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The modularity conjecture for rigid Calabi-Yau threefolds over Q

By

Masa-Hiko SAITO and Noriko YUI

Abstract

We formulate the modularity conjecture for rigid Calabi-Yau threefolds defined over the field $\mathbf{Q}$ of rational numbers. We establish the modularity for the rigid Calabi-Yau threefold arising from the root lattice A_3. Our proof is based on geometric analysis.

1. The L-series of Calabi-Yau threefolds

Let $\mathbf{Q}$ be the field of rational numbers, and let $\bar{\mathbf{Q}}$ be its algebraic closure with Galois group $\mathcal{G} := \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$. Let X be a smooth projective threefold defined over $\mathbf{Q}$ or more generally over a number field.

Definition 1.1. X is a *Calabi-Yau* threefold if it satisfies the following two conditions:
(a) $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$, and
(b) The canonical bundle is trivial, i.e., $K_X \simeq \mathcal{O}_X$.

The numerical invariants of Calabi-Yau threefolds

Let X be a Calabi-Yau threefold defined over $\mathbf{Q}$, and let $\bar{X} = X \times_{\mathbf{Q}} \bar{\mathbf{Q}}$. The (i,j)-th *Hodge* number $h^{i,j}(X)$ of X is defined by

$$h^{i,j}(X) = \dim_{\bar{\mathbf{Q}}} H^j(\bar{X}, \Omega^i_{\bar{X}}).$$

The condition (a) implies that $h^{1,0}(X) = h^{2,0}(X) = 0$, and the condition (b) that $h^{3,0}(X) = h^{0,3}(X) = 1$. The number $h^{2,1}(X)$ represents the number of deformations of complex structures on X, and $h^{1,1}(X)$ is the number of Hodge $(1,1)$-cycles on X. By using Hodge symmetry and Serre duality, we obtain

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$h^{2,1}(X) = h^{1,2}(X)$ and $h^{1,1}(X) = h^{2,2}(X)$ and finaly the *Hodge diamond* can be given as follows:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{1,2} & & 1 \\ & 0 & & h^{2,2} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

The n-th *Betti* number of X is also defined by

$$B_n(X) = \dim_{\mathbf{Q}_\ell} H^n_{\text{et}}(\bar{X}, \mathbf{Q}_\ell) = \dim_{\mathbf{C}} H^n(X \otimes \mathbf{C}, \mathbf{C})$$

where the cohomology groups are, respectively, the ℓ-adic étale and singular cohomology groups. The Hodge decomposition theorem

$$H^n(X \otimes \mathbf{C}, \mathbf{C}) = \oplus_{i+j=n} H^j(X \otimes \mathbf{C}, \Omega^i_{X \otimes \mathbf{C}})$$

ensures that $B_n(X) = \sum_{i+j=n} h^{i,j}(X)$, and hence one can compute the Betti numbers of Calabi-Yau threefolds:

$$B_0(X) = B_6(X) = 1, \quad B_1(X) = B_5(X) = 0,$$

$$B_2(X) = h^{1,1}(X) = B_4(X) = h^{2,2}(X) \quad \text{and} \quad B_3(X) = 2(1 + h^{2,1}(X)).$$

Note also that the (topological) Euler characteristic is given by $\chi(X) = 2(h^{1,1}(X) - h^{2,1}(X))$.

Definition 1.2. A smooth projective Calabi-Yau threefold X over $\mathbf{Q}$ (or over any field) is called *rigid* if $h^{2,1}(X) = h^{1,2}(X) = 0$ so that $B_3(X) = 2$.

Definition 1.3. Let X be a Calabi-Yau threefold defined over $\mathbf{Q}$. Assume that X has a suitable integral model. The L-series of X is defined to be the L-series of the (semi-simplification of the) Galois representation on $H^3_{\text{et}}(\bar{X}, \mathbf{Q}_\ell)$. That is,

$$L(X, s) := L(H^3_{\text{et}}(\bar{X}, \mathbf{Q}_\ell), s).$$

Digression. Let X be a Calabi-Yau threefold defined over $\mathbf{Q}$ with a suitable integral model. Let p be a *good* prime for X, that is, the reduction X (mod p) defines a smooth projective variety over the prime field $\mathbf{F}_p$. Let Frob_p denote the (geometric) Frobenius morphism of X at p. We consider the action of Frob_p on the ℓ-adic etale cohomology group $H^3_{\text{et}}(\bar{X}, \mathbf{Q}_\ell)$, and let $t_3(p)$ denote the trace of the Frob_p on $H^3_{\text{et}}(\bar{X}, \mathbf{Q}_\ell)$. By the Lefschetz fixed point formula, $t_3(p)$ can be determined by counting the number of $\mathbf{F}_p$-rational points on X:

$$t_3(p) = 1 + p^3 + (1 + p)t_2(p) - \#X(\mathbf{F}_p).$$

Here $t_2(p) = p\,\mathrm{Trace}(\rho(\mathrm{Frob}_p))$ where ρ is an Artin representation of dimension $h^{1,1}(X)$ so in particular its trace is an integer which is a sum of $h^{1,1}(X)$ roots of unity. This implies that $|t_2(p)| \leq p\,h^{1,1}(X)$, and for a set of primes of positive density, we get $t_2(p) = ph^{1,1}(X)$. Define

$$P_{3,p}(T) := \det(1 - \mathrm{Frob}_p\,T \mid H^3_{\mathrm{et}}(\bar{X}, \mathbf{Q}_\ell)).$$

If X is a rigid Calabi-Yau threefold, then $P_{3,p}(T)$ is an integral polynomial of degree $\deg(P_{3,p}) = 2$; it is of the form:

$$P_{3,p}(T) = 1 - t_3(p)T + p^3T^2 \in 1 + T\mathbf{Z}[T]$$

where $t_3(p)$ is subject to the Riemann Hypothesis: $|t_3(p)| \leq 2p^{3/2}$. The L-series $L(H^3_{\mathrm{et}}(\bar{X}, \mathbf{Q}_\ell), s)$ is then given by

$$L(H^3_{\mathrm{et}}(\bar{X}, \mathbf{Q}_\ell), s) = (*)\prod_p P_{3,p}(p^{-s})^{-1}$$

where p runs over good primes, and $(*)$ is the Euler factor corresponding to bad primes.

2. The modularity conjecture for rigid Calabi-Yau threefolds over Q

Let $k \geq 1$ be an integer. Let Γ be an arithmetic subgroup of $\mathrm{SL}_2(\mathbf{Z})$ of finite index. We denote by $S_k(\Gamma)$ the complex vector space of all cusp forms of weight k with respect to Γ. We now formulate the modularity conjecture for rigid Calabi-Yau threefolds defined over $\mathbf{Q}$.

Conjecture 2.1. The modularity conjecture: Any rigid Calabi-Yau threefold X defined over $\mathbf{Q}$ is modular in the sense that its L-series of X coincides with the Mellin transform of the L-series of a modular (cusp) form f of weight 4 on $\Gamma_0(N)$. Here N is a positive integer divisible by the primes of bad reduction. More precisely, we have, up to a finite Euler factors,

$$L(X, s) = L(f, s) \quad \text{for} \quad f \in S_4(\Gamma_0(N)).$$

Here are some justifications for formulating the modularity conjecture for rigid Calabi-Yau threefolds over $\mathbf{Q}$.

Remark 2.1. The conjecture of Taniyama-Shimura-Weil that every elliptic curve over $\mathbf{Q}$ is modular, has been established by Wiles and his former students in totality (see Breuil, Conrad, Diamond and Taylor [BCDT]). Noting that elliptic curves are dimension one Calabi-Yau varieties, our modularity Conjecture 2.1, may be regarded a dimension three generalization of the Taniyama-Shimura-Weil conjecture to rigid Calabi-Yau threefolds over $\mathbf{Q}$.

Remark 2.2. Livné [Li2] considered a rank 2 motive $\mathcal{M}$ over $\mathbf{Q}$ with Hodge numbers $h^{p,q}(\mathcal{M}) = h^{q,p}(\mathcal{M}) = 1$ where $p > q$, which respects an orthogonal form up to similitudes. Livné showed how to express the L-series of $\mathcal{M}$ in terms of Hecke characters.

The examples of rank 2 motives $\mathcal{M}$ considered by Livné arose from an elliptic curve E over $\mathbf{Q}$ or from a singular $K3$ surface X, i.e., $\mathcal{M} = H^1(E) \otimes \mathbf{Q}$ or $H^2(X, \mathbf{Z})/\mathrm{Pic}(X) \otimes \mathbf{Q}$. In the latter case, Shioda and Inose [SI] determined the L-series of a singular $K3$ surface X (up to a finite numbers of Euler factors) passing to some finite extension of $\mathbf{Q}$:

$$L(X, s) = L(\psi_1^2, s)L(\bar{\psi}_1^2, s)$$

where ψ_1 is the Hecke character associated to an elliptic curve with complex multiplication.

Remark 2.3. Fontaine and Mazur [FM] have conjectured that all irreducible odd 2-dimensional Galois representations "coming from geometry" should be modular, up to a Tate twist. Our modularity Conjecture 2.1, may be regarded as a concrete realization of the Fontaine-Mazur conjecture. For a recent progress on a conjecture of Fontaine-Mazur, the reader is referred to a paper of Richard Taylor [T].

Remark 2.4. Let X be a projective variety of odd dimension m over $\mathbf{Q}$ such that the m-th Betti number $B_m = \dim H^m(X \otimes \mathbf{C}, \mathbf{C}) = 2$ and $H^m(X \otimes \mathbf{C}, \mathbf{C})$ has the Hodge decomposition of type $(m, 0) + (0, m)$. Serre [Sr] has formulated a modularity conjecture for the residual mod p 2-dimensional Galois representation attached to X for all primes. In particular, our modularity Conjecture 2.1 is a special case of the conjectural Theorem 6 of Serre [Sr] for $m = 3$. Serre has informed us about this in his e-mail dated June 23, 2000. We are thankful to him for pointing this out.

3. A rigid Calabi-Yau threefold arising from the root lattice A_3

This example of a rigid Calabi-Yau threefold is constructed from the root lattice A_3 via a toric construction.

3.1. Toric construction. For the general backgrounds on toric varieties, the reader is referred to Batyrev [Bat], Fulton [Fu] and Fulton and Harris [FH].

We consider a root system $\mathcal{R}$ of rank r. Let $\mathcal{L}_{\mathcal{R}}$ be the root lattice generated by $\mathcal{R}$, and let $\mathcal{L}_{\mathcal{R}}^*$ be its dual lattice. Let $\Sigma_{\mathcal{R}}$ be the fan in $\mathcal{L}_{\mathcal{R}}^* \otimes \mathbf{Q}$. The fan $\Sigma_{\mathcal{R}}$ gives rise to a toric variety, which is denoted by $X(\Sigma_{\mathcal{R}})$. Let $\Delta_{\mathcal{R}}$ be a polyhedron in $\mathcal{L}_{\mathcal{R}} \otimes \mathbf{Q}$ with vertices in $\mathcal{R}$, and let $L(\Delta_{\mathcal{R}})$ denote the space of Laurent polynomials with support in $\Delta_{\mathcal{R}}$. For each root $r \in \mathcal{R}$, assign a monomial $\mathbf{e}^r$, and define a Laurent polynomial

$$\chi_{\mathcal{R}} := \sum_{r \in \mathcal{R}} \mathbf{e}^r \in L(\Delta_{\mathcal{R}}).$$

Then we have a rational function $\chi_{\mathcal{R}} : X(\Sigma_{\mathcal{R}}) \to \mathbf{P}^1$. For each $\lambda \in \mathbf{P}^1$, define

$$X_\lambda := \overline{\chi_{\mathcal{R}}^{-1}(\lambda)} \subset X(\Sigma_{\mathcal{R}}).$$

There is a base locus $\mathcal{B}$. Let $\mathcal{B} : X_{\mathcal{R}} \to X(\Sigma_{\mathcal{R}})$ be the blow up of the base locus. Then we obtain a pencil of varieties

$$\mathcal{X}_{\mathcal{R}} := \{(\lambda, x) \in \mathbf{P}^1 \times X(\Sigma_{\mathcal{R}}) \,|\, \overline{\chi_{\mathcal{R}}(x) = \lambda}\}.$$

We are interested in root lattices $\mathcal{R}$ which give rise to Calabi-Yau threefolds $\mathcal{X}_{\mathcal{R}}$.

3.2. The root lattice A_3 and the construction of a Calabi-Yau threefold. Verrill [V] constructed a Calabi-Yau threefold associated to the root lattice A_3. We will briefly describe Verrill's construction. Let $\{E_i \,|\, i = 1, 2, 3, 4\}$ be the standard basis for $\mathbf{R}^4$. The root lattice A_3 is a sublattice of $\mathbf{R}^4$ of rank 3 generated by $v_1 := E_1 - E_2$, $v_2 := E_2 - E_3$, $v_3 := E_3 - E_4$ and the collection of all roots is given by the set

$$\{E_i - E_j \,|\, 1 \leq i, j \leq 4,\ i \neq j\}.$$

To a root $E_i - E_j$, we associate a monomial $X_i X_j^{-1}$ by putting $X_i = \mathbf{e}^{E_i}$. Then the character of the adjoint representation is given by

$$\begin{aligned} \chi_{A_3} &= X_1X_2^{-1} + X_1X_3^{-1} + X_1X_4^{-1} + X_2X_3^{-1} + X_2X_4^{-1} + X_3X_4^{-1} \\ &\quad + X_2X_1^{-1} + X_3X_1^{-1} + X_4X_1^{-1} + X_3X_2^{-1} + X_4X_2^{-1} + X_4X_3^{-1} \\ &= (X_1 + X_2 + X_3 + X_4)(X_1^{-1} + X_2^{-1} + X_3^{-1} + X_4^{-1}) - 4. \end{aligned}$$

Let $X(\Sigma_{A_3}) \to \mathbf{P}^1$ denote the toric variety defined by $\chi_{A_3} = \lambda \in \mathbf{P}^1$. It is a rational variety. Now we take a double cover of $\mathcal{X}(\Sigma_{A_3})$ by putting $\lambda = (t-1)^2/t$ with $t \in \mathbf{P}^1$. Then a desingularization X of the double cover is a Calabi-Yau threefold defined over $\mathbf{Q}$ whose generic fiber X_λ is a $K3$ surface.

Lemma 3.1. *The numerical characters of X are given as follows*:

$$h^{1,2} = 0, \quad h^{3,0} = 1, \quad h^{1,1} = 50, \quad \chi(X) = 100$$

and hence X is a rigid Calabi-Yau threefold.

We can see from the defining equation of X that 2 and 3 are bad primes. For this rigid Calabi-Yau threefold X over $\mathbf{Q}$, we can establish the modularity conjecture. We will give a proof of the following theorem in Section 6 below. Verrill [V] also proved the theorem with totally different method (à la Livné [Li1]) from ours.

Theorem 3.1. *Let X be the rigid Calabi-Yau threefold constructed by Verrill. Then L-series of X coincides, up to the Euler factors corresponding to primes 2 and 3, with the Mellin transform of a weight 4 newform on $\Gamma_0(6)$. That is,*

$$L(X,s) = L(f,s) \quad \text{for} \quad f \in S_4(\Gamma_0(6)).$$

The cusp form f has the q-expansion of the form:

$$f(q) = \eta(q)^2\eta(q^2)^2\eta(q^3)^2\eta(q^6)^2.$$

4. The self-products of elliptic modular surfaces

Let Γ be an arithmetic subgroup of $\mathrm{SL}_2(\mathbf{Z})$ with no torsion elements and let $\mathcal{H}$ be the upper half complex plane. Then we can define the modular curve $C^0_\Gamma := \mathcal{H}/\Gamma$ and its compactification $C_\Gamma := \overline{\mathcal{H}/\Gamma} = \mathcal{H}/\Gamma \cup \{\text{cusps of } \Gamma\}$. It is known (cf. [D]) that the universal family of elliptic curves

$$f^0 : S^0_\Gamma \longrightarrow C^0_\Gamma = \mathcal{H}/\Gamma$$

exists, and that its Néron model has the unique smooth minimal compactification

$$f := f_\Gamma : S_\Gamma \longrightarrow C_\Gamma = \overline{\mathcal{H}/\Gamma}.$$

The fibration $f : S_\Gamma \longrightarrow C_\Gamma$ is called the *elliptic modular surface associated to* Γ.

Let us now recall some basic facts about elliptic modular surfaces, which are relevant to our subsequent discussions. Consider the following diagram

$$\begin{array}{ccc} S^0_\Gamma & \hookrightarrow & S_\Gamma \\ f^0 \downarrow & & f \downarrow \\ C^0_\Gamma & \overset{j}{\hookrightarrow} & C_\Gamma . \end{array}$$

Making use of the invariant cycle theorem, we obtain an isomorphism of sheaves

$$R^1 f_* \mathbf{Q}_{S_\Gamma} \simeq j_* R^1 f^0_* \mathbf{Q}_{S^0_\Gamma} .$$

In fact, we have the natural map

$$R^1 f_* \mathbf{Q}_{S_\Gamma} \longrightarrow j_* R^1 f^0_* \mathbf{Q}_{S^0_\Gamma} \tag{4.1}$$

which is an isomorphim over $C^0_\Gamma = C_\Gamma \setminus \Sigma$ where $\Sigma = \{p_1, \ldots, p_r\}$ is the set of cusps of Γ. Looking at the stalks at a cusp p_i, the morphism

$$(R^1 f_* \mathbf{Q}_{S_\Gamma})_{p_i} \longrightarrow (j_* R^1 f^0_* \mathbf{Q}_{S^0_\Gamma})_{p_i} \tag{4.2}$$

is surjective by local invariant cycle theorem (cf. e.g. [Proposition 15.12 [Z]]). On the other hand, it is easy to see that both of the stalks in (4.2) are isomorphic to a one-dimensional $\mathbf{Q}$-vector space. Hence the linear map (4.2) is an isomorphism.

Let V denote the natural representation of $SL_2(\mathbf{Q})$, and let $S^k(V)$ be the k-th symmetric tensor representation of V. Then it is known ([Z], Proposition 12.5) that for each $k \geq 0$ there exists an isomorphism

$$\tilde{H}^1(\Gamma, S^k(V)) \simeq H^1(C_\Gamma, j_*(S^k(R^1 f^0_* \mathbf{Q}_{S^0_\Gamma}))) \quad \text{for each} \quad k \geq 0. \tag{4.3}$$

Here $\tilde{H}^1(\Gamma, S^k(V))$ is the parabolic cohomology group associated to the representation of Γ on $S^k(V)$. Let $\omega_{S_\Gamma/C_\Gamma} = K_{S_\Gamma} \otimes f^*(K^{-1}_{C_\Gamma})$ be the relative canonical bundle of f, and we set

$$\omega = f_*(\omega_{S_\Gamma/C_\Gamma}).$$

Then we see that ω is an invertible sheaf on C_Γ.

Now we are ready to describe an isomorphism, the so-called *Shimura isomorphism* and its far-reaching consequences.

Proposition 4.1 ([Sh], [D], [Z]). *There exists a natural isomorphism* (*the Shimura isomorphism*)

$$H^0(C_\Gamma, \Omega^1_{C_\Gamma} \otimes \omega^k) \simeq S_{k+2}(\Gamma) \tag{4.4}$$

and it gives rise to the following commutative diagram:

$$\begin{array}{ccc} \tilde{H}^1(\Gamma, S^k(V)) \otimes_{\mathbf{Q}} \mathbf{C} & \simeq & S_{k+2}(\Gamma) \oplus \overline{S_{k+2}(\Gamma)} \\ \downarrow & & \downarrow \\ H^1(C_\Gamma, j_*(S^k(R^1 f^0_* \mathbf{C}_{S^0_\Gamma}))) & \simeq & H^0(C_\Gamma, \Omega^1_{C_\Gamma} \otimes \omega^k) \oplus \overline{H^0(C_\Gamma, \Omega^1_{C_\Gamma} \otimes \omega^k)} \end{array}$$

The first vertical arrow is the isomorphism induced by (4.3).

With the Shimura isomorphism at our disposal, we may now characterize rational elliptic modular surfaces.

Proposition 4.2. *The elliptic modular surface associated to* Γ, $f : S_\Gamma \longrightarrow C_\Gamma$ *is a rational elliptic surface, if and only if*

$$\dim S_2(\Gamma) = \dim S_3(\Gamma) = 0.$$

Proof. From the formula for the canonical bundle of the elliptic fibration $f : S_\Gamma \longrightarrow C_\Gamma$, we see that

$$K_{S_\Gamma} = f^*(\Omega^1_{C_\Gamma} \otimes \omega). \tag{4.5}$$

The formula (4.5) combined with the Shimura isomorphism (4.4) then give rise to the isomorphisms

$$H^0(C_\Gamma, \Omega^1_{C_\Gamma}) \simeq S_2(\Gamma), \quad H^0(S_\Gamma, K_{S_\Gamma}) \simeq H^0(C_\Gamma, \Omega^1_{C_\Gamma} \otimes \omega) \simeq S_3(\Gamma). \tag{4.6}$$

Suppose that S_Γ is a rational surface. Then C_Γ is also a rational curve, so that $\dim S_2(\Gamma) = 0$. We also have $\dim H^0(S_\Gamma, K_{S_\Gamma}) = 0$, and hence $\dim S_3(\Gamma) = 0$.

Conversely, assume that $\dim S_k(\Gamma) = 0$ for $k = 2, 3$. Then again the Shimura isomorphism (4.4) implies that $H^0(C_\Gamma, \Omega^1_{C_\Gamma}) = 0$. Hence $C_\Gamma \simeq \mathbf{P}^1$. Since f is not isotrivial, the formula for the canonical bundle asserts that we may write ω as $\omega = \mathcal{O}_{\mathbf{P}^1}(a)$ and $K_{S_\Gamma} = f^*(\mathcal{O}_{\mathbf{P}^1}(a-2))$ with a suitable positive integer a. Note that the positivity of a follows from the positivity of the direct image of the relative canonical sheaf $\omega = f_*(\omega_{S_\Gamma/C_\Gamma})$.

Under this situation, $\dim S_3(\Gamma) = \dim H^0(S_\Gamma, K_{S_\Gamma}) = \dim H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(a -2)) = 0$ implies that $a = 1$. We also have the isomorphisms

$$f_*\mathcal{O}_{S_\Gamma} \simeq \mathcal{O}_{\mathbf{P}^1}, \quad R^1 f_*\mathcal{O}_{S_\Gamma} \simeq \omega^\vee \simeq \mathcal{O}_{\mathbf{P}^1}(-1).$$

Further the Leray spectral sequence shows that $H^1(S_\Gamma, \mathcal{O}_{S_\Gamma}) = 0$. Moreover, for every $k > 0$, $K^k_{S_\Gamma} = f^*(\mathcal{O}_{\mathbf{P}^1}(-k))$ has no section. These facts finally imply that S_Γ is a rational surface. □

Next, let us consider the self-fiber product of the elliptic modular surface associated to Γ

$$f^2 = f \times f : S^2_\Gamma = S_\Gamma \times_{C_\Gamma} S_\Gamma \longrightarrow C_\Gamma.$$

In general, S^2_Γ has singularities which arise from the critical points of the map $f : S_\Gamma \longrightarrow C_\Gamma$. However, if f has only semistable fibers (that is, the singular fibers are all reduced and have only nodal singularities), then the fiber product S^2_Γ has only ordinary double points as its singularities.

Proposition 4.3. *Suppose that $f : S_\Gamma \longrightarrow C_\Gamma$ has only semistable fibers. Then there exists a small projective resolution $\pi : \tilde{S}^2_\Gamma \longrightarrow S^2_\Gamma$. Moreover, if we set $h = f^2 \circ \pi$, then the canonical bundle of $\tilde{S}^2_\Gamma$ can be written as*

$$K_{\tilde{S}^2_\Gamma} \simeq (h)^*(\Omega^1_{C_\Gamma} \otimes \omega^2). \tag{4.7}$$

Proof. The first assertion follows from Lemma 3.1 [Sch2]. Since π is a small resolution, $K_{\tilde{S}^2_\Gamma} = \pi^*(K_{S^2_\Gamma})$. On the other hand, since S^2_Γ is a hypersurface in $S_\Gamma \times S_\Gamma$ (i.e., it is the pull-back of the diagonal $\triangle : C_\Gamma \hookrightarrow C_\Gamma \times C_\Gamma$), we can derive from the adjunction formula that

$$\begin{aligned} K_{S^2_\Gamma} &= (K_{S_\Gamma \times S_\Gamma} + f^{-1}(\triangle(C_\Gamma)))_{|f^{-1}(\triangle(C_\Gamma))} \\ &= (f^2)^*((\Omega^1_{C_\Gamma})^{\otimes 2} \otimes \omega^2 \otimes (\Omega^1_{C_\Gamma})^{-1}) \\ &= (f^2)^*(\Omega^1_\Gamma \otimes \omega^2). \end{aligned}$$

This gives rise to the formula (4.7). □

Corollary 4.1 (cf. [Sok]). *Under the assumption of Proposition 4.3, we have a canonical isomorphism*

$$H^0(\tilde{S}^2_\Gamma, K_{\tilde{S}^2_\Gamma}) \simeq H^0(C_\Gamma, \Omega^1_{C_\Gamma} \otimes \omega^2) \simeq S_4(\Gamma).$$

For the full cohomology group $H^3(\tilde{S}^2_\Gamma, \mathbf{Q})$, the following fact holds (cf. Section 1, [Sch1]).

Lemma 4.1 (Lemma 1.7, [Sch1]). *The rational Hodge structure of $H^3(\tilde{S}^2_\Gamma, \mathbf{Q})$ is isomorphic to the direct sum of three copies of $H^1(C_\Gamma, \mathbf{Q})[-1]$ and a piece of pure type $(3,0), (0,3)$. In particular, if C_Γ is a rational curve, then $H^3(\tilde{S}^2_\Gamma, \mathbf{Q})$ is of pure type $(3,0), (0,3)$.*

Corollary 4.2. *Assume that $S_\Gamma \longrightarrow C_\Gamma$ is a rational elliptic modular surface associated to Γ with only semistable singular fibers. Then any small projective resolution $\tilde{S}^2_\Gamma$ of the self-product S^2_Γ is a rigid Calabi-Yau threefold.*

Proof. Since $f : S_\Gamma \longrightarrow C_\Gamma$ is a rational surface, it is immediate that $C_\Gamma \simeq \mathbf{P}^1$. Let $h : \tilde{S}^2_\Gamma \longrightarrow \mathbf{P}^1$ be the natural fibration. We may write ω as $\omega = \mathcal{O}_{\mathbf{P}^1}(1)$ using the formula (4.7). Then we see that

$$K_{\tilde{S}^2_\Gamma} = h^*(\Omega^1_{\mathbf{P}^1} \otimes \omega^2) = h^*(\mathcal{O}_{\mathbf{P}^1}(-2+2)) = \mathcal{O}_{\tilde{S}^2_\Gamma}.$$

Recall from Lemma (4.1) that the rational Hodge structure $H^3(\tilde{S}^2_\Gamma, \mathbf{Q})$ is of pure type $(3,0), (0,3)$. So it follows that $H^1(\tilde{S}^2_\Gamma, \Omega^2_{\tilde{S}^2_\Gamma}) = H^1(\Theta_{\tilde{S}^2_\Gamma}) = 0$. We also see (cf. [Sch1]) that $h^1(\mathcal{O}_{\tilde{S}^2_\Gamma}) = h^2(\mathcal{O}_{\tilde{S}^2_\Gamma}) = 0$. These facts then imply that $\tilde{S}^2_\Gamma$ is a rigid Calabi-Yau threefold. □

Remark 4.1. There are in total six arithmetic subgroups $\Gamma \subset \mathrm{SL}_2(\mathbf{Z})$ which give rise to rational elliptic modular surfaces with only semistable fibers. The list of such groups coincides with the list of Beauville [B]. (Beauville classified elliptic surfaces with exactly four semistable fibers (i.e., of type I_b, $b \geq 1$) over $\mathbf{P}^1$). In Table 1, groups Γ are listed in the first column, 4 singular fibers of type I_b, $b \geq 1$ in the second column and $h^{1,1}$ (resp. the Euler characteristic χ) of the corresponding Calabi-Yau threefolds $\tilde{S}^2_\Gamma$ in the third (resp. fourth) column. Note that the Euler characteristic $\chi(\tilde{S}^2_\Gamma)$ is exactly equal to $2h^{1,1}$. This is because $\tilde{S}^2_\Gamma$ is rigid. Also note that for any group Γ in Table 1, $\dim S_4(\Gamma) = 1$. This follows from the isomorphism $S_4(\Gamma) \simeq H^{3,0}(\tilde{S}^2_\Gamma) \simeq \mathbf{C}$.

Table 1:

Γ					$h^{1,1}(\tilde{S}^2_\Gamma)$	$\chi(\tilde{S}^2_\Gamma)$
$\Gamma(3)$	I_3	I_3	I_3	I_3	36	72
$\Gamma_1(4) \cap \Gamma(2)$	I_4	I_4	I_2	I_2	40	80
$\Gamma_1(5)$	I_5	I_5	I_1	I_1	52	104
$\Gamma_1(6)$	I_6	I_3	I_2	I_1	50	100
$\Gamma_0(8) \cap \Gamma_1(4)$	I_8	I_2	I_1	I_1	70	140
$\Gamma_0(9) \cap \Gamma_1(3)$	I_9	I_1	I_1	I_1	84	168

Now we quote a theorem which is a special case of the fundamental result due to Sato-Kuga-Shimura ([KS]; cf. [D]). We make use the argument of [D], where Deligne worked over the case when $\Gamma = \Gamma(N)$ (full modular case). However his argument works for other cases like $\Gamma = \Gamma_1(N)$ as far as the universal family of generalized elliptic curves exists.

Theorem 4.1. *Let Γ be an arithmetic subgroup of $\mathrm{SL}_2(\mathbf{Z})$ listed in Table 1, and let $f : S_\Gamma \longrightarrow \mathbf{P}^1$ be the rational elliptic surface associated to Γ. Let $f^2 : \tilde{S}^2_\Gamma \longrightarrow \mathbf{P}^1$ be the smooth rigid Calabi-Yau threefold arising from the self-product S^2_Γ. Then $\tilde{S}^2_\Gamma$ is modular, that is, the L-series of $\tilde{S}^2_\Gamma$ coincides with the Mellin transform of a weight 4 newform of Γ up to finite Euler factors coming from the bad primes*:

$$L(\tilde{S}^2_\Gamma, s) = L(f, s) \quad \textit{for} \quad f \in S_4(\Gamma).$$

Remark 4.2. The bad primes for $\tilde{S}^2_\Gamma$ depends on the level of the discrete group Γ. For example, we can easily see that only 2 and 3 are the bad primes for $\Gamma_1(6)$.

5. The geometry of Verrill's Calabi-Yau threefold

In this section, we will look into the geometric structure of Verrill's rigid Calabi-Yau threefold X constructed in Section 3. Note that from now on all varieties are defined over $\mathbf{Q}$.

We denote by $[x : y : z : w]$ the homogeneous coordinate of $\mathbf{P}^3$ and by t the inhomogeneous coordinate of $\mathbf{P}^1$. Then Verrill's rigid Calabi-Yau threefold X is a desingularization of the hypersurface in $\mathbf{P}^3 \times \mathbf{P}^1$:

$$(5.1) \quad \left\{(x+y+z+w)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{w}\right) - \frac{(t-1)^2}{t} - 4 = 0\right\} \subset \mathbf{P}^3 \times \mathbf{P}^1.$$

It is easily seen that this equation is equivalent to the following equation:

$$(5.2) \quad \begin{aligned} F(x,y,z,w,t) = (x+y+z+w) \cdot [(yz+zx+xy)w + xyz] \cdot t \\ - (t+1)^2 \cdot xyzw = 0. \end{aligned}$$

The elliptic modular surface associated to $\Gamma_1(6)$

Let us consider the hypersurface S in $\mathbf{P}^1 \times \mathbf{P}^2$ defined by

$$(5.3) \quad H(x,y,z,s) = (s+1)xyz - (x+y+z)(yz+zx+xy) = 0.$$

The hypersurface S has three singular points at

$$[s, [x : y : z]] = [\infty, [1 : 0 : 0]], \ [\infty, [0 : 1 : 0]], \ [\infty, [0 : 0 : 1]].$$

Let

$$\pi : \tilde{S} \longrightarrow S$$

be the blowing up of these three points. Then it is easy to see that $\tilde{S}$ is the minimal resolution of S and $f = p_1 \circ \pi : \tilde{S} \longrightarrow \mathbf{P}^1$ induces the structure of the

Table 2:

s	8	-1	0	∞
singular fiber	I_1	I_2	I_3	I_6

elliptic surface. Moreover the rational elliptic surface $f : \tilde{S} \longrightarrow \mathbf{P}^1$ has only four semistable singular fibers of types I_1, I_2, I_3, I_6 (see Table 2):

By [B], $f : \tilde{S} \longrightarrow \mathbf{P}^1$ is an integral model of the elliptic modular surface

$$f : S_{\Gamma_1(6)} \longrightarrow \overline{H/\Gamma_1(6)} \simeq \mathbf{P}^1$$

of $\Gamma_1(6)$.

Theorem 5.1. *Verrill's Calabi-Yau threefold X defined in* (5.1) *is birationally equivalent over $\mathbf{Q}$ to a crepant resolution $\tilde{S}^2_{\Gamma_1(6)}$ of the self-product of the elliptic modular surface $S_{\Gamma_1(6)}$. More precisely, there exists a birational map defined over $\mathbf{Q}$ between X and $\tilde{S}^2_{\Gamma_1(6)}$.*

Remark 5.1. There are two fibrations, one is the fibration of $X \longrightarrow \mathbf{P}^1$ of K3 surfaces and the other is the fibration $f^2 : \tilde{S}^2_{\Gamma_1(6)} \longrightarrow \mathbf{P}^1$ of abelian surfaces. We will show below that these two fibrations are not equal. (However, we do not know any relation (apart from being non-equal) between these two fibrations.)

Proof of Theorem 5.1: *Birational transformations.* To prove Theorem 5.1, we will construct a birational map over $\mathbf{Q}$ between the two varieties explicitly. We define the birational map:

$$\begin{array}{ccc} \pi : \mathbf{P}^2 \times \mathbf{P}^2 & \longrightarrow & \mathbf{P}^3 \times \mathbf{P}^1 \\ [x : y : z] \ \times \ [T : W : U] & \mapsto & [x : y : z : w] \times (t) \end{array}$$

by putting

$$x = x, \ y = y, \ z = z, \ t = T/U, \ w = W \cdot (x + y + z)/U.$$

Pulling back the equation $F(x, y, z, w, t)$ in (5.2) via π, we obtain the equation

$$\begin{aligned} \tilde{F} := {} & (x + y + z)^2(yz + xz + xy)TW^2 \\ & - (x + y + z) \cdot [(x + y + z)(yz + xz + xy) - xyz]TWU \\ & + xyz(x + y + z)TU^2 - (x + y + z)xyz[T^2 + U^2]W = 0. \end{aligned}$$

Consider the hypersurface $X'' \subset \mathbf{P}^2 \times \mathbf{P}^2$ defined by $\tilde{F} = 0$. Of course, X'' is birationally equivalent to the original hypersurface X' in (5.1). We have the following commutative diagram:

$$\begin{array}{ccc} X'' & \stackrel{\iota}{\hookrightarrow} & \mathbf{P}^2 \times \mathbf{P}^2 \\ f_1 \downarrow & \swarrow p_1 & \\ \mathbf{P}^2 & & \end{array}$$

Now recall the hypersurface $S \subset \mathbf{P}^1 \times \mathbf{P}^2$ defined in (5.3). Considering the birational map $\mu = (p_2)_{|S} : S \longrightarrow \mathbf{P}^2$ and taking the pull-back morphism f_1 by μ, we obtain the following commutative diagram:

$$\begin{array}{ccccc} \mathbf{P}^1 \times \mathbf{P}^2 \times \mathbf{P}^2 & \supset & X''' & \stackrel{\nu}{\longrightarrow} & X'' \\ \downarrow p_{12} & & f_1' \downarrow & & f_1 \downarrow \\ \mathbf{P}^1 \times \mathbf{P}^2 & \supset & S & \stackrel{\mu}{\longrightarrow} & \mathbf{P}^2 \end{array}$$

Here we set $X''' := S \times_{\mathbf{P}^2} X''$. Note that the natural induced map $\nu : X''' \longrightarrow X''$ is a birational morphism. Now we see that X''' is a complete intersection defined by the equations

$$\begin{array}{rcl} \tilde{F} & = & 0 \\ H(x,y,z) & = & (s+1)(xyz) - (x+y+z)(yz+zx+xy) = 0. \end{array}$$

Now using the second equation, $\tilde{F} = 0$ can be transformed into the following equation:

$$\begin{aligned} \tilde{F} :&= (x+y+z)\cdot(s+1)xyz \cdot TW^2 \\ &\quad + (x+y+z)\cdot[(s+1)xyz - xyz]TWU \\ &\quad + xyz(x+y+z)TU^2 - (x+y+z)[T^2+U^2]xyzW \\ &= (x+y+z)xyz \times \{(s+1)\cdot TW^2 - sTWU + TU^2 - T^2W - WU^2\} \end{aligned}$$

Forgetting the factor $(x+y+z)xyz$ from $\tilde{F}$, we obtain the equation

$$G(T,W,U) = (s+1)\cdot TW^2 + sTWU + TU^2 - T^2W - WU^2 = 0.$$

Therefore we see that the subvariety X''' is isomorphic to the complete intersection in $\mathbf{P}^1 \times \mathbf{P}^2 \times \mathbf{P}^2$ defined by the equations

$$\begin{aligned} G(T,W,U) &= (s+1)TW^2 + sTWU + TU^2 - T^2W - WU^2 = 0, \\ H(x,y,z) &= (s+1)xyz - (x+y+z)(yz+zx+xy) = 0. \end{aligned}$$

Next, considering the the birational transformation of $\mathbf{P}^2$ given by

$$T = Z\cdot(X+Y+Z), \quad W = -XY, \quad U = Y(X+Y+Z).$$

Then the polynomial $G(T,W,U)$ above can be transformed into the polynomial

$$\tilde{G}(X,Y,Z) = (s+1)XYZ - (X+Y+Z)(YZ+ZX+XY),$$

which is nothing but $H(X,Y,Z)$. Hence the complete intersection $X''' \subset \mathbf{P}^1 \times \mathbf{P}^2 \times \mathbf{P}^2$ is isomorphic to the fiber product $S \times_{\mathbf{P}^1} S \longrightarrow \mathbf{P}^1$ of the elliptic surface $f : S \longrightarrow \mathbf{P}^1$ defined by the equation $H(x,y,z) = 0$. Hence, X' defined in (5.1) is birationally equivalent to the self-product $S \times_{\mathbf{P}^1} S$ of S. Since the minimal resolution of S is isomorphic to the elliptic modular surface $S_{\Gamma_1(6)}$, X is birationally equivalent to the self product $\tilde{S}^2_{\Gamma_1(6)}$. Observing that the birational maps discussed above are all defined over $\mathbf{Q}$, this completes the proof of Theorem 5.1. □

6. Proof for the modularity of X (Theorem 3.1)

Since the birational map between Verrill's example X and $\tilde{S}^2_{\Gamma_1(6)}$ constructed in Section 5 is defined over $\mathbf{Q}$, we obtain the following Lemma.

Lemma 6.1. *The birational map constructed in Section* 5 *gives rise to the following isomorphism compatible with the Galois action up to bad primes* 2 *and* 3

$$H^3_{et}(\bar{X}, \mathbf{Q}_\ell) \simeq H^3_{et}(\bar{S}^2_{\Gamma_1(6)}, \mathbf{Q}_\ell).$$

Consequently, we have

$$L(X,s) = L(\tilde{S}^2_{\Gamma_1(6)}, s)$$

up to Euler factors at bad primes 2 *and* 3.

For a subgroup $\Gamma \subset \mathrm{SL}_2(\mathbf{Z})$, its projectivization $\Gamma/\{\pm I_2\}$ will be denoted by $P\Gamma$. By virtue of Theorems 4.1 and 5.1, Remark 4.2 and Lemma 6.1, the proof of Theorem 3.1 (the modularity of Verrill's Calabi-Yau threefold X) is reduced to the proof of the following lemma.

Lemma 6.2. *The projective image of* $\Gamma_1(N)$ *and* $\Gamma_0(N)$ *are the same in* $\mathrm{PSL}_2(\mathbf{Z})$ *if and only if* N *is a divisor of* 4 *or a divisor of* 6. *In particular,* $P\Gamma_1(6) = P\Gamma_0(6)$ *and so it follows that* $S_4(P\Gamma_1(6)) \simeq S_4(P\Gamma_0(6))$.

Proof. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix in $\Gamma_0(N)$ or $\Gamma_1(N)$. Then $ad \equiv 1 \pmod N$. The first assertion follows from the fact that the only positive integers N for which $ad \equiv 1 \pmod N$ implies that $a \equiv d \equiv \pm 1 \pmod N$ are the divisors of 4 and 6. □

Remark 6.1. Verrill [V] has an alternative proof of the modularity for X. Her proof is along the lines of Livné's paper [Li1]. It makes use of the Serre criterion [Sr] and Faltings results [F] to prove equality of two L-series. The main point was to show that *finitely many* Euler factors of two L-series coincide.

Remark 6.2. Our geometric proof of the modularity conjecture for X guarantees that the 2-dimensional Galois representations associated to X and that attached to the modular form are indeed isomorphic (i.e., have the same semi-simplification). This provides a confirmation to the conjecture of Fontaine and Mazur [FM].

We close this section by posing an open problem.

Problem 6.1. May one use the method of Wiles to establish the modularity of X? More concretely, find a single good prime ℓ and establish the modularity for the residual mod ℓ Galois representation associated to the rigid Calabi-Yau threefold X in question.

7. The intermediate Jacobians of rigid Calabi-Yau threefolds

In this section, we shall define the intermediate Jacobian for Calabi-Yau threefolds, following the exposition of Bloch [Bl].

Definition 7.1. Let X be a smooth projective Calabi-Yau threefold defined over $\mathbf{C}$. There is a Hodge filtration

$$H^3(X, \mathbf{C}) = F^0 \supset F^1 \supset F^2 \supset F^3 \supset (0)$$

where

$$\begin{aligned} F^1 &= H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \\ F^2 &= H^{3,0} \oplus H^{2,1} \\ F^3 &= H^{3,0}. \end{aligned}$$

This complex vector space also has a $\mathbf{Z}$-structure defined by the image $H^3(X, \mathbf{Z}) \to H^3(X, \mathbf{C})$.

Now we recall some key properties of Hodge filtrations not only for H^3 but for more general cases. We write $\mathcal{F}$ for generic filtrations. We have

$$\mathcal{F}^i \oplus \overline{\mathcal{F}^{2r-i}} \simeq H^{2r-1}(X, \mathbf{C}), \quad \mathcal{F}^i \cap \overline{\mathcal{F}^{2r-i-1}} \simeq H^{i,2r-i-1}$$

where $H^{i,j} \simeq H^j(X, \Omega^i_X)$ and $r \in \{1, 2, 3\}$. From this, one can compute that

$$\dim_{\mathbf{C}} \mathcal{F}^r = \frac{1}{2} \dim_{\mathbf{C}} H^{2r-1}(X, \mathbf{C})$$

and

$$\mathcal{F}^r \cap \mathrm{Image}(H^{2r-1}(X, \mathbf{Z}) \to H^{2r-1}(X, \mathbf{C})) = (0).$$

Therefore, the quotient

$$J^r(X) = H^{2r-1}(X, \mathbf{C})/(\mathcal{F}^r + H^{2r-1}(X, \mathbf{Z}))$$

is a compact complex torus, called the *intermediate Jacobian*. By the Poincaré duality, we have

$$H^{2r-1}(X, \mathbf{C})/\mathcal{F}^r \simeq \mathcal{F}^{4-r}\, H^{7-2r}(X, \mathbf{C})^*$$

where $*$ denotes $\mathbf{C}$-linear dual. It then follows that

$$J^r(X) \simeq \mathcal{F}^{4-r}\, H^{7-2r}(X, \mathbf{C})^*/H_{7-2r}(X, \mathbf{Z}).$$

Digression. Now assume that X is rigid, and take $r = 2$ and $\mathcal{F} = F$. Then we have

$$F^1 = H^{3,0} = F^2 = F^3$$

and

$$J^2(X) \simeq H^3(X, \mathbf{C})/(F^2 + H^3(X, \mathbf{Z})) \simeq H^{3,0}(X)^*/H_3(X, \mathbf{Z})$$

is a complex torus of dimension one. This means that there is an elliptic curve E such that $E(\mathbf{C}) \simeq J^2(X)$. We will formulate the following question.

Question 7.1. *Is it true that a rigid Calabi-Yau threefold defined over* $\mathbf{Q}$ *is modular if and only if the intermediate Jacobian is an elliptic curve defined over* $\mathbf{Q}$*?*

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Department of Mathematics
Faculty of Science, Kobe University
Rokko, Kobe 657-8501 Japan

Department of Mathematics
Queen's University
Kingston, Ontario K7L 3N6 Canada

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