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Stabilization of weak solutions to compressible Navier-Stokes equations*

By

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Abstract

In [17] the present authors investigated the stabilization of the weak solutions to space periodic problem for barotropic compressible Navier-Stokes equations. The main goal of this paper is to show the power of the method introduced in [17] by treating other boundary conditions. In fact, the only limitation of the method is potential external force and the validity of the Poincaré inequality for the velocity.

1. Introduction

We consider the barotropic Navier-Stokes equations in a bounded domain $\Omega \subset R^N$

$$(\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) - \mu_1 \Delta u - \mu_2 \nabla(\operatorname{div} u) = \rho f, \tag{1.1}$$

$$\rho_t + \operatorname{div}(\rho u) = 0, \qquad x \in \Omega, \qquad t > 0, \tag{1.2}$$

The unknown quantities are the density ρ and velocity u. The given data are the functional dependence $p = p(\rho)$ between the density ρ and the pressure p, viscosity constants $\mu_1 > 0$, $\mu_2 \geq \frac{N-2}{N}\mu_1$ and external forces density $f = f(x)$.

Along with the equations (1.1), (1.2) we consider such a boundary condition that the Poincaré inequality for velocity holds true. As a typical boundary condition of this type we consider the Dirichlet boundary condition

$$u(x,t) = 0, \qquad x \in \partial\Omega. \tag{1.3}$$

Other boundary conditions can be considered as for example periodic boundary conditions with certain symmetry (see our previous paper [17]) or no-stick boundary conditions. We explain all the modifications required by the latter case in Section 6. As usual we impose the initial conditions

$$u(x,0) = u_0(x), \qquad \rho(x,0) = \rho_0(x), \qquad x \in \Omega \tag{1.4}$$

with the given functions u_0, ρ_0.

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By stabilization of solutions in this context we mean that given weak solution of the problem (1.1)–(1.4) satisfying the energy inequality below, for any sequence $t_n \to \infty$ and some $r \geq 1$ there is a subsequence s_n and a function ρ_∞ such that

$$\lim_{n\to\infty} \int_\Omega |\rho(x, s_n) - \rho_\infty(x)|^r \, dx = 0. \tag{1.5}$$

It appears that if $f = \nabla g$ with a given smooth function g then (1.5) holds true with an equilibrium density ρ_∞, that is a solution of the rest state equations

$$\nabla p(\rho_\infty) = \rho_\infty f, \qquad x \in \Omega, \tag{1.6}$$

$$\int_\Omega \rho_\infty \, dx = M_0 := \int_\Omega \rho_0 \, dx, \qquad \rho_\infty \geq 0. \tag{1.7}$$

If (1.5) is proved and the rest state defined by (1.6), (1.7) is uniquely determined then, of course, (1.5) holds without restriction to subsequences.

There are many results in this direction in one space variable (see e.g. [1], [3], [20], [23]). In several space dimensions a related result has been proved in [13] and [19], when the data is a small perturbation of a constant equilibrium. Stability of stationary solutions is investigated in [21], and the convergence of global strong solution to a stationary one for small data is obtained in [22]. Stability of stationary solutions for the case of large external forces has been proved in [14], while a similar problem has been solved for heat-conducting compressible fluid in [18]. The uniform stability under permanently acting disturbances has been tackled in [15], while in [10] it is shown that all smooth, small-amplitude solutions are asymptotically incompressible. In [17] we started to investigate the unconditional stabilization of solutions to (1.1), (1.2) to test our technique on the space periodic problem with a certain symmetry. This paper was followed by [7], where the Dirichlet boundary conditions were considered and a different method was used.

Finally, let us note that in comparison to [17] the procedure had to be totally revisited since otherwise the Dirichlet boundary conditions produce additional boundary terms hardly to control. So the function ψ_ε below was redefined by (3.19) which on one hand eliminated the uncontrollable boundary terms but produced another term in (3.38) below (the integral $I_7^\varepsilon(t)$). This term can be estimated thanks to careful considerations on the boundary of Ω allowed by Lemma 3.7. Also we had to go into detailed analysis of the problem (3.19) (see the considerations after (3.46)).

2. Preliminaries

We adopt the usual notation, namely, $C^k(\cdot)$ for spaces of k-times continuously differentiable functions, $W^{k,q}(\cdot)$ for the Sobolev spaces of k-th order and power q, $L^q(\cdot)$ for Lebesgue spaces with power q. The norm in L^q will be denoted by $|\cdot|_q$ and in $W^{k,q}$ by $\|\cdot\|_{k,q}$. The outer normal to $\partial\Omega$ is denoted by ν. We also denote by $\mathscr{L}(X, Y)$ the space of linear continuous operators from a Banach space X into a Banach space Y. In what follows we will use the usual mollifier with respect to the variable t given by

$$(R_\varepsilon z)(t) := \int_{-\infty}^{\infty} \varphi_\varepsilon(t-s)z(s)ds := \frac{1}{\varepsilon}\int_{-\infty}^{\infty} \varphi_0\Big(\frac{t-s}{\varepsilon}\Big)z(s)ds,$$

where $\operatorname{supp}\varphi_0 \subset (-1, 1)$, $\int_{-\infty}^{\infty} \varphi_0(s)ds = 1$.

Note that for vector functions u satisfying (1.3) we have the *Poincaré inequality*

$$\int_\Omega |u|^2\, dx \leq \text{const.} \int_\Omega |\nabla u|^2\, dx. \tag{2.1}$$

Finally, let us point out that given a smooth function θ, and if ρ satisfying (1.2) is sufficiently regular then the so-called renormalized equation of continuity

$$\theta(\rho)_t + \operatorname{div}(\theta(\rho)u) + (\rho\theta'(\rho) - \theta(\rho))\operatorname{div} u = 0, \tag{2.2}$$

holds true in the weak sense. It can be proved via regularization of ρ and [12], Vol. I, Lemma 2.3 that if θ' is bounded and $\rho \in L^2_{\text{loc}}(\bar\Omega \times [0, \infty))$ then the result holds (for $p(\rho) \sim \rho^\gamma$ this corresponds to the requirement $\gamma \geq 2$). In [4] it is shown that there is a weak solution of (1.1)–(1.4) for which the density ρ satisfies better global integral estimate making it possible to relax the growth condition for the state equation function $p(\cdot)$, ($p \sim \rho^\gamma$ requires $\gamma > 1$ for $N = 2$ and $\gamma > 3/2$ for $N = 3$).

3. Global uniform estimates

In this section we derive some global uniform estimates of a weak solution to the problem (1.1)–(1.4). We begin with the fundamental assumptions:

(i) $f = \nabla g$, $g \in W^{2,\infty}(\Omega)$, $\partial\Omega \in C^2$;

(ii) $p(\cdot) \in C^1([0, \infty))$, $p(0) = 0$, $p'(r) > 0$ for $r > 0$, $\liminf_{r\to\infty} p(r) = \infty$ and there exist constants $\gamma > 1$ and C such that

$$r^\gamma \leq C(1 + P(r)), \qquad \text{where } P(r) := \int_1^r \int_1^s \frac{p'(\tau)}{\tau}\, d\tau ds, \tag{3.1}$$

$\gamma > 1$ for $N = 2$, $\gamma > 3/2$ for $N = 3$, in particular, in what follows, for the sake of simplicity, we assume $p(r) = Ar^\gamma$ with a positive constant $A := p(1)$;

(iii) $\rho_0 \in L^\gamma(\Omega)$, $\sqrt{\rho_0}u_0 \in L^2(\Omega)$, $\rho_0 \geq 0$.

In the forthcomming text we use the following definition of the weak solution:

Definition 3.1 (weak solution). By a weak solution of (1.1)–(1.4) we call a couple (u, ρ) such that putting $Q_T = \Omega \times (0, T)$, for any $T > 0$ and any $\eta \in C^1(0, T; C_0^\infty(\Omega; R^N))$, $\zeta \in C^1(0, T; C^\infty(\Omega; R))$ such that $\eta(x, T) \equiv 0$, $\zeta(x, T) \equiv 0$ and we have $\rho, p(\rho), \rho|u|^2$, $|\nabla u|^2 \in L^1_{\text{loc}}(\bar\Omega \times [0, \infty))$ and

$$\int_{Q_T} (\rho u\eta_t + \rho((u\cdot\nabla)\eta\cdot u) - \mu_1\nabla u\cdot\nabla\eta - \mu_2 \operatorname{div} u \operatorname{div}\eta$$

$$+ p(\rho)\operatorname{div}\eta + \rho f\cdot\eta)dxdt + \int_\Omega \rho_0(x)u_0(x)\eta(x,0)dx = 0, \tag{3.2}$$

$$\int_{Q_T} (\rho\zeta_t + \rho(u\cdot\nabla)\zeta)dxdt + \int_\Omega \rho_0(x)\zeta(x,0)dx = 0. \tag{3.3}$$

Remark 3.2. Let us note that due to assumed regularity of (ρ, u) and the assumption (3.1) the smoothness of the test functions η, ζ in the definition 3.1 can be relaxed to the following inclusions:

for $N = 2$:

$$\eta \in W^{1,2}(0,T; L^{\gamma'+\delta}(\Omega)) \cap L^\infty(0,T; W^{1,\gamma'+\delta}(\Omega)) \cap L^1(0,T; W^{1,\infty}(\Omega)),$$

$$\zeta \in W^{1,2}(0,T; L^{\gamma'}(\Omega)) \cap L^2(0,T; W^{1,\gamma'}(\Omega)), \quad \text{with } \delta > 0 \text{ arbitrarily small.}$$

$$\left(\gamma' = \frac{\gamma}{\gamma-1}\right); \tag{3.4}$$

for $N = 3$:

$$\eta \in W^{1,2}(0,T; L^{6\gamma/(5\gamma-6)}(\Omega)) \cap L^\infty(0,T; W^{1,3\gamma/(2\gamma-3)}(\Omega)) \cap L^1(0,T; W^{1,\infty}(\Omega)),$$

$$\zeta \in L^1(0,T; L^{\gamma/(\gamma-1)}(\Omega)) \cap L^2(0,T; W^{1,6\gamma/(5\gamma-6)}(\Omega)). \tag{3.5}$$

We shall make further basic assumptions:

(iv) *We are given a globally defined weak solution* (ρ, u) *of* (1.1)–(1.4) *satisfying the energy inequality*

$$\frac{d}{dt}\int_\Omega \left(\frac{1}{2}\rho|u|^2 + P(\rho) - \rho g\right)dx + \mu_1\int_\Omega |\nabla u|^2\,dx + \mu_2\int_\Omega |\operatorname{div} u|^2\,dx \le 0. \tag{3.6}$$

in the sense of distributions;

(v) *for a function* θ *constructed in the proof of Lemma* 3.4 *below the equation* (2.2) *holds in the weak sense up to the boundary, i.e.*

$$\int_{Q_T} (\zeta_t\theta(\rho) + \theta(\rho)(u\cdot\nabla)\zeta + (\theta(\rho) - \rho\theta'(\rho))\operatorname{div} u\zeta)dxdt = 0$$

$$\textit{for all } \zeta \in C_0^\infty((0,T); C^\infty(\Omega; R)).$$

The definition 3.1 and energy inequality (3.6) imply the following

Proposition 3.3. *Let* $f \in W^{1,2}(\Omega)$, $f = \nabla g$ *and let the assumptions* (iii) *and* (iv) *be satisfied. Then for this globally defined weak solution of* (1.1)–(1.4) *we have*

$$\rho, P(\rho) \in L^\infty(0,\infty; L^1(\Omega)), \quad u \in L^2(0,\infty; W^{1,2}(\Omega)). \tag{3.7}$$

In particular,

$$\lim_{t\to\infty}\int_{t-a}^{t+a}|\nabla u(s)|_2^2\,ds=0 \qquad \textit{for any } a>0.$$

For the existence of the global weak solution of the problem (1.1)–(1.4) we refer to [11], [12] and [6]. For our given solution we want to prove convergence of $\rho(t)$ as t tends to infinity in $L^r(\Omega)$ for some $r\geq 1$ either on subsequences or completely (see Theorem 4.4). This requires a series of global estimates which will form the main body of this section.

For the technical reasons let us prolong the state equation function $p(\cdot)$ to the negative part of real axis by

$$p(r)=-p(-r). \tag{3.8}$$

Denote

$$M_0:=\int_\Omega \rho_0\,dx. \tag{3.9}$$

By the use of appropriate test functions in the equation of continuity and a certain limiting argument it can be shown that (1.2) implies

$$M_0=\int_\Omega \rho(t)dx, \qquad t>0. \tag{3.10}$$

A similar argument is used below and so we do not present the proof here.

Regularize the density in t by means of the usual mollifier (see Section 2)

$$\rho_\varepsilon(x,t):=R_\varepsilon(\rho(x,\cdot))(t), \qquad x\in\Omega, t>0, \varepsilon>0. \tag{3.11}$$

For each $s>0$ define $w_\varepsilon(s)$ as a (unique) generalized solution of the Neumann problem

$$\int_\Omega \nabla w_\varepsilon(s)\cdot\nabla\eta\,dx=\int_\Omega \rho_\varepsilon(s)f\cdot\nabla\eta\,dx, \qquad \eta\in W^{1,\gamma/(\gamma-1)}(\Omega),$$

$$\int_\Omega w_\varepsilon(x,s)dx=0. \tag{3.12}$$

Classical results on the generalized elliptic boundary-value problems together with Proposition 3.3 and assumption (3.1) give us existence of $w_\varepsilon(s)$ for which there is a constant independent of ε and s such that the following estimate is satisfied:

$$\begin{aligned}\|w_\varepsilon(s)\|_{1,\gamma}&\leq C|\rho_\varepsilon(s)f|_\gamma\leq C\sup_{\tau>0}|\rho_\varepsilon(\tau)|_\gamma\,|f|_\infty\\ &\leq C\sup_{\tau>0}|\rho(\tau)|_\gamma\leq C<\infty, \qquad s>0, \varepsilon>0\end{aligned} \tag{3.13}$$

(C is always a generic constant). Note that by (ii) it follows from (3.13)

$$\sup\{|w_\varepsilon(s)|_{\gamma^*}: s\geq 0\}\leq C<\infty, \qquad \varepsilon>0, \tag{3.14}$$

where $\gamma^* := \dfrac{N\gamma}{N-\gamma}$ if $\gamma < N$, $\gamma^* := q > 1$ sufficiently large if $\gamma = N$ and $\gamma^* = \infty$ if $\gamma > N$.

In the following lemma an auxiliary function θ is introduced which will be helpful later.

Lemma 3.4. *There exists a positive constant c_0 and a bounded increasing continuously differentiable function θ on R with $\lim_{r\to\infty} r\theta'(r) = 0$, such that*

$$(p(r_1) - p(r_2))(\theta(r_1) - \theta(r_2)) \geq c_0(\theta(r_1) - \theta(r_2))^2 \qquad \textit{for all } r_1, r_2 \in R.$$

Proof. Let $r_0 > 0$, $\theta(r) = p(r)$ for $0 \leq r \leq r_0$ and θ be concave, bounded with $\theta'(r) < p'(r)$ in (r_0, ∞), $\lim_{r\to\infty} r\theta'(r) = 0$. Extend θ to $(-\infty, 0)$ by $\theta(r) = -\theta(-r)$ for $r < 0$. Clearly θ has the desired property.

Now, let us introduce

$$G_\varepsilon(s,m) := \int_\Omega (\theta \circ p^{-1})(w_\varepsilon(s) + m)dx, \qquad s > 0, m \in R, \varepsilon > 0. \tag{3.15}$$

Clearly, for each $s > 0$, $\varepsilon > 0$, we have $\lim_{m\to\pm\infty} G_\varepsilon(s,m) = \pm|\Omega|\sup_{r\geq 0}\theta(r)$ and $\dfrac{\partial G_\varepsilon}{\partial m}(s,m) > 0$ for $m \in R$. Since for almost all (x,s), $\rho(x,s)$ is finite we have

$$\begin{aligned}\int_\Omega R_\varepsilon\theta(\rho(x,\cdot))(s)dx &= \int_\Omega\int_{-\infty}^{\infty} \varphi_\varepsilon(s-\tau)\theta(\rho(x,\tau))d\tau dx \\ &< \theta(\infty)\int_\Omega\int_{-\infty}^{\infty} \varphi_\varepsilon(s-\tau)d\tau dx = \theta(\infty)|\Omega|.\end{aligned}$$

So $\int_\Omega R_\varepsilon\theta(\rho(x,\cdot))(s)dx$ lies in the range of $G_\varepsilon(s,\cdot)$ and for any fixed $s \geq 0$ and $\varepsilon > 0$ the equation

$$G_\varepsilon(s, m_\varepsilon(s)) = \int_\Omega R_\varepsilon\theta(\rho(x,\cdot))(s)dx, \qquad (s > 0, \varepsilon > 0), \tag{3.16}$$

has a unique solution $m = m_\varepsilon(s)$. Now, let us define

$$\bar{\rho}_\varepsilon(s) := p^{-1}(w_\varepsilon(s) + m_\varepsilon(s)). \tag{3.17}$$

Then by (3.15), (3.16) we have

$$\int_\Omega \theta(\bar{\rho}_\varepsilon(s))dx = \int_\Omega R_\varepsilon\theta(\rho(\cdot))(s)dx, \qquad s > 0, \varepsilon > 0. \tag{3.18}$$

Finally, we define another auxiliary function $\psi_\varepsilon(s)$ as a solution of

$$\begin{aligned}&\operatorname{div}\psi_\varepsilon(s) = R_\varepsilon\theta(\rho)(s) - \theta(\bar{\rho}_\varepsilon(s)) \qquad \text{in } \Omega, \\ &\psi_\varepsilon(x,s) = 0, \qquad x \in \partial\Omega, s > 0, \varepsilon > 0.\end{aligned} \tag{3.19}$$

This problem is not uniquely solvable but it is known (see e.g. [9], (3.8)) that one possible solution operator is given by

$$\psi_\varepsilon(x,s) := S(R_\varepsilon\theta(p)(s) - \theta(\bar{p}_\varepsilon(s)))(x) = \int_\Omega K(x,y)(R_\varepsilon\theta(p)(s) - \theta(\bar{p}_\varepsilon(s))dy, \quad (3.20)$$

where K is an explicitly defined weakly singular kernel (see [9], (3.8)). By [9], Theorem 3.1, $S \in \mathscr{L}(L^q(\Omega), W^{1,q}(\Omega))$ for any $q \in (1,\infty)$. With our particular choice of θ, for any $q \in [1,\infty)$ we get

$$\|\psi_\varepsilon(s)\|_{1,q} \leq C < \infty \qquad \text{with } C \text{ independent of } \varepsilon \text{ and } s. \quad (3.21)$$

Further, by (3.18) we have

$$\int_\Omega (\theta \circ p^{-1})(w_\varepsilon(s) + m_\varepsilon(s))dx = \int_\Omega R_\varepsilon\theta(p(\cdot))(s)dx.$$

Show that

$$|m_\varepsilon(s)| \leq C < \infty, \qquad \varepsilon > 0, s > 0. \quad (3.22)$$

If we had

$$|\Omega|\theta(-\infty) + \delta < \int_\Omega R_\varepsilon\theta(p(\cdot))(s)dx < |\Omega|\theta(\infty) - \delta \qquad \text{for } \varepsilon \in (0,1), s > 0 \quad (3.23)$$

with some positive δ then (3.22) would clearly follow. Denoting $\Omega(s) := \{x \in \Omega: p(x,s) \leq M\}$ we have

$$\int_\Omega R_\varepsilon\theta(p)dx = \int_{\Omega(s)} R_\varepsilon\theta(p)dx + \int_{\Omega\setminus\Omega(s)} R_\varepsilon\theta(p)dx$$
$$\leq \theta(M)|\Omega(s)| + \theta(\infty)(|\Omega| - |\Omega(s)|).$$

Show that there exists $M > 0$ such that $\inf_{s\in R}|\Omega(s)| > 0$. If this is not the case then there are s_n such that $|\Omega(s_n)| \to 0$ and $\rho(x,s_n) \geq n$ for any $x \in \Omega\setminus\Omega(s_n)$. Hence

$$n|\Omega\setminus\Omega(s_n)| \leq \int_{\Omega\setminus\Omega(s_n)} \rho(x,s_n)dx \leq |\Omega|^{(\gamma-1)/\gamma}|\rho(\cdot,s_n)|_\gamma \leq C\sup_s|\rho(\cdot,s)|_\gamma < \infty,$$

which is a contradiction when $n \to \infty$. So we have proved the right-hand part of the inequality (3.23). Since the left part is trivial due to positivity of ρ, (3.22) readily follows.

Lemma 3.5. *Under the assumptions* (i)–(iv) *there exist the limits*

$$\lim_{\varepsilon\to 0+} w_\varepsilon = \overline{w} \qquad \text{in } L^r_{\mathrm{loc}}([0,\infty); W^{1,\gamma}(\Omega)) \qquad \text{for any } r > 1$$
$$\lim_{n\to\infty} m_{\varepsilon_n} = m, \qquad \text{in } L^\infty_{\mathrm{loc}}(0,\infty) \quad (3.24)$$

for some $\varepsilon_n \to 0+$. *In particular,*

$$p(\bar{\rho}_{\varepsilon_n}) \to \bar{w} + m =: p(\bar{\rho}) \tag{3.25}$$

in the above sense.

Proof. By (3.12), (3.13) we can write

$$w_\varepsilon(s) = A\rho_\varepsilon(s), \qquad \text{where } A \in \mathscr{L}(L^\gamma(\Omega), W^{1,\gamma}(\Omega)). \tag{3.26}$$

Since $\rho \in L^\infty(0,\infty; L^\gamma(\Omega))$, by the continuity of mollifiers we have

$$\lim_{\varepsilon \to 0+} \rho_\varepsilon = \rho \qquad \text{in any } L^r_{\mathrm{loc}}([0,\infty); L^\gamma(\Omega)), \qquad r \in [1,\infty). \tag{3.27}$$

Then using (3.26) we get $(3.24)_1$ with

$$\bar{w}(s) = A\rho(s), \qquad s > 0. \tag{3.28}$$

Show that $\{m_\varepsilon\}_{\varepsilon \in (0,1)}$ are bounded in $W^{1,q}(0,T)$ for any $T > 0$ and some $q > 1$. Indeed, by (3.52), (3.54) below we have

$$|m'_\varepsilon(s)| \le C\left(R_\varepsilon(|(\rho u)(s)|_r) + |\nabla u(s)|_2 + \frac{1}{\varepsilon}\varphi_0(s/\varepsilon)\right) \to C(|(\rho u)|_r + |\nabla u(s)|_2)$$

in $L^q(0,T)$ for some $q > 1$ and $r > 1$. Since by (3.22) $m_\varepsilon(s)$ are uniformly bounded, $(3.24)_2$ follows by embedding theorem.

Put now

$$Q(t) := \int_{t-1}^{t}\int_\Omega (p(\rho(s)) - p(\bar{\rho}(s)))(\theta(\rho(s)) - \theta(\bar{\rho}(s)))dxds, \qquad t \ge 1. \tag{3.29}$$

By monotonicity of $p(\cdot)$ and θ we have $Q(t) \ge 0$. Our intention now is to prove the following global property of Q.

Lemma 3.6. *Let $\bar{\rho}$ be a function defined in* (3.25) *of Lemma* 3.4. *Then the function $Q(t)$ defined by* (3.29) *satisfies*

$$\lim_{t \to \infty} Q(t) = 0. \tag{3.30}$$

Proof. Let $a > 1$, $\varphi \in C_0^\infty(-a,a)$, $\varphi \ge 0$, $\varphi(\sigma) = 1$ for $\sigma \in (-1,0)$. Put

$$Q_a^\varepsilon(t) := \int_{t-a}^{t+a} \varphi(s-t)\int_\Omega (p(\rho(s)) - p(\bar{\rho}_\varepsilon(s)))(R_\varepsilon\theta(\rho(s)) - \theta(\bar{\rho}_\varepsilon(s)))dxds. \tag{3.31}$$

Then clearly

$$\begin{aligned} Q_a^\varepsilon(t) = &\int_{t-a}^{t+a} \varphi(s-t)\int_\Omega (p(\rho(s)) - p(\bar{\rho}_\varepsilon(s)))(\theta(\rho(s)) - \theta(\bar{\rho}_\varepsilon(s)))dxds \\ &+ \int_{t-a}^{t+a} \varphi(s-t)\int_\Omega (p(\rho(s)) - p(\bar{\rho}_\varepsilon(s)))(R_\varepsilon\theta(\rho(s)) - \theta(\rho(s)))dxds, \end{aligned} \tag{3.32}$$

where the last term on the right-hand side of (3.32) tends to zero as $\varepsilon \to 0+$. By Lemma 3.5

$$\lim_{n\to\infty}\int_{t-1}^{t}\int_{\Omega}(p(\rho(s)) - p(\bar{\rho}_{\varepsilon_n}(s)))(\theta(\rho(s)) - \theta(\bar{\rho}_{\varepsilon_n}(s)))dxds = Q(t), \qquad t > 1 \quad (3.33)$$

for some $\varepsilon_n \downarrow 0$. Now we wish to estimate $Q_a^{\varepsilon}(t)$. Denote

$$V_a(t) := \{(x, s) : x \in \Omega, t - a < s < t + a\}. \tag{3.34}$$

By Definition 3.1 and Remark 3.2, and, with regard to the definition and smoothness of ψ_ε (we shall return to the differentiability of ψ_ε with respect to t later in more detail), we can write

$$\begin{aligned}
&\int_{V_a(t)}\varphi(s-t)p(\rho(s))(R_\varepsilon\theta(\rho)(s) - \theta(\bar{\rho}_\varepsilon(s)))dxds \\
&= \int_{V_a(t)}\varphi(s-t)p(\rho(s))\mathrm{div}\,\psi_\varepsilon(s)dxds \\
&= \int_{V_a(t)}\varphi(s-t)(-\rho u(s)\psi_{\varepsilon t}(s) - \rho u\cdot(u\cdot\nabla)\psi_\varepsilon(s) + \mu_1\nabla u(s)\nabla\psi_\varepsilon(s) \\
&\quad + \mu_2\,\mathrm{div}\,u(s)(R_\varepsilon\theta(\rho)(s) - \theta(\bar{\rho}_\varepsilon(s)) - \rho(s)f\cdot\psi_\varepsilon(s))dxds \\
&\quad - \int_{V_a(t)}\varphi'(s-t)\rho u(s)\psi_\varepsilon(s)dxds.
\end{aligned} \tag{3.35}$$

Now, take the Helmholtz decomposition of $\psi_\varepsilon(s)$, that is

$$\psi_\varepsilon(s) = \nabla z_\varepsilon(s) + v_\varepsilon(s), \qquad \mathrm{div}\,v_\varepsilon(s) = 0 \text{ in } \Omega, \qquad v_\varepsilon(s)\cdot\nu = 0 \text{ in } \partial\Omega.$$

By the usual construction of the decomposition and by (3.21) we have $\int_\Omega z_\varepsilon\,dx = 0$, $z_\varepsilon \in W^{2,q}(\Omega)$, $\left.\dfrac{dz_\varepsilon}{d\nu}\right|_{\partial\Omega} = 0$, $v_\varepsilon \in W^{1,q}(\Omega)$, ($q \in [1, \infty)$ arbitrary). Take into account the generalized formulation (3.12), namely

$$\int_\Omega(\nabla w_\varepsilon(s) - \rho_\varepsilon(s)f)\nabla\eta\,dx = 0 \qquad \text{for } \eta \in W^{1,\gamma/(\gamma-1)}(\Omega). \tag{3.36}$$

Then we find

$$\begin{aligned}
&\int_{V_a(t)}\varphi(s-t)p(\bar{\rho}_\varepsilon(s))(R_\varepsilon\theta(\rho)(s) - \theta(\bar{\rho}_\varepsilon(s)))dxds \\
&= \int_{V_a(t)}\varphi(s-t)w_\varepsilon(s)\mathrm{div}\,\psi_\varepsilon(s)dxds = -\int_{V_a(t)}\varphi(s-t)\nabla w_\varepsilon(s)\psi_\varepsilon(s)dxds \\
&= -\int_{V_a(t)}\varphi(s-t)\nabla w_\varepsilon(\nabla z_\varepsilon(s) + v_\varepsilon(s))dxds \\
&= -\int_{V_a(t)}\varphi(s-t)\nabla w_\varepsilon\nabla z_\varepsilon(s)dxds = -\int_{V_a(t)}\varphi(s-t)\rho_\varepsilon(s)f\cdot\nabla z_\varepsilon(s)dxds
\end{aligned} \tag{3.37}$$

Subtracting (3.35) and (3.37) we obtain

$$\int_{V_a(t)} \varphi(s-t)(p(\rho(s)) - p(\bar{\rho}_\varepsilon(s)))(R_\varepsilon\theta(\rho(s)) - \theta(\bar{\rho}_\varepsilon(s)))dxds$$

$$= \mu_1 \int_{V_a(t)} \varphi(s-t)\nabla u(s)\nabla\psi_\varepsilon(s)dxds$$

$$+ \mu_2 \int_{V_a(t)} \varphi(s-t)\operatorname{div} u(s)(R_\varepsilon\theta(\rho(s)) - \theta(\bar{\rho}_\varepsilon(s)))dxds$$

$$- \int_{V_a(t)} \varphi(s-t)\rho u(s)\cdot(u(s)\cdot\nabla)\psi_\varepsilon(s)dxds$$

$$- \int_{V_a(t)} \varphi'(s-t)\rho u(s)\psi_\varepsilon(s)dxds - \int_{V_a(t)} \varphi(s-t)\rho u(s)\psi_{\varepsilon t}(s)dxds$$

$$+ \int_{V_a(t)} \varphi(s-t)(\rho_\varepsilon(s) - \rho(s))f\cdot\nabla z_\varepsilon(s)dxds$$

$$- \int_{V_a(t)} \varphi(s-t)\rho(s)f\cdot v_\varepsilon(s)dxds =: \sum_{j=1}^{7} I_j^\varepsilon(t). \tag{3.38}$$

Estimate the integrals $I_j^\varepsilon(t)$ in (3.38) one by one. Starting with $I_1^\varepsilon(t)$ and taking into account (3.21), (3.13) we find

$$|I_1^\varepsilon(t)| \le C|\varphi|_\infty \int_{t-a}^{t+a} |\nabla u(s)|_2 \|\psi_\varepsilon(s)\|_{1,2}\, ds$$

$$\le C\int_{t-a}^{t+a} |\nabla u(s)|_2\, ds \le C\sigma_a(t), \tag{3.39}$$

where we denote

$$\sigma_a(t) := \left(\int_{t-a}^{t+a} |\nabla u(s)|_2^2\, ds\right)^{1/2}. \tag{3.40}$$

Quite analogously we get

$$|I_2^\varepsilon(t)| \le C\sigma_a(t). \tag{3.41}$$

To estimate $I_3^\varepsilon(t)$, notice that having

$$\gamma > 1 \text{ for } N = 2, \qquad \gamma > \frac{3}{2} \text{ for } N = 3, \tag{3.42}$$

we have

$$|I_3^\varepsilon(t)| = \left|\int_{t-a}^{t+a} \varphi(s-t)\int_\Omega \rho u_j u_k \partial_k \psi_{\varepsilon j}(s)dxds\right|$$

$$\le C|\varphi|_\infty \int_{t-a}^{t+a} |\rho(s)|_\gamma |u(s)|_{2q}^2 \|\psi_\varepsilon(s)\|_{1,r}\, ds$$

$$\le C\int_{t-a}^{t+a} |\nabla u(s)|_2^2\, ds \le C\sigma_a(t)^2 \tag{3.43}$$

with r arbitrarily large and $q = r\gamma(r\gamma - r - \gamma)^{-1}$ which requires $\gamma > 1$ for $N = 2$ and $\gamma \geq 3/2$ for $N = 3$. Continuing with $I_4^\varepsilon(t)$ we notice that if $\gamma \geq 6/5$ in the case $N = 3$, then

$$|I_4^\varepsilon(t)| \leq |\varphi'|_\infty \int_{t-a}^{t+a} |\rho(s)|_\gamma |u(s)|_{\gamma/(\gamma-1)} |\psi_\varepsilon(s)|_\infty \, ds$$

$$\leq C \int_{t-a}^{t+a} |\nabla u(s)|_2 |\rho(s)|_\gamma \, ds \leq C\sigma_a(t). \tag{3.44}$$

Next we estimate $I_5^\varepsilon(t)$. First, by (3.20) and the properties of the kernel K it is clear that the function $\psi_{\varepsilon t}$ exists and is sufficiently regular so that the integral $I_5^\varepsilon(t)$ is well defined. In addition, $\psi_{\varepsilon t} \in W^{1,r}(\Omega)$ for any $r > 1$. Denote $(D_h z)(t) := \frac{1}{h}(z(t+h) - z(t))$ for any Banach space valued function $z = z(t)$. The linearity of (3.12), the estimate (3.13) and regularity of ρ_ε provide that $(D_{1/n}\rho_\varepsilon)(s)$ is a Cauchy sequence in $L^\gamma(\Omega)$ for each fixed s, and consequently $D_{1/n}w_\varepsilon$ is a Cauchy sequence in $W^{1,\gamma}(\Omega)$. Thus $(\partial w_\varepsilon/\partial t)(t)$ exists and belongs to $W^{1,\gamma}(\Omega)$ for each $\varepsilon > 0$, $t > 0$. Further, the regularity of $w_\varepsilon(s)$ implies continuous differentiability of the function $G_\varepsilon(s, m)$ given by (3.15) with respect to s and since, as we have already proved, $\partial G_\varepsilon/\partial m$ is positive, the function $m_\varepsilon(s)$ given as unique solution of (3.16) is continuously differentiable. It follows that the function $\theta(\bar{\rho}_\varepsilon)$ with $\bar{\rho}_\varepsilon$ given by (3.17) has derivative with respect to s in $L^\infty_{\mathrm{loc}}([0,\infty); W^{1,\gamma}(\Omega))$. Then, by the same way as we proved the existence and regularity of $\partial w_\varepsilon/\partial t$ we can show that $\partial\psi_\varepsilon/\partial t(s)$ exists in the sense of $W^{1,r}(\Omega)$ with $r > 1$ arbitrary and for each s and $\varepsilon > 0$ it is an element of the same space.

Now we turn to the estimate for $\psi_{\varepsilon t}$. Let $\zeta \in W^{1,q/(q-1)}(\Omega)$. Then by the weak version of renormalized continuity equation we have (φ_0 is a regularization kernel)

$$\int_\Omega R_\varepsilon\theta(\rho)_t(s)\zeta \, dx = \frac{1}{\varepsilon^2}\int_\Omega \int_0^\infty \varphi_0'\left(\frac{s-\tau}{\varepsilon}\right)\theta(\rho(\tau))\zeta \, d\tau dx$$

$$= \frac{1}{\varepsilon}\int_0^\infty \int_\Omega \varphi_0\left(\frac{s-\tau}{\varepsilon}\right)\theta(\rho(\tau))u(\tau)\cdot\nabla\zeta \, dx d\tau + \frac{1}{\varepsilon}\varphi_0(s/\varepsilon)\int_\Omega \theta(\rho_0)\zeta \, dx$$

$$- \frac{1}{\varepsilon}\int_0^\infty \int_\Omega \varphi_0\left(\frac{s-\tau}{\varepsilon}\right)(\rho\theta'(\rho) - \theta(\rho))\mathrm{div}\, u(\tau)\zeta \, dx d\tau$$

$$= \int_\Omega R_\varepsilon(\theta(\rho)u(\cdot))(s)\cdot\nabla\zeta \, dx - \int_\Omega R_\varepsilon((\rho\theta'(\rho) - \theta(\rho))\mathrm{div}\, u)(\cdot)(s)\zeta \, dx$$

$$+ \frac{1}{\varepsilon}\varphi_0(s/\varepsilon)\int_\Omega \rho_0\zeta \, dx, \qquad s > 0, \qquad \varepsilon \text{ small}.$$

So, in the sense of distributions we have

$$(R_\varepsilon\theta(\rho))_t = -\text{div } R_\varepsilon(\theta(\rho)u) - R_\varepsilon((\rho\theta'(\rho) - \theta(\rho))\text{div } u) + \frac{1}{\varepsilon}\varphi_0(s/\varepsilon)\rho_0; \tag{3.45}$$

(notice that div $R_\varepsilon(\theta(\rho)u) \in C^\infty_{\text{loc}}(0, \infty; L^r(\Omega))$ for some $r > 1$). So by (3.20) we have

$$\psi_{\varepsilon t}(s) = S \text{ div } z + Sg - S\theta(\bar{\rho}_\varepsilon)_t, \tag{3.46}$$

$$z = -R_\varepsilon(\theta(\rho)u), \qquad g = R_\varepsilon((\theta(\rho) - \rho\theta'(\rho))\text{div } u) + \frac{1}{\varepsilon}\varphi_0(s/\varepsilon)\rho_0. \tag{3.47}$$

Firstly, it can be easily checked that z belongs to $\{w \in C^\infty(0, \infty; L^r(\Omega)),$ div $w \in C^\infty(0, \infty; L^s(\Omega)),\ w \cdot \nu|_{\partial\Omega} = 0\}$, where $r \in [1, 2)$, $s \in [1, 6)$ are such that $\frac{1}{6} + \frac{\alpha}{\gamma} = \frac{1}{r}$, $\frac{1}{2} + \frac{\alpha}{\gamma} = \frac{1}{s}$. Secondly, if $y = \text{div } \eta$, where $\eta \in L^r(\Omega)$ with $q \le r < \infty$ and $\eta \cdot \nu|_{\partial\Omega} = 0$ then $|S_y|_r \le \text{const } |\eta|_r$, (cf. [9], Lemma 3.5). Thus we have

$$|S \text{ div } z|_q + |Sg|_q \le C(|z|_q + |g|_{Nq/(N+q)})$$

whenever the norms on the right have sense. By (3.47) we have for $N = 2$, $|z|_q \le C|\nabla u|_2$, $|g|_{2q/(q+2)} \le C|g|_2 \le C(|\nabla u|_2 + \varepsilon^{-1}\varphi_0(s/\varepsilon)|\rho_0|_2)$, with $q > 1$ arbitrary, and, for $N = 3$, $|z|_6 \le C|\nabla u|_2$, $|g|_2$ as above. So we have proved

$$|S \text{ div } z|_q + |Sg|_q \le C\left(|\nabla u|_2 + \frac{1}{\varepsilon}\varphi_0(s/\varepsilon)\right) \qquad \text{with } q > 1 \text{ arbitrary if } N = 2$$
$$\text{and } q \le 6 \text{ if } N = 3. \tag{3.48}$$

We need here $\rho_0 \in L^2(\Omega)$ which is not assumed if $\gamma < 2$ but this can be overcome by regularization of ρ_0, since later the ε-dependent term disappears in the limit $\varepsilon \to 0+$. Further, we have

$$\frac{\partial G_\varepsilon}{\partial m}(s, m) = \int_\Omega (\theta \circ p^{-1})'(w_\varepsilon(s) + m)dx. \tag{3.49}$$

We claim that

$$c_0 := \inf_{\varepsilon, s} \int_\Omega (\theta \circ p^{-1})'(w_\varepsilon(s) + m_\varepsilon(s))dx > 0. \tag{3.50}$$

Indeed, if this was not true then there were $w_n \to \bar{w}$ weak in $W^{1,\gamma}(\Omega)$, $m_n \to \bar{m} \in R$ such that, by Fatou lemma

$$0 < \int_\Omega (\theta \circ p^{-1})'(\bar{w} + \bar{m})dx \le \liminf_{n \to \infty} \int_\Omega (\theta \circ p^{-1})'(w_n + m_n)dx = 0,$$

which is a contradiction. Consequently, by Implicit Function Theorem there exists the derivative $m_\varepsilon'(s)$ and we have

$$m_\varepsilon'(s) = \left(\int_\Omega (\theta \circ p^{-1})'(w_\varepsilon(s) + m_\varepsilon(s))dx\right)^{-1}$$
$$\times \left(\int_\Omega (R_\varepsilon\theta(\rho))_t - \int_\Omega (\theta \circ p^{-1})'(w_\varepsilon(x, s) + m_\varepsilon(s))w_{\varepsilon t}(x, s)dx\right), \tag{3.51}$$

which, with the help of (3.45), (3.48) yields

$$|m_\varepsilon'(s)| \le c_0^{-1}\left(\sup_{r\in R}(\theta\circ p^{-1})'(r)|w_{\varepsilon t}(s)|_1 + \left|\int_\Omega (R_\varepsilon\theta(\rho))_t\,dx\right|\right)$$

$$\le C\left(|w_{\varepsilon t}(s)|_1 + |\nabla u(s)|_2 + \frac{1}{\varepsilon}\varphi_0(s/\varepsilon)\right), \qquad s>0, \varepsilon>0. \tag{3.52}$$

Now, putting together (3.46), (3.47) and (3.52) we obtain

$$|\psi_{\varepsilon t}(s)|_q \le C\left(|\nabla u(s)|_2 + |w_{\varepsilon t}(s)|_1 + \frac{1}{\varepsilon}\varphi_0(s/\varepsilon) + |\theta(\bar\rho_\varepsilon(s))_t|_{Nq/(N+q)}\right). \tag{3.53}$$

By differentiation we get $|\theta(\bar\rho_\varepsilon(s))_t|_{Nq/(N+q)} \le C(|w_\varepsilon(s)|_{Nq/(N+q)} + |m_\varepsilon'(s)|)$. Show that

$$|w_{\varepsilon t}(s)|_r \le C\left(R_\varepsilon(|\rho u(s)|_r) + \frac{1}{\varepsilon}\varphi_0(s/\varepsilon)\right). \tag{3.54}$$

for any r for which the right-hand side has sense. To this purpose, let $\zeta\in C_0^\infty(\Omega)$, $\eta=\Delta\zeta$. Note that functions of this type are dense in any $L^r(\Omega)$, $1<r<\infty$. Then by the differentiability of w_ε and (3.12) we have

$$\int_\Omega(\rho_{\varepsilon t}f - \nabla w_{\varepsilon t})\nabla\zeta\,dx = 0.$$

Consequently by (3.45) with $\theta(\rho)=\rho$,

$$\int_\Omega w_{\varepsilon t}\eta\,dx = -\int_\Omega \rho_{\varepsilon t}f\cdot\nabla\zeta\,dx \le \|\rho_{\varepsilon t}\|_{-1,r}\|f\cdot\nabla\zeta\|_{1,r'}$$

$$\le C\|f\|_{1,\infty}\|\rho_{\varepsilon t}\|_{-1,r}|\eta|_{r'} \le C\left(R_\varepsilon(|\rho u(s)|_r) + \frac{1}{\varepsilon}\varphi_0(s/\varepsilon)\right)|\eta|_{r'},$$

which yields (3.54). So we have

$$|\psi_{\varepsilon t}(s)|_q \le C\left(R_\varepsilon(|\rho u(s)|_{Nq/(N+q)}) + |\nabla u(s)|_2 + \frac{1}{\varepsilon}\varphi_0(s/\varepsilon)\right), \qquad s,\varepsilon>0 \tag{3.55}$$

with q as in (3.48). With (3.55) in hands we estimate $I_5^\varepsilon(t)$:

$$|I_5^\varepsilon(t)| \le |\varphi|_\infty \sup_{s\ge 0}|\rho(s)|_\gamma \int_{t-a}^{t+a}|u(s)|_r|\psi_{\varepsilon t}(s)|_q ds$$

$$\le C\int_{t-a}^{t+a}|\nabla u(s)|_2|\psi_{\varepsilon t}(s)|_q ds, \tag{3.56}$$

where $\frac{1}{r}+\frac{1}{q}<\frac{\gamma-1}{\gamma}$ and, for $N=2$ we can take $r\in\left(\frac{\gamma}{\gamma-1},\infty\right)$ arbitrarily large so that $q>\frac{\gamma}{\gamma-1}$ can be chosen arbitrarily close to $\frac{\gamma}{\gamma-1}$, and for $N=3$, assuming

$\gamma > 3/2$ we fix $r = 6$, $q = \dfrac{6\gamma}{5\gamma - 6}$. Putting together (3.38), (3.39), (3.41), (3.43), (3.44), (3.56) and (3.55) we find

$$\left| \int_{V_a(t)} \varphi(s-t)(p(\rho(s)) - p(\bar{\rho}_\varepsilon(s)))(R_\varepsilon\theta(\rho(s)) - \theta(\bar{\rho}_\varepsilon(s)))dx \right|$$

$$\leq C\left(\sigma_a(t) + \int_{t-a}^{t+a} \left(R_\varepsilon(|\rho u(s)|_{Nq/(N+q)}) + |\nabla u(s)|_2 + \frac{1}{\varepsilon}\varphi_0(s/\varepsilon) \right) |\nabla u(s)|_2 \, ds \right)$$

$$+ |I_6^\varepsilon(t)| + |I_7^\varepsilon(t)| \tag{3.57}$$

with $\sigma_a(t)$ given by (3.40). Since we assume $t > a$ and φ_0 has finite support we have

$$\lim_{\varepsilon \to 0+} \frac{1}{\varepsilon}\varphi_0(s/\varepsilon) = 0 \qquad \text{for } s \geq t - a.$$

Moreover, $\dfrac{1}{\varepsilon}\varphi_0(s/\varepsilon) \leq C < \infty$ for $\varepsilon > 0$, $s \geq t - a$. By continuity of mollifier R_ε we have

$$R_\varepsilon(|\rho u(\cdot)|_{Nq/(N+q)}) \to |\rho u(\cdot)|_{Nq/(N+q)} \qquad \text{as } \varepsilon \to 0+ \ \text{ in } L^1(t-a, t+a; R^+)$$

Since, by our choice of q, $\dfrac{Nq}{N+q} < 2$, we have $|\rho u|_q \leq |\rho|^{1/2}_{Nq/(2N+(2-N)q)} |\sqrt{\rho}u|_2$, where $\dfrac{Nq}{2N + (2-N)q}$ is less than $\dfrac{11}{5}\gamma - 1$ if $N = 2$ and less than $\dfrac{5}{3}\gamma - 1$ if $N = 3$, so that Lemma 5.1 below applies. As

$$|I_6^\varepsilon(t)| \leq C|\varphi|_\infty |f|_\infty \sup_{s,\varepsilon} |\psi_\varepsilon(s)|_\infty \int_{t-a}^{t+a} |\rho_\varepsilon(s) - \rho(s)|_1 \, ds,$$

we have $\lim_{\varepsilon\to 0+} I_6^\varepsilon(t) = 0$. In addition, there exist $\varepsilon_n \downarrow 0$ such that (3.33) holds true. So putting in (3.57) $\varepsilon := \varepsilon_n$ and $n \to \infty$ we finally obtain

$$0 \leq \int_{V_a(t)} (p(\rho(s)) - p(\bar{\rho}(s)))(\theta(\rho(s)) - \theta(\bar{\rho}(s)))dx \leq C\Bigg(\sigma_a(t)$$

$$+ \sigma_a(t) \sup_{s>0} |\sqrt{\rho}u(s)|_2 (\sup_{s \geq a} \int_{s-a}^{s+a} |\rho(\tau)|_{Nq/(2N+(2-N)q)} \, d\tau)^{1/2} \Bigg)$$

$$+ \limsup_{n\to\infty} |I_7^{\varepsilon_n}(t)|. \tag{3.58}$$

It remains to estimate

$$I_7^\varepsilon(t) = \int_{V_a(t)} \varphi(s-t)\rho(s) f \cdot v_\varepsilon(s) dx ds. \tag{3.59}$$

where we know that

$$\sup_{\varepsilon,s}\|v_\varepsilon(s)\|_{1,q}<\infty,\qquad \operatorname{div} v_\varepsilon(s)=0 \text{ and } v_\varepsilon(s)\cdot\nu=0 \text{ on } \partial\Omega.$$

Let $\eta>0$ and $\kappa\in C_0^\infty(\Omega)$ be such that $|\operatorname{supp}(1-\kappa)|\le\eta$. Then $I_7^\varepsilon(s)$ can be decomposed as

$$\begin{aligned}I_7^\varepsilon(t)&=\int_{t-a}^{t+a}\varphi(s-t)\int_\Omega\rho(s)fv_\varepsilon(s)\kappa\,dxds\\&\quad+\int_{t-a}^{t+a}\varphi(s-t)\int_\Omega\rho(s)fv_\varepsilon(s)(1-\kappa)dxds=:J_1+J_2,\end{aligned}\tag{3.60}$$

and clearly

$$|J_2|\le 2a|\varphi|_\infty|f|_\infty\sup_{\varepsilon,s}|v_\varepsilon(s)||\operatorname{supp}(1-\kappa)|\le C\eta.\tag{3.61}$$

Since (ρ,u) is a weak solution of (1.1)–(1.4) we can write J_1 in the form

$$\begin{aligned}J_1&=\int_{t-a}^{t+a}\varphi(s-t)\int_\Omega(\mu_1\nabla u\nabla(\kappa v_\varepsilon)+\mu_2\operatorname{div}u\operatorname{div}(\kappa v_\varepsilon)-p(\rho)\operatorname{div}(\kappa v_\varepsilon)\\&\quad-\rho u\cdot((u\cdot\nabla)(\kappa v_\varepsilon))-\rho u\kappa u_{\varepsilon t})dxds-\int_{t-a}^{t+a}\varphi'(s-t)\int_\Omega\kappa(\rho uv_\varepsilon)(s)dxds\\&=\int_{t-a}^{t+a}\varphi(s-t)\int_\Omega(\mu_1\kappa\nabla u\cdot\nabla v_\varepsilon-\kappa\rho u\cdot((u\cdot\nabla)v_\varepsilon)-\rho u\kappa v_{\varepsilon t})dxds\\&\quad-\int_{t-a}^{t+a}\varphi'(s-t)\int_\Omega\kappa(\rho uv_\varepsilon)(s)dxds+\int_{t-a}^{t+a}\varphi(s-t)\int_\Omega(\mu_1(\nabla\kappa\cdot\nabla u)\cdot v_\varepsilon\\&\quad+\mu_2(\nabla\kappa\cdot v_\varepsilon)\operatorname{div}u-\rho(u\cdot\nabla\kappa)(u\cdot v_\varepsilon)-p(\rho)(\nabla\kappa\cdot v_\varepsilon))dxds=:\sum_{k=1}^8\bar J_k.\end{aligned}\tag{3.62}$$

The integrals $\bar J_1,\dots,\bar J_4$ can be estimated quite analogously as the integrals $I_1^\varepsilon(t),\dots,I_4^\varepsilon(t)$ in (3.38) and we leave the details to the reader. To estimate the remaining terms $\bar J_5$ to $\bar J_8$ we need the following lemma.

Lemma 3.7. *Let Ω be of class C^2. Then for any sufficiently small $\eta>0$ there exists a domain $\Omega_\eta\subset\Omega$ such that $\bar\Omega_\eta\subset\Omega$, $|\Omega\backslash\Omega_\eta|\le C\eta$ and if $x\in\partial\Omega$ then there is a unique $y=y(x)\in\partial\Omega_\eta$ such that $\nu(x)=\nu(y(x))$ and $|x-y(x)|=\eta$ for all $x\in\partial\Omega$. In addition, there is a function $\kappa\in W^{1,\infty}(\Omega)$ such that $\kappa(x)=1$ for $x\in\Omega_\eta$, $\kappa(x)=0$ for $x\in\Omega\backslash\Omega_{\eta/2}$, $|\nabla\kappa(x)|\le C\eta^{-1}$ for $x\in\Omega_{\eta/2}\backslash\Omega_\eta$ and $\left.\frac{d\kappa}{d\tau}\right|_{\partial\Omega_\eta}=0$, where τ is the tangential unit vector to the boundary.*

Proof. Asume that Ω is of class C^2, i.e. there are open sets $D_r\subset R^{N-1}$, functions $a_r\in C^2(D_r,R)$ and coordinate systems (x_r',x_{rN}), $r=1,\dots,m$, such that $\partial\Omega=\bigcup_{r=1}^m\{x\in R^N: x_{rN}=a_r(x_r')\}$, where $x_r'=(x_{r1},\dots,x_{r,N-1})$. Further, we assume that there exist $\beta_r > 0$, $r = 1, \dots, m$ such that $\{x \in R^N; x'_r \in D_r, a_r(x'_r) < x_{rN} < a_r(x'_r) + \beta_r\} \subset \Omega$. In the sequel we shall omit the indeces when working on a local part of the boundary. For $x \in \partial\Omega$ the unit normal vector $\nu(x) = (1 + |\nabla a(x')|^2)^{-1/2}(-\nabla a(x'), 1)^T$ points out of Ω. Let η be small enough (i.e. so small that it still holds $x_N - \delta < a_r(x' - \delta\nu(x)') + \beta$ for all $\delta \in (0, \eta)$; by continuity, η can be chosen uniformly with respect to $x' \in D_r$, $r = 1, \dots, m$). Then $S_\eta := \{y = x - \delta\nu(x); 0 < \delta \le \eta, x \in \partial\Omega\} \subset \Omega$. Define $\Omega_\eta := \Omega \backslash S_\eta$. Then $\partial\Omega_\eta = \{y \in R^N; y = x - \eta\nu(x), x \in \partial\Omega\}$. This can be proved by rather lengthy but routine argument and so we do not present it in detail.

Denote by $\nu^\eta(y)$ the unit exterior normal vector at y to $\partial\Omega_\eta$. Show that $\nu_\eta(y) = \nu(x)$, where $y = x - \eta\nu(x)$. Indeed, solving the system $y' = x' - \eta\nu(x)'$, we get $x' = \phi(y')$ and by differentiation

$$\frac{\partial\phi_k}{\partial y_j} = \delta_{kj} + \eta\frac{\partial}{\partial x_l}\left(\frac{\partial a}{\partial x_k}(1 + |\nabla a|^2)^{-1/2}\right)\frac{\partial\phi_l}{\partial y_j}.$$

Further, the local equation for $\partial\Omega_\eta$ near y is

$$y_N = a(\phi(y')) - \eta(1 + |\nabla a(\phi(y'))|^2)^{-1/2}.$$

So by an elementary calculation after evident cancellations we obtain

$$\frac{\partial y_N}{\partial y_j} = \frac{\partial a}{\partial x_j}(\phi(y')) = \frac{\partial a}{\partial x_j}(x'),$$

and the assertion is proved. Now, let k be a smooth function such that $k(\xi) = 0$ for $\xi \le \eta/2$ and $k(\xi) = 1$ for $\xi \ge \eta$. Define $\kappa(x) = 1$ on Ω_η and for $x \in \Omega\backslash\Omega_\eta$, $\kappa(x) := k(|x - y(x)|)$, where $(y(x), \mu(x))$ is a solution of the system

$$F_j(y, \mu) := y_j - x_j - \mu\nu_j(y) = 0, \qquad j = 1, \dots, N, \qquad F_{N+1}(y, \mu) := y_N - a(y') = 0$$

near the point $(y^0, \mu_0) := (x, 0)$. By Implicit Function Theorem the solution exists since

$$\det\frac{\partial F}{\partial x}\bigg|_{(x,0)} = (1 + |\nabla a(x)|^2)^{1/2}\det\begin{pmatrix} I_N & -\nu^T(x) \\ \nu(x) & 0 \end{pmatrix}$$

and the rows in the matrix can be shown linearly independent.

Finally, prove that $\dfrac{d\kappa}{d\tau}\bigg|_{\partial\Omega_\eta} = 0$, where τ is any vector orthogonal to $\nu^\eta(x)$ with $x \in \partial\Omega_\eta$. Since

$$\frac{\partial\kappa}{\partial x_j} = k'(|x - y(x)|)\frac{x_k - y_k}{|x - y(x)|}\left(\frac{\partial x_k}{\partial x_j} - \frac{\partial y_k}{\partial x_j}\right) = k'(|x - y(x)|)\left(\nu_k\frac{\partial y_k}{\partial x_j} - \nu_j\right),$$

by orthogonality of $\nu(y(x))$ and $\tau = \tau^\eta(x)$ we have

$$\frac{d\kappa}{d\tau} = \frac{\partial\kappa}{\partial x_j}\tau_j = k'(|x - y(x)|)\frac{\partial y_k}{\partial x_j}\nu_k\tau_j.$$

Differentiating the relations

$$y_k = x_k - \eta \frac{\partial a}{\partial x_k}(1+|\nabla a|^2)^{-1/2}, k = 1, \dots, N-1, \qquad y_N = x_N + \eta(1+|\nabla a|^2)^{-1/2}$$

by an elementary calculation we obtain

$$\frac{\partial y_k}{\partial x_j} v_k \tau_j = 0.$$

Finally, $|\nabla \kappa(x)| \le C(1+\sup\{|\nabla_x y(\xi)|; \xi \in \Omega_{\eta/2} \backslash \Omega_\eta\}) \sup_{\xi \in (\eta/2, \eta)} |k'(\xi)|$ for $x \in \Omega_{\eta/2} \backslash \Omega_\eta$ and it is clear that k can be chosen in such a way that $\sup_{\xi \in (\eta/2, \eta)} |k'(\xi)| \le C\eta^{-1}$. Since Ω is C^2, by preceding formulas we have $|\nabla \kappa(x)| \le C\eta^{-1}$ for all $x \in \Omega$.

All integrals $\bar{J}_5$ to $\bar{J}_8$ are of the type

$$\int_{t-a}^{t+a} \int_\Omega a(x,s) \nabla \kappa(x) v_\varepsilon(x,s)\,dx\,ds. \tag{3.63}$$

Given $x \in \Omega_{\eta/2} \backslash \Omega_\eta$ issue from x the ray which is a normal to $\partial\Omega_\eta$ at x_1 and to $\partial\Omega$ at x_2. Then $|x - x_2| \le \eta$. Further, since $v(x_2) \cdot \nu(x_2) = 0$, $\nu(x_2) = \nu(x_1)$ and $\nabla\kappa(x_1) \perp \nu(x_2)$, by Lemma 3.7, we have $\nabla\kappa(x) \cdot v(x_2) = 0$. Indeed, we might construct Ω_α with $\alpha = |x - x_2|$ and use the same argument as in Lemma 3.7 for Ω_η to show that $\nabla\kappa(x) \cdot \tau(x) = 0$ for any vector τ tangential to $\partial\Omega_\alpha$ at x. Consequently, by embedding theorem, we find

$$\begin{aligned} |\nabla\kappa(x) \cdot v_\varepsilon(x,s)| &= |\nabla\kappa(x)(v_\varepsilon(x,s) - v_\varepsilon(x_2,s))| \\ &\le \frac{C}{\eta} \|v_\varepsilon(s)\|_{1,q} |x - x_2|^{1-(N/q)} \le C\eta^{-N/q}, \qquad x \in \Omega_\eta \backslash \Omega_{\eta/2} \end{aligned} \tag{3.64}$$

with $q > 1$ arbitrary but fixed. So the integral (3.63) is estimated as follows

$$\left| \int_{t-a}^{t+a} \int_\Omega a(x,s) \nabla\kappa(x) v_\varepsilon(x,s)\,dx\,ds \right| \le C\eta^{-N/q} \int_{V_a(t)} |a(x,s)|\,dx\,ds,$$

and we get

$$\max\{|\bar{J}_5|, |\bar{J}_6|\} \le C\eta^{-N/q} \int_{t-a}^{t+a} \int_{\Omega_{\eta/2} \backslash \Omega_\eta} |\nabla u|\,dx\,ds \le C\eta^{(1/2)-(N/q)} \sigma_a(t), \tag{3.65}$$

$$\begin{aligned} |\bar{J}_7| &\le C\eta^{-N/q} \int_{t-a}^{t+a} \int_{\Omega_{\eta/2} \backslash \Omega_\eta} \rho |u|^2\,dx\,ds \\ &\le C\eta^{-N/q} \sup_s |\rho(s)|_\gamma |1|_{L^\alpha(\Omega_{\eta/2} \backslash \Omega_\eta)} \sigma_a(t)^2 \le C\eta^{(1/\alpha)-(N/q)} \sigma_a(t)^2, \end{aligned} \tag{3.66}$$

where $\alpha > \dfrac{\gamma}{\gamma - 1}$ for $N = 2$ and $\alpha = \dfrac{6\gamma}{5\gamma - 6}$, $(\gamma > 6/5)$ for $N = 3$. Taking q sufficiently large we see that $\bar{J}_k$, $k = 5, 6, 7$ tend to zero as $\eta \to 0+$. To estimate the last integral $\bar{J}_8$ in (3.62) we use Lemma 5.1 in Appendix. We have

$$|\bar{J}_8| \leq C\eta^{-N/q}\int_{t-a}^{t+a}\int_{\Omega_{\eta/2}\setminus\Omega_\eta} \rho^\gamma \, dxds$$

$$\leq C\eta^{-N/q}\left(\int_{V_a(t)} \rho^\beta \, dxds\right)^{\gamma/\beta} |\Omega_{\eta/2}\setminus\Omega_\eta|^{(\beta-\gamma)/\gamma} \leq C\eta^{(\beta-\gamma)/\gamma-(N/q)} \quad (3.67)$$

whenever

$$\int_{V_a(t)} \rho^\beta \, dxds \leq C < \infty. \quad (3.68)$$

Since the lemma 5.1 guarrantees the existence of $\beta > \gamma$ such that the estimate (3.68) holds true, we conclude that due to estimates for $\bar{J}_1$ to $\bar{J}_4$, (3.60)–(3.67) we have $|I_7^\varepsilon(t)| \leq C\sigma_a(t) + \omega(\eta)$, where $\lim_{\eta\to 0+}\omega(\eta) = 0$ and in particular, $\limsup_{n\to\infty} |I_7^{\varepsilon_n}(t)| \leq C\sigma_a(t)$. By (3.58), (3.32), (3.29), regularity of $\bar{\rho}_\varepsilon(s)$ and Lemma 3.5 combined with (3.52), (3.54) we have

$$0 \leq Q(t) \leq \lim_{n\to\infty} Q_a^{\varepsilon_n}(t) \leq C\sigma_a(t)$$

and (3.30) follows from Proposition 3.3.

4. Convergence of density

In this section we show that under the assumptions of Sec. 3, for any sequence $\{t_n\}$ there is a subsequence $\{s_n\} \subset \{t_n\}$ such that there exists an equilibrium state $\rho_\infty \geq 0$ satisfying

$$\lim_{n\to\infty} |\rho(s_n) - \rho_\infty|_r = 0. \quad (4.1)$$

with $r \in [1, \gamma)$ arbitrary. Hence, if the equilibrium state is unique then it follows that

$$\lim_{t\to\infty} |\rho(t) - \rho_\infty|_r = 0. \quad (4.2)$$

We start with the following trivial observation.

Proposition 4.1. *Let $q \in W^{1,1}_{\mathrm{loc}}(a, \infty)$, $(a \in R)$ be such that $q(s) \geq 0$ for $s \geq a$ and $\lim_{t\to\infty} \int_{t-1}^t (q(s) + |q'(s)|)ds = 0$. Then*

$$\lim_{t\to\infty} q(t) = 0. \quad (4.3)$$

Proof. Since

$$q(t) = q(s) + \int_s^t q'(\tau)d\tau$$

for $a + 1 \leq s < t < \infty$, by integration $\int_{t-1}^t ds$ we get

$$q(t) \leq \int_{t-1}^{t} q(s)ds + \int_{t-1}^{t}\int_{s}^{t} |q'(\tau)|d\tau ds \leq \int_{t-1}^{t} (q(s) + |q'(s)|)ds,$$

and (4.3) follows immediately.

Put

$$q(t) := |\theta(\rho(t)) - \theta(\bar{\rho}(t))|_2^2, \qquad t \geq 1, \tag{4.4}$$

where $\bar{\rho}(t)$ is the function from Lemma 3.5. Then by Lemma 3.4 we have

$$q(t) \leq C\int_{\Omega} (p(\rho(t)) - p(\bar{\rho}(t)))(\theta(\rho(t)) - \theta(\bar{\rho}(t)))dx \qquad \text{for } t \geq 0. \tag{4.5}$$

So it follows from Lemma 3.6 that

$$\lim_{t\to\infty} \int_{t-1}^{t} q(s)ds = 0. \tag{4.6}$$

Now we are going to prove that

$$\lim_{t\to\infty} \int_{t-1}^{t} |q'(s)|ds = 0. \tag{4.7}$$

This will be a consequence of the following Lemma.

Lemma 4.2. *Let $\bar{\rho}(t) = \lim_{n\to\infty} p^{-1}(w_{\varepsilon_n} + m_{\varepsilon_n})$ (cf. Lemma 3.4). Then we have*

$$\int_{1}^{\infty} \left| \frac{d}{dt}(|\theta(\rho(t)) - \theta(\bar{\rho}(t))|_2^2) \right|^2 dt \leq C\int_{1}^{\infty} |\nabla u(t)|_2^2 \, dt < \infty. \tag{4.8}$$

Proof. It suffices to prove that

$$\left| \int_{1}^{\infty} \eta'(s) \int_{\Omega} (\theta(\rho(s)) - \theta(\bar{\rho}(s)))^2 \, dxds \right| \leq C\|\eta\|_{L^2(0,\infty)} \|\nabla u\|_{L^2(0,\infty;L^2(\Omega))}, \tag{4.9}$$

for any $\eta \in C_0^{\infty}(1,\infty)$. First, by the renormalized equation of continuity we have

$$\begin{aligned} &\left| \int_{1}^{\infty} \eta'(s) \int_{\Omega} \theta(\rho(s))^2 \, dxds \right| \\ &\quad = \left| \int_{1}^{\infty} \eta(s) \int_{\Omega} (2\rho(s)\theta(\rho(s))\theta'(\rho(s)) - \theta(\rho(s))^2)\text{div } u(s)dxds \right| \\ &\quad \leq C\int_{1}^{\infty} |\eta(s)| \, |\nabla u(s)|_2 \, ds \leq C\|\eta\|_{L^2(0,\infty)} \|\nabla u\|_{L^2(0,\infty;L^2(\Omega))}. \end{aligned} \tag{4.10}$$

Second, we know that $\theta(\bar{\rho}_\varepsilon) = (\theta \circ p^{-1})(w_\varepsilon + m_\varepsilon)$, $w_{\varepsilon_n} \to \bar{w}$ in $L^r(\Omega)$ with $r < \dfrac{N\gamma}{N-\gamma}$ and weakly in $W^{1,\gamma}(\Omega)$, $m_{\varepsilon_n} \to \bar{m}$, $(n \to \infty)$. Hence $\bar{w} \in W^{1,\gamma}(\Omega)$ and consequently $(\theta \circ p^{-1})(\bar{w} + \bar{m}) = \theta(\bar{\rho}) \in W^{1,\gamma}(\Omega)$ as well. Now, again by renormalized equation of continuity, taking $\phi := \eta(s)\theta(\bar{\rho}(x,s))$ as a test function we get

$$
\begin{aligned}
&\left| \int_1^\infty \eta'(s) \int_\Omega \theta(\rho(s))\theta(\bar{\rho}(s))dxds \right| \\
&\quad = \left| \int_1^\infty \eta(s) \int_\Omega \theta(\rho(s))(u \cdot \nabla)\theta(\bar{\rho}(s))dxds \right. \\
&\qquad + \int_1^\infty \eta(s) \int_\Omega \theta(\rho(s))\theta(\bar{\rho}(s))_t \, dxds \\
&\qquad \left. + \int_1^\infty \eta(s) \int_\Omega (\rho(s)\theta'(\rho(s)) - \theta(\rho(s)))\mathrm{div}\, u(s)\theta(\bar{\rho}(s))dxds \right| \\
&\quad \le C \int_1^\infty |\eta(s)|(|u(s)|_6 |\nabla\theta(\bar{\rho}(s))|_{6/5}| + |\theta(\bar{\rho}(s))_t|_1 \\
&\qquad + |\nabla u(s)|_1)ds \le C\|\eta\|_{L^2(0,\infty)}(\|\nabla u\|_{L^2(0,\infty;L^2(\Omega))} + \|\theta(\bar{\rho}(s))_t\|),
\end{aligned} \tag{4.11}
$$

(for the estimate of the last term see an analogous estimate below). Finally, we have to estimate

$$
\int_1^\infty \eta'(s) \int_\Omega \theta(\bar{\rho}(s))^2 \, dxds = \lim_{n\to\infty} \int_1^\infty \eta'(s) \int_\Omega \theta(\bar{\rho}_{\varepsilon_n}(s))^2 \, dxds. \tag{4.12}
$$

It is clear that

$$
\begin{aligned}
\left| \int_1^\infty \eta'(s) \int_\Omega \theta(\bar{\rho}_\varepsilon(s))^2 \, dxds \right| &= 2\left| \int_1^\infty \eta(s) \int_\Omega \theta(\bar{\rho}_\varepsilon(s))\theta(\bar{\rho}_\varepsilon(s))_t \, ds \right| \\
&\le C \int_1^\infty |\eta(s)| \, |\theta(\bar{\rho}_\varepsilon(s))_t|_1 \, ds.
\end{aligned} \tag{4.13}
$$

By (3.52), (3.54) we have ($\delta > 0$ small enough)

$$
\begin{aligned}
|\theta(\bar{\rho}_\varepsilon(s))_t|_1 &\le C\left(|\rho u(s)|_{1+\delta} + |\nabla u(s)|_2 + \frac{1}{\varepsilon}\varphi_0(s/\varepsilon) \right) \\
&\le C\left(|\nabla u(s)|_2 + \frac{1}{\varepsilon}\varphi_0(s/\varepsilon) \right)
\end{aligned} \tag{4.14}
$$

having thus still the only restriction

$$
\gamma > \frac{3}{2} \qquad \text{for } N = 3. \tag{4.15}
$$

Thus by (4.12), (4.13), (4.14) we find

$$\left|\int_1^\infty \eta'(s)\int_\Omega \theta(\bar\rho(s))^2\,dxds\right| \le \limsup_{n\to\infty}\left|\int_1^\infty \eta'(s)\int_\Omega \theta(\bar\rho_{\varepsilon_n}(s))^2\,dxds\right|$$

$$\le C\int_1^\infty \eta(s)|\nabla u(s)|_2\,ds \le C\|\eta\|_{L^2(0,\infty)}\|\nabla u\|_{L^2(0,\infty;L^2(\Omega))}.$$

This completes the proof.

Now, by (4.4), (4.5), (4.6), (4.8) and Proposition 4.1 we get the following result. (Observe that $\int_{t-1}^t |q'(s)|ds \le (\int_{t-1}^t |q'(s)|^2\,ds)^{1/2} \to 0$ as $t\to\infty$ in virtue of (4.8).)

Lemma 4.3. *Let $\gamma > 1$ for $N = 2$ and $\gamma > 3/2$ for $N = 3$. Then for any $r \in [1,\infty)$,*

$$\lim_{t\to\infty} |\theta(\rho(t)) - \theta(\bar\rho(t))|_r = 0, \tag{4.16}$$

where θ is a function defined in Lemma 3.4.

Proof. The convergence of $q(t)$ given by (4.4) to zero follows from Lemma 3.6, Proposition 4.1 and Lemma 4.2. Since θ is bounded, (4.16) follows immediately.

Finally, we shall prove the following main theorem.

Theorem 4.4. *Under the assumptions of Lemma 4.3 (and assumptions (i)–(v) from Section 3) for any sequence $t_n\to\infty$ and any $r\in[1,\gamma)$ there exists a subsequence $\{s_n\}_{n=1}^\infty \subset \{t_n\}_{n=1}^\infty$ and a function $\rho_\infty \in L^\gamma(\Omega)$ satisfying (4.22) below with $\int_\Omega \rho_\infty\,dx = \int_\Omega \rho_0\,dx$ such that*

$$\lim_{n\to\infty}\int_\Omega |\rho(s_n) - \rho_\infty|^r\,dx = 0. \tag{4.17}$$

If, moreover, the above equilibrium is uniquely determined then

$$\lim_{t\to\infty}\int_\Omega |\rho(t) - \rho_\infty|^r\,dx = 0. \tag{4.18}$$

Proof. In (3.26) we have defined operator $A\in\mathscr{L}(L^\gamma(\Omega), W^{1,\gamma}(\Omega))$, and by (3.28) we have $\bar w(t) = A\rho(t)$ for $t\ge 0$. Let $t_n\uparrow\infty$ be arbitrary. Then we can select $\{s_n\}\subset\{t_n\}$, such that $\rho(s_n)\to\rho_\infty$ weakly in $L^\gamma(\Omega)$, $w(s_n)\to w_\infty = A(\rho_\infty)\in W^{1,\gamma}(\Omega)$ weakly in $W^{1,\gamma}(\Omega)$, strongly in $L^q(\Omega)$ with $q < \dfrac{N\gamma}{N-\gamma}$, and almost everywhere in Ω and also such that $m(s_n)\to m_\infty$ (since $m(\cdot)$ is bounded) and by Lemma 4.3 $\theta(\rho(s_n)) - \theta(\bar\rho(s_n))\to 0$ a.e. in Ω. Hence given $r\ge 1$, $\theta(\bar\rho(s_n))\to\theta\circ$

$p^{-1}(w_\infty + m_\infty)$, $\theta(\rho(s_n)) \to \theta \circ p^{-1}(w_\infty + m_\infty)$ a.e. in Ω and in $L^r(\Omega)$, and $\rho(s_n) \to p^{-1}(w_\infty + m_\infty)$ a.e. in Ω and by boundedness of $|\rho(s_n)|_\gamma$ also in $L^q(\Omega)$, $1 \le q < \gamma$. Since $\rho(s_n) \to \rho_\infty$ weakly in $L^\gamma(\Omega)$ we find $\rho_\infty = p^{-1}(A(\rho_\infty) + m_\infty)$, or

$$p(\rho_\infty) = A(\rho_\infty) + m_\infty. \tag{4.19}$$

Since $w(s_n) \to w_\infty$ weakly in $W^{1,\gamma}(\Omega)$ and $w(t_n)$ satisfy (3.36), we obtain

$$\int_\Omega (\nabla w_\infty - \rho_\infty f)\nabla\Theta\, dx = 0 \qquad \text{for all } \Theta \in C^\infty(\Omega),$$

or, since $w_\infty = p(\rho_\infty) - m_\infty$,

$$\int_\Omega (\nabla p(\rho_\infty) - \rho_\infty f)\nabla\Theta\, dx = 0 \qquad \text{for all } \Theta \in C^\infty(\Omega). \tag{4.20}$$

Now, given z in $C_0^\infty(\Omega)$ arbitrary, we have $z = \nabla\Theta + v$, where Θ is a solution of the problem

$$\begin{aligned} \Delta\Theta &= \operatorname{div} z \qquad \text{in } \Omega \\ \frac{d\Theta}{d\nu} &= 0 \qquad \text{on } \partial\Omega, \\ \int_\Omega \Theta\, dx &= 0, \end{aligned} \tag{4.21}$$

v is smooth enough, and $\operatorname{div} v = 0$ in Ω and $v \cdot \nu = 0$ on $\partial\Omega$. Let us prove

$$\nabla p(\rho_\infty) = \rho_\infty f. \tag{4.22}$$

Since (4.20) is valid it is only to prove that

$$\int_\Omega (\nabla p(\rho_\infty) - \rho_\infty f)v\, dx = 0. \tag{4.23}$$

As we have

$$\int_\Omega \nabla p(\rho_\infty) v\, dx = -\int_\Omega p(\rho_\infty) \operatorname{div} v\, dx = 0, \tag{4.24}$$

it remains to show

$$\int_\Omega \rho_\infty f \cdot v\, dx = 0. \tag{4.25}$$

It is clear that the proof will be complete when we show that

$$\lim_{n\to\infty} \int_\Omega \rho(s_n) f \cdot v\, dx = 0, \tag{4.26}$$

since then $0 = \lim_{n\to\infty} \int_\Omega \rho(s_n) f \cdot v\, dx = \int_\Omega \rho_\infty f \cdot v\, dx$ by the weak convergence of $\rho(s_n)$ to ρ_∞ in $L^\gamma(\Omega)$. To prove (4.26) it suffices to show that $\int_\Omega \rho(\cdot) f \cdot v\, dx \in W^{1,2}(0,\infty)$, i.e.

$$\left| \int_0^\infty \eta(s) \int_\Omega \rho(s) f \cdot v\, dx ds \right| + \left| \int_0^\infty \eta'(s) \int_\Omega \rho(s) f \cdot v\, dx ds \right|$$
$$\le C \|\eta\|_{W^{1,2}(0,\infty)} \qquad \text{for any } \eta \in C_0^\infty(0,\infty). \tag{4.27}$$

The following estimate is almost a repetition of the estimate of the integral $I_7^\varepsilon(t)$ given by (3.60) with the function κ defined in Lemma 3.7. By this technique we obtain the estimate

$$\left| \int_0^\infty \eta(s) \int_\Omega \rho(s) f \cdot v\, dx ds \right|$$
$$\le \mu_1 |\nabla v|_2 \|\nabla u\|_{L^2(0,\infty;L^2(\Omega))} \|\eta\|_{L^2(0,\infty)}$$
$$+ \int_0^\infty |\eta(s)|\, |\nabla v|_\infty |\rho(s)|_\gamma^{1/2} |\sqrt{\rho}(s) u(s)|_2 |u(s)|_6 ds + |v|_\infty \int_0^\infty |\eta'(s)| |\rho u(s)|_1\, ds$$
$$\le C \|\eta\|_{W^{1,2}(0,\infty)} \qquad (\text{if } \gamma > 3/2).$$

Further, from the weak equation of continuity

$$\left| \int_0^\infty \eta'(s) \int_\Omega \rho(s) f \cdot v\, dx ds \right| = \left| \int_0^\infty \eta(s) \int_\Omega \rho(s)(u(s) \cdot \nabla)(f \cdot v) dx \right|$$
$$\le C \|\eta\|_{L^2(0,\infty)} \qquad \text{if } \gamma \ge 6/5.$$

So, (4.26) is proved and this yields (4.25). Thus (4.23) is established. This completes the proof of the theorem.

Remark 4.5. If considering strictly positive equilibria densities only, the necessary and sufficient condition for uniqueness may be derived from [2]. If this condition is not satisfied then necessarily, $\{\rho_\infty(x) = 0\} \ne \emptyset$ and the optimal condition for uniqueness is given in [8].

5. Global estimate of density

The purpose of this section is to prove the global estimate (3.68) for certain $\beta > \gamma$. Let us formulate it as the following

Lemma 5.1. *Let $\gamma > 1$ for $N = 2$ and $\gamma > 3/2$ if $N = 3$. Then for any global weak solution (u, ρ) we have $\rho \in L^{\gamma+\alpha}_{\mathrm{loc}}(\bar{\Omega} \times [0,\infty))$ for some $\alpha > 0$. More precisely, if $N = 2$ then $\alpha \le \gamma - 1$ for $\gamma \in (1,2]$, $\alpha \le \gamma/2$ for $\gamma > 2$, and if $N = 3$ then $\alpha \le \frac{2}{3}\gamma - 1$ for $\gamma \in (1,6]$, $\alpha \le \gamma/2$ for $\gamma \ge 6$. To achieve $\alpha > 0$ we need $\gamma > 1$ for $N = 2$ and $\gamma > 3/2$ for $N = 3$.*

Proof. Let $\theta = \theta(r)$, $r \in R$ be a C^1, unbounded, nondecreasing function, $\theta(r) \geq 0$ for $r \geq 0$. For example, $\theta(r) = r^\alpha$ with α specified above. For $\delta > 0$ let θ_δ be a familly of positive smooth cut-off functions such that $\theta_\delta(r) \nearrow \theta(r)$ as $\delta \to 0+$ for $r \geq 0$ and $\theta_\delta'(r) \leq \theta'(r)$ for $r \geq 0$. Define $\psi_{\varepsilon,\delta}$ as a solution of

$$\begin{aligned} &\operatorname{div} \psi_{\varepsilon,\delta} = R_\varepsilon \theta_\delta(\rho) - MR_\varepsilon \theta_\delta(\rho), \qquad x \in \Omega, \\ &\psi_{\varepsilon,\delta}(x,t) = 0, \qquad x \in \partial\Omega, \\ &\int_\Omega \psi_{\varepsilon,\delta}\, dx = 0, \qquad t > 0, \end{aligned} \tag{5.1}$$

where $Mv := |\Omega|^{-1} \int_\Omega v\, dx$ for $v \in L^1(\Omega)$, $(R_\varepsilon z)(t) = \int_0^\infty \varphi_\varepsilon(t-s) z(s) ds$ is the mollifier in t. We consider again the solution of the type $\psi_{\varepsilon,\delta}(x,s) = S(R_\varepsilon \theta_\delta(\rho(\cdot,s)) - MR_\varepsilon \theta_\delta(\cdot,s))(x)$, where S is as in (3.20). Write in short $\psi_{\varepsilon,\delta} = \psi$. From the properties of the operator S listed in Section 3 we have $\psi \in W^{1,q}(\Omega)$ for any $q \in [1,\infty)$ and

$$\|\psi\|_{1,q} \leq \text{const}|\theta_\delta(\rho)|_q.$$

If $1 < q \leq \gamma/\alpha$ then

$$\|\psi\|_{1,q} \leq \text{const}|\rho|_\gamma^\alpha \leq C < \infty \qquad \text{uniformly for all } \varepsilon,\delta > 0, q \in (1,\gamma/\alpha]. \tag{5.2}$$

Hence ψ can be used as a test function in the definition of the weak solution. Let $a > 1$, $\varphi \in C_0^\infty(-a,a)$, $\varphi \geq 0$, $\varphi(\sigma) = 1$ for $\sigma \in (-1,0)$. Now, set in a weak formulation $\eta(x,s) = \varphi(s-t)\psi(x,s)$ for test function. We get

$$\begin{aligned} Q_a^\varepsilon(t) &:= \int_{t-a}^{t+a} \varphi(s-t) \int_\Omega p(\rho)(R_\varepsilon \theta_\delta(\rho) - M\theta_\delta(\rho)) dx ds \\ &= \int_{t-a}^{t+a} \varphi(s-t) \int_\Omega p(\rho) \operatorname{div} \psi(s) dx ds = \int_{t-a}^{t+a} \varphi(s-t) \int_\Omega (\mu_1 \nabla u \cdot \nabla \psi(s) \\ &\quad + \mu_2 \operatorname{div} u \operatorname{div} \psi - \rho((u\cdot\nabla)\psi \cdot u) - \rho f \cdot \psi - \rho u \cdot \psi_t) dx ds \\ &\quad - \int_{t-a}^{t+a} \varphi'(s-t) \int_\Omega \rho u \psi\, dx ds = \sum_{j=1}^6 I_j(t). \end{aligned} \tag{5.3}$$

Estimate integrals $I_1, \ldots, I_6$ one by one. Then we have

$$|I_1(t)| \leq \mu_1 |\varphi|_\infty \sigma_a(t) \left(\int_{V_a(t)} |\nabla\psi(s)|^2\, dx ds \right)^{1/2} \leq C\sigma_a(t) \tag{5.4}$$

as soon as $\gamma/\alpha \geq 2$. The same holds for $I_2(t)$:

$$|I_2(t)| \leq C\sigma_a(t). \tag{5.5}$$

For $I_3(t)$ we get

$$|I_3(t)| \leq |\varphi|_\infty \int_{t-a}^{t+a} |\rho|_\gamma |u|_6^2 |\nabla\psi|_{3\gamma/(2\gamma-3)}\, ds \leq C\sigma_a(t)^2 \tag{5.6}$$

if $\alpha \leq \gamma - 1$ for $N = 2$ and $\alpha \leq \frac{2}{3}\gamma - 1$ for $N = 3$. Further,

$$|I_4(t)| \leq |\varphi|_\infty |f|_\infty \int_{t-a}^{t+a} |\rho|_\gamma |\psi|_{\gamma/(\gamma-1)}\, ds \leq C < \infty \tag{5.7}$$

if $\alpha \leq \frac{N+1}{N}\gamma - 1$, and $I_6(t)$ can be estimated analogously as $I_3(t)$ even with the milder restriction $\alpha < \frac{4N-5}{4N-6}\gamma - 1$. To estimate $|I_5(t)|$ we notice that

$$\begin{aligned} |I_5(t)| &\leq |\varphi|_\infty \int_{t-a}^{t+a} |\rho|_\gamma |u|_r |\psi_t|_q ds \\ &\leq C \int_{t-a}^{t+a} |\nabla u|_2 |\psi_t|_q ds, \end{aligned} \tag{5.8}$$

where $q > \frac{\gamma}{\gamma - 1}$ if $N = 2$ and $q = \frac{6\gamma}{5\gamma - 6}$ if $N = 3$. As in (3.46), we have $\psi_t(s) = S \operatorname{div} z + Sg - S(R_\varepsilon M\theta_\delta(\rho))_t$, where z and g are given by (3.47) with θ_δ in place of θ. In Section 3 we have derived the estimate $|S \operatorname{div} z + Sg|_q \leq C(|z|_q + |g|_{Nq/(N+q)})$ for any $q \in (1, \infty)$ for which the norms on the right have sense. So we get

$$|\psi_t|_q \leq R_\varepsilon |\theta(\rho)u|_q + R_\varepsilon |(\theta(\rho) - \rho\theta'(\rho))\operatorname{div} u|_{Nq/(N+q)} + |M(R_\varepsilon\theta(\rho))_t|_{Nq/(N+q)}. \tag{5.9}$$

Clearly,

$$|\theta(\rho)u|_q \leq C|\rho^\alpha u|_q \leq C|\rho|_\gamma^\alpha |u|_{\gamma q/(\gamma - \alpha q)}, \tag{5.10}$$

$$\begin{aligned} &|R_\varepsilon((\theta(\rho) - \rho\theta'(\rho))\operatorname{div} u)|_r \leq CR_\varepsilon |\rho^\alpha \operatorname{div} u|_r \\ &\qquad \leq CR_\varepsilon(|\rho|_\gamma^\alpha |\operatorname{div} u|_{\gamma r/(\gamma - \alpha r)}) \leq CR_\varepsilon |\operatorname{div} u|_{\gamma r/(\gamma - \alpha r)}, \qquad r = \frac{Nq}{N+q}, \end{aligned} \tag{5.11}$$

and, by (3.45)

$$\begin{aligned} |M(R_\varepsilon\theta(\rho))_t &\leq C\left(R_\varepsilon |\rho^\alpha \operatorname{div} u|_1 + \frac{1}{\varepsilon}\varphi_0(s/\varepsilon)\right) \\ &\leq C\left(R_\varepsilon |\operatorname{div} u|_{\gamma/(\gamma-\alpha)} + \frac{1}{\varepsilon}\varphi_0(s/\varepsilon)\right). \end{aligned} \tag{5.12}$$

If $N = 2$ then we must satisfy the restrictions $q > \frac{\gamma}{\gamma - 1}$ and $\frac{r\gamma}{\gamma - r\alpha} \leq 2$, which leads to $0 < \alpha < \gamma - 1$. If $N = 3$, then $q = \frac{6\gamma}{5\gamma - 6}$ and the restrictions $\frac{q\gamma}{\gamma - q\alpha} \leq 6$ and $\frac{r\gamma}{\gamma - r\alpha} \leq 2$ lead to the requirement $\alpha \leq \frac{2}{3}\gamma - 1$. In both cases (5.4), (5.5) gives the restriction $\alpha \leq \frac{\gamma}{2}$, so in conclusion, we can allow

$$\alpha < \gamma - 1 \ \& \ \alpha \leq \gamma/2 \ \text{if} \ N = 2 \ \text{and} \ \alpha \leq \frac{2}{3}\gamma - 1 \ \& \ \alpha \leq \gamma/2 \ \text{if} \ N = 3. \tag{5.13}$$

Then we have in (5.8) $|\psi_t|_q \leq CR_\varepsilon |\nabla u|_2$ and consequently

$$|I_5(t)| \leq C \int_{t-a}^{t+a} |\nabla u|_2 R_\varepsilon |\nabla u|_2 \, ds.$$

Returning back to (5.3), we have

$$\begin{aligned}
\int_{t-1}^{t} \int_\Omega p(\rho) R_\varepsilon \theta_\delta(\rho) dx ds &\leq \int_{V_a(t)} \varphi(s-t) p(\rho) R_\varepsilon \theta_\delta(\rho) dx ds \\
&= \int_{V_a(t)} \varphi(s-t) M R_\varepsilon \theta_\delta(\rho(s)) |p(\rho(s))|_1 \, dx ds + Q_a^\varepsilon(t) \\
&\leq |\varphi|_\infty \sup_s (|\theta(\rho(s))|_1 |p(\rho(s))|_1) + |Q_a^\varepsilon(t)| \leq C < \infty. \quad (5.14)
\end{aligned}$$

By the continuity of mollifier we have $\lim_{\varepsilon \to 0+} R_\varepsilon \theta_\delta(\rho) = \theta_\delta(\rho)$ in $L^q(V(t))$ with $V(t) = \Omega \times (t-1, t)$ and $q \in (1, \infty)$ arbitrary and $R_{\varepsilon_n} \theta_\delta(\rho) \to \theta_\delta(\rho)$ a.e. in $V(t)$ for some $\varepsilon_n \to 0+$. It follows by Fatou lemma and (5.14) that

$$\int_{V(t)} p(\rho) \theta_\delta(\rho) dx ds \leq C < \infty \qquad \text{for any } \delta > 0.$$

Since $\lim_{\delta \to 0+} \theta_\delta(\rho) = \theta(\rho)$ a.e. in $V(t)$, we may use Fatou lemma again to conclude

$$\int_{V(t)} p(\rho) \theta(\rho) dx ds \leq \liminf_{\delta \to 0+} \int_{V(t)} p(\rho) \theta_\delta(\rho) dx ds \leq C < \infty.$$

This completes the proof of Lemma.

6. Other boundary conditions

In this section we discuss the applicability of our method to the problem (1.1), (1.2), (1.4) with other boundary conditions, namely, the no-stick boundary conditions. To this purpose we write (1.1) rather in the form

$$(\rho u)_t + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(2\mu_1 D(u)) - (\mu_2 - \mu_1) \nabla \operatorname{div} u + \nabla p(\rho) = \rho f, \tag{6.1}$$

where $D_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i})$. We consider the system (6.1), (1.2), (1.4) equipped with the so-called no-stick boundary conditions

$$u \cdot \nu = 0, \qquad \tau_i D_{ij}(u) \nu_j = 0 \ \text{in} \ \partial\Omega \tag{6.2}$$

(ν and τ denotes respectively the normal and tangential vector to $\partial\Omega$) in a domain Ω which is *not* rotationally symmetric. In this case, several modifications have to be made:

In the definition 3.1, it is obvious to change the set of test functions η to $\eta \in C^1(0, T; C^\infty(\Omega, R^N))$ with $\eta \cdot \nu = 0$ on $\partial\Omega$. Clearly, under $(6.2)_1$, the Poincaré inequality (2.1) holds.

The energy inequality now reads

$$\frac{d}{dt}\int_\Omega \left(\frac{1}{2}|\rho|u^2 + P(\rho) - \rho g\right)dx + 2\mu_1 \int_\Omega |D(u)|^2\, dx + (\mu_2 - \mu_1)\int_\Omega |\operatorname{div} u|^2\, dx \leq 0. \tag{6.3}$$

It implies (3.6) due to the inequalities $N|Du| \geq |\operatorname{div} u|$, $\mu_2 > \dfrac{N-2}{N}\mu_1$, and the Korn inequality $|\nabla u|_2 \leq C|D(u)|_2$, which holds for all functions from $W^{1,2}(\Omega)$ satisfying $(6.2)_1$ since Ω is not rotationally symmetric.

Now, we are in position to state the corresponding Theorem:

Theorem 6.1. *Let* (ρ, u) *be a weak solution to the problem given by* (6.1), (1.2), (6.2), (1.4) *with* g, ρ_0, u_0, Ω *satisfying assumptions* (i)–(iii), (v) *from Section* 3, $\mu_2 > \dfrac{N-2}{N}\mu_1$, *and* (6.3). *Then under the hypotheses of Lemma* 4.3, *for any* $r \in [1, \gamma)$ *and any sequence* $t_n \to \infty$, *there exists a subsequence* $\{s_n\}_{n=1}^\infty$ *and a function* $\rho_\infty \in L^\gamma(\Omega)$ *satisfying* (4.22) *and* $\int_\Omega \rho_\infty\, dx = \int_\Omega \rho_0\, dx$ *such that*

$$\lim_{n\to\infty}\int_\Omega |\rho(s_n) - \rho_\infty|^r\, dx = 0.$$

If, moreover, the equilibrium is uniquely determined then

$$\lim_{t\to\infty}\int_\Omega |\rho(t) - \rho_\infty|^r\, dx = 0.$$

Proof. The proof of Theorem 6.1 follows precisely the same lines up to (3.28). The proof of (3.30) requires a slight modification: The formula (3.38) remains valid as well as all estimates concerning $\{I_j^\varepsilon\}_{j=1}^6$ (see (3.39)–(3.58)). Estimate of I_7^ε can be done in a more simple way. One uses the fact that (3.59), when using the weak formulation of the momentum equation, becomes

$$\int_{t-a}^{t+a} \varphi(s-t)\int_\Omega (\mu_1 D(u)D(v) - \rho u\cdot(u\cdot\nabla)v - \rho u v_t)dxds$$
$$- \int_{t-a}^{t+a} \varphi'(s-t)\rho u v\, dxds.$$

Clearly, it tends to zero as $t \to \infty$. This observation completes the proof of (3.30) in the case of no-stick boundary conditions. The reasoning of Section 4 remains without changes. This completes the proof.

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