On the infinite dimensional Laplacian operator

By

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§0. Presentation of the problem.

It is well known that the whole of rotationally invariant, symmetric operators on $L^2(R^n)$ can be generated by the following two operators:

$$r: \varphi(x) \rightarrow r\varphi(x) \quad \text{where} \quad r = \sqrt{\sum_{k=1}^n x_k^2}$$

$$\Delta: \varphi(x) \rightarrow \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2} \varphi(x).$$

We shall discuss a similar problem on R^∞ . Namely, let L be a nuclear space, and L^* be its dual space. With respect to a Gaussian measure on L^* , we consider the Hilbert space $L^2(L^*)$, and investigate the whole of rotationally invariant, symmetric operators on $L^2(L^*)$. (Note: It seems natural to consider a Gaussian measure on L^* , because it is essentially unique as a rotationally invariant measure^[1]).

§1. Definition of the infinite dimensional Laplacian operator.

At first we shall show that for the infinite dimensional case, neither of two operators r and Δ mentioned above can be defined, but some particular linear combination of them has a meaning.

(1) Consider a triplet $L \subset \mathfrak{H} \simeq \mathfrak{H}^* \subset L^*$, where $\mathfrak{H}$ is a Hilbert space and L is a nuclear space. For a fixed complete orthonormal system (=C.O.N.S.) $\{\xi_k\}$ of L (it means that $\{\xi_k\}$ is a C.O.N.S.

of $\mathfrak{H}$ and each ξ_k belongs to L), we define $r_n(x) = \sqrt{\sum_{k=1}^n (\xi_k, x)^2}$ for $x \in L^*$. Letting $n \rightarrow \infty$, it is easily seen that $\{r_n\}$ is convergent if and only if $x \in \mathfrak{H}$. However, since the Gaussian measure is zero on $\mathfrak{H}$, we see that for almost all $x \in L^*$ the function $r(x) = \lim_n r_n(x)$ can not be defined, hence the multiplication operator r can not have any meaning.

(2) Let $d^n x$ be the Lebesgue measure on R^n , and $\mu_{c, n}$ be a Gaussian measure on R^n with variance c^2 .

We shall rewrite $\mathcal{A}$ on $L^2(R^n, d^n x)$ as a symmetric operator on $L^2(R^n, \mu_{c, n})$. By the natural isomorphism $\mathcal{O}$ from $L^2(R^n, d^n x)$ onto $L^2(R^n, \mu_{c, n})$, a function $\varphi(x)$ is mapped to

$$\left(\frac{1}{\sqrt{2\pi}c}\right)^{-n/2} \exp\left[\frac{1}{4c^2} \sum_{k=1}^n x_k^2\right] \varphi(x).$$

Consequently, the Laplacian operator $\mathcal{A} = \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2}$ on $L^2(R^n, d^n x)$ is transformed to $\mathcal{O}\mathcal{A}\mathcal{O}^{-1}$ on $L^2(R^n, \mu_{c, n})$, which we denote as $\mathcal{A}_{c, n}$. By elementary calculations, we see that

$$\mathcal{A}_{c, n} = \sum_{k=1}^n \left(\frac{\partial^2}{\partial x_k^2} - \frac{x_k}{c^2} \frac{\partial}{\partial x_k} - \frac{1}{2c^2} + \frac{x_k^2}{4c^4} \right).$$

Let $\{\xi_k\}$ be a C.O.N.S. of L , and put $(\xi_k, x) = x_k$ for $x \in L^*$. Let M_n be the subspace of $L^2(L^*)$ which consists of all tame functions dependent only on $x_1, x_2, \dots, x_n$, then M_n is isomorphic with $L^2(R^n, \mu_{c, n})$. Being $M_n \subset M_{n'}$ for $n < n'$, we can operate $\mathcal{A}_{c, n'}$ on M_n . Thus, on the dense subspace $M \equiv \bigcup_n M_n$ of $L^2(L^*)$, we can consider formally $\lim_{n \rightarrow \infty} \mathcal{A}_{c, n}$.

In order to check the convergency of $\mathcal{A}_{c, n}$ as $n \rightarrow \infty$, we shall operate $\mathcal{A}_{c, n}$ on $\exp[i(\xi, x)] \in M_N$ where $\xi = \sum_{k=1}^N \alpha_k \xi_k \in L$. It is easy to see that for $n \geq N$,

$$\begin{aligned} & \mathcal{A}_{c, n} \exp[i(\xi, x)] \\ &= \left[-\sum_{k=1}^N \alpha_k^2 - \frac{i}{c^2} \sum_{k=1}^N \alpha_k x_k - \frac{n}{2c^2} + \frac{1}{4c^4} \sum_{k=1}^n x_k^2 \right] \exp[i(\xi, x)]. \end{aligned}$$

The first term in the square bracket is equal to $-\|\xi\|^2$, while the

second term is equal to $-\frac{i}{c^2}(\xi, x)$, but the third and fourth terms are divergent as $n \rightarrow \infty$.

Thus, we have found that in the case of $L^2(L^*)$, among all linear combinations of Δ , r^2 and I (=identity), only the following one is convergent.

$$\lim_{n \rightarrow \infty} \left[\Delta_c, n - \frac{r_n^2}{4c^4} + \frac{n}{2c^2} I \right] = \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\frac{\partial^2}{\partial x_k^2} - \frac{x_k}{c^2} \frac{\partial}{\partial x_k} \right).$$

We shall call this limit operator as the infinite dimensional Laplacian operator and denote it with Δ_c .

Here, we must mention one more remark. For the definition of Δ_c , we have fixed a C.O.N.S. $\{\xi_k\}$ of L , but it can be defined uniquely independent from the choice of $\{\xi_k\}$. We shall give here a more precise definition of Δ_c .

Let $\xi_1, \xi_2, \dots, \xi_n$ be a finite orthonormal system (=F.O.N.S.) of L . Let $\tau_{\xi_1, \xi_2, \dots, \xi_n}$ be a subspace of $L^2(L^*)$ which consists of all tame functions dependent only on $(\xi_1, x), (\xi_2, x), \dots, (\xi_n, x)$.

Namely;

$$(1) \quad \tau_{\xi_1, \xi_2, \dots, \xi_n} = \{F(x) \in L^2(L^*) \mid \exists f(t_1, \dots, t_n); \\ F(x) = f((\xi_1, x), \dots, (\xi_n, x))\}.$$

Moreover we put

$$(2) \quad \mathfrak{U}_{\xi_1, \xi_2, \dots, \xi_n} = \left\{ F(x) \in \tau_{\xi_1, \xi_2, \dots, \xi_n} \mid \sqrt{\frac{d\mu_{c,n}}{dt}}(t_1, \dots, t_n) f(t_1, \dots, t_n) \in (S)_{R^n} \right\},$$

where $\frac{d\mu_{c,n}}{dt} = \left(\frac{1}{\sqrt{2\pi}c} \right)^n \exp \left[-\frac{1}{2c^2} \sum_{k=1}^n t_k^2 \right]$ and $(S)_{R^n}$ means the whole of rapidly decreasing C^∞ -functions on R^n as defined by L. Schwartz.

Furthermore, we consider the union $\mathfrak{U} = \bigcup_{\text{all F.O.N.S.}} \mathfrak{U}_{\xi_1, \xi_2, \dots, \xi_n}$.

Definition

The infinite dimensional Laplacian operator Δ_c is an operator

defined on $\mathfrak{A}$ as follows:

If $F(x) \in \mathfrak{A}_{\xi_1, \xi_2, \dots, \xi_n}$, then

$$(3) \quad \Delta_c F(x) = \sum_{k=1}^n \left(\frac{\partial^2}{\partial t_k^2} - \frac{t_k}{c^2} \frac{\partial}{\partial t_k} \right) f(t_1, \dots, t_n) \Big|_{t_i = (\xi_i, x)}$$

where $|_{t_i = (\xi_i, x)}$ means that after the calculation of the right hand side, we replace t_i by (ξ_i, x) .

It can be easily checked that even if $F(x) \in \mathfrak{A}_{\xi_1, \xi_2, \dots, \xi_n}$ and also $\in \mathfrak{A}_{\xi'_1, \xi'_2, \dots, \xi'_m}$, the image $\Delta_c F$ is defined consistently by the above formula (3). For the proof, it is sufficient to consider only two cases:

- (i) The case that $n < m$ and $\xi_i = \xi'_i$ for $1 \leq i \leq n$. In this case, the function $f(t_1, t_2, \dots, t_m)$ actually does not depend on $t_{n+1}, t_{n+2}, \dots, t_m$.
- (ii) The case that $n = m$, and $(\xi_1, \dots, \xi_n)$ and $(\xi'_1, \dots, \xi'_n)$ span the same n -dimensional subspace L_n of L . This case means an orthogonal transformation of bases of L_n .

(Note: For any $\xi \in L$, $\exp[i(\xi, x)] \in \mathfrak{A}$).

§2. The eigen values and eigen spaces of Δ_c .

Consider an Hermite polynomial:

$$H_n(t) = (-1)^n \exp t^2 \frac{d^n}{dt^n} \exp(-t^2).$$

As well known, it satisfies the differential equation:

$$\left[\frac{d^2}{dt^2} - 2t \frac{d}{dt} \right] H_n(t) = -2n H_n(t),$$

so that we get

$$(4) \quad \left[\frac{d^2}{dt^2} - \frac{t}{c^2} \frac{d}{dt} \right] H_n\left(\frac{t}{\sqrt{2}c}\right) = -\frac{n}{c^2} H_n\left(\frac{t}{\sqrt{2}c}\right).$$

Now, for a fixed C.O.N.S. $\{\xi_k\}$ of L , we put

$$(5) \quad \phi_{(n_1, n_2, \dots, n_k, \dots)}(x) = \prod_{k=1}^{\infty} H_{n_k} \left(\frac{(\xi_k, x)}{\sqrt{2}c} \right)$$

where $n_1 + n_2 + \dots + n_k + \dots < \infty$, thus the infinite product in the right

hand side is actually a finite product. Then, it is evident that

$$(6) \quad \Delta_c \Phi_{(n_1, n_2, \dots, n_k, \dots)}(x) = -\frac{n_1 + n_2 + \dots + n_k + \dots}{c^2} \Phi_{(n_1, n_2, \dots, n_k, \dots)}(x).$$

It is also clear that if $(n_1, n_2, \dots, n_k, \dots) \neq (n'_1, n'_2, \dots, n'_k, \dots)$, then

$$\Phi_{(n_1, n_2, \dots, n_k, \dots)} \perp \Phi_{(n'_1, n'_2, \dots, n'_k, \dots)} \text{ in } L^2(L^*).$$

Let M_m be a subspace of $L^2(L^*)$ which consists of all tame functions dependent only on (ξ_1, x) , (ξ_2, x) , $\dots$, (ξ_m, x) , then the union $M \equiv \bigcup_{m=1}^{\infty} M_m$ is dense in $L^2(L^*)$.

On the other hand, the system;

$$\left\{ \prod_{k=1}^m H_{n_k} \left(\frac{(\xi_k, x)}{\sqrt{2}c} \right) \middle| n_1, n_2, \dots, n_m = 0, 1, 2, \dots \right\}$$

forms a complete orthogonal system (=C.O.S.) of M_m , since the system of Hermite polynomials is complete with respect to a Gaussian measure as well known. Therefore, the system;

$$(7) \quad \{ \Phi_{(n_1, n_2, \dots, n_k, \dots)}(x) \mid n_1 + n_2 + \dots + n_k + \dots < \infty \}$$

forms a C.O.S. of $L^2(L^*)$.

Let $\mathfrak{H}'_n$ be the eigen space of Δ_c which belongs to an eigen value $-\frac{n}{c^2}$, then its closure $\mathfrak{H}_n$ is spanned by the subsystem of (7):

$$(8) \quad \{ \Phi_{(n_1, n_2, \dots, n_k, \dots)}(x) \mid n_1 + n_2 + \dots + n_k + \dots = n \},$$

and we get an orthogonal decomposition of $L^2(L^*)$ into the sum of $\mathfrak{H}_n$;

$$(9) \quad L^2(L^*) = \sum_{n=0}^{\infty} \mathfrak{H}_n$$

(Note: Each $\mathfrak{H}_n$ is infinite dimensional except for $n=0$).

In above discussions, at first step we have fixed a C.O.N.S. $\{\xi_k\}$ of L , but it is evident that if we adopt another C.O.N.S. $\{\xi'_k\}$, we would have a similar base of $\mathfrak{H}_n$. Thus, we see that $\mathfrak{H}_n$ (and $\mathfrak{H}'_n$ also) is invariant under any rotation of L^* , and corresponding to another C.O.N.S. of L , (8) gives us another C.O.S. of $\mathfrak{H}_n$.

§3. Uniqueness of rotationally invariant, symmetric operators.

In this section, we shall show that the whole of rotationally invariant, symmetric operators on $L^2(L^*, \mu_c)$ can be generated by only one operator $\mathcal{A}_c$ defined in §1.

Assumption; *Let A be a symmetric operator on $L^2(L^*, \mu_c)$ which is invariant under any rotation of L^* , and assume that for any $\xi \in L$, $\exp[i(\xi, x)]$ belongs to the domain of A .*

We shall show that such an operator A can be expressed as a function of $\mathcal{A}_c$ in some sense.

Put $\varphi_\xi(x) \equiv A \exp[i(\xi, x)] \in L^2(L^*)$, and we shall determine the form of $\varphi_\xi(x)$.

(I) Rotationally invariance.

The condition of rotationally invariance is written as follows:

$$(10) \quad \forall u \in O(L^*); \quad \varphi_\xi = T_u^{-1} \varphi_{u\xi},$$

where $O(L^*)$ is the rotation group of L^* .

Because for any rotation u of L^* , we have

$$\begin{aligned} AT_u \exp[i(\xi, x)] &= A \exp[i(\xi, u^{-1}x)] \\ &= A \exp[i(u\xi, x)] = \varphi_{u\xi}(x), \end{aligned}$$

so that the condition $T_u^{-1}AT_u = A$ implies that $T_u^{-1}\varphi_{u\xi} = \varphi_\xi$.

As a corollary of (10), we remark that whenever $\xi = u\xi$, we have $\varphi_\xi = T_u^{-1}\varphi_\xi$.

(i) a) *In the case that $\xi = 0$, $\varphi_0(x)$ must be constant.*

Because for any $u \in O(L^*)$, being $u0 = 0$, we get $\varphi_0 = T_u\varphi_0$.

Since μ_c is O -ergodic^{[1], [2]}, any rotationally invariant function must be constant, so that

$$(11) \quad \varphi_0(x) = \text{constant}.$$

b) *In the case that $\xi \neq 0$, we shall prove the following result.*

$$\forall \xi \in L, \exists \phi_\xi(t) \quad (= \text{a function of a real variable});$$

$$(12) \quad \varphi_{\xi}(x) = \phi_{\xi} \left(\frac{(\xi, x)}{\|\xi\|} \right).$$

For the proof, put $\xi_1 = \frac{\xi}{\|\xi\|}$ and choose $\xi_2, \xi_3, \dots$ suitably to form a C.O.N.S. of L together with ξ_1 . It is sufficient to show that $\varphi_{\xi}(x)$ depends only on (ξ_1, x) .

Let M_m be the space of all tame functions dependent only on $(\xi_1, x), (\xi_2, x), \dots, (\xi_m, x)$. Since M_m is a closed subspace of $L^2(L^*)$, we can define the projection operator P_m which maps $L^2(L^*)$ onto M_m ;

$$P_m \varphi_{\xi} \in M_m, \quad (\varphi_{\xi} - P_m \varphi_{\xi}) \perp M_m.$$

As $M \equiv \bigcup_{m=1}^{\infty} M_m$ is dense in $L^2(L^*)$, we have $\lim_{m \rightarrow \infty} P_m \varphi_{\xi} = \varphi_{\xi}$ in $L^2(L^*)$.

If $\xi = u\xi$, being $\varphi_{\xi} = T_u \varphi_{\xi}$, we have

$P_m \varphi_{\xi} - T_u P_m \varphi_{\xi} = P_m \varphi_{\xi} - \varphi_{\xi} + T_u (\varphi_{\xi} - P_m \varphi_{\xi})$, so that:

$$(13) \quad \|P_m \varphi_{\xi} - T_u P_m \varphi_{\xi}\|_{\mu_{\epsilon}} \leq 2 \|P_m \varphi_{\xi} - \varphi_{\xi}\|_{\mu_{\epsilon}} \xrightarrow{(m \rightarrow \infty)} 0.$$

Consider a rotation u which maps ξ_k as follows;

$$u; \begin{cases} \xi_1 \rightarrow \xi_1, & \xi_{m+1} \rightarrow \xi_{m+1} \\ \xi_2 \rightarrow \xi_{m+2}, & \xi_{m+2} \rightarrow \xi_2 \\ \vdots & \vdots \\ \xi_m \rightarrow \xi_{2m}, & \xi_{2m} \rightarrow \xi_m. \end{cases}$$

Such a rotation u exists, because any rotation of a finite dimensional subspace can be extended to a rotation of L , hence to a rotation of L^* .

For this u , we shall calculate $\|P_m \varphi_{\xi} - T_u P_m \varphi_{\xi}\|_{\mu_{\epsilon}}$.

$$(14) \quad \|P_m \varphi_{\xi} - T_u P_m \varphi_{\xi}\|_{\mu_{\epsilon}}^2 = 2 \|P_m \varphi_{\xi}\|_{\mu_{\epsilon}}^2 - 2 \Re \langle P_m \varphi_{\xi}, T_u P_m \varphi_{\xi} \rangle.$$

Since $P_m \varphi_{\xi} \in M_m$, there exists a function $f_{\xi, m}(t_1, t_2, \dots, t_m)$ of m real variables such that $P_m \varphi_{\xi}(x) = f_{\xi, m}(x_1, x_2, \dots, x_m)$ where $x_k = (\xi_k, x)$. Then it is clear that $T_u P_m \varphi_{\xi}(x) = P_m \varphi_{\xi}(u^{-1}x) = f_{\xi, m}(x_1, x_{m+2}, \dots, x_{2m})$.

Consequently,

$$(15) \quad \langle P_m \varphi_{\xi}, T_u P_m \varphi_{\xi} \rangle = \int f_{\xi, m}(x_1, x_2, \dots, x_m) \overline{f_{\xi, m}(x_1, x_{m+2}, \dots, x_{2m})} d\mu_{\epsilon, 2m}(x_1, \dots, x_{2m})$$

$$= \int |\phi_{\xi, m}(x_1)|^2 d\mu_{c, 1}(x_1).$$

where $\mu_{c, k}$ is a k -dimensional Gaussian measure with variance c^2 , and

$$\phi_{\xi, m}(x_1) = \int f_{\xi, m}(x_1, x_2, \dots, x_m) d\mu_{c, m-1}(x_2, \dots, x_m).$$

It is evident that $\phi_{\xi, m}((\xi_1, x)) = P_1 P_m \varphi_{\xi}(x) = P_1 \varphi_{\xi}(x)$. Thus, substituting (15) into (14), we get

$$(16) \quad \|P_m \varphi_{\xi} - T_u P_m \varphi_{\xi}\|_{\mu_c}^2 = 2\|P_m \varphi_{\xi}\|_{\mu_c}^2 - 2\|P_1 \varphi_{\xi}\|_{\mu_c}^2 \\ = 2\|P_m \varphi_{\xi} - P_1 \varphi_{\xi}\|_{\mu_c}^2.$$

In the limit of $m \rightarrow \infty$, the left hand side of (16) converges to 0 because of (13), so that $P_m \varphi_{\xi} - P_1 \varphi_{\xi} \rightarrow 0$ in $L^2(L^*)$.

On the other hand, since $P_m \varphi_{\xi} \rightarrow \varphi_{\xi}$ in $L^2(L^*)$, we see that $P_1 \varphi_{\xi} = \varphi_{\xi}$, hence $\varphi_{\xi} \in M_1$. Thus, we have proved (12). (q.e.d.)

(ii) Next we shall prove that:

There exists a family of functions $\psi_{\lambda}(t)$ of a real variable t with a non-negative parameter λ , such that

$$(17) \quad \varphi_{\xi}(x) = \psi_{\|\xi\|}((\xi, x)).$$

Because if $\|\xi\| = \|\eta\|$, for a suitable rotation u of L we have $\eta = u\xi$, hence $\varphi_{\eta} = T_u \varphi_{\xi}$ from the relation (10). Then, in virtue of (12), we get

$$\phi_{\eta}\left(\frac{(\eta, x)}{\|\eta\|}\right) = (T_u \varphi_{\xi})(x) = \varphi_{\xi}(u^{-1}x) \\ = \phi_{\xi}\left(\frac{(\xi, u^{-1}x)}{\|\xi\|}\right) = \phi_{\xi}\left(\frac{(\eta, x)}{\|\eta\|}\right).$$

This means $\phi_{\eta} = \phi_{\xi}$, thus actually the mapping $\xi \rightarrow \phi_{\xi}$ depends only on the norm, so that we have the expression (17).

(Note; The result of case (i) a) is included in (17)).

(II) Symmetricity.

In order that A is symmetric on the dense subspace of $L^2(L^*)$ which is spanned by $\{\exp[i(\xi, x)] \mid \xi \in L\}$, the following equality

is necessary and sufficient.

$$\langle \exp[i(\xi, x)], A \exp[i(\eta, x)] \rangle = \langle A \exp[i(\xi, x)], \exp[i(\eta, x)] \rangle.$$

This equality is written in the integral form as;

$$(18) \quad \int \psi_{\|\eta\|}(\langle \eta, x \rangle) \exp[-i(\xi, x)] d\mu_c(x) \\ = \int \overline{\psi_{\|\xi\|}(\langle \xi, x \rangle)} \exp[i(\eta, x)] d\mu_c(x).$$

From this equation, we shall prove that:

In order that A is symmetric, the following condition is necessary and sufficient.

Let $f_\lambda(\alpha)$ be the Fourier transform of $\psi_\lambda(\lambda t) \exp\left(-\frac{t^2}{2c^2}\right)$, then

$$(19) \quad \exists h(\alpha) \text{ (=entire function);} \\ f_\lambda(\alpha) = \exp\left[-\frac{c^2}{2}(\lambda^2 + \alpha^2)\right] h(\lambda\alpha).$$

For the proof, in the left hand side of (18), considering a suitable C.O.N.S. we substitute $\eta = \|\eta\|\xi_1$ and $\xi = \|\xi\|(\ell\xi_1 + m\xi_2)$, while in the right hand side, using another C.O.N.S. we substitute $\xi = \|\xi\|\xi'_1$ and $\eta = \|\eta\|(\ell\xi'_1 - m\xi'_2)$, where $\ell^2 + m^2 = 1$.

Then, the equality (18) is rewritten as;

$$\int \psi_{\|\eta\|}(\|\eta\|x_1) \exp\left[-i\|\xi\|(\ell x_1 + m x_2) - \frac{1}{2c^2}(x_1^2 + x_2^2)\right] dx_1 dx_2 \\ = \int \overline{\psi_{\|\xi\|}(\|\xi\|x_1)} \exp\left[i\|\eta\|(\ell x_1 - m x_2) - \frac{1}{2c^2}(x_1^2 + x_2^2)\right] dx_1 dx_2.$$

After the integration with x_2 , we get

$$\exp\left[-\frac{c^2}{2}\|\xi\|^2 m^2\right] \int \psi_{\|\eta\|}(\|\eta\|t) \exp\left[-i\|\xi\|\ell t - \frac{t^2}{2c^2}\right] dt \\ = \exp\left[-\frac{c^2}{2}\|\eta\|^2 m^2\right] \int \overline{\psi_{\|\xi\|}(\|\xi\|t)} \exp\left[i\|\eta\|\ell t - \frac{t^2}{2c^2}\right] dt.$$

Let $f_\lambda(\alpha)$ be the Fourier transform of $\psi_\lambda(\lambda t) \exp\left(-\frac{t^2}{2c^2}\right)$, then f_λ is an entire function of α , because $\varphi_\xi(x) \in L^2(L^*)$ implies that

$\psi_\lambda(\lambda t) \exp\left(-\frac{t^2}{4c^2}\right)$ is square integrable with t .

Thus, the condition (18) is written as a functional equation of $f_\lambda(\alpha)$ as follows;

$$(20) \quad \exp\left[-\frac{c^2}{2}\|\xi\|^2 m^2\right] f_{\|\eta\|}(-l\|\xi\|) \\ = \exp\left[-\frac{c^2}{2}\|\eta\|^2 m^2\right] \overline{f_{\|\xi\|}(-l\|\eta\|)}.$$

Especially, putting $\|\xi\| = \|\eta\|$, we see that $f_{\|\xi\|}(\alpha) = \overline{f_{\|\xi\|}(\bar{\alpha})}$ for $-\|\xi\| < \alpha < \|\xi\|$, but since $f_{\|\xi\|}(\alpha)$ is an entire function of α , we have;

$$(21) \quad f_{\|\xi\|}(\alpha) = \overline{f_{\|\xi\|}(\bar{\alpha})} \quad \text{for any } \alpha.$$

Substituting (21) into (20), we get

$$\exp\left[-\frac{c^2}{2}\|\xi\|^2 m^2\right] f_{\|\eta\|}(-l\|\xi\|) \\ = \exp\left[-\frac{c^2}{2}\|\eta\|^2 m^2\right] f_{\|\xi\|}(-l\|\eta\|).$$

Remark that $m^2 = 1 - l^2$, and put $g(\lambda, \alpha) = \exp\left[\frac{c^2}{2}(\lambda^2 + \alpha^2)\right] f_\lambda(\alpha)$, then we have a functional equation of $g(\lambda, \alpha)$;

$$g(\|\eta\|, -l\|\xi\|) = g(\|\xi\|, -l\|\eta\|).$$

(Since g is entire with respect to α , this equality holds even if $|l| > 1$).

Consequently, there exists an entire function $h(\alpha)$ such that $g(\lambda, \alpha) = h(\lambda\alpha)$, thus we have $f_\lambda(\alpha) = \exp\left[-\frac{c^2}{2}(\lambda^2 + \alpha^2)\right] h(\lambda\alpha)$. This is just the condition (19). (q.e.d.)

Evidently, the condition (19) can be replaced by;

$$(19)' \quad \exists h_1(\alpha) \quad (= \text{entire function}); \\ f_\lambda(\alpha) = \exp\left[-\frac{c^2}{2}(\lambda + \alpha)^2\right] h_1(\lambda\alpha).$$

From these investigations, we see that the whole of rotationally invariant, symmetric operators on $L^2(L^*)$ can be mapped linearly

in a one-to-one way into the space of real entire functions of one variable.

For a remark, we shall show that $h_1(\alpha) = \alpha$ for the operator $\mathcal{A}_c$.

$$\begin{aligned}\mathcal{A}_c \exp[i(\xi, x)] &= \left(-\frac{\partial^2}{\partial x_1^2} - \frac{x_1}{c^2} \frac{\partial}{\partial x_1} \right) \exp[i\|\xi\|x_1] \\ &= \left(-\|\xi\|^2 - i\frac{\|\xi\|x_1}{c^2} \right) \exp[i\|\xi\|x_1] \\ &= \left(-\|\xi\|^2 - i\frac{(\xi, x)}{c^2} \right) \exp[i(\xi, x)]\end{aligned}$$

$$\therefore \psi_\lambda(t) = \left(-\lambda^2 - i\frac{t}{c^2} \right) \exp(it)$$

$$\therefore \psi_\lambda(\lambda t) \exp\left(-\frac{t^2}{2c^2}\right) = -\left(\lambda^2 + i\frac{\lambda t}{c^2}\right) \exp\left(i\lambda t - \frac{t^2}{2c^2}\right).$$

The Fourier transform of this function is $\lambda\alpha \exp\left[-\frac{c^2}{2}(\lambda + \alpha)^2\right]$,

so that we obtain $h_1(\alpha) = \alpha$.

In a similar way we can show that for the operator $\mathcal{A}_c^n$ ($=$ n -th power of $\mathcal{A}_c$), $h_1(\alpha)$ is a polynomial of α with degree n .

Return to general A . Since $h_1(\alpha)$ is an entire function, it can be expanded into a power series, so that we can approximate it by some polynomial of α , thus we can approximate A by some polynomial of $\mathcal{A}_c$. In this sense, the whole of rotationally invariant, symmetric operators on $L^2(L^*)$ can be generated by only one operator $\mathcal{A}_c$.

Unfortunately, because of rather complicated correspondence between $h_1(\alpha)$ and A , it is difficult to discuss the topological properties of this correspondence, so that the sense of the approximation is not so clear. We can not mention exactly in what meaning A is approximated by a polynomial of $\mathcal{A}_c$.

We have only the following result;

$$\forall \varepsilon > 0, \forall \xi, \eta \in L, \exists P (= \text{real polynomial});$$

$$|\langle \exp[i(\xi, x)], (P(\mathcal{A}_c) - A) \exp[i(\eta, x)] \rangle| < \varepsilon.$$

This means the weak convergence with respect to a dense subspace

of $L^2(L^*)$. The discussion on the uniformity of the convergence seems to be rather difficult and we can not expect any useful result along this line.

§4. Eigen spaces of A .

The fact: "*The whole of rotationally invariant, symmetric operators on $L^2(L^*)$ can be generated by only one operator A_c* " has been shown in §3 but not in any rigorous sense.

In this section, for the expression of this fact we shall give another form which has an exacter sense. Namely, we shall prove that the space $\mathfrak{H}_n$ defined in §2 is an eigen space of any A , or in the other words, A is a constant multiplier on each $\mathfrak{H}_n$.

Conversely, it is evident that any operator which is a real constant multiplier on each $\mathfrak{H}_n$ is rotationally invariant and symmetric.

(I) First we shall show that:

$$\forall n \exists \lambda_n (= \text{real number}), \forall \xi \in L;$$

$$(22) \quad Q_n A \exp[i(\xi, x)] = \lambda_n Q_n \exp[i(\xi, x)]$$

where Q_n is the projection operator onto $\mathfrak{H}_n$.

Since the condition $\xi \perp \eta$ in L implies that

$$\exp[i(\xi, x)] \perp H_n \left(\frac{(\eta, x)}{\sqrt{2} c \|\eta\|} \right) \text{ in } L^2(L^*) \text{ for any } n \neq 0,$$

it is easily seen that

$$Q_n \exp[i(\xi, x)] = \alpha_{n, \xi} H_n \left(\frac{(\xi, x)}{\sqrt{2} c \|\xi\|} \right),$$

for some complex number $\alpha_{n, \xi}$. In a similar way, since also $A \exp[i(\xi, x)]$ depends only on (ξ, x) as shown by (17), we see that

$$Q_n A \exp[i(\xi, x)] = \beta_{n, \xi} H_n \left(\frac{(\xi, x)}{\sqrt{2} c \|\xi\|} \right),$$

for some complex number $\beta_{n, \xi}$.

Thus, for the proof of (22) it is sufficient to show that the ratio $\beta_{n,\xi}/\alpha_{n,\xi}$ is real and independent from ξ .

Neglecting the effect of the normalization constant of Hermite polynomial, we have

$$\begin{aligned}\alpha_{n,\xi} &= \int \exp[i(\xi, x)] H_n\left(\frac{(\xi, x)}{\sqrt{2}c\|\xi\|}\right) d\mu_c(x) \\ &= \int_{-\infty}^{\infty} \exp\left[i\|\xi\|t - \frac{t^2}{2c^2}\right] H_n\left(\frac{t}{\sqrt{2}c}\right) dt,\end{aligned}$$

and

$$\begin{aligned}\beta_{n,\xi} &= \int A \exp[i(\xi, x)] H_n\left(\frac{(\xi, x)}{\sqrt{2}c\|\xi\|}\right) d\mu_c(x) \\ &= \int_{-\infty}^{\infty} \psi_{\|\xi\|}(\|\xi\|t) \exp\left(-\frac{t^2}{2c^2}\right) H_n\left(\frac{t}{\sqrt{2}c}\right) dt.\end{aligned}$$

Into these integrals, substitute the formula:

$$H_n\left(\frac{t}{\sqrt{2}c}\right) = (-c)^n \exp\left(\frac{t^2}{2c^2}\right) \frac{d^n}{dt^n} \exp\left(-\frac{t^2}{2c^2}\right),$$

then we get

$$\begin{aligned}\alpha_{n,\xi} &= (-c)^n \int_{-\infty}^{\infty} \exp(i\|\xi\|t) \frac{d^n}{dt^n} \exp\left(-\frac{t^2}{2c^2}\right) dt \\ &= (ic\|\xi\|)^n \exp\left(-\frac{c^2\|\xi\|^2}{2}\right),\end{aligned}$$

and

$$\begin{aligned}\beta_{n,\xi} &= (-c)^n \int_{-\infty}^{\infty} \psi_{\|\xi\|}(\|\xi\|t) \frac{d^n}{dt^n} \exp\left(-\frac{t^2}{2c^2}\right) dt \\ &= (-ic)^n \int_{-\infty}^{\infty} \psi_{\|\xi\|}(\|\xi\|t) \frac{\partial^n}{\partial \alpha^n} \exp\left(-\frac{(t-i\alpha)^2}{2c^2}\right) dt \Big|_{\alpha=0} \\ &= (-ic)^n \frac{d^n}{d\alpha^n} \int_{-\infty}^{\infty} \psi_{\|\xi\|}(\|\xi\|t) \exp\left[-\frac{t^2}{2c^2} + \frac{i\alpha t}{c^2} + \frac{\alpha^2}{2c^2}\right] dt \Big|_{\alpha=0} \\ &= (-ic)^n \frac{d^n}{d\alpha^n} \left[\exp\left(\frac{\alpha^2}{2c^2}\right) \int_{-\infty}^{\infty} \psi_{\|\xi\|}(\|\xi\|t) \exp\left(-\frac{t^2}{2c^2} + \frac{i\alpha t}{c^2}\right) dt \right] \Big|_{\alpha=0} \\ &= (-ic)^n \frac{d^n}{d\alpha^n} \left[\exp\left(\frac{\alpha^2}{2c^2}\right) \exp\left\{-\frac{c^2}{2}\left(\|\xi\|^2 + \frac{\alpha^2}{c^4}\right)\right\} h\left(\frac{\alpha\|\xi\|}{c^2}\right) \right] \Big|_{\alpha=0} \\ &= (-ic)^n \frac{d^n}{d\alpha^n} \left[\exp\left(-\frac{c^2}{2}\|\xi\|^2\right) h\left(\frac{\alpha\|\xi\|}{c^2}\right) \right] \Big|_{\alpha=0}\end{aligned}$$

$$= (-ic\|\xi\|)^n \exp\left(-\frac{c^2}{2}\|\xi\|^2\right) \frac{1}{c^{2n}} h^{(n)}(0).$$

Comparing $\alpha_{n,\xi}$ with $\beta_{n,\xi}$, we have the ratio:

$$\frac{\beta_{n,\xi}}{\alpha_{n,\xi}} = (-1)^n \frac{h^{(n)}(0)}{c^{2n}},$$

which is independent from $\xi (\in L)$. Thus, putting λ_n as;

$$(23) \quad \lambda_n = (-1)^n \frac{h^{(n)}(0)}{c^{2n}},$$

the equality (22) has been proved. (q.e.d.)

In terms of $h_1(\alpha)$, λ_n is written as;

$$(23)' \quad \lambda_n = \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{h_1^{(k)}(0)}{c^{2k}},$$

where $\binom{n}{k}$ is the binomial coefficient.

(II) Let $\mathfrak{D}$ be the domain of A . From the assumption at the beginning of §3, $\mathfrak{D}$ includes the space spanned by $\{\exp[i(\xi, x)] \mid \xi \in L\}$.

Here, we shall show that A can be extended on $\mathfrak{D} \cup (\bigcup_{n=0}^{\infty} \mathfrak{H}_n)$, and that each $\mathfrak{H}_n$ becomes an eigen space of the extended symmetric operator $\tilde{A}$. ($\bigcup_{n=0}^{\infty} \mathfrak{H}_n$ means the smallest linear space which includes all $\mathfrak{H}_n$, and becomes a dense subspace of $L^2(L^*)$).

i) If $F(x) \in \bigcup_{n=0}^{\infty} \mathfrak{H}_n$, $F(x)$ can be expressed as a finite sum of the elements of $\mathfrak{H}_n$, namely

$$F = \sum_{n=0}^N F_n \quad (\text{each } F_n \in \mathfrak{H}_n).$$

For such F , we define the operator A' as;

$$A'F = \sum_{n=0}^N \lambda_n F_n,$$

where λ_n is given by (23). Evidently, $\mathfrak{H}_n$ is the eigen space of A' with an eigen value λ_n .

ii) We shall prove that if $F(x) \in \mathfrak{D} \cap (\bigcup_{n=0}^{\infty} \mathfrak{H}_n)$, then $AF = A'F$. For any $\xi \in L$, we have

$$\begin{aligned} \langle \exp[i(\xi, x)], AF(x) \rangle &= \langle A \exp[i(\xi, x)], F(x) \rangle \\ &= \langle \sum_{n=0}^{\infty} Q_n A \exp[i(\xi, x)], F(x) \rangle \\ &= \langle \sum_{n=0}^{\infty} \lambda_n Q_n \exp[i(\xi, x)], F(x) \rangle \\ &= \langle \exp[i(\xi, x)], \sum_{n=0}^{\infty} \lambda_n Q_n F(x) \rangle \\ &= \langle \exp[i(\xi, x)], \sum_{n=0}^N \lambda_n F_n(x) \rangle \\ &= \langle \exp[i(\xi, x)], A'F(x) \rangle. \end{aligned}$$

Since the set $\{\exp[i(\xi, x)] \mid \xi \in L\}$ spans a dense subspace of $L^2(L^*)$, this means $AF(x) = A'F(x)$.

iii) We shall extend A on $\mathfrak{D} \cup (\bigcup_{n=0}^{\infty} \mathfrak{H}_n)$.

For any $F(x) \in \mathfrak{D} \cup (\bigcup_{n=0}^{\infty} \mathfrak{H}_n)$, we decompose it into the form:

$$(24) \quad F(x) = F_1(x) + F_2(x), \quad F_1 \in \mathfrak{D}, \quad F_2 \in \bigcup_{n=0}^{\infty} \mathfrak{H}_n$$

and we define the operator $\tilde{A}$ as follows;

$$\tilde{A}F = AF_1 + A'F_2.$$

Then, $\tilde{A}$ is an extension of A , and each $\mathfrak{H}_n$ is an eigen space of $\tilde{A}$. (Note: In virtue of ii), we see that $\tilde{A}$ is defined consistently independent from the way of decomposition (24)).

From these investigations, we have concluded *that as far as the domain of A includes the set $\{\exp[i(\xi, x)] \mid \xi \in L\}$, the spaces $\{\mathfrak{H}_n\}$ become eigen spaces of an extension of A* . Thus, in an exacter way than in §3, we have established the uniqueness of A_c as a rotationally invariant, symmetric operator on $L^2(L^*)$.

As an application of this theorem, we shall consider the representation of the infinite dimensional rotation group $O(L^*)$.

On each $\mathfrak{H}_n$, the operator $T_u: F(x) \rightarrow F(u^{-1}x)$ gives an unitary representation of $O(L^*)$, because $\mathfrak{H}_n$ is an $O(L^*)$ -invariant space. Furthermore, this representation is irreducible. Because, if it were reducible, there would exist a proper subspace $\mathfrak{M}$ of $\mathfrak{H}_n$ such that $\mathfrak{M}$ is $O(L^*)$ -invariant. Let P be the projection operator onto $\mathfrak{M}$, then P would be rotationally invariant and symmetric but not a constant multiplier on $\mathfrak{H}_n$. This is a contradiction against the main result of this section.

In many other aspects, the results in this paper have close relations with the representation of the infinite dimensional rotation group. For instance, in the limit of $n \rightarrow \infty$, the uniform measure on n -dimensional sphere tends to an infinite dimensional Gaussian measure, and n -dimensional Laplacian operator on the sphere tends to the infinite dimensional Laplacian operator $\mathcal{A}_c$ defined in this paper. Corresponding to these facts, we see that n -dimensional spherical harmonics tend to Hermite polynomials which are eigen functions of $\mathcal{A}_c$. In near future, the author will publish a joint paper with N. Kono to discuss these topics.

We would also like to mention a report by M. Oshikawa [4] which is useful for the information on the general Minlos theory and our problem.

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