \underline{\mathbb{R}}\}_{i=1}^{\nu} \cup \{L_i^{-1} \oplus \underline{\mathbb{R}}\}_{i=1}^{\nu}$ are the set of all topological S^1-reductions of $E = L \oplus \underline{\mathbb{R}}$ (, where $L = L_1$), we may assume that

$$c_1(L) - c_1(L_i) \in \operatorname{Tor} H^2(D(X); \mathbb{Z}) \quad (i = 1, \dots, \nu).$$

Then, $c_1(L)|_{L(\alpha_j, (\alpha c_j^*)/\alpha_j)} = c_1(L_i)|_{L(\alpha_j, (\alpha c_j^*)/\alpha_j)}$, because

$$\iota^*(\operatorname{Tor} H^2(D(X); \mathbb{Z})) = 0 \ (j = 1, \dots, n).$$

(Here, $\iota : \partial D(X) \to D(X)$ is the inclusion.)

Hence, by the index theorem [K], we see that

$$\operatorname{ind}(-\bar{\partial}^* \oplus \bar{\partial})_{L^{-1}} = \operatorname{ind}(-\bar{\partial}^* \oplus \bar{\partial})_{L_i^{-1}}.$$

Thus, we have

Theorem 5.2. *In the situation above, the orientation which $o(L)$ assigns to $\Lambda_E \to \mathcal{B}_E$ and the one which $o(L_i)$ assigns to $\Lambda_E \to \mathcal{B}_E$ coincide $(i = 1, \dots, \nu)$.*

This is translated in terms of an orientable manifold $\mathfrak{M}_E^*$ as follows.

Theorem 5.3. *We assume that the singular points $\mathfrak{M}_E \setminus \mathfrak{M}_E^*$ of the moduli space $\mathfrak{M}_E$ is $\{[\nabla^{L_i} \oplus d]\}_{i=1}^{\nu}$, where $\nabla^{L_i} \oplus d$ is an S^1-reducible A.S.D. V-connection on E, compatible with a decomposition*

$$E \cong L_i \oplus \underline{\mathbb{R}},$$

which satisfies

$$c_1(L) - c_1(L_i) \in \operatorname{Tor} H^2(D(X); \mathbb{Z}) \quad (i = 1, \dots, \nu).$$

Then, we have the following:

(i) *The 1st cohomology of the Atiyah-Hitchin-Singer complex (modulo the S^1-action) associated to $\nabla^{L_i} \oplus d$ is identified with a neighbourhood U_i of $[\nabla^{L_i} \oplus d]$ in $\mathfrak{M}_E$, and $U_i \setminus [\nabla^{L_i} \oplus d]$ is equipped with an orientation naturally induced by the complex structure of L_i. Moreover, $\{U_i \setminus [\nabla^{L_i} \oplus d]\}_{i=1}^{\nu}$ with these orientations belong to an oriented local coordinate system of $\mathfrak{M}_E^*$.*

(ii) *U_i is the open cone on $\mathbb{CP}^{(R(M)-1)/2}$. If we remove these cones from $\mathfrak{M}_E$, we obtain a compact manifold $\hat{\mathfrak{M}}_E$ whose boundary consists of ν disjoint copies of $\mathbb{CP}^{(R(M)-1)/2}$. Then, the orientations of the boundary components $\mathbb{CP}^{(R(M)-1)/2}$'s which are induced by an orientation of $\mathfrak{M}_E^*$ coincide.*

6. Proof of Theorem 1.1

As we have shown in Sections 2 through 5, $\mathfrak{M}_E$ has the following properties:

(1) $\mathfrak{M}_E^*$ has a natural structure of $R(M)$-dimensional smooth manifold, where $R(M)$ is a positive odd integer.

(2) $\mathfrak{M}_E$ is compact.

(3) Singular points of $\mathfrak{M}_E$ correspond to the set of gauge equivalence classes of S^1-reducible A.S.D. V-connections on E, which is non-empty and finite. Each singular points has a neighbourhood of the cone on $\mathbb{CP}^{(R(M)-1)/2}$.

(4) $\mathfrak{M}_E^*$ is orientable. If we choose and fix an orientation of $\mathfrak{M}_E^*$, then the orientations of boundary components $\mathbb{CP}^{(R(M)-1)/2}$'s which are induced by that of $\mathfrak{M}_E^*$ coincide.

If $R(M) \equiv 1 (\mathrm{mod}\ 4)$, this is a contradiction. For, by Tom's theorem,

$$\mathbf{\Omega}^* \otimes \mathbb{Q} = \mathbb{Q}[\mathbb{CP}^2, \mathbb{CP}^4, \mathbb{CP}^6, \dots],$$

where $\mathbf{\Omega}^*$ is the oriented cobordism ring.

In general, we argue as follows.

Let Σ be an oriented closed surface in $D(X)$. Fix a spin structure on Σ and let V_+ and V_- be the complex spinor bundles of $\pm(1/2)$-spinors on Σ.

For any connection on A on $L \otimes \underline{\mathbb{R}}|_\Sigma \to \Sigma$, we can define the twisted Dirac operator

$$\mathcal{D}_{\Sigma,A} \;:\; \Gamma(V_+ \otimes_{\mathbb{R}} (L \oplus \underline{\mathbb{R}})|_\Sigma) \to \Gamma(V_- \otimes_{\mathbb{R}} (L \oplus \underline{\mathbb{R}})|_\Sigma).$$

We can construct the determinant index bundle $\det \operatorname{ind} \mathcal{D}_\Sigma \to \mathcal{B}^*_\Sigma$, where $\mathcal{B}^*_\Sigma$ is the set of gauge equivalence classes of irreducible connections on $L \oplus \underline{\mathbb{R}}|_\Sigma$.

Pulling this back by the restriction map $r : \mathfrak{M}^*_E \to \mathcal{B}^*_\Sigma$, we obtain a complex line bundle $r^*(\det \operatorname{ind} \mathcal{D}_\Sigma) \to \mathfrak{M}^*_E$, which we denote as $\det \operatorname{ind} \mathcal{D}_\Sigma \to \mathfrak{M}^*_E$, for simplicity.

Let $\{[\nabla^{L_i} \otimes d]\}_{i=1}^{\nu}$ be the set of singular points of $\mathfrak{M}_E$. Here $\nabla^{L_i} \otimes d$ is an S^1 reducible A.S.D. V-connection on E, compatible with a decomposition

$$E \cong L_i \oplus \underline{\mathbb{R}}.$$

We may assume

$$c_1(L) - c_1(L_i) \in \operatorname{Tor} H^2(D(X);\mathbb{Z}) \quad (i = 1, \dots, \nu).$$

Proposition 6.1 (cf. [D2, Lemma 2.28]). *When we restrict* $\det \operatorname{ind} \mathcal{D}_\Sigma \to \mathfrak{M}^*_E$ *to a link* $\mathbb{CP}^{(R(M)-1)/2}$ *of* $[\nabla^{L_i} \oplus d]$*, we have*

$$c_1((\det \operatorname{ind} \mathcal{D}_\Sigma)|_{\mathbb{CP}^{\frac{R(M)-1}{2}}}) = 2\langle c_1(L_i), [\Sigma]\rangle \cdot h,$$

where $h \in H^2(\mathbb{CP}^{(R(M)-1)/2};\mathbb{Z})$ *is the* 1*st Chern class of Hopf line bundle over* $\mathbb{CP}^{(R(M)-1)/2}$.

Proof. Complex line bundle over $\mathbb{CP}^{(R(M)-1)/2}$ are in one-to-one correspondence with S^1-equivariant complex line bundles over $\mathbb{C}^{(R(M)+1)/2}$, where the 1st Chern class of a line bundle over $\mathbb{CP}^{(R(M)-1)/2}$ corresponds to the weight of the S^1-action on the fibre $0 \times \mathbb{C}$ of $\mathbb{CP}^{(R(M)+1)/2} \times \mathbb{C} \to \mathbb{C}^{(R(M)+1)/2}$ over $0 \in \mathbb{C}^{(R(M)+1)/2}$. So, the 1st Chern class of $(\det \operatorname{ind} \mathcal{D}_\Sigma)|_{\mathbb{CP}^{(R(M)-1)/2}} \to \mathbb{CP}^{(R(M)-1)/2}$ corresponds to the weight of the S^1-action on $0 \times \mathbb{C}$, which is nothing but the weight of the action of $\Gamma_{\nabla^{L_i} \oplus d}$ on $\det \operatorname{ind} \mathcal{D}_{\Sigma, \nabla^{L_i} \oplus d|_\Sigma}$.

Since $\Gamma_{\nabla^{L_i} \oplus d} \cong S^1$ acts on L_i with weight 1, $\Gamma_{\nabla^{L_i} \oplus d}$ acts on

$$\begin{aligned} V_\pm \otimes_{\mathbb{R}} (L_i \oplus \underline{\mathbb{R}})|_\Sigma &= V_\pm \otimes_{\mathbb{C}} (L_i \oplus L_i^{-1} \oplus \underline{\mathbb{C}})|_\Sigma \\ &= V_\pm \otimes_{\mathbb{C}} L_i \oplus V_\pm \otimes_{\mathbb{C}} L_i^{-1} \oplus V_\pm \otimes_{\mathbb{C}} \underline{\mathbb{C}}|_\Sigma \end{aligned}$$

with weight 1, -1 and 0, respectively.

Hence, $\Gamma_{\nabla^{L_i} \oplus d}$ acts on

$$\det \operatorname{ind} \mathcal{D}_{\Sigma, \nabla^{L_i} \oplus d|_\Sigma} = \det \operatorname{ind} \mathcal{D}_{\Sigma, L_i} \otimes_{\mathbb{C}} \det \operatorname{ind} \mathcal{D}_{\Sigma, L_i^{-1}} \otimes_{\mathbb{C}} \det \operatorname{ind} \mathcal{D}_\Sigma$$

with weight $\operatorname{index} \mathcal{D}_{\Sigma, L_i} - \operatorname{index} \mathcal{D}_{\Sigma, L_i^{-1}}$. By the index theorem,

$$\begin{aligned} \operatorname{index} \mathcal{D}_{\Sigma, L_i} - \operatorname{index} \mathcal{D}_{\Sigma, L_i^{-1}} &= \langle ch(L_i)\hat{A}(\Sigma), [\Sigma]\rangle - \langle ch(L_i^{-1})\hat{A}(\Sigma), [\Sigma]\rangle \\ &= 2\langle c_1(L_i), [\Sigma]\rangle. \end{aligned}$$

Hence, we have

$$c_1((\det \operatorname{ind} \mathcal{D}_\Sigma)|_{\mathbb{CP}^{\frac{R(M)-1}{2}}}) = 2\langle c_1(L_i), [\Sigma]\rangle \cdot h.$$

□

Recall that

$$c_1(L) - c_1(L_i) \in \operatorname{Tor} H^2(D(X);\mathbb{Z}) \quad (i = 1, \ldots, \nu).$$

So, if we choose Σ to be an oriented closed surface which represents the homology class dual to $c_1(L) \in H^2(D(X);\mathbb{Z})$, then for each boundary component $\mathbb{CP}^{(R(M)-1)/2}$ of $\hat{\mathfrak{M}}_E$ we have

$$c_1((\det \operatorname{ind} \mathcal{D}_\Sigma)|_{\mathbb{CP}^{\frac{R(M)-1}{2}}}) = 2 \cdot h,$$

which is independent of the choice of boundary components.

Thus we have

$$\begin{aligned}
&\langle c_1(\det \operatorname{ind} \mathcal{D}_\Sigma)^{\frac{R(M)-1}{2}}|_{\partial\hat{\mathfrak{M}}_E}, [\partial\hat{\mathfrak{M}}_E]\rangle \\
&\quad = \sum_{i=1}^{\nu} \langle c_1(\det \operatorname{ind} \mathcal{D}_\Sigma)^{\frac{R(M)-1}{2}}|_{\mathbb{CP}^{\frac{R(M)-1}{2}}}, [\mathbb{CP}^{\frac{R(M)-1}{2}}]\rangle \\
&\quad = \sum_{i=1}^{\nu} \langle 2^{\frac{R(M)-1}{2}} h^{\frac{R(M)-1}{2}}, [\mathbb{CP}^{\frac{R(M)-1}{2}}]\rangle \\
&\quad = \nu \cdot 2^{\frac{R(M)-1}{2}} > 0.
\end{aligned}$$

On the other hand, by the general theory of algebraic topology, we have

$$\langle c_1(\det \operatorname{ind} \mathcal{D}_\Sigma)^{\frac{R(M)-1}{2}}|_{\partial\hat{\mathfrak{M}}_E}, \partial[\hat{\mathfrak{M}}_E]\rangle = \langle i^* c_1(\det \operatorname{ind} \mathcal{D}_\Sigma)^{\frac{R(M)-1}{2}}, \partial[\hat{\mathfrak{M}}_E]\rangle,$$

where $i; \partial\hat{\mathfrak{M}}_E \hookrightarrow \hat{\mathfrak{M}}_E$ is the inclusion,

$$\begin{aligned}
&= \langle c_1(\det \operatorname{ind} \mathcal{D}_\Sigma)^{\frac{R(M)-1}{2}}, i^*\partial[\hat{\mathfrak{M}}_E]\rangle \\
&= 0, \quad \text{since} \quad i^*\partial = 0.
\end{aligned}$$

This is a contradiction.

Hence, we have proved Theorem 1.1.

Department of Mathmatical Science
University of Tokyo
Komaba, Meguro-ku
Tokyo 153-8914, Japan

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