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Toward a definition of moduli of complexes of coherent sheaves on a projective scheme

By

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Introduction

Let A be an abelian variety, $\hat{A}$ its dual abelian variety and $\mathcal{P}$ the normalized Poincaré bundle on $A \times \hat{A}$. Define the functor $\hat{\Phi}$ of $\mathcal{O}_A$-modules M into the category of $\mathcal{O}_{\hat{A}}$-modules by

$$\hat{\Phi}(M) := p_{2*}(p_1^*(M) \otimes \mathcal{P}).$$

S. Mukai proved in [7] that the derived functor $\mathbf{R}\hat{\Phi}$ of $\hat{\Phi}$ gives an equivalence of categories between two derived categories $D(A)$ and $D(\hat{A})$. The functor $\mathbf{R}\hat{\Phi}$ is called a Fourier-Mukai transform and has useful applications to the moduli theory of sheaves on abelian varieties. Let Spl_A denote the moduli space of simple sheaves on A. When $\mathbf{R}\hat{\Phi}$ determines an isomorphism of a component M_1 of Spl_A to a component M_2 of $\mathrm{Spl}_{\hat{A}}$, the condition (W.I.T) must be satisfied, that is, for any member $F \in M_1$, $R^i\hat{\Phi}(F) = 0$ for all but one i. So it is important to search for a component of Spl_A which satisfies (W.I.T) if one wants to apply Fourier-Mukai transform to the moduli theory of sheaves. Indeed there are many interesting examples of such a moduli correspondence. However, if a “moduli space of complexes” is defined, Fourier-Mukai transform can be considered as a correspondence of moduli spaces of complexes. So it is desirable to define a “moduli space of complexes”.

T. Bridgeland constructed the projective moduli scheme of perverse point sheaves in [3] and applied it to construct a flop. Perverse point sheaves are something like objects in the derived category which are obtained by deforming structure sheaves of points. The work of [3] also gives a significance of considering moduli spaces of complexes for the purpose of taking several types of compactifications.

Throughout this paper, we let $f : X \to S$ denote a flat projective morphism of noetherian schemes. Fix an S-very ample line bundle $\mathcal{O}_X(1)$ on X.

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Definition 0.1. Define a functor $\mathrm{Splcpx}_{X/S}$ of the category of locally noetherian schemes over S to the category of sets by putting

$$\mathrm{Splcpx}_{X/S}(T) := \left\{ E^{\bullet} \;\middle|\; \begin{array}{l} E^{\bullet} \text{ is a bounded complex of coherent sheaves} \\ \text{on } X_T \text{ such that each } E^i \text{ is flat over } T \text{ and} \\ \text{for any } t \in T,\ \mathrm{Ext}^0_{X_t}(E^{\bullet}(t), E^{\bullet}(t)) \cong k(t) \\ \text{and } \mathrm{Ext}^{-1}_{X_t}(E^{\bullet}(t), E^{\bullet}(t)) = 0 \end{array} \right\} \Big/ \sim$$

for any locally noetherian scheme T over S, where $E^{\bullet} \sim F^{\bullet}$ if there exist a line bundle L on T, a bounded complex $Q^{\bullet}$ of quasi-coherent sheaves on X_T and quasi-isomorphisms $Q^{\bullet} \to E^{\bullet}, Q^{\bullet} \to F^{\bullet} \otimes L$.

Note that $E^{\bullet}(t)$ denotes the complex $E^{\bullet} \otimes k(t)$. We let $\mathrm{Splcpx}^{\text{ét}}_{X/S}$ denote the associated sheaf of $\mathrm{Splcpx}_{X/S}$ in the étale topology. Our main theorem is the following:

Theorem 0.2. $\mathrm{Splcpx}^{\text{ét}}_{X/S}$ *is represented by a locally separated algebraic space over* S.

Remark 0.3. $\mathrm{Splcpx}^{\text{ét}}_{X/S}$ contains the moduli space $\mathrm{Spl}^{\text{ét}}_{X/S}$ of simple sheaves as an open subscheme. We can see from the definition that Fourier-Mukai transform $\mathbf{R}\hat{\Phi}$ induces an isomorphism $\mathrm{Splcpx}^{\text{ét}}_{A} \xrightarrow{\sim} \mathrm{Splcpx}^{\text{ét}}_{\hat{A}}$ of moduli spaces. Of course, the other types of Fourier-Mukai transforms also induce moduli correspondences of complexes.

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1. Representability of the equivalence relation in the derived category

The essential part of the following proposition is proved in [4, III, (7.7.5)]. We shall give a proof again.

Proposition 1.1. *Let* T *be a locally noetherian scheme over* S *and* $E^{\bullet}, F^{\bullet}$ *be bounded complexes of coherent* $\mathcal{O}_{X_T}$*-modules flat over* T*. Let* i_0 *be an integer. Assume that for any* $t \in T$, $\mathrm{Ext}^{i_0-1}_{X_t}(E^{\bullet}(t), F^{\bullet}(t)) = 0$*. Then there is a coherent* $\mathcal{O}_T$*-module* $\mathcal{H}$ *such that* $\mathrm{Ext}^{i_0}_{X_{T'}/T'}(E^{\bullet}_{T'}, F^{\bullet}_{T'} \otimes \mathcal{M}) \cong \mathcal{H}om(\mathcal{H}_{T'}, \mathcal{M})$ *for any* $T' \to T$ *and any coherent sheaf* $\mathcal{M}$ *on* T'.

Proof. Let l, l' be integers such that $E^i = 0$ for $i > l$ and $F^j = 0$ for $j < l'$. Take any affine open set U of T. Take an integer m_l such that for any $t \in U$, $E^l(m_l)(t)$ is globally generated and $H^i(E^l(m_l)(t)) = H^i(F^j(m_l)(t)) = 0$ for any $i > 0$ and $j \geq l'$. We put $K^l := E^l_U$ and

$$K^{l-1} := \ker\left(E^{l-1}_U \oplus H^0(E^l(m_l)_U) \otimes \mathcal{O}_{X_U}(-m_l) \longrightarrow E^l_U\right).$$

Then there is a quasi-isomorphism of complexes

$$\begin{array}{ccccccc} E_U^{l'} & \longrightarrow \cdots \longrightarrow & E_U^{l-2} & \longrightarrow & K^{l-1} & \longrightarrow & H^0(E^l(m_l)_U)\otimes\mathcal{O}_{X_U}(-m_l) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ E_U^{l'} & \longrightarrow \cdots \longrightarrow & E_U^{l-2} & \longrightarrow & E_U^{l-1} & \longrightarrow & E_U^l. \end{array}$$

If K^i, m_{i+1} are defined for $i \geq p$, then take an integer m_p such that for any $t \in U$, $K^p(m_p)(t)$ is generated by global sections and $H^i(K^p(m_p)(t)) = H^i(F^j(m_p)(t)) = 0$ for $i > 0$ and $j \geq l'$. We put

$$K^{p-1} := \ker\left(E_U^{p-1} \oplus H^0(K^p(m_p)) \otimes \mathcal{O}_{X_U}(-m_p) \longrightarrow K^p\right).$$

By descending induction, K^i, m_i are defined for all $i \leq l$. Then consider the complex $V^{\cdot} = (V^i, d^i_{V^{\cdot}})$ defined by

$$V^i = \begin{cases} 0 & \text{if} \quad i > l \\ H^0(K^i(m_i)) \otimes \mathcal{O}_{X_U}(-m_i) & \text{if} \quad i \leq l, \end{cases}$$

$$d^i_{V^{\cdot}} : H^0(K^i(m_i)) \otimes \mathcal{O}_{X_U}(-m_i) \to K^i \to H^0(K^{i+1}(m_{i+1})) \otimes \mathcal{O}_{X_U}(-m_{i+1}).$$

Then there is a quasi-isomorphism $V^{\cdot} \to E_U^{\cdot}$.

Consider the canonical homomorphism

$$(f_{T'})_* \mathcal{H}om^{\cdot}(V^{\cdot}_{T'}, F^{\cdot}_{T'} \otimes \mathcal{M}) \longrightarrow \mathbf{R}(f_{T'})_* \mathbf{R}\mathcal{H}om^{\cdot}(E^{\cdot}_{T'}, F^{\cdot}_{T'} \otimes \mathcal{M}). \tag{1.1}$$

for any $T' \to U$ and any coherent sheaf $\mathcal{M}$ on T'. Here $\mathcal{H}om^{\cdot}(V^{\cdot}_{T'}, F^{\cdot}_{T'} \otimes \mathcal{M})$ is the complex of $\mathcal{O}_{X_{T'}}$-modules defined as in [[5], II, Section 3]. Note that there is a canonical spectral sequence

$$H^p(R^q(f_{T'})_* \mathcal{H}om^{\cdot}(V^{\cdot}_{T'}, F^{\cdot}_{T'} \otimes \mathcal{M})) \Rightarrow \operatorname{Ext}^{p+q}_{X_{T'}/T'}(E^{\cdot}_{T'}, F^{\cdot}_{T'} \otimes \mathcal{M}).$$

Here $R^q(f_{T'})_* \mathcal{H}om^{\cdot}(V^{\cdot}_{T'}, F^{\cdot}_{T'} \otimes \mathcal{M})$ is the complex of $\mathcal{O}_{T'}$-modules whose i-th component is $R^q(f_{T'})_* \mathcal{H}om^i(V^{\cdot}_{T'}, F^{\cdot}_{T'} \otimes \mathcal{M})$ and the left hand side is the p-th cohomology of this complex. Since $H^i(X_t, \mathcal{H}om(V^k(t), F^j(t))) = 0$ for $i > 0$ and for any k, j and $t \in T'$, we have $R^i(f_{T'})_* \mathcal{H}om(V^k_{T'}, F^j_{T'} \otimes \mathcal{M}) = 0$ for $i > 0$. Therefore we have $H^p(R^q(f_{T'})_* \mathcal{H}om^{\cdot}(V^{\cdot}_{T'}, F^{\cdot}_{T'} \otimes \mathcal{M})) = 0$ for any $q > 0$. Hence we have an isomorphism

$$H^p((f_{T'})_* \mathcal{H}om^{\cdot}(V^{\cdot}_{T'}, F^{\cdot}_{T'} \otimes \mathcal{M})) \xrightarrow{\sim} \operatorname{Ext}^p_{X_{T'}/T'}(E^{\cdot}_{T'}, F^{\cdot}_{T'} \otimes \mathcal{M})$$

for each p, which means that the homomorphism (1.1) is a quasi-isomorphism. Note that each $f_{U*}\mathcal{H}om^i(V^{\cdot}, F^{\cdot}_U)$ is a vector bundle and there is an isomorphism $f_{U*}\mathcal{H}om^i(V^{\cdot}, F^{\cdot}_U) \otimes \mathcal{M} \xrightarrow{\sim} f_{T'*}\mathcal{H}om^i(V^{\cdot}_{T'}, F^{\cdot}_{T'} \otimes \mathcal{M})$ for any $T' \to U$ and any coherent sheaf $\mathcal{M}$ on T'. Let us consider the complex

$$f_{U*}\mathcal{H}om^{i_0-2}(V^{\cdot}, F^{\cdot}_U) \xrightarrow{d^{i_0-2}} f_{U*}\mathcal{H}om^{i_0-1}(V^{\cdot}, F^{\cdot}_U) \xrightarrow{d^{i_0-1}} f_{U*}\mathcal{H}om^{i_0}(V^{\cdot}, F^{\cdot}_U).$$

The homomorphism $\operatorname{coker} d^{i_0-2}\otimes k(t)\to f_{U*}\mathcal{H}om^{i_0}(V^\cdot,F^\cdot_U)\otimes k(t)$ induced by d^{i_0-1} is injective, because $\operatorname{Ext}^{i_0-1}_{X_t}(V^\cdot(t),F^\cdot(t))=0$ for any $t\in U$. So we see that $\operatorname{im} d^{i_0-1}$ is a subbundle of $f_{U*}\mathcal{H}om^{i_0}(V^\cdot,F^\cdot_U)$. If we put

$$\mathcal{H}^{(U)} := \operatorname{coker}\left((f_{U*}\mathcal{H}om^{i_0+1}(V^\cdot,F^\cdot_U))^\vee \to (f_{U*}\mathcal{H}om^{i_0}(V^\cdot,F^\cdot_U)/\operatorname{im} d^{i_0-1})^\vee\right),$$

we have a functorial isomorphism $\operatorname{Ext}^{i_0}_{X_{T'}/T'}(E^\cdot_{T'},F^\cdot_{T'}\otimes\mathcal{M})\cong\mathcal{H}om(\mathcal{H}^{(U)}_{T'},\mathcal{M})$ for any $T'\to U$ and any coherent sheaf $\mathcal{M}$ on T'.

By the universality of $\mathcal{H}^{(U)}$, we can glue $\{\mathcal{H}^{(U)}\}$ to obtain a coherent sheaf $\mathcal{H}$ on T. Then there is a functorial isomorphism $\operatorname{Ext}^{i_0}_{X_{T'}/T'}(E^\cdot_{T'},F^\cdot_{T'}\otimes\mathcal{M})\cong\mathcal{H}om(\mathcal{H}_{T'},\mathcal{M})$ for any $T'\to T$ and any coherent sheaf $\mathcal{M}$ on T'. □

Proposition 1.2. *Let Y be a locally noetherian scheme over S. Assume that morphisms of functors $\varphi_i : h_Y\to \mathrm{Splcpx}^{\text{ét}}_{X/S}$ are given for $i=1,2$. Consider the subfunctor R of h_Y defined by $R(T):=\{x\in Y(T)\,|\varphi_1x=\varphi_2x\}$. Then R is represented by a subscheme of Y.*

Proof. It suffices to show that $h_U\cap R$ is represented by a subscheme of U for any noetherian open subscheme U of Y. So we may assume that Y is noetherian. By an easy descent argument, we may assume that φ_i factors through $\mathrm{Splcpx}_{X/S}$. So φ_i is represented by a member $[E^\cdot_i]\in\mathrm{Splcpx}_{X/S}(Y)$ for each $i=1,2$.

If we put

$$\tilde{T} := \left\{t\in Y\,\middle|\operatorname{Ext}^{-1}_{X_t}(E^\cdot_1(t),E^\cdot_2(t))=0\right\},$$

then $\tilde{T}$ is an open subscheme of Y. By Proposition 1.1, there exists a coherent sheaf $\mathcal{H}$ on $\tilde{T}$ such that $\operatorname{Ext}^0_{X_{T'}/T'}((E^\cdot_1)_{T'},(E^\cdot_2)_{T'}\otimes\mathcal{M})\cong\mathcal{H}om(\mathcal{H}_{T'},\mathcal{M})$ for any $T'\to\tilde{T}$ and any coherent sheaf $\mathcal{M}$ on T'. Put

$$U := \left\{t\in\tilde{T}\,\middle|\,\operatorname{rank}(\mathcal{H}\otimes k(t))\le 1\right\}.$$

Then U is an open subscheme of $\tilde{T}$ and $\mathbf{P}(\mathcal{H}|_U)$ is a closed subscheme of U. Let us consider the universal element

$$\tilde{u}\in H^0(\operatorname{Ext}^0_{X_{\mathbf{P}(\mathcal{H}|_U)}/\mathbf{P}(\mathcal{H}|_U)}((E^\cdot_1)_{\mathbf{P}(\mathcal{H}|_U)},(E^\cdot_2)_{\mathbf{P}(\mathcal{H}|_U)}\otimes\mathcal{O}_{\mathbf{P}(\mathcal{H})}(1)))$$

which corresponds to the canonical surjection $\mathcal{H}\otimes\mathcal{O}_{\mathbf{P}(\mathcal{H}|_U)}\to\mathcal{O}_{\mathbf{P}(\mathcal{H})}(1)_U$. Put

$$Z := \left\{t\in\mathbf{P}(\mathcal{H}|_U)\,\middle|\,\tilde{u}(t)\in\operatorname{Hom}_{D(X_t)}(E^\cdot_1(t),E^\cdot_2(t))\text{ is a quasi-isomorphism}\right\}.$$

We will show that Z is an open subscheme of $\mathbf{P}(\mathcal{H}|_U)$. It is sufficient to show that $Z\cap W$ is open in W for any affine open set W of $\mathbf{P}(\mathcal{H}|_U)$. As in the proof of Proposition 1.1, there exist a complex $V^\cdot$ of locally free $\mathcal{O}_{X_W}$-modules and a quasi-isomorphism $V^\cdot\to(E^\cdot_1)_W$ such that $\tilde{u}_W$ is represented by a homomorphism $v:V^\cdot\to(E^\cdot_2)_W\otimes\mathcal{O}_{\mathbf{P}(\mathcal{H})}(1)$. Consider the mapping cone $U^\cdot:=(E^\cdot_2)_W\otimes\mathcal{O}_{\mathbf{P}(\mathcal{H})}(1)\oplus V^\cdot[1]$ of v. Then for any $t\in W$, $H^i(U^\cdot(t))=0$ except for finitely many i. Thus $Z\cap W=\{t\in W|H^i(U^\cdot(t))=0\text{ for any }i\}$ is an open set of W. By construction we can see that $h_Z=R$. □

Remark 1.3. Let $E^{\cdot}$ be a bounded complex of coherent $\mathcal{O}_{X_T}$-modules flat over T. Then

$$U := \left\{ t \in T \,\middle|\, \mathrm{Ext}^{-1}_{X_t}(E^{\cdot}(t), E^{\cdot}(t)) = 0 \text{ and } \mathrm{Ext}^0_{X_t}(E^{\cdot}(t), E^{\cdot}(t)) \cong k(t) \right\}$$

is an open subset of T.

Proof. $Z := \left\{ t \in T \,\middle|\, \mathrm{Ext}^{-1}_{X_t}(E^{\cdot}(t), E^{\cdot}(t)) = 0 \right\}$ is an open subscheme of T. Take a coherent sheaf $\mathcal{H}$ on Z such that for any $T' \to Z$ and any coherent sheaf $\mathcal{M}$ on T', there is a functorial isomorphism

$$\mathrm{Ext}^0_{X_{T'}/T'}(E^{\cdot}_{T'}, E^{\cdot}_{T'} \otimes \mathcal{M}) \xrightarrow{\sim} \mathcal{H}om(\mathcal{H}_{T'}, \mathcal{M}).$$

Consider the open subscheme

$$Z' := \{ t \in Z \mid \mathrm{rank}(\mathcal{H} \otimes k(t)) \leq 1 \}$$

of Z. Let $\theta : \mathcal{H}|_{Z'} \to \mathcal{O}_{Z'}$ be the homomorphism corresponding to the identity $1 \in \mathrm{Hom}_{D(X_{Z'})}(E^{\cdot}_{Z'}, E^{\cdot}_{Z'})$. Then

$$\tilde{Z} := \{ t \in Z' \,|\, \theta \otimes k(t) : \mathcal{H} \otimes k(t) \to k(t) \text{ is an isomorphism} \}$$

is an open subscheme of Z' and we have $\tilde{Z} = U$. □

2. Deformation of complexes

Next we consider deformations of complexes. First we prove a lemma which is often needed in the sequel.

Lemma 2.1. *Let A be a noetherian ring over S. Let $V^{\cdot}$, $E^{\cdot}$ and $W^{\cdot} = (W_i \otimes \mathcal{O}_{X_A}(-m_i), d^i_{W^{\cdot}})$ be complexes of coherent $\mathcal{O}_{X_A}$-modules bounded above, where W_i are free A-modules of finite rank.*

(i) *Assume that $E^{\cdot}$ is bounded and that the condition*

(∗) *$H^p(X_A, E^j(m_i)) = 0$ for any $p > 0$ and any i, j*

is satisfied. Then the canonical homomorphisms

$$H^p(\mathrm{Hom}^{\cdot}(W^{\cdot}, E^{\cdot})) \to \mathrm{Ext}^p_{X_A}(W^{\cdot}, E^{\cdot})$$

are bijective for any p. Here $\mathrm{Hom}^{\cdot}(W^{\cdot}, E^{\cdot})$ is the complex of A-modules defined as in [[5], I, Section 6] *and the left hand side is the p-th cohomology of this complex.*

(ii) *Assume that a quasi-isomorphism $\varphi^{\cdot} : V^{\cdot} \to E^{\cdot}$ is given. Consider the mapping cone $U^{\cdot} := E^{\cdot} \oplus V^{\cdot}[1]$ of $\varphi^{\cdot}$ and put $F^i := \ker(U^i \to U^{i+1})$. Let n be an integer. Assume that $(V^{\cdot} \xrightarrow{\varphi^{\cdot}} E^{\cdot}, W^{\cdot})$ satisfies the following condition:*

(L_n) *$H^0(X_A, U^{j-1}(m_{j-n})) \to H^0(X_A, F^j(m_{j-n}))$ is surjective for any j and $H^p(X_A, F^j(m_i)) = 0$ for $i \leq j - n$ and $p > 0$.*

Then the canonical homomorphisms

$$H^p(\mathrm{Hom}^{\cdot}(W^{\cdot},V^{\cdot})) \to H^p(\mathrm{Hom}^{\cdot}(W^{\cdot},E^{\cdot}))$$

are surjective for $p \geq n$ and bijective for $p > n$.

Proof. (i) Note that there is a spectral sequence

$$H^p(H^q(X_A,\mathcal{H}om^{\cdot}(W^{\cdot},E^{\cdot}))) \Rightarrow \mathrm{Ext}^{p+q}_{X_A}(W^{\cdot},E^{\cdot}),$$

where $H^q(X_A,\mathcal{H}om^{\cdot}(W^{\cdot},E^{\cdot}))$ is the complex of A-modules whose i-th component is $H^q(X_A,\mathcal{H}om^i(W^{\cdot},E^{\cdot}))$ and the left hand side is the p-th cohomology of this complex. Since the condition $(*)$ is satisfied, $H^q(X_A,\mathcal{H}om^i(W^{\cdot},E^{\cdot}))=0$ for any $q>0$ and any i. Thus we have an isomorphism $H^p(\mathrm{Hom}^{\cdot}(W^{\cdot},E^{\cdot})) \xrightarrow{\sim} \mathrm{Ext}^p_{X_A}(W^{\cdot},E^{\cdot})$.

(ii) Since $\varphi^{\cdot}: V^{\cdot}\to E^{\cdot}$ is quasi-isomorphic, the sequence

$$0 \longrightarrow F^{j-1} \longrightarrow U^{j-1} \longrightarrow F^j \longrightarrow 0$$

is exact. Thus the condition (L_n) implies that the homomorphisms

$$\mathrm{Hom}(W^i,U^{j-1}) \longrightarrow \mathrm{Hom}(W^i,F^j) \tag{2.1}$$

are surjective for $i \leq j-n$. Take an integer $p\geq n$ and consider the complex

$$\mathrm{Hom}^{p-1}(W^{\cdot},U^{\cdot}) \xrightarrow{d^{p-1}} \mathrm{Hom}^p(W^{\cdot},U^{\cdot}) \xrightarrow{d^p} \mathrm{Hom}^{p+1}(W^{\cdot},U^{\cdot}).$$

Take $\{(f^i,g^i)\}\in \mathrm{Hom}^p(W^{\cdot},U^{\cdot})$ such that $d^p\{(f^i,g^i)\}=0$ in $\mathrm{Hom}^{p+1}(W^{\cdot},U^{\cdot})$, where $f^i\in \mathrm{Hom}(W^i,E^{i+p})$ and $g^i\in\mathrm{Hom}(W^i,V^{i+p+1})$. One can check that $d^p\{(f^i,g^i)\}$ is given by

$$\{(f^{i+1}d^i+(-1)^{p+1}d^{i+p}f^i+(-1)^{p+1}\varphi^{i+p+1}g^i, g^{i+1}d^i+(-1)^p d^{i+p+1}g^i)\}.$$

Since $E^{\cdot}$ and $V^{\cdot}$ are bounded above, there is an integer l such that $f^i=0$, $g^i=0$ for all $i>l$. Then we have $(d^{l+p}f^l+\varphi^{l+p+1}g^l, -d^{l+p+1}g^l)=(0,0)$, which means that $(f^l,g^l)\in\mathrm{Hom}(W^l,F^{l+p})$. By the surjectivity of (2.1), there exists $(a^l,b^l)\in \mathrm{Hom}(W^l,E^{l+p-1}\oplus V^{l+p})$ such that $(d^{l+p-1}a^l+\varphi^{l+p}b^l,-d^{l+p}b^l)=(f^l,g^l)$. Put

$$\{(f^i_{l-1},g^i_{l-1})\} := \{(f^i,g^i)\}-(-1)^p d^{p-1}(a^l,b^l),$$

where (a^l,b^l) is regarded as an element of $\mathrm{Hom}^{p-1}(W^{\cdot},U^{\cdot})$ by the canonical inclusion $\mathrm{Hom}(W^l,U^{l+p-1})\hookrightarrow \mathrm{Hom}^{p-1}(W^{\cdot},U^{\cdot})$. Then $f^i_{l-1}=0$, $g^i_{l-1}=0$ for $i>l-1$. By descending induction, we define $\{(f^i_j,g^i_j)\}\in\mathrm{Hom}^p(W^{\cdot},F^{\cdot})$ such that $d^p\{(f^i_j,g^i_j)\}=0$ and $f^i_j=0$, $g^i_j=0$ for $i>j$ as follows. If $\{(f^i_j,g^i_j)\}$ is defined, then $(d^{j+p}f^j_j+\varphi^{j+p+1}g^j_j, -d^{j+p+1}g^j_j)=(0,0)$, and so $(f^j_j,g^j_j)\in\mathrm{Hom}(W^j,F^{j+p})$. By the surjectivity of (2.1), there exists $(a^j,b^j)\in$

$\mathrm{Hom}(W^j, E^{j+p-1} \oplus V^{j+p})$ such that $(d^{j+p-1}a^j + \varphi^{j+p}b^j, -d^{j+p}b^j) = (f^j_j, g^j_j)$. Then we put

$$\{(f^i_{j-1}, g^i_{j-1})\} := \{(f^i_j, g^i_j)\} - (-1)^p d^{p-1}(a^j, b^j).$$

Then $\{(a^i, b^i)\}$ defines an element of $\mathrm{Hom}^{p-1}(W^{\cdot}, U^{\cdot})$ and we can see that $(-1)^p d^{p-1}\{(a^i, b^i)\} = \{(f^i, g^i)\}$. Thus we have

$$H^p(\mathrm{Hom}^{\cdot}(W^{\cdot}, U^{\cdot})) = 0 \tag{2.2}$$

for $p \geq n$. On the other hand, there is an isomorphism $\theta : \mathrm{Hom}^{\cdot}(W^{\cdot}, U^{\cdot}) \xrightarrow{\sim} \mathrm{Hom}^{\cdot}(W^{\cdot}, E^{\cdot}) \oplus \mathrm{Hom}^{\cdot}(W^{\cdot}, V^{\cdot})[1]$, where $\mathrm{Hom}^{\cdot}(W^{\cdot}, E^{\cdot}) \oplus \mathrm{Hom}^{\cdot}(W^{\cdot}, V^{\cdot})[1]$ is the mapping cone of the homomorphism $\mathrm{Hom}^{\cdot}(W^{\cdot}, V^{\cdot}) \to \mathrm{Hom}^{\cdot}(W^{\cdot}, E^{\cdot})$ induced by $\varphi^{\cdot}$. Thus we obtain a long exact sequence

$$\begin{aligned} \cdots \longrightarrow H^{p-1}(\mathrm{Hom}^{\cdot}(W^{\cdot}, U^{\cdot})) \longrightarrow H^p(\mathrm{Hom}^{\cdot}(W^{\cdot}, V^{\cdot})) \longrightarrow \\ H^p(\mathrm{Hom}^{\cdot}(W^{\cdot}, E^{\cdot})) \longrightarrow H^p(\mathrm{Hom}^{\cdot}(W^{\cdot}, U^{\cdot})) \longrightarrow \cdots . \end{aligned} \tag{2.3}$$

and by (2.2) we conclude the assertion. □

Remark 2.2. Assume that $[E^{\cdot}] \in \mathrm{Splcpx}_{X/S}(A)$ is given where A is a noetherian ring over S. Then there are a complex $V^{\cdot}$ of the form $V^i = V_i \otimes \mathcal{O}_{X_A}(-m_i)$ where V_i are free A-modules, and a quasi-isomorphism $\varphi^{\cdot} : V^{\cdot} \to E^{\cdot}$ such that $(V^{\cdot} \to E^{\cdot}, V^{\cdot})$ satisfies the conditions (L_0) and ($*$) of Lemma 2.1.

Proof. Let l be an integer such that $E^i = 0$ for $i > l$. As in the proof of Proposition 1.1, we take a sufficiently large integer m_l and put $V^l := H^0(E^l(m_l)) \otimes \mathcal{O}_{X_A}(-m_l)$, $F^{l-1} := \ker(V^l \oplus E^{l-1} \to E^l)$. Inductively we put $V^i := H^0(F^i(m_i)) \otimes \mathcal{O}_{X_A}(-m_i)$ and $F^{i-1} := \ker(V^i \oplus E^{i-1} \to F^i)$, where m_i is a sufficiently large integer. Here we may assume that $H^p(X_A, E^j(m_i)) = 0$ for any i, j and any $p > 0$ and that $H^p(X_A, F^j(m_i)) = 0$ for $i \leq j$ and $p > 0$. There is a quasi-isomorphism $\varphi^{\cdot} : V^{\cdot} \to E^{\cdot}$. If we consider the mapping cone $U^{\cdot} := E^{\cdot} \oplus V^{\cdot}[1]$, then we have $F^i = \ker(U^i \to U^{i+1})$. Thus the conditions ($\mathrm{L}_0$) and ($*$) are satisfied. □

Proposition 2.3. *Let A be an artinian local ring over S with residue field $A/m = k$ and I an ideal of A with $mI = 0$. Let $r : \mathrm{Splcpx}_{X/S}(A) \to \mathrm{Splcpx}_{X/S}(A/I)$ be the canonical map. Assume that $[E^{\cdot}] \in \mathrm{Splcpx}_{X/S}(A/I)$ is given and put $E^{\cdot}_0 := E^{\cdot} \otimes k$.*

(i) *There is an element $\omega(E^{\cdot}) \in \mathrm{Ext}^2(E^{\cdot}_0, E^{\cdot}_0) \otimes_k I$ such that $\omega(E^{\cdot}) = 0$ if and only if $[E^{\cdot}]$ can be lifted to an A-valued point of $\mathrm{Splcpx}_{X/S}$.*

(ii) *If there is a lift $[\tilde{E}^{\cdot}]$ of $[E^{\cdot}]$ to $\mathrm{Splcpx}_{X/S}(A)$, then there is a bijection $r^{-1}([E^{\cdot}]) \cong \mathrm{Ext}^1(E^{\cdot}_0, E^{\cdot}_0) \otimes_k I$.*

Proof. (i) Let $l' < l$ be integers such that $E^i = 0$ if $i < l'$ or $i > l$. By Remark 2.2, we can construct a complex $V^\cdot = (V_i \otimes \mathcal{O}_{X_{A/I}}(-m_i), d^i_{V^\cdot})$ and a quasi-isomorphism $V^\cdot \to E^\cdot$ such that $(V^\cdot \otimes k \to E^\cdot \otimes k, V^\cdot)$ satisfies the conditions (L_0) and ($*$) of Lemma 2.1, where V_i are free A-modules of finite rank and $V^i = 0$ for $i > l$. We may also assume that $H^i(\mathcal{O}_X(m_p - m_{p+1}) \otimes k) = 0$ for any p and any $i > 0$. By Lemma 2.1, there are isomorphisms

$$H^i(\mathrm{Hom}^\cdot(V^\cdot, I \otimes V^\cdot)) \xrightarrow{\sim} H^i(\mathrm{Hom}^\cdot(V^\cdot, I \otimes E^\cdot)) \xrightarrow{\sim} \mathrm{Ext}^i(E^\cdot_0, I \otimes E^\cdot_0)$$

for $i \geq 1$.

Let $\tilde{d}^i_{V^\cdot} : V_i \otimes \mathcal{O}_{X_A}(-m_i) \to V_{i+1} \otimes \mathcal{O}_{X_A}(-m_{i+1})$ be a lift of the homomorphism $d^i_{V^\cdot} : V_i \otimes \mathcal{O}_{X_{A/I}}(-m_i) \to V_{i+1} \otimes \mathcal{O}_{X_{A/I}}(-m_{i+1})$. Put $\delta^i := \tilde{d}^{i+1}_{V^\cdot} \circ \tilde{d}^i_{V^\cdot}$. Then the image of δ^i is contained in $I \otimes V_{i+2} \otimes \mathcal{O}_{X \times A}(-m_{i+2})$. we have $\tilde{d}^{i+2}_{V^\cdot} \circ \delta^i - \delta^{i+1} \circ \tilde{d}^i_{V^\cdot} = 0$ by definition. Thus $\{\delta^i\}$ defines an element $\omega(E^\cdot) \in H^2(\mathrm{Hom}^\cdot(V^\cdot, I \otimes V^\cdot)) = \mathrm{Ext}^2(E^\cdot_0, I \otimes_k E^\cdot_0)$.

We will show that $\omega(E^\cdot)$ is independent of the choice of the representative $E^\cdot$ of $[E^\cdot]$, $V^\cdot$ and $\tilde{d}^i_{V^\cdot}$. Let $Q^\cdot$ be a bounded complex of coherent $\mathcal{O}_{X_{A/I}}$-modules flat over A/I such that $[Q^\cdot] = [E^\cdot]$ in $\mathrm{Splcpx}_{X/S}(A/I)$. Take a complex $W^\cdot$ of the form $W^i = W_i \otimes \mathcal{O}_{X_{A/I}}(-m'_i)$ and a quasi-isomorphism $W^\cdot \to Q^\cdot$ such that $(W^\cdot \otimes k \to Q^\cdot \otimes k, W^\cdot)$ satisfies the conditions (L_0) and ($*$). Take a lift $\tilde{d}^i_{W^\cdot}$ of the derivation $d^i_{W^\cdot}$. Since $V^\cdot \cong W^\cdot$ in $D(X_{A/I})$, there are a complex $U^\cdot$ of coherent $\mathcal{O}_{X_{A/I}}$-modules and quasi-isomorphisms $U^\cdot \to V^\cdot$, $U^\cdot \to W^\cdot$. We may assume that $U^\cdot$ is of the form $U^i = U_i \otimes \mathcal{O}_{X_{A/I}}(-n_i)$ such that $(U^\cdot \otimes k \to V^\cdot \otimes k, U^\cdot)$ and $(U^\cdot \otimes k \to W^\cdot \otimes k, U^\cdot)$ satisfy the condition (L_0). Take a lift $\tilde{d}^i_{U^\cdot}$ of $d^i_{U^\cdot}$. Then we can check that

$$[\{\tilde{d}^{i+1}_{U^\cdot} \circ \tilde{d}^i_{U^\cdot}\}] = [\{\tilde{d}^{i+1}_{V^\cdot} \circ \tilde{d}^i_{V^\cdot}\}]$$

in $H^2(\mathrm{Hom}^\cdot(U^\cdot, U^\cdot \otimes I)) = H^2(\mathrm{Hom}^\cdot(V^\cdot, V^\cdot \otimes I)) = \mathrm{Ext}^2(E^\cdot_0, E^\cdot_0 \otimes I)$. Similarly we have

$$[\{\tilde{d}^{i+1}_{U^\cdot} \circ \tilde{d}^i_{U^\cdot}\}] = [\{\tilde{d}^{i+1}_{W^\cdot} \circ \tilde{d}^i_{W^\cdot}\}]$$

in $H^2(\mathrm{Hom}^\cdot(U^\cdot, U^\cdot \otimes I)) = H^2(\mathrm{Hom}^\cdot(W^\cdot, W^\cdot \otimes I)) = \mathrm{Ext}^2(E^\cdot_0, E^\cdot_0 \otimes I)$. Hence we have $\omega(E^\cdot) = \omega(Q^\cdot)$.

Assume that $\omega(E^\cdot) = 0$. Then there exists $\alpha^i \in \mathrm{Hom}(V^i, I \otimes V^{i+1})$ such that $\delta^i = \alpha^{i+1} d^i + d^{i+1} \alpha^i$. If we put

$$\bar{d}^i := \tilde{d}^i_{V^\cdot} - \alpha^i : V_i \otimes \mathcal{O}_{X_A}(-m_i) \to V_{i+1} \otimes \mathcal{O}_{X_A}(-m_{i+1}),$$

then $\bar{d}^{i+1}\bar{d}^i = \delta^i - \tilde{d}^{i+1}_{V^\cdot} \circ \alpha^i - \alpha^{i+1} \circ \tilde{d}^i_{V^\cdot} = 0$. Hence $\tilde{V}^\cdot := (V_i \otimes \mathcal{O}_{X/A}(-m_i), \bar{d}^i)$ is a complex on X_A which is a lift of $V^\cdot$. Consider the complex

$$\sigma_{\geq l'}(\tilde{V}^\cdot) : \cdots \longrightarrow 0 \longrightarrow \mathrm{coker}\, \bar{d}^{l'-1} \longrightarrow \tilde{V}^{l'+1} \longrightarrow \cdots \longrightarrow \tilde{V}^l \longrightarrow 0 \longrightarrow \cdots .$$

Then there is a canonical quasi-isomorphism $\tilde{V}^{\cdot} \to \sigma_{\geq l'}(\tilde{V}^{\cdot})$ and $\operatorname{coker} \overline{d}^{l'-1}$ is flat over A [4, IV, Proposition 11.3.7]. Thus $[\sigma_{\geq l'}(\tilde{V}^{\cdot})]$ is a lift of $[E^{\cdot}]$ to an A-valued point of $\mathrm{Splcpx}_{X/S}$.

Assume that there is a lift $[Q^{\cdot}] \in \mathrm{Splcpx}_{X/S}(A)$ of $[E^{\cdot}]$. Take a complex $W^{\cdot}$ of the form $W^i = W_i \otimes \mathcal{O}_{X_A}(-n_i)$ and a quasi-isomorphism $W^{\cdot} \to Q^{\cdot}$ such that $(W^{\cdot} \otimes k \to Q^{\cdot} \otimes k, W^{\cdot})$ satisfies the conditions (L_0) and $(*)$. Then $\omega(E^{\cdot}) = \omega(Q^{\cdot})$ is defined by $\{(d^{i+1}_{W^{\cdot}} \circ d^i_{W^{\cdot}})\}$, which is obviously zero.

(ii) Take a complex $V^{\cdot} = (V_i \otimes \mathcal{O}_{X_A}(-m_i), d^i_{V^{\cdot}})$ with $V_i = 0$ for $i > l$ and a quasi-isomorphism $V^{\cdot} \to \tilde{E}^{\cdot}$ such that $(I \otimes V^{\cdot} \to I \otimes \tilde{E}^{\cdot}, V^{\cdot})$ satisfies the conditions (L_0) and $(*)$. Assume that $v \in \mathrm{Ext}^1(E^{\cdot}_0, I \otimes_k E^{\cdot}_0)$ is given. Since $\mathrm{Ext}^1(E^{\cdot}_0, I \otimes_k E^{\cdot}_0) \cong H^1(\mathrm{Hom}^{\cdot}(V^{\cdot}, I \otimes V^{\cdot}))$, v can be considered as an element $[\{v^i\}]$ of $H^1(\mathrm{Hom}^{\cdot}(V^{\cdot}, I \otimes V^{\cdot}))$. If we put $V^{\cdot}_v := (V_i \otimes \mathcal{O}_{X_A}(-m_i), d^i_{V^{\cdot}} + v^i)$, then $V^{\cdot}_v$ is a complex on X_A, which is a lift of $V^{\cdot} \otimes A/I$. Let us consider the complex

$$\sigma_{\geq l'}(V^{\cdot}_v) : \cdots \longrightarrow 0 \longrightarrow \operatorname{coker} d^{l'-1}_{V^{\cdot}_v} \longrightarrow V^{l'+1}_v \longrightarrow \cdots \longrightarrow V^l_v \longrightarrow 0 \to \cdots .$$

Then there is a canonical quasi-isomorphism $V^{\cdot}_v \to \sigma_{\geq l'}(V^{\cdot}_v)$ and $\operatorname{coker} d^{l'-1}_{V^{\cdot}_v}$ is flat over A. Thus $[\sigma_{\geq l'}(V^{\cdot}_v)]$ is a member of $r^{-1}([E^{\cdot}])$. It can be checked that $[\sigma_{\geq l'}(V^{\cdot}_v)] \in r^{-1}([E^{\cdot}])$ is independent of the choice of $V^{\cdot}$ and the representative $\{v^i\}$ of v. Thus we can define a map

$$\sigma : \mathrm{Ext}^1(E^{\cdot}_0, I \otimes_k E^{\cdot}_0) \longrightarrow r^{-1}([E^{\cdot}]); \quad v \mapsto [\sigma_{\geq l'}(V^{\cdot}_v)].$$

Conversely assume that an element $[Q^{\cdot}] \in r^{-1}([E^{\cdot}])$ is given. We may assume that there is a quasi-isomorphism $Q^{\cdot} \otimes A/I \to \tilde{E}^{\cdot} \otimes A/I$. Then there is a complex $W^{\cdot} = (W_i \otimes \mathcal{O}_{X_{A/I}}(-n_i), d^i_{W^{\cdot}})$ and a quasi-commutative diagram of quasi-isomorphisms

$$\begin{array}{ccc} W^{\cdot} & \longrightarrow & V^{\cdot} \otimes A/I \\ \downarrow & & \downarrow \\ Q^{\cdot} \otimes A/I & \longrightarrow & \tilde{E}^{\cdot} \otimes A/I \end{array}$$

such that both $(W^{\cdot} \otimes k \to Q^{\cdot} \otimes k, W^{\cdot})$ and $(W^{\cdot} \otimes k \to V^{\cdot} \otimes k, W^{\cdot})$ satisfy the condition (L_0). Then there is a complex $\tilde{W}^{\cdot} = (W_i \otimes \mathcal{O}_{X_A}(-n_i), d^i_{\tilde{W}^{\cdot}})$ and a quasi-isomorphism $\tilde{W}^{\cdot} \to V^{\cdot}$ which is a lift of the given quasi-isomorphism $W^{\cdot} \to V^{\cdot} \otimes A/I$. Similarly there is a complex $\tilde{W}^{\cdot}_{Q^{\cdot}} = (W_i \otimes \mathcal{O}_{X_A}(-n_i), d^i_{\tilde{W}^{\cdot}_{Q^{\cdot}}})$ and a lift $\tilde{W}^{\cdot}_{Q^{\cdot}} \to Q^{\cdot}$ of the given quasi-isomorphism $W^{\cdot} \to Q^{\cdot} \otimes A/I$. If we put

$$v^i := d^i_{\tilde{W}^{\cdot}_{Q^{\cdot}}} - d^i_{\tilde{W}^{\cdot}} : W_i \otimes \mathcal{O}_{X_A}(-n_i) \to W_{i+1} \otimes \mathcal{O}_{X_A}(-n_{i+1}),$$

then the image of v^i is contained in $I \otimes W_{i+1} \otimes \mathcal{O}_{X \otimes k}(-n_{i+1})$ and $d^{i+1}_{\tilde{W}^{\cdot}_{Q^{\cdot}}} \circ v^i + v^{i+1} d^i_{\tilde{W}^{\cdot}} = 0$. Then $\{v^i\}$ defines an element $v_{Q^{\cdot}}$ of $H^1(\mathrm{Hom}^{\cdot}(W^{\cdot}, I \otimes W^{\cdot})) =$

$\mathrm{Ext}^1(E_0^{\cdot}, I \otimes_k E_0^{\cdot})$. It can be shown that $v_{Q^{\cdot}}$ is independent of the choice of the representative $Q^{\cdot}$ of $[Q^{\cdot}]$, $W^{\cdot}$, $\tilde{W}^{\cdot}_{Q^{\cdot}}$ and $\tilde{W}^{\cdot}$. Then $Q^{\cdot} \mapsto v_{Q^{\cdot}}$ gives the inverse of σ. □

3. Proof of the main theorem

Now we prove the main theorem.

Theorem 0.2. $\mathrm{Splcpx}^{\text{ét}}_{X/S}$ *is represented by a locally separated algebraic space over* S*.*

Proof. Let Y_1, Y_2 be schemes locally of finite type over S. Assume that there are morphisms of functors $\phi_i : h_{Y_i} \to \mathrm{Splcpx}^{\text{ét}}_{X/S}$ for $i = 1, 2$. Then from Proposition 1.2, one sees that the functor $h_{Y_1} \times_{\mathrm{Splcpx}^{\text{ét}}_{X/S}} h_{Y_2}$ is represented by a subscheme of $Y_1 \times_S Y_2$.

Thus it suffices to show that there exist a scheme Z locally of finite type over S and a smooth surjective morphism $h_Z \to \mathrm{Splcpx}^{\text{ét}}_{X/S}$. For this it is sufficient to show that for any geometric point $x \in \mathrm{Splcpx}^{\text{ét}}_{X/S}(K)$, there exist a scheme Z of finite type over S and a smooth morphism $\phi : h_Z \to \mathrm{Splcpx}^{\text{ét}}_{X/S}$ such that x is contained in the image of $\phi(K)$. Take any geometric point $x \in \mathrm{Splcpx}^{\text{ét}}_{X/S}(K)$. Then x is represented by a complex $E^{\cdot} \in \mathrm{Splcpx}_{X/S}(K)$. There exist integers $l' < l$ such that $E^i = 0$ if $i < l'$ or $i > l$. Then there exist a complex $V^{\cdot} = (V_i \otimes \mathcal{O}_{X_K}(-m_i), d^i_{V^{\cdot}})$ with $V_i = 0$ for $i > l$ and a quasi-isomorphism $V^{\cdot} \to E^{\cdot}$ such that $(V^{\cdot} \to E^{\cdot}, V^{\cdot})$ satisfies the conditions (L_1) and $(*)$ of Lemma 2.1, where V_i are free sheaves of finite rank on S. We may assume that $H^i(X_K, \operatorname{coker} d^j_{V^{\cdot}}(m_k)) = 0$ for $i > 0$, $k \le j \le l' - 1$ and $H^i(X_s, \mathcal{O}_{X_s}(m_{k-1} - m_k)) = 0$ for any $i > 0$, $k \le l$ and $s \in S$. Let us consider the two complexes

$$\sigma_{\ge l'}(V^{\cdot}) : \operatorname{coker} d^{l'-1}_{V^{\cdot}} \longrightarrow V^{l'+1} \longrightarrow \cdots \longrightarrow V^l,$$
$$\tau_{\ge l'-2}(V^{\cdot}) : V^{l'-2} \longrightarrow V^{l'-1} \longrightarrow \cdots \longrightarrow V^l.$$

Then there is a canonical composition of quasi-isomorphisms $V^{\cdot} \to \sigma_{\ge l'}(V^{\cdot}) \to E^{\cdot}$. By assumption $(\sigma_{\ge l'}(V^{\cdot}) \to E^{\cdot}, \tau_{\ge l'-2}(V^{\cdot}))$ also satisfies the conditions (L_1) and $(*)$ of Lemma 2.1. Thus the canonical homomorphisms

$$H^i(\mathrm{Hom}^{\cdot}(\tau_{\ge l'-2}(V^{\cdot}), \sigma_{\ge l'}(V^{\cdot}))) \longrightarrow \mathrm{Ext}^i_{X_K}(\tau_{\ge l'-2}(V^{\cdot}), E^{\cdot})$$

are surjective for $i \ge 1$ and bijective for $i > 1$. Consider the scheme

$$P := \prod_{i=l'-2}^{l-1} \mathbf{V}\left((V_i^{\vee} \otimes V_{i+1} \otimes f_*(\mathcal{O}_X(m_i - m_{i+1})))^{\vee} \right)$$

over S. Let

$$\tilde{V}^{\cdot}_{\tau} : V_{l'-2} \otimes \mathcal{O}_{X_P}(-m_{l'-2}) \xrightarrow{\tilde{d}^{l'-2}} V_{l'-1} \otimes \mathcal{O}_{X_P}(-m_{l'-1}) \xrightarrow{\tilde{d}^{l'-1}}$$
$$\cdots \xrightarrow{\tilde{d}^{l-2}} V_{l-1} \otimes \mathcal{O}_{X_P}(-m_{l-1}) \xrightarrow{\tilde{d}^{l-1}} V_l \otimes \mathcal{O}_{X_P}(-m_l)$$

be the universal family. Put

$$\tilde{K}_{l'} := \operatorname{coker}\left(V_{l'-1} \otimes \mathcal{O}_{X_P}(-m_{l'-1}) \stackrel{\tilde{d}^{l'-1}}{\to} V_{l'} \otimes \mathcal{O}_{X_P}(-m_{l'}) \right).$$

Let $\bar{Z}$ be the subscheme of P such that for any $T \to S$,

$$\bar{Z}(T) = \left\{ g \in P(T) \;\middle|\; \begin{array}{l} (1_X \times g)^* \tilde{V}^{\bullet}_{\tau} \text{ is a complex and for any } t \in T, \\ \tilde{V}^{\bullet}_{\tau}(t) \text{ is exact at } \tilde{V}^{l'-1}_{\tau}(t) \end{array} \right\}.$$

One sees from [4, IV, Proposition 11.3.7], that $(\tilde{K}_{l'})_{\bar{Z}}$ is flat over $\bar{Z}$ and the sequence

$$\begin{aligned} \tilde{V}^{\bullet}_{\sigma} : (\tilde{K}_{l'})_{\bar{Z}} &\stackrel{\tilde{\iota}}{\longrightarrow} V_{l'+1} \otimes \mathcal{O}_{X_{\bar{Z}}}(-m_{l'+1}) \stackrel{\tilde{d}^{l'+1}}{\longrightarrow} V_{l'+2} \otimes \mathcal{O}_{X_{\bar{Z}}}(-m_{l'+2}) \stackrel{\tilde{d}^{l'+2}}{\longrightarrow} \\ \cdots &\stackrel{\tilde{d}^{l-2}}{\longrightarrow} V_{l-1} \otimes \mathcal{O}_{X_{\bar{Z}}}(-m_{l-1}) \stackrel{\tilde{d}^{l-1}}{\longrightarrow} V_l \otimes \mathcal{O}_{X_{\bar{Z}}}(-m_l) \end{aligned}$$

becomes a complex. Consider the open subscheme

$$Z := \left\{ t \in \bar{Z} \;\middle|\; \begin{array}{l} \operatorname{Ext}^0((\tilde{V}^{\bullet}_{\sigma})(t), (\tilde{V}^{\bullet}_{\sigma})(t)) \cong k(t), \operatorname{Ext}^{-1}(\tilde{V}^{\bullet}_{\sigma}(t), \tilde{V}^{\bullet}_{\sigma}(t)) = 0, \\ H^i(\operatorname{Hom}^{\bullet}(\tilde{V}^{\bullet}_{\tau}(t), \tilde{V}^{\bullet}_{\sigma}(t))) \to \operatorname{Ext}^i(\tilde{V}^{\bullet}_{\tau}(t), \tilde{V}^{\bullet}_{\sigma}(t)) \\ \text{are surjective for } i \geq 1 \text{ and bijective for } i > 1 \text{ and} \\ H^i(\operatorname{coker} \tilde{d}^j(t)(m_k)) = 0 \text{ for } l'-2 \leq k \leq j \leq l'-1,\ i > 0 \end{array} \right\}$$

of $\bar{Z}$. Then $(\tilde{V}^{\bullet}_{\sigma})_Z$ defines a morphism $\phi : h_Z \to \mathrm{Splcpx}^{\text{ét}}_{X/S}$. By construction x is contained in the image of $\phi(K)$. We only have to show that ϕ is smooth.

We have to show that $Z \times_{\mathrm{Splcpx}^{\text{ét}}_{X/S}} T \to T$ is smooth for any locally noetherian scheme T and any morphism $T \to \mathrm{Splcpx}^{\text{ét}}_{X/S}$. There exists an étale covering $T' \to T$ such that the composite $T' \to T \to \mathrm{Splcpx}^{\text{ét}}_{X/S}$ factors through $\mathrm{Splcpx}_{X/S}$. It suffices to show that $Z \times_{\mathrm{Splcpx}^{\text{ét}}_{X/S}} T' \to T'$ is smooth. However, we have $(Z \times_{\mathrm{Splcpx}^{\text{ét}}_{X/S}} T')(A) = (Z \times_{\mathrm{Splcpx}_{X/S}} T')(A)$ for any artinian ring A. So it suffices to show that $\phi' : Z \to \mathrm{Splcpx}_{X/S}$ is formally smooth.

Let (A, m) be an artinian local ring with residue field k and I be an ideal of A such that $mI = 0$. Assume that a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec}(A/I) & \hookrightarrow & \operatorname{Spec}(A) \\ \varsigma \downarrow & & \eta \downarrow \\ Z & \stackrel{\phi'}{\longrightarrow} & \mathrm{Splcpx}_{X/S} \end{array} \tag{3.1}$$

is given. Take a complex $Q^{\bullet}$ on X_A which represents η. There is a complex $V^{\bullet}_{\varsigma} = (V'_i \otimes \mathcal{O}_{X_{A/I}}(-m'_i), d^i_{V^{\bullet}_{\varsigma}})$ exact at V^i_{ς} for $i \leq l'-1$ such that $V'_i = V_i, m'_i = m_i$ and $d^i_{V^{\bullet}_{\varsigma}} = \tilde{d}^i \otimes A/I$ for $i \geq l'-2$. Then there is a canonical quasi-isomorphism $V^{\bullet}_{\varsigma} \to (\tilde{V}_{\sigma})_{A/I}$. We may assume that $H^i(\operatorname{coker} d^j_{V^{\bullet}_{\varsigma}}(m'_k)) = 0$ for $k \leq j \leq l'-1$, $i > 0$. Since the homomorphism

$$H^i(\operatorname{Hom}^{\bullet}(\tilde{V}^{\bullet}_{\tau} \otimes k, \tilde{V}^{\bullet}_{\sigma} \otimes k)) \longrightarrow \operatorname{Ext}^i(\tilde{V}^{\bullet}_{\tau} \otimes k, \tilde{V}^{\bullet}_{\sigma} \otimes k)$$

is surjective for $i=1$ and bijective for $i=2$, one can check that the canonical homomorphism

$$H^i(\mathrm{Hom}^{\cdot}(V^{\cdot}_{\zeta}\otimes k, V^{\cdot}_{\zeta}\otimes k)) \longrightarrow \mathrm{Ext}^i(\tilde{V}^{\cdot}_{\sigma}\otimes k, \tilde{V}^{\cdot}_{\sigma}\otimes k)$$

is surjective for $i=1$ and injective for $i=2$.

From the commutativity of the diagram (3.1), there exists an isomorphism θ of $(\tilde{V}^{\cdot}_{\sigma})_{A/I}$ to $Q^{\cdot}\otimes A/I$ in the derived category $D(X_{A/I})$. We can take a complex $W^{\cdot}=(W_i\otimes\mathcal{O}_{X_{A/I}}(-n_i), d^i_{W^{\cdot}})$ and a quasi-isomorphism $W^{\cdot}\to V^{\cdot}_{\zeta}$ such that $(W^{\cdot}\otimes k\to V^{\cdot}_{\zeta}\otimes k, W^{\cdot})$ satisfies the condition (L_0) of Lemma 2.1, where W_i are free A-modules of finite rank. If we choose each n_i sufficiently large, then we have a quasi-isomorphism

$$\mathrm{Hom}^{\cdot}(W^{\cdot}, Q^{\cdot}\otimes A/I) \longrightarrow \mathbf{R}\,\mathrm{Hom}^{\cdot}((\tilde{V}^{\cdot}_{\sigma})_{A/I}, Q^{\cdot}\otimes A/I).$$

Thus there exists a morphism $\theta' : W^{\cdot}\to Q^{\cdot}\otimes A/I$ which represents θ.

By replacing $W^{\cdot}$, we may assume that both $(W^{\cdot}\otimes k\to V^{\cdot}_{\zeta}\otimes k, W^{\cdot})$ and $(W^{\cdot}\otimes k\to Q^{\cdot}\otimes k, W^{\cdot}\otimes k)$ satisfy the condition (L_0) of Lemma 2.1. Then there is a complex $\tilde{W}^{\cdot}_{Q^{\cdot}}=(W_i\otimes\mathcal{O}_{X_A}(-n_i), d^i_{\tilde{W}^{\cdot}_{Q^{\cdot}}})$ and a quasi-isomorphism $\tilde{W}^{\cdot}_{Q^{\cdot}}\to Q^{\cdot}$ which is a lift of the given quasi-isomorphism $W^{\cdot}\to Q^{\cdot}\otimes A/I$. Since $[Q^{\cdot}]$ is a lift of $[(\tilde{V}^{\cdot}_{\sigma})_{A/I}]$ to $\mathrm{Splcpx}_{X/S}(A)$, the obstruction class $\omega((\tilde{V}^{\cdot}_{\sigma})_{A/I})$ vanishes. Thus, by the proof of Proposition 2.3 (i), there is a complex $\tilde{V}^{\cdot}_{\zeta}=(V'_i\otimes\mathcal{O}_{X_A}(-m'_i), d^i_{\tilde{V}^{\cdot}_{\zeta}})$ which is a lift of $V^{\cdot}_{\zeta}$, because

$$H^2(\mathrm{Hom}^{\cdot}(V^{\cdot}_{\zeta}\otimes k, V^{\cdot}_{\zeta}\otimes k)) \longrightarrow \mathrm{Ext}^2(\tilde{V}^{\cdot}_{\sigma}\otimes k, \tilde{V}^{\cdot}_{\sigma}\otimes k)$$

is injective. Since $(W^{\cdot}\otimes k\to V^{\cdot}_{\zeta}\otimes k, W^{\cdot})$ satisfies the condition (L_0), there is a complex $\tilde{W}^{\cdot}=(W_i\otimes\mathcal{O}_{X_A}(-n_i), d^i_{\tilde{W}^{\cdot}})$ and a quasi-isomorphism $g^{\cdot}:\tilde{W}^{\cdot}\to\tilde{V}^{\cdot}_{\zeta}$ which is a lift of the given quasi-isomorphism $W^{\cdot}\to V^{\cdot}_{\zeta}$. As in Proposition 2.3, $\{d^i_{\tilde{W}^{\cdot}_{Q^{\cdot}}}-d^i_{\tilde{W}^{\cdot}}\}$ induces an element v of $\mathrm{Ext}^1((\tilde{V}^{\cdot}_{\sigma})_{A/I}, I\otimes_{A/I}(\tilde{V}^{\cdot}_{\sigma})_{A/I})$. Since the canonical homomorphism

$$H^1(\mathrm{Hom}^{\cdot}(\tilde{V}^{\cdot}_{\zeta}, I\otimes\tilde{V}^{\cdot}_{\zeta})) \longrightarrow \mathrm{Ext}^1((\tilde{V}^{\cdot}_{\sigma})_{A/I}, I\otimes_{A/I}(\tilde{V}^{\cdot}_{\sigma})_{A/I})$$

is surjective, there is a lift $\tilde{v}=[\{\tilde{v}^i\}]\in H^1(\mathrm{Hom}^{\cdot}(\tilde{V}^{\cdot}_{\zeta}, I\otimes\tilde{V}^{\cdot}_{\zeta}))$ of v. Then $\tilde{V}^{\cdot}_{v}:=(V'_i\otimes\mathcal{O}_{X_A}(-m'_i), d^i_{\tilde{V}^{\cdot}_{\zeta}}+\tilde{v}^i)$ is a complex and there is a quasi-isomorphism $\tilde{W}^{\cdot}_{Q^{\cdot}}\to\tilde{V}^{\cdot}_{v}$. Consider the complex

$$\tau_{\geq l'-2}(\tilde{V}^{\cdot}_{v}) : \tilde{V}^{l'-2}_{v} \xrightarrow{d^{l'-2}} \tilde{V}^{l'-1}_{v} \longrightarrow \cdots \longrightarrow \tilde{V}^{l}_{v}.$$

Then $\tau_{\geq l'-2}(\tilde{V}^{\cdot}_{v})$ determines a morphism $\xi : \mathrm{Spec}(A)\to Z$ which makes the diagram

$$\begin{array}{ccc} \mathrm{Spec}(A/I) & \hookrightarrow & \mathrm{Spec}(A) \\ \zeta\downarrow & \xi\swarrow & \eta\downarrow \\ Z & \xrightarrow{\phi'} & \mathrm{Splcpx}_{X/S} \end{array}$$

commute. Hence $\phi : Z \to \mathrm{Splcpx}_{X/S}$ is formally smooth. □

Example 3.1.

(i) Let C be a smooth projective curve over an algebraically closed field k. It can be shown that for any member $E^{\cdot}$ of $\mathrm{Splcpx}_{C/k}(k)$, there exist a simple sheaf F on C and an integer i such that $E^{\cdot}$ is equivalent to $F[i]$, where $F[i]$ denotes the i-th shift of F.

(ii) Let X be a smooth projective variety over k of dimension $d \geq 2$ and E be a torsion free simple sheaf on X which is not locally free. Then $\mathbf{R}\mathcal{H}om^{\cdot}(E, \mathcal{O}_X)$ is a member of $\mathrm{Splcpx}_{X/k}(k)$ which is not a shift of any simple sheaf on X.

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