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Multi-dimensional diffusion and the Markov process on the boundary

Dedicated to the memory of the late Y. Taniyama

By

Ken-iti SATO and Tadashi UENO

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W. Feller [5, 6, 7] determined all the diffusion processes in one dimension, since A. Kolmogorov introduced the diffusion equation in 1931. Stimulated by his work, which is of analytic character, followed the works of E. B. Dynkin, K. Ito, H. P. McKean, Jr., and D. Ray, and completely determined one dimensional diffusion in a satisfactory correspondence between probabilistic and analytic properties. The study of the Brownian motion by P. Lévy and the rigorous set-ups for probabilistic treatment by J. L. Doob seem to have had prepared a necessary background for these works.

Approaches to such a solution have been tried in the case of multi-dimensional diffusion on the basis of these researches, though it seems to be far from completion in any sense. A. D. Wentzell [36] tried to find all the diffusions determined by the equation of type

$$(0.1) \qquad \frac{\partial u}{\partial t}(t, x) = Au(t, x), \quad x \in D, \ t \in [0, \infty),$$

where D is a domain in a sufficiently smooth manifold of N dimensions[1] and A is an elliptic operator on $\overline{D}$, both D and A having sufficient regularities. He proved that any sufficiently smooth function

1) Wentzell assumed that D is a domain in the N-dimensional Euclidean space R^N. But the same treatment is possible in sufficiently smooth manifolds without any change.

u in the domain of the generator, which is a contraction of $\bar{A}$, of a strongly continuous semigroup of nonnegative linear operators $\{T_t, t \geqslant 0\}$ on $C(\bar{D})$ with norm $\|T_t\| \leqslant 1$ necessarily satisfies a boundary condition of type[2)]

$$(0.2) \qquad Lu(x) = 0, \quad x \in \partial D,$$

$$(0.3) \qquad \begin{aligned} Lu(x) = & \sum_{i,j=1}^{N-1} \alpha^{ij}(x) \frac{\partial^2 u}{\partial \xi_x^i \partial \xi_x^j}(x) + \sum_{i=1}^{N-1} \beta^i(x) \frac{\partial u}{\partial \xi_x^i}(x) \\ & + \gamma(x) u(x) + \delta(x) \lim_{y \in D, y \to x} Au(y) + \mu(x) \frac{\partial u}{\partial n}(x) \\ & + \int_{\bar{D}} \left\{ u(y) - u(x) - \sum_{i=1}^{N-1} \frac{\partial u}{\partial \xi_x^i}(x) \xi_x^i(y) \right\} \nu_x(dy). \end{aligned}$$[3)]

Moreover, he proved that this type of boundary conditions are also sufficient to determine all the rotation invariant diffusions on a solid sphere in R^3 or a circular disc in R^2.

Here, in this paper, we will first prove that we can obtain the semigroup $\{T_t, t \geqslant 0\}$ on $C(\bar{D})$ determined by the diffusion equation (0.1) and Wentzell's boundary condition (0.2), if the equation of type

$$(0.4) \qquad \begin{aligned} & (\alpha - A) u(x) = 0, \quad x \in D \\ & (\beta - L) u(x) = \varphi(x), \quad x \in \partial D \end{aligned}$$

is solved for sufficiently many functions φ on the boundary ∂D, where L is taken to be an operator given by (0.3). This equation will be reduced to an *integro-differential equation on the boundary*:[4)]

$$\beta \cdot \psi(x) - \left(\sum_{i,j=1}^{N-1} \alpha^{ij}(\alpha, x) \frac{\partial^2 \psi}{\partial \xi_x^i \partial \xi_x^j}(x) + \sum_{i=1}^{N-1} \beta^i(\alpha, x) \frac{\partial \psi}{\partial \xi_x^i}(x) \right.$$

2) Precise assumptions on D and A and definitions of $C(\bar{D})$, $C(\partial D)$ and $C^2(\bar{D})$ etc. are referred to §2. $\bar{A}$ is the closure of A in $C(\bar{D})$, where A is considered to have $C^2(\bar{D})$ as its domain.

3) Precise conditions on terms in (0.3) are referred to §4.

4) This method is an exension of Feller's idea in [5], where D is an interval and the integro-differential equation (0.5) is replaced by a system of two linear equations of two unknowns.

$$(0.5)\qquad +\gamma(\alpha, x)\psi(x)+\int_{\partial D}\Big\{\psi(y)-\psi(x)-\sum_{i=1}^{N-1}\frac{\partial\psi}{\partial\xi_x^i}(x)\xi_x^i(y)\Big\}$$

$$\nu_x(\alpha, dy)\Big)=\varphi(x), \quad x\in\partial D.$$

This type of equations are known to be solved in some special cases.

In constructing the semigroup on $C(\bar{D})$, which is essentially the diffusion on $\bar{D}$, we make use of a certain class of semigroups $\{S_t^\alpha, t\geqslant 0\}$, $\alpha\geqslant 0$, of non-negative linear operators on $C(\partial D)$ with norm $\|S_t^\alpha\|\leqslant 1$. This means that there is a class of *Markov processes on the boundary* ∂D corresponding to these semigroups. Moreover, there is a kind of *duality* in appearance between one of these semigroups, that is, $\{S_t^0, t\geqslant 0\}$, and the semigroup $\{T_t, t\geqslant 0\}$ on $C(\bar{D})$, which corresponds to the diffusion on $\bar{D}$. This duality naturally leads to a conjecture that the Markov process on the boundary corresponding to $\{S_t^0, t\geqslant 0\}$ is *the trace on* ∂D *of the diffusion on* $\bar{D}$ and other semigroups on $C(\partial D)$ in the class are Markov procseses of some such kind.[5] This probabilistic interpretation will be justified in a special case (the reflecting diffusion), where L is given to be the inward-directed normal derivative $\partial/\partial n$, by introducing an additive functional named *the local time on the boundary*, which plays, in some respects, a similar rôle to that of the local time of P. Lévy [20], H. F. Trotter [31] and K. Ito-H. P. McKean, Jr. [16] in one dimension. In fact, it will be proved that the diffusion on $\bar{D}$ determined by the equation (0.1) and the boundary condition of type

$$\gamma(x)u(x)+\delta(x)\lim_{y\in D, y\to x}Au(y)+\frac{\partial u}{\partial n}(x)=0, \quad x\in\partial D,$$

can be constructed by a modification of the reflecting diffusion making use of this local time on the boundary.

The analytic construction of semigroups on $C(\bar{D})$ and $C(\partial D)$

5) This duality will be naturally extended to the class of semigroups $\{S_t^\alpha, t\geqslant 0\}$, $\alpha\geqslant 0$, and the class of diffusions on $\bar{D}$ associated with semigroups $\{T_t^\beta, t\geqslant 0\}$, $\beta\geqslant 0$, where T_t^β has boundary condition $(L-\beta)u=0$. Duality for their Green operators is expressed by (5.8)—(5.9). A more intuitive explanation will be found in [33].

will be treated in Chapter II (§4–§6), and probabilistic justifications will be contained in Chapter III (§7–§9). Chapter I (§1–§3) is de voted to the preliminary results which are used in these chapters. The main part of this paper consists of the rigorous proofs of the statements in [28, 33] and a part of [30], while some developments in details are added anew.

In two dimensions, the diffusion satisfying Wentzell's boundary condition was constructed by Wentzell himself [37] under some additional conditions, and by N. Ikeda [12] using stochastic defferential equations. M. Fukushima and N. Ikeda investigated the connection between diffusions and Markov processes on the boundary from a more probabilistic point of view.[6] A decomposition of a certain class of Markov processes to the minimal process and the Markov process on the boundary will be made in [29]. A deeper analysis of the Markov process on the boundary will be done by M. Motoo in [24] and subsequent papers using the notion of the sweeping-out of additive functionals. Under some conditions, he has succeeded a probabilistic construction of diffusions from the minimal process and the Markov process on the boundary. We remark that M. I. Višik, in a series of papers including [34]. made an investigation for elliptic equations, which is similar to Wentzell [36] and a part of this paper, in the Hilbert space set-up independently.

The authors wish to note here that early in 1957–8 K. Ito and H. P. McKean, Jr. made a series of instructive lectures and discussions, which, together with the book [14] of K. Ito, really brought about the flavour of the new trends in the theory of Markov processes at that time. Friends in the Seminar on Probability, especially N. Ikeda, M. Motoo and H. Tanaka willingly joined in discussions with the authors during the research of this problem. S. Ito and A. Orihara kindly answered questions about differential equations and Lie groups, respectively. The authors express their thanks to them all.

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6) Their results can be seen in [8].

Chapter I. Preliminary

§1. The Hille-Yosida theorem

First, we restate a version of the Hille-Yosida theorem [10, 39] for our present use. Let K be a compact metric space and $C(K)$ be the space of all real valued continuous functions defined on K with norm $\|f\|=\max_{x\in K}|f(x)|$. We call a system of linear operators $\{T_t, t\geqslant 0\}$ acting on $C(K)$, simply, a *semigroup on* $C(K)$ if it satisfies $T_t T_s = T_{t+s}$, $T_0=$the identity, $\|T_t\|\leqslant 1$, $\lim_{t\to 0}\|T_t f-f\|=0$ for any $f\in C(K)$, and $T_t f\geqslant 0$ for any $f\geqslant 0$. The generator $\mathfrak{G}$ of a semigroup is defined for such f that the right hand side of

$$\mathfrak{G}f=\lim_{t\to 0}\frac{1}{t}(T_t f-f)^{7)}$$

exists. The domain of $\mathfrak{G}$ is denoted by $\mathfrak{D}(\mathfrak{G})$.

Theorem 1.1. *If* $\{T_t, t\geqslant 0\}$ *is a semigroup on* $C(K)$, *then the generator* $\mathfrak{G}$ *satisfies the following conditions.*

(1.1) $\mathfrak{D}(\mathfrak{G})$ *is a dense subspace of* $C(K)$.

Let $\alpha>0$.

(1.2) *For any* $f\in C(K)$ *there is a unique element* u *of* $\mathfrak{D}(\mathfrak{G})$ *satisfying* $(\alpha-\mathfrak{G})u=f$.

Hence, $(\alpha-\mathfrak{G})^{-1}$ *is defined on* $C(K)$ for $\alpha>0$.

(1.3) $\|(\alpha-\mathfrak{G})^{-1}\|\leqslant 1/\alpha$.

(1.4) $(\alpha-\mathfrak{G})^{-1}f\geqslant 0$ *for* $f\geqslant 0$.

Conversely, if $\mathfrak{G}$ *is a linear operator satisfying* (1.1) *and if there is a non-negative number* α_0 *such that* (1.2)-(1.4) *hold for all* $\alpha>\alpha_0$, *then* $\mathfrak{G}$ *is the generator of a semigroup on* $C(K)$, *which is uniquely determined by* $\mathfrak{G}$.

$(\alpha-\mathfrak{G})^{-1}$ is sometimes denoted by G_α and is called the *Green*

7) By $\lim_{n\to\infty} u_n=u$ or $u_n\to u$ in $C(K)$ we mean $\lim_{n\to\infty}\|u_n-u\|=0$.

operator of the semigroup. It is given by

$$G_\alpha f=\int_0^\infty e^{-\alpha t}T_t f dt.$$

Corollary. *Let $\mathfrak{G}$ be the generator of a semigroup on $C(K)$. If constant* 1 *is in $\mathfrak{D}(\mathfrak{G})$ and if $\mathfrak{G}1\leqslant -c$ for some constant c, then $\mathfrak{G}'=\mathfrak{G}+c$ is the generator of a semigroup on $C(K)$, where $\mathfrak{D}(\mathfrak{G})=\mathfrak{D}(\mathfrak{G}')$.*

Proof. Since $\mathfrak{G}$ is a generator, $\{(\alpha-c)-\mathfrak{G}\}^{-1}=\{\alpha-(\mathfrak{G}+c)\}^{-1}=(\alpha-\mathfrak{G}')^{-1}$ is defined on $C(K)$ and non-negative for any sufficiently large α. Condition $\mathfrak{G}1\leqslant -c$ implies $(\alpha-\mathfrak{G}')1=(\alpha-(\mathfrak{G}+c))1=\alpha-(\mathfrak{G}1+c)\geqslant\alpha$, and hence $\alpha(\alpha-\mathfrak{G}')^{-1}1\leqslant(\alpha-\mathfrak{G}')^{-1}(\alpha-\mathfrak{G}')1=1$, implying $\|(\alpha-\mathfrak{G}')^{-1}\|\leqslant 1/\alpha$. Thus, $\mathfrak{G}'$ is the generator of a semigroup on $C(K)$ by Theorem 1. 1.

Theorem 1. 2.[8] i) *Let B be a linear operator defined on a subspace $\mathfrak{D}(B)$ of $C(K)$ taking values in $C(K)$ and satisfy the following conditions.*

(1. 5) *$\mathfrak{D}(B)$ is dense in $C(K)$.*

(1. 6) *If f in $\mathfrak{D}(B)$ takes a positive maximum at x in K_0, then there is a point x' in K such that $f(x')=f(x)$ and $Bf(x')\leqslant 0$, where K_0 is a fixed open and dense subset of K.*

Then, there is a closed extension of B, and hence there is the smallest closed extension $\overline{B}$. $\overline{B}$ also satisfies (1. 6).

ii) *Let B satisfy* (1. 5) *and the following two conditions.*

(1. 7) *If f in $\mathfrak{D}(B)$ takes a positive maximum at x in K, then there is a point x' in K such that $f(x')=f(x)$ and $Bf(x')\leqslant 0$;*

(1. 8) *The range of α_0-B is dense in $C(K)$ for some $\alpha_0\geqslant 0$.*

8) This has been proved essentially by Wentzell [36, Lemma 1 and Theorem 2] and K. Ito [14, Theorem 39. 1].

Then, $\overline{B}$ is the generator of a semigroup on $C(K)$, which is uniquely determined by B.

Proof. i) Let $\{u_n\}$ be a sequence in $\mathfrak{D}(A)$ such that $\lim_{n\to\infty} u_n=0$ and $\lim_{n\to\infty} Bu_n$ exists. In order to show that B has a closed extension, it is sufficient to prove $\lim_{n\to\infty} Bu_n=0$. Assume that $\lim_{n\to\infty} Bu_n$ takes a positive value. K_0 being dense in K, there is an $x_0\in K_0$ such that $\lim_{n\to\infty} Bu_n(x_0)>0$. Since K_0 is open and the convergence in $C(K)$ is uniform, there are an open neighbourhood U of x_0 contained in K_0 and positive numbers ε and n_0 such that $Bu_n(x)>\varepsilon$ for $x\in U$ and $n\geqslant n_0$. By (1.5) we can take an $h\in\mathfrak{D}(B)$ such that $h(x_0)>1$ and $h(x)<0$ for $x\in K-U$. Put $u_n'=u_n+\varepsilon(1+\|Bh\|)^{-1}h$. Since $\lim_{n\to\infty} u_n=0$, $u_n'(x_0)$ is positive and greater than $u_n'(x)$ for all $x\in K-U$ and sufficiently large n, and hence, u_n' takes a positive maximum at some ponit in U, and never in $K-U$. By (1.6), there is a point x_n at which u_n' takes the positive maximum and $Bu_n'(x_n)\leqslant 0$. But, since

$$Bu_n'(x)=Bu_n(x)+\varepsilon(1+\|Bh\|)^{-1}Bh(x)>Bu_n(x)-\varepsilon>0$$

for $x\in U$ and $n\geqslant n_0$, x_n is not contained in U, which is absurd. Therefore, $\lim_{n\to\infty} Bu_n$ can take a positive value nowhere. Similarly, we can prove that $-\lim_{n\to\infty} Bu_n$ can not take a positive value, and hence $\lim_{n\to\infty} Bu_n=0$.

Now, we prove that $\overline{B}u(x_0)\leqslant 0$ when $u\in\mathfrak{D}(\overline{B})$ takes a positive maximum at point x_0 in K_0. Assume $\overline{B}u(x_0)>0$, take a sequence $\{u_n\}$ in $\mathfrak{D}(B)$ such that $\lim_{n\to\infty} u_n=u$ and $\lim_{n\to\infty} Bu_n=\overline{B}u$, and put $u_n'=u_n+\varepsilon(1+\|Bh\|)^{-1}h$ by taking ε, U and h in the same way as above. Using $\lim_{n\to\infty} u_n=u$ and $u(x_0)=\max_{x\in K_0} u(x)$ in place of $\lim_{n\to\infty} u_n=0$, we can make a similar argument as above, leading to a contradiction.

ii) We prove that $f\geqslant 0$ and $(\alpha_0-B)u=f$ imply $u\geqslant 0$. By (1.8) we can take $v\in\mathfrak{D}(B)$ and g such that $g\geqslant 1$ and $(\alpha_0-B)v=g$. Since $(\alpha_0-B)(u+\varepsilon v)=f+\varepsilon g>0$ for any $\varepsilon>0$, $-(u+\varepsilon v)$ does not take positive maximum by (1.7), and hence $u+\varepsilon v\geqslant 0$, which implies

$u \geqslant 0$. Thus, $(\alpha_0 - B)u = 0$ implying $u = 0$, $\alpha_0 - B$ maps $\mathfrak{D}(B)$ one to one and onto the range $\mathfrak{R}(\alpha_0 - B)$. Then, $(\alpha_0 - B)^{-1}$ is defined on $\mathfrak{R}(\alpha_0 - B)$, linear and non-negative. Moreover, it is bounded because $-\|f\|g \leqslant f \leqslant \|f\|g$ and $(\alpha_0 - B)^{-1} \geqslant 0$ imply $-\|f\|(\alpha_0 - B)^{-1}g \leqslant (\alpha_0 - B)^{-1}f \leqslant \|f\|(\alpha_0 - B)^{-1}g$, that is, $\|(\alpha - B)^{-1}f\| \leqslant \|(\alpha_0 - B)^{-1}g\| \cdot \|f\|$. $\mathfrak{R}(\alpha_0 - B)$ being dense by (1.8) and $(\alpha_0 - B)^{-1}$ being bounded, $(\alpha_0 - \overline{B})u = f$ has a solution u for any $f \in C(K)$. By the former half of the proof with K_0 replaced by K, $\overline{B}$ satisfies (1.7). Thus, $(\alpha_0 - \overline{B})^{-1}$ is defined on $C(K)$, non-negative, and maps $C(K)$ one to one and onto $\mathfrak{D}(\overline{B})$. Clearly, $\|(\alpha_0 - \overline{B})^{-1}\| = \|(\alpha_0 - B)^{-1}\| < \infty$. Write G_{α_0} for $(\alpha_0 - \overline{B})^{-1}$.

Let α_1 satisfy $0 < (\alpha_1 - \alpha_0)\|G_{\alpha_0}\| < 1$. For any $f \in C(K)$ and α satisfying $\alpha_0 < \alpha \leqslant \alpha_1$ the Neumann series $u = G_{\alpha_0}f + \sum_{n=1}^{\infty}(\alpha_0 - \alpha)^n G_{\alpha_0}^{n+1}f$ is a solution of $u + (\alpha - \alpha_0)G_{\alpha_0}u = G_{\alpha_0}f$. But, applying $(\alpha_0 - \overline{B})$ to the both hand sides, we know that u is also a solution of $(\alpha - \overline{B})u = f$. Besides, $\overline{B}$ satisfying (1.7), u is the unique solution, and hence $(\alpha - \overline{B})^{-1}$ is defined on $C(K)$, and maps $\mathfrak{D}(\overline{B})$ one to one and onto $C(K)$. It is also non-negative and bounded. We write G_α for $(\alpha - \overline{B})^{-1}$. Moreover, $\|G_\alpha\| \leqslant 1/\alpha$. In fact, if $u = (\alpha - \overline{B})^{-1}f$ takes a positive value, there is an $x_0 \in K$ such that $\max_{x \in K} u(x) = u(x_0)$ and $\overline{B}u(x_0) \leqslant 0$, and hence

$$\max_{x \in K} u(x) = u(x_0) \leqslant \frac{1}{\alpha}(\alpha - \overline{B})u(x_0) = \frac{1}{\alpha}f(x_0) \leqslant \frac{1}{\alpha}\|f\|,$$

where $f \in C(K)$ and $\alpha_0 < \alpha \leqslant \alpha_1$. Similarly, $\min_{x \in K} u(x) \geqslant -(1/\alpha)\|f\|$, if u takes a negative value. These inequalities imply $\|u\| \leqslant (1/\alpha)\|f\|$ and hence $\|G_\alpha\| = \|(\alpha - \overline{B})^{-1}\| \leqslant 1/\alpha$.

We obtain a similar result if we replaee α_0 and α_1 by α_1 and α_2 $(\alpha_1 < \alpha_2 < 2\alpha_1)$ respectively, because $(\alpha - \alpha_1)\|G_{\alpha_1}\| < (\alpha - \alpha_1)/\alpha_1 \leqslant (\alpha_2 - \alpha_1)/\alpha_1 < 1$ for $\alpha_1 < \alpha \leqslant \alpha_2$, and the corresponding Neumann series converges again. Taking, for instance, $\alpha_n = 2\alpha_{n-1} - (\alpha_1/2)$ $(n = 2, 3, \cdots)$ and repeating similar observations, we have a system of operators $\{G_\alpha, \alpha \geqslant \alpha_0\}$ on $C(K)$ satisfying (1.1)-(1.4) with $\mathfrak{G}$ replaced by $\overline{B}$.

Thus, $\bar{B}$ is the generator of a semigroup on $C(K)$ by Theorem 1.1.

Remark. In the above proof we have proved that (1.5) and (1.6) imply that

(1.9) if f in $\mathfrak{D}(B)$ takes a positive maximum at x in K_0, then $Bf(x)\leqslant 0$.

Similarly (1.5) and (1.7) imply (1.9) with K_0 replaced by K.

Corollary.[9] *Suppose that* $\mathfrak{G}$ *is the generator of a semigroup on* $C(K)$ *and that* M *is a bounded operator on* $C(K)$. *Define* $\mathfrak{G}'$ *on* $\mathfrak{D}(\mathfrak{G})$ by $\mathfrak{G}'f=\mathfrak{G}f+Mf$. *If either* M *or* $\mathfrak{G}'$ *satisfies* (1.7), *then* $\mathfrak{G}'$ *generates a semigroup on* $C(K)$.

Proof. Clearly $\mathfrak{G}$ satisfies (1.7), and hence, if M satisfies (1.7), so does $\mathfrak{G}+M$ by the preceding remark. Since $\mathfrak{D}(\mathfrak{G}')=\mathfrak{D}(\mathfrak{G})$ is dense in $C(K)$, it is sufficient to prove that $\mathfrak{G}'$ satisfies (1.8). M being bounded, we can find a positive α_0 such that $\|G_{\alpha_0}M\|\leqslant \|G_{\alpha_0}\|\cdot\|M\|<1$. Then, a Neumann series $u=G_{\alpha_0}f+\sum_{n=1}^{\infty}(G_{\alpha_0}M)^n G_{\alpha_0}f$ is a solution of $u-G_{\alpha_0}Mu=G_{\alpha_0}f$ and hence it is also a solution of $(\alpha_0-\mathfrak{G}-M)u=(\alpha_0-\mathfrak{G}')u=f$ for any $f\in C(K)$, completing the proof.

§2. Solutions of parabolic and elliptic equations

Let D be a domain in an N-dimensional orientable manifold of class C^∞ and have compact closure $\bar{D}$. The boundary ∂D of D is assumed to be non-empty and to consist of a finite number of connected components, which are $(N-1)$-dimensional hypersurfaces of class C^3. Let an elliptic differential operator A be given on $\bar{D}$ by

$$
\begin{aligned}
Au(x)=&\sum_{i,j=1}^{N}\frac{1}{\sqrt{a(x)}}\frac{\partial}{\partial x^i}\left(a^{ij}(x)\sqrt{a(x)}\frac{\partial u}{\partial x^j}(x)\right)\\
&+\sum_{i=1}^{N}b^i(x)\frac{\partial u}{\partial x^i}(x)+c(x)u(x),
\end{aligned}
\tag{2.1}
$$

9) Similar results are found in [10] and [27].

where $a^{ij}(x)$ is a contravariant tensor of order 2, of class $C^{2,\kappa}$,[10] symmetric and strictly positive definite at each point of $\bar{D}$, $b^i(x)$ is a contravariant tensor of order 1 and of class $C^{1,\kappa}$, $c(x)$ is a non-positive function in $C^{0,\kappa}(\bar{D})$, and $a(x)=\det(a^{ij}(x))^{-1}$. $x^1, \cdots, x^N$ are local coordinates of x in a coordinate neighborhood U. The volume measure m on $\bar{D}$ is given by

$$m(E)=\int_E \sqrt{a(x)}\, dx^1\cdots dx^N, \quad E\subset U\cap\bar{D}.$$

The (inward-directed) normal derivative $\partial u/\partial n$ and the surface measure $\tilde{m}$ on ∂D are also associated with a^{ij}. That is, if we take such a coordinate system $\bar{\sigma}(x)=(\bar{x}^1, \cdots, \bar{x}^N)$ that ∂D is characterized by $\bar{x}^N=0$ and D by $\bar{x}^N>0$ in U, and if the values of a^{ij} and a in the system $\bar{\sigma}$ are denoted by $\bar{a}^{ij}$ and $\bar{a}$, respctively, we have

$$\frac{\partial u}{\partial n}(x)=\sum_{i=1}^N \frac{1}{\sqrt{\bar{a}^{NN}(x)}}\bar{a}^{iN}(x)\frac{\partial u}{\partial \bar{x}^i}(x), \quad x\in\partial D\cap U,$$

$$\tilde{m}(E)=\int_E \sqrt{\bar{a}(x)}\ \sqrt{\bar{a}^{NN}(x)}\, d\bar{x}^1\cdots d\bar{x}^{N-1}, \quad E\subset\partial D\cap U.$$

The definitions of A, m, $\tilde{m}$, and $\partial/\partial n$ do not depend on the choice of local coordinates, and m and $\tilde{m}$ can be extended for any measurable subset of D and ∂D respectively, uniquely in a natural way.

Now, consider the Cauchy problem for parabolic differential equation

$$\text{(2. 2)} \qquad \frac{\partial u}{\partial t}(t, x)=Au(t, x) \quad t>0,\ x\in D$$

with boundary condition

$$\text{(2. 3)} \qquad \gamma(x)u(t, x)+\frac{\partial u}{\partial n}(t, x)=0 \quad t>0,\ x\in\partial D$$

10) We say that a function or a tensor is of class C^n, if it is n-times continuously differentiable. Moreover, if its n-th derivatives satisfy the uniform Hölder condition with exponent κ $(0<\kappa\leq 1)$ in a set E, it is called to be of class $C^{n,\kappa}$ in E. $C^n(E)$ and $C^{n,\kappa}(E)$ are the sets of all real-valued functions of class C^n and of class $C^{n,\kappa}$ in E, respectively.

where $\gamma(x)$ is a non-positive function in $C^{2,\kappa}(\partial D)$. The fundamental solution of this problem has been constructed by S. Ito [17, 18] extending the method of E. E. Levi. We cite a part of his results in the following Theorems 2.1–2.3.

Theorem 2.1. i) *There is a function* $p(t, x, y)$ *defined on* $(0, \infty)\times\bar{D}\times\bar{D}$, *and continuous in* y *for fixed* $(t, x)\in(0, \infty)\times\bar{D}$. *For any* $f\in C(\bar{D})$

$$u(t, x)=\int_{\bar{D}}p(t, x, y)f(y)m(dy)^{11)}$$

is continuous in $(0, \infty)\times\bar{D}$, *continuously differentiable in* $t>0$, *belongs to* $C^2(\bar{D})$ *as a function of* x *and is the unique solution of the equation* (2.2) *satisfying the boundary condition* (2.3) *and the initial condition*

$$\lim_{t\to 0}u(t, x)=f(x), \quad \textit{uniformly in } x\in\bar{D}.^{12)}$$

Such a function $p(t, x, y)$ *is unique.*

ii) $p(t, x, y)$ *is non-negative and satisfies*

$$p(t+s, x, z)=\int_{\bar{D}}p(t, x, y)p(s, y, z)m(dy),$$

$$\int_{\bar{D}}p(t, x, y)m(dy)\leqslant e^{Ct}, \quad \textit{where } C=\max_{x\in\bar{D}}c(x).$$

Moreover, $c(x)\equiv 0$ *and* $\gamma(x)\equiv 0$ *imply*

$$\int_{\bar{D}}p(t, x, y)m(dy)=1.$$

The function $p(t, x, y)$ is called the *fundamental solution* of the Cauchy problem for the equation (2.2) with boundary condition (2.3). Making use of this function we can solve a more general equation in the following.

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11) Since $m(\partial D)=0$, we may write $\int_D$ in stead of $\int_{\bar{D}}$.

12) This holds since the coefficient of $\partial u/\partial n$ in (2.3) is positive on ∂D and $\bar{D}$ is compact.

Theorem 2.2. *Let $f(x)$, $h(t, x)$ and $\varphi(t, x)$ be bounded continuous functions on $\overline{D}$,* $(0, \infty)\times\overline{D}$ and $(0, \infty)\times\partial D$, *respectively. If there exists a number $\kappa=\kappa(t_1, t_2)$ for any positive t_1 and t_2 $(t_1<t_2)$ such that $h\in C^{0,\kappa}([t_1, t_2]\times\overline{D})$ and $\varphi\in C^{0,\kappa}([t_1, t_2]\times\partial D)$, then*

$$
\begin{aligned}
u(t, x) &= \int_{\overline{D}} p(t, x, y) f(y) m(dy) \\
&\quad + \int_0^t ds \int_{\overline{D}} p(s, x, y) h(t-s, y) m(dy) \qquad (2.4) \\
&\quad + \int_0^t ds \int_{\partial D} p(s, x, y) \varphi(t-s, y) \tilde{m}(dy)
\end{aligned}
$$

exists and is

(2.5) *continuous on* $(0, \infty)\times\overline{D}$, *continuously differentiable in $t>0$, and of class $C^2(D)$ and $C^1(\overline{D})$ as a function of x.*

$u(t, x)$ satisfies

$$(2.6) \qquad \left(\frac{\partial}{\partial t}-A\right)u(t, x)=h(t, x), \qquad t>0,\ x\in D.$$

$$(2.7) \qquad -\left(\gamma+\frac{\partial}{\partial n}\right)u(t, x)=\varphi(t, x), \qquad t>0,\ x\in\partial D,$$

$$(2\cdot 8) \qquad \lim_{t\to 0} u(t, x)=f(x), \qquad \text{boundedly in } x\in D.$$ [13]

Conversely, a function $u(t, x)$ on $(0, \infty)\times\overline{D}$ satisfying (2.5)–(2.8) is necessarily represented by (2.4).

We remark that the above theorems are sharpened in the following lemmas, which we prove in the appendix.

Lemma 2.1. *$p(t, x, y)$ is continuous in (t, x, y) on $(0, \infty)\times\overline{D}\times\overline{D}$.*

Lemma 2.2. *If $h(t, x)$ is bounded and measurable on $(0, \infty)\times\overline{D}$,*

$$u(t, x)=\int_0^t ds\int_{\overline{D}} p(s, x, y) h(t-s, y) m(dy)$$

13) Actually, (2.8) holds uniformly on $\overline{D}$.

is continuously differentiable in $x \in \overline{D}$. *Moreover, we have, for any positive* T,

$$(2.9) \qquad \max_{i,j} \sup_{x \in U_j, 0 < t \leq T} \left| \frac{\partial u}{\partial x^i_{(j)}}(t, x) \right| < \infty,$$

where $\sigma_j(x) = (x^1_{(j)}, \cdots, x^N_{(j)})$ *is a coordinate system in* U_j $(1 \leq j \leq M)$, *and* $\bigcup_{j=1}^{M} U_j = \overline{D}$.

Integrating $p(t, x, y)$ in t, we have the Green function for an elliptic equation.

Theorem 2.3. *If at least one of* $\min_{x \in \overline{D}} c(x) < 0$ *and* $\min_{x \in \partial D} \gamma(x) < 0$ *holds, then*

$$g(x, y) = \int_0^\infty p(t, x, y) dt$$

is finite unless $x = y$. *For any* $f \in C^{0,\kappa}(\overline{D})$ *and* $\varphi \in C^{0,\kappa}(\partial D)$

$$(2.10) \quad u(x) = \int_{\overline{D}} g(x, y) f(y) m(dy) + \int_{\partial D} g(x, y) \varphi(y) \widetilde{m}(dy)$$

exists and satisfies

$$(2.11) \qquad u \in C^1(\overline{D}) \cap C^2(D)$$

$$(2.12) \qquad -Au(x) = f(x), \quad x \in D, \cdot$$

$$(2.13) \qquad -\left(\gamma + \frac{\partial}{\partial n}\right) u(x) = \varphi(x), \quad x \in \partial D.$$

Conversely, a function u *which satisfies* (2.11)-(2.13) *is necessarily represented by* (2.10).

Corollary. *Let* c *(or* γ*),* f *and* φ *satisfy the condition in Theorem* 2.3. *If* $v(t, x)$ *is a function satisfying* (2.5) *and if*

$$\left(\frac{\partial}{\partial t} - A\right) v(t, x) = f(x), \qquad t > 0, \ x \in D,$$

$$-\left(\gamma + \frac{\partial}{\partial n}\right) v(t, x) = \varphi(x), \quad t > 0, \ x \in \partial D,$$

$$\lim_{t \to 0} v(t, x) = 0, \qquad \text{boundedly in } x \in D,$$

then $\lim_{t \to \infty} v(t, x) = u(x)$ *exists and* $u(x)$ *satisfies* (2.11)-(2.13).

Now we consider the solution of two typical types of problems for elliptic equations and introduce some notations for the use in Chapter II. The results stated below are found in the standard references such as [2] and [22]. They are also found in S. Ito [18].

Theorem 2.4. i) *For any constant* $\alpha \geqslant 0$ *and* $f \in C^{0,\kappa}(\bar{D})$, *the solution* u *in* $C^2(\bar{D})$ *of*

$$(\alpha - A)u(x) = f(x), \quad x \in D$$
$$u(x) = 0, \quad x \in \partial D$$

exists. Such u *is unique and we denote it by* $G_\alpha^{\min} f$.[14][15]

ii) $G_\alpha^{\min}$, *considered as an operator in* $C(\bar{D})$, *is linear, non-negative and bounded. If* α *is positive,* $\|G_\alpha^{\min}\| \leqslant 1/\alpha$. *Since the domain of definition* $C^{0,\kappa}(\bar{D})$ *is dense in* $C(\bar{D})$, $G_\alpha^{\min}$ *is uniquely extended to the whole* $C(\bar{D})$. *Henceforth* $G_\alpha^{\min}$ *denotes this extension.*

iii) $G_\alpha^{\min}$ *is non-negative and* $G_\alpha^{\min} f$ *vanishes on* ∂D *for each* $f \in C(\bar{D})$.

iv) *For* $\{G_\alpha^{\min}, \alpha \geqslant 0\}$ *the resolvent equation holds, that is,*

$$G_\alpha^{\min} - G_\beta^{\min} + (\alpha - \beta) G_\alpha^{\min} G_\beta^{\min} = 0 \quad \text{for any } \alpha, \beta \geqslant 0.$$

v) *For any* $f \in C(\bar{D})$ *and* $x \in D$

$$\lim_{\alpha \to \infty} \alpha G_\alpha^{\min} f(x) = f(x).$$

Moreover, the convergence is uniform in $x \in \bar{D}$ *if* f *vanishes on* ∂D.

Corollary to iv). *The range of* $G_\alpha^{\min}$, *that is,* $\{G_\alpha^{\min} f \mid f \in C(\bar{D})\}$ *does not depend on the choice of* $\alpha \geqslant 0$.

Theorem 2.5. i) *For any constant* $\alpha \geqslant 0$ *and* $\varphi \in C(\partial D)$, *the solution* u *in* $C(D) \cap C^2(D)$ *of*

$$(\alpha - A)u(x) = 0, \quad x \in D$$
$$u(x) = \varphi(x), \quad x \in \partial D$$

exists. Such u *is unique and we denote it by* $H_\alpha \varphi$.

---

14) min indicates that this is the minimal resolvent in the sense of Feller.

15) More precisely, it can be proved that $G_\alpha^{\min} f \in C^{2,\kappa}(\bar{D})$.

ii) H_α, *considered as a mapping from* $C(\partial D)$ *into* $C(\bar{D})$, *is linear, non-negative and bounded with norm one.*

iii) $H_\alpha\varphi$ *does not take a non-negative maximum [non-positive minimum] in* D, *unless it is a constant function.*

iv) *If* $H_\alpha\varphi$ *is not constant and takes a non-negative maximum [non-positive minimum] at point* x_0 *on* ∂D, *and if* $H_\alpha\varphi$ *is differentiable at* x_0, *then*

$$\frac{\partial}{\partial n}H_\alpha\varphi(x_0)<0 \quad [>0].$$ [16]

v) *If* φ *is in* $C^{2,\kappa}(\partial D)$, $H_\alpha\varphi$ *is in* $C^2(\bar{D})$.[17]

The property iv) is due to Giraud, Hopf, and Oleĭnik in case $H_\alpha\varphi(x_0)>0$ $[<0]$. But the proof in [26] can be applied in this case without any change. v) is reduced to Theorem 2. 4, i) by an appropriate extension of φ to $\bar{D}$. We remark that v) is a special case of Theorem 9. 3 in Agmon-Donglis-Nirenberg [1].

Now, we add some properties of the closure of A.

Lemma 2. 3.[18] A, *considered as a linear operator defined on* $\mathfrak{D}(A)=C^2(\bar{D})$, *has the smallest closed extension* $\bar{A}$. *If* u *in* $\mathfrak{D}(\bar{A})$ *is twice continuously differentiable in some neighbourhood of* x *in* D, *then* $\bar{A}u(x)=Au(x)$.

Proof. Since A satisfies (1. 5) and (1. 6) in Theorem 1. 2 with K and K_0 replaced by $\bar{D}$ and D, respectively, it has the smallest closed extension $\bar{A}$. For u in $\mathfrak{D}(\bar{A})$ there is a sequence $\{u_n\in\mathfrak{D}(A)=C^2(\bar{D})\}$ satisfying $u_n\to u$ and $Au_n\to\bar{A}u$. If u is twice continuously differentiable in a neighbourhood U of x, then we have, for any h in $C^2(\bar{D})$ with support contained in U,

$$\int_U h(y)Au_n(y)m(dy)=\int_U A'h(y)u(y)m(dy),$$

16) This remains true if $H_\alpha\varphi$ is replaced by a function u in $C(\bar{D})\cap C^2(D)$ satisfying $(\alpha-A)u\leqslant 0$ $[\geqslant 0]$.

17) More precisely, we have $H_\alpha\varphi\in C^{2,\kappa}(\bar{D})$.

18) Cf. Wentzell [36, Lemma 1].

where A' is the formal adjoint of A. Letting $n\to\infty$, we have

$$\int_U h(y)\bar{A}u(y)m(dy)=\int_U A'h(y)u(y)m(dy)$$
$$=\int_U h(y)Au(y)m(dy),$$

implying that $\bar{A}u(y)=Au(y)$ in U.

Lemma 2.4. *For any $f\in C(\bar{D})$, $\varphi\in C(\partial D)$ and $\alpha\geqslant 0$, $G_\alpha^{\min}f$ and $H_\alpha\varphi$ belong to $\mathfrak{D}(\bar{A})$, and satisfy*

(2.14) $\qquad (\alpha-\bar{A})G_\alpha^{\min}f=f,$

(2.15) $\qquad (\alpha-\bar{A})H_\alpha\varphi=0.$

Proof. For f in $C^{0,\kappa}(\bar{D})$, $G_\alpha^{\min}f$ belongs to $C^2(\bar{D})=\mathfrak{D}(A)$ and hence $(\alpha-\bar{A})G_\alpha^{\min}f=(\alpha-A)G_\alpha^{\min}f=f$ by Theorem 2.4. But, $C^{0,\kappa}(\bar{D})$ being dense in $C(\bar{D})$ and $G_\alpha^{\min}$ being bounded, (2.14) holds for any $f\in C(\bar{D})$. Simirarly, we can prove (2.15) for any $\varphi\in C(\partial D)$, since (2.15) holds for all φ in $C^{2,\kappa}(\partial D)$ by Theorem 2.5, and H_α is bounded and maps a dense subset $C^{2,\kappa}(\partial D)$ of $C(\partial D)$ into $C^2(\bar{D})$.

Making use of $\bar{A}$ we determine the range $\mathfrak{R}(G_\alpha^{\min})$ in the following. This is mainly for its own sake, rather than for later use.

Proposition 2.1. *$\mathfrak{R}(G_\alpha^{\min})$ is equal to $\{u\in\mathfrak{D}(\bar{A})\mid[u]_{\partial D}=0\}$, where $[u]_{\partial D}$ is the restriction of $u\in C(\bar{D})$ on the boundary ∂D.*

Proof. Since $\mathfrak{R}(G_\alpha^{\min})$ does not depend on $\alpha\geqslant 0$, we fix a positive α. We note that v in $\mathfrak{D}(\bar{A})$ is constant 0 if it satifies $(\alpha-\bar{A})v=0$ and vanishes on ∂D. In fact, at point $x\in D$ where v takes a positive maximum [negative minimum] $\bar{A}v(x)\leqslant 0$ [$\geqslant 0$] by Theorem 1.2 and the remark, and hence $(\alpha-\bar{A})v(x)>0$ [<0], contradicting to $(\alpha-\bar{A})v=0$. $u\in\mathfrak{R}(G_\alpha^{\min})$ belongs to $\mathfrak{D}(\bar{A})$ and satisfies $[u]_{\partial D}=0$ by Theorem 2.4 and Lemma 2.4. Conversely, let u belong to $\mathfrak{D}(\bar{A})$ and satisfy $[u]_{\partial D}=0$. $v=G_\alpha^{\min}(\alpha-\bar{A})u-u$ satisfies $(\alpha-\bar{A})v=0$ by Lemma 2.4 and vanishes on ∂D, and hence $v=0$, implying $u=G_\alpha^{\min}(\alpha-\bar{A})u\in\mathfrak{R}(G_\alpha^{\min})$.

§3. Approximation of the integral $\int_{\partial D} f(x)\widetilde{m}(dx)$

The inverse matrix $(a_{ij}(x))$ of $(a^{ij}(x))$ is a symmetric, strictly positive definite covariant tensor of order 2. The length of a curve C, which is of class C^1 piecewise, is defined by

$$(3.1) \qquad l(C)=\int_0^1\left\{\sum_{i,j=1}^N (a_{ij}(x(\lambda))\frac{dx^i(\lambda)}{d\lambda}\frac{dx^j(\lambda)}{d\lambda}\right\}^{1/2}d\lambda,$$

where C is given by

$$C: \lambda\in[0,\ 1]\rightarrow x(\lambda)\in\overline{D}.$$

$l(C)$ does not depend on the choice of coordinate sytem $(x^1(\lambda), x^2(\lambda), \cdots, x^N(\lambda))$. The infimum $d(x, y)$ of the lengths of all curves contained in $\overline{D}$ which connect x and y and of class C^1 piecewise, satisfies the postulates for distance and is called the distance between x and y determined by a^{ij}. We write $d(x, \partial D)=\inf_{y\in\partial D} d(x, y)$ and $D_\rho=\{x\in\overline{D}\,|\,d(x, \partial D)<\rho\}$. The purpose in this section is to prove that $\frac{1}{\rho}\int_{D\rho} f(x)m(dx)$ approximates $\int_{\partial D} f(x)\widetilde{m}(dx)$ when $\rho\rightarrow 0$.

S. Ito [17] proved that for any $x_1\in\partial D$ there is a neighbourhood[19] U of x_1 and a coordinate system $\bar{\sigma}(x)=(\bar{x}^1, \cdots, \bar{x}^N)$ in U satisfying the following conditions: $\partial D\cap U$ and $D\cap U$ are characterized by $\bar{x}^N=0$ and $\bar{x}^N>0$, respectively; $\bar{a}^{Ni}(x)=\bar{a}^{iN}(x)=\bar{a}_{Ni}(x)=\bar{a}_{iN}(x)=1$ or 0 on $\partial D\cap U$ according as $i=N$ or $i\neq N$, where $\bar{a}^{ij}(x)$ and $\bar{a}_{ij}(x)$ are the values of a^{ij} and a_{ij} in the system $\bar{\sigma}$. Such $\bar{\sigma}$ is called a *canonical coordinate system*, and U a *cadonical coordinate neighbourhood.*

The topology given by the distance $d(x, y)$ is the same as the original one. Namely we can prove

Lemma 3.1. *Let* $\{x_n;\ n=1, 2, \cdots\}\subset\overline{D}$ *and* $x_0\in\overline{D}$. *We have* $d(x_n, x_0)\rightarrow 0$ *if and only if* $x_n^i\rightarrow x_0^i$ $(1\leqslant i\leqslant N)$ *for their local coordinates.*

Proof. Let U be a coordinate neighbourhood of x_0, and $\sigma(x)$

19) In the topology relative to $\overline{D}$.

$=(x^1, \cdots, x^N)$ be a canonical coordinate system in U. For each $y \in U \cap \bar{D}$ satisfying $|y^i - x_0^i| \geqslant \varepsilon$ for some i, we have $d(y, x_0) \geqslant K\varepsilon$ where K is a positive constant independent of y. In fact, for each curve

$$C : \lambda \in [0, 1] \rightarrow x(\lambda) \in \bar{D}$$

from x_0 to y and of class C^1 piecewise, we have

$$l(C) \geqslant \int_0^1 \left\{ \sum_{j,k=1}^N a_{jk}(x(\lambda)) \frac{dx^j(\lambda)}{d\lambda} \frac{dx^k(\lambda)}{d\lambda} \right\}^{1/2} d\lambda$$

$$\geqslant K \int_0^1 \left| \frac{dx^i(\lambda)}{dx} \right| d\lambda \geqslant K|y^i - x_0^i| \geqslant K\varepsilon.$$

Thus, $d(x_n, x_0) \rightarrow 0$ implies that x_n is contained in U for all sufficiently large n, and that $x_n^i \rightarrow x_0^i$ for $i=1, 2, \cdots, N$. To prove the converse, put $V = \{z \| z^i - x_0^i| < \delta,\ 1 \leqslant i \leqslant N\}$ for sufficiently small δ and let $V \subset U$. For each $z \in V \cap \bar{D}$ we have

$$d(z, x_0) \leqslant K' \left(\sum_{i=1}^N (z^i - x_0^i)^2\right)^{1/2}$$

where K' is a constant independent of z. For, if we define a curve C' from x_0 to z by

$$C' : \lambda \in [0, 1] \rightarrow x(\lambda) = \sigma^{-1}(x_0^i + \lambda(z^i - x_0^i),\ 1 \leqslant i \leqslant N)$$

then

$$d(z, x_0) \leqslant l(C') = \int_0^1 \left\{ \sum_{i,j=1}^N a_{ij}(x(\lambda))(z^i - x_0^i)(z^j - x_0^j) \right\}^{1/2} d\lambda$$

$$\leqslant K' \left(\sum_{i=1}^N (z^i - x_0^i)^2 \right)^{1/2}.$$

Here we should take σ as a canonical coordinate system in case $x_0 \in \partial D$, so that C' is a curve contained in $\bar{D}$. Thus $x_n^i \rightarrow x_0^i$, $1 \leqslant i \leqslant N$, imply $d(x_n, x_0) \rightarrow 0$.

Let U be a canonical coordinate neighbourhood and $\bar{\sigma}(x) = (\bar{x}^1, \cdots, \bar{x}^N)$ be a canonical coordinate system. We write

$$V(\rho, r, x_0; \bar{\sigma}) = \left\{ x \in \bar{D} \cap U \,\middle|\, \sum_{i=1}^{N-1} (\bar{x}^i - x_0^i)^2 < r^2,\ 0 \leqslant \bar{x}^N < \rho \right\} \tag{3.2}$$

for $x_0 \in \partial D \cap U$ and sufficiently small ρ and $r > 0$.

Lemma 3.2. *For any $V(\rho, r, x_0; \bar{\sigma})$ we have*

$$(3.3)\qquad \lim_{d(x,\partial D)\to 0} \frac{d(x,\partial D)}{\bar{x}^N} = \lim_{\bar{x}^N\to 0} \frac{d(x,\partial D)}{\bar{x}^N} = 1 \quad \textit{for } x\in V.$$

Proof. Since $\bar{a}_{NN}(x)=1$ for $x\in\partial D$ and $\bar{a}_{NN}(x)$ is continuous, there is, for any $\varepsilon>0$, a δ such that $\bar{x}^N<\delta$ and $x\in U$ imply $\bar{a}_{NN}(x)<1+\varepsilon$. Define a curve $C(x)$ for $x\in V$ by

$$C(x) : \lambda\in[0,1]\to y(\lambda)=\bar{\sigma}^{-1}(\bar{y}^1(\lambda),\cdots,\bar{y}^N(\lambda)),$$
$$\bar{y}^1(\lambda)=\bar{x}^1,\cdots,\bar{y}^{N-1}(\lambda)=\bar{x}^{N-1}, \bar{y}^N(\lambda)=\lambda\bar{x}^N.$$

We have

$$(3.4)\qquad d(x,\partial D)\leqslant l(C(x))=\int_0^1 \bar{a}_{NN}(y(\lambda))^{1/2}\bar{x}^N d\lambda<(1+\varepsilon)^{1/2}\bar{x}^N,$$

if $x\in V$ and $\bar{x}^N<\delta$.

Let $\bar{\bar{\sigma}}(x)=(\bar{\bar{x}}^1,\cdots,\bar{\bar{x}}^N)$ be a coordinate system in U defined by

$$\bar{\bar{x}}^i=k\bar{x}^i \quad (i\leqslant i\leqslant N-1),\ \bar{\bar{x}}^N=\bar{x}^N,$$

where k is a positive number. The values of a_{ij} in the system $\bar{\bar{\sigma}}$ at $x\in\partial D\cap U$ are

$$\bar{\bar{a}}_{ij}(x)=k^{-2}\bar{a}_{ij}(x) \qquad (i,j\neq N),$$
$$\bar{\bar{a}}_{iN}(x)=\bar{a}_{iN}(x)=0 \quad (i\neq N),$$
$$\bar{\bar{a}}_{NN}(x)=\bar{a}_{NN}(x)=1.$$

Thus, the eigenvalues $\bar{\bar{\alpha}}_1(x),\cdots,\bar{\bar{\alpha}}_N(x)$ of the matrix $(\bar{\bar{a}}_{ij}(x))$ at $x\in\partial D\cap U$ are given by

$$(3.5)\qquad \bar{\bar{\alpha}}_i(x)=k^{-2}\bar{\alpha}_i(x)\ (1\leqslant i\leqslant N-1),\ \bar{\bar{\alpha}}_N(x)=\bar{\alpha}_N(x)=1,$$

where $\bar{\alpha}_1(x),\cdots,\bar{\alpha}_N(x)$ are the eigenvalues of $(\bar{a}_{ij}(x))$. Let V_1 be a set of type (3.2) satisfying $\bar{V}\subset V_1$.[20] For a sufficiently small k we have $\bar{\bar{\alpha}}_i(x)\geqslant 1$, $1\leqslant i\leqslant N$, $x\in\partial D\cap V_1$ by (3.5). Thus for any $\varepsilon>0$ there is such a $\delta'>0$ that $\bar{y}^N<\delta'$ and $y\in V_1$ imply $\bar{\bar{\alpha}}_i(y)\geqslant 1-\varepsilon$, $1\leqslant i\leqslant N$. Hence, the following estimation holds for any curve $C'(x)$, which is contained in $V_1\cap\{y\mid\bar{y}^N<\delta'\}$, starts at x and ends at a point on ∂D.

20) $\bar{V}$ is the closure of set V.

$$l(C'(x))=\int_0^1\left\{\sum_{i,j=1}^N \bar{a}_{ij}(y(\lambda))\frac{d\bar{y}^i(\lambda)}{d\lambda}\frac{d\bar{y}^j(\lambda)}{d\lambda}\right\}^{1/2}d\lambda$$

$$\geqslant(1-\varepsilon)^{1/2}\int_0^1\left\{\sum_{i=1}^N\left(\frac{d\bar{y}^i(\lambda)}{d\lambda}\right)^2\right\}^{1/2}d\lambda$$

$$\geqslant(1-\varepsilon)^{1/2}\int_0^1\left|\frac{d\bar{y}^N(\lambda)}{d\lambda}\right|d\lambda$$

$$\geqslant(1-\varepsilon)^{1/2}\bar{\bar{x}}^N=(1-\varepsilon)^{1/2}\bar{x}^N.$$

On the other hand, we can prove that $d(x, \partial D)$ is the infimum of $l(C'(x))$ of such $C'(x)$ that we have mentioned above when $x\in V$ and $\bar{x}^N<\delta''$ by a sufficiently small $\delta''>0$. Thus we have

$$d(x, \partial D)\geqslant(1-\varepsilon)^{1/2}\bar{x}^N$$

if $x\in V$ and $\bar{x}<\delta''$. This combined with (3. 4) imply the second equality in (3. 3).

By Lemma 3. 1, $d(x, \partial D)\rightarrow 0$ implies $\bar{x}^N\rightarrow 0$, and hence we have also the first equality in (3. 3).

Corollary. *If* $V=V(\rho_0, r_0, x_0; \bar{\sigma})$ *is a set of type* (3. 2), *then for sufficiently small* $\rho>0$ *there are* $\rho'=\rho'(\rho)$ *and* $\rho''=\rho''(\rho)$ *such that*

(3. 6) $$V(\rho', r_0, x_0; \bar{\sigma})\subset D_\rho\cap V\subset V(\rho'', r_0, x_0; \bar{\sigma})$$

and

(3. 7) $$\lim_{\rho\to 0}\frac{\rho'}{\rho}=\lim_{\rho\to 0}\frac{\rho''}{\rho}=1.$$

Proof. If we write $B_\rho=\{x\in \bar{V}\,|\, d(x, \partial D)=\rho\}$ and define $\rho'=\min_{x\in B_\rho}\bar{x}^N$ and $\rho''=\max_{x\in B_\rho}\bar{x}^N$, (3. 6) is clear. (3. 7) follows from Lemma 3. 2.

Lemma 3. 3.

(3. 8) $$\lim_{\rho\to 0}\frac{1}{\rho}\int_{D_\rho}f(x)m(dx)=\int_{\partial D}f(x)\tilde{m}(dx),$$

for $f\in C(\bar{D})$. *Moreover, for any compact subset* $\{f_\lambda, \lambda\in\Lambda\}$ *of* $C(\bar{D})$ *the convergence of* $\lim_{\rho\to 0}\frac{1}{\rho}\int_{D_\rho}f_\lambda(x)m(dx)$ *is uniform in* $\lambda\in\Lambda$.

Proof. Since ∂D is compact, we can take such non-negative functions $h_j(x)$, $1 \leqslant j \leqslant M$, in $C(\overline{D})$ that $\sum_{j=1}^{M} h_j(x)=1$ in a neighbourhood of ∂D and that each h_j has support contained in a set $V_j=V(\rho_j, r_j, x_j; \bar{\sigma}_j)$ of type (3.2). We write $\bar{\sigma}_j(x)=(\bar{x}^1, \cdots, \bar{x}^N)$, fixing j for a while. Taking $\rho'=\rho'(\rho)$ and $\rho''=\rho''(\rho)$ as in Corollary to Lemma 3.2 and writing $V_j(\rho')=V(\rho', r_j, x_j; \bar{\sigma}_j)$, we have

$$\begin{aligned}
&\left|\frac{1}{\rho}\int_{D_\rho} f(x)h_j(x)m(dx)-\int_{\partial D} f(x)h_j(x)\widetilde{m}(dx)\right| \\
&\quad\leqslant\left|\frac{1}{\rho}\int_{D_\rho} f(x)h_j(x)m(dx)-\frac{1}{\rho}\int_{V_j(\rho')} f(x)h_j(x)m(dx)\right| \\
&\quad+\left|\frac{1}{\rho}\int_{V_j(\rho')} f(x)h_j(x)m(dx)-\frac{1}{\rho'}\int_{V_j(\rho')} f(x)h_j(x)m(dx)\right| \\
&\quad+\left|\frac{1}{\rho'}\int_{V_j(\rho')} f(x)h_j(x)m(dx)-\int_{\partial D} f(x)h_j(x)\widetilde{m}(dx)\right|.
\end{aligned}$$

Writing $I_1(f, \rho)$, $I_2(f, \rho)$ and $I_3(f, \rho)$ for the first, second and third summands of the righthand side, and putting

$$R_j=\left\{(\bar{x}^1, \cdots, \bar{x}^{N-1}) \mid \sum_{i=1}^{N-1}(\bar{x}^i-\bar{x}_j^i)^2<r_j^2\right\},$$

we have

$$\begin{aligned}
I_1(f, \rho)&\leqslant\frac{1}{\rho}\|f\| m(V_j(\rho'')-V_j(\rho')) \\
&=\frac{1}{\rho}\|f\|\int_{\rho'}^{\rho''} d\bar{x}^N\int\cdots\int_{R_j}\sqrt{\bar{a}(x)}\, d\bar{x}^1\cdots d\bar{x}^{N-1}, \\
I_2(f, \rho)&\leqslant\left(1-\frac{\rho'}{\rho}\right)\frac{1}{\rho'}\int_{V_j(\rho')}|f(x)|h_j(x)m(dx) \\
&\leqslant\left(1-\frac{\rho'}{\rho}\right)\|f\|\frac{1}{\rho'}\int_0^{\rho'} d\bar{x}^N\int\cdots\int_{R_j}\sqrt{\bar{a}(x)}\, d\bar{x}^1\cdots d\bar{x}^{N-1}, \\
I_3(f, \rho)&\leqslant\int\cdots\int_{R_j}\Big|f(P_j(x))h_j(P_j(x))\sqrt{\bar{a}(P_j(x))} \\
&\quad-\frac{1}{\rho'}\int_0^{\rho'} f(x)h_j(x)\sqrt{\bar{a}(x)}\, d\bar{x}^N\Big|\, d\bar{x}^1\cdots d\bar{x}^{N-1},
\end{aligned}$$

where $P_j(x)=\bar{\sigma}_j^{-1}(\bar{x}^1, \cdots, \bar{x}^{N-1}, 0)$. Hence I_1, I_2 and I_3 vanish when $\rho\to 0$. Since

$$\sum_{j=1}^{M}\int_{\partial D} f(x)h_j(x)\widetilde{m}(dx)=\int_{\partial D} f(x)\widetilde{m}(dx)$$

and $$\sum_{j=1}^{M}\int_{D_\rho} f(x)h_j(x)m(dx)=\int_{D_\rho} f(x)m(dx)$$

for sufficiently small ρ, we have (3.8).

If $\{f_\lambda, \lambda\in\Lambda\}$ is a compact subset of $C(\bar{D})$, $\{f_\lambda, \lambda\in\Lambda\}$ is equi-continuous uniformly on $\bar{D}$, and hence $I_3(f_\lambda, \rho)$ converges to 0 uniformly in $\lambda\in\Lambda$. The convergence of $I_1(f_\lambda, \rho)$ and $I_2(f_\lambda, \rho)$ are also uniform in $\lambda\in\Lambda$ because $\{\|f\|, \lambda\in\Lambda\}$ is bounded.

Lemma 3.4. *Suppose that* $\{V(\rho_j, r_j, x_j; \bar{\sigma}_j), 1\leqslant j\leqslant M\}$ *cover* ∂D. *If we denote the projection (relative to* $\bar{\sigma}_j$*) of point* $x\in V_j$ *to* ∂D *by* $P_j(x)=\bar{\sigma}_j^{-1}(\bar{x}, \cdots, \bar{x}^{N-1}, 0)$ *and if a class of functions* $\{f_\rho\in C(\bar{D}), \rho>0\}$ *is bounded and satisfies*

$$(3\cdot 9)\qquad \lim_{\rho\to 0}\ \sup_{x\in V(\rho, r_j, x_j; \bar{\sigma}_j)} |f_\rho(x)-f_\rho(P_j(x))|=0,\quad 1\leqslant j\leqslant M,$$

then we have

$$\lim_{\rho\to 0}\left(\frac{1}{\rho}\int_{D_\rho} f_\rho(x)m(dx)-\int_{\partial D} f_\rho(x)\tilde{m}(dx)\right)=0.$$

The proof is similar to that of Lemma 3.3, where $\{f_\lambda, \lambda\in\Lambda\}$ is replaced by $\{f_\rho, \rho>0\}$. Since $\{f_\rho, \rho>0\}$ is bounded, $I_1(f_\rho, \rho)$ and $I_2(f_\rho, \rho)$ tend to 0 when $\rho\to 0$, and so does $I_3(f_\rho, \rho)$ by (3.9). Thus we have the conclusion.

Let $p(t, x, y)$ be the fundamental solution in Theorem 2.1. Concerning the integral of $p(t, x, y)$ on the strip D_ρ, the following estimation holds.

Lemma 3.5. *There is a positive* ρ_0 *such that for any* $T>0$

$$\frac{1}{\rho}\int_{D_\rho} p(t, x, y)m(dy)\leqslant Kt^{-1/2}$$

and

$$\int_{\partial D} p(t, x, y)\tilde{m}(dy)\leqslant Kt^{-1/2}$$

hold uniformly in $0<t\leqslant T$, $0<\rho\leqslant\rho_0$ *and* $x\in\bar{D}$.

The proof is referred to the appendix. Combining this lemma and Lemma 3.3, we have

Lemma 3.6. *For any $T>0$, we have*

$$\lim_{\rho\to 0}\frac{1}{\rho}\int_0^t ds\int_{D_\rho} p(s, x\, y)\, m(dy)=\int_0^t ds\int_{\partial D} p(s, x, y)\,\tilde{m}(dy), \tag{3.10}$$

where the convergence is uniform in $0\leqslant t\leqslant T$ and $x\in\bar{D}$.

Proof. For any $\varepsilon>0$ we can take positive numbers t_0 and ρ_0 such that

$$\frac{1}{\rho}\int_0^{t_0} ds\int_{D_\rho} p(s, x, y)\, m(dy)<\varepsilon,$$

and

$$\int_0^{t_0} ds\int_{\partial D} p(s, x, y)\,\tilde{m}(dy)<\varepsilon$$

uniformly in $0<\rho\leqslant\rho_0$ and $x\in\bar{D}$ by Lemma 3.5. On the other hand, $p(s, x, y)$ being continuous in (s, x, y) on $(0, \infty)\times\bar{D}\times\bar{D}$ by Lemma 2.1, $\{p(s, x, \cdot), s\in[t_0, T], x\in\bar{D}\}$ is a compact set in $C(\bar{D})$, and hence

$$\lim_{\rho\to 0}\frac{1}{\rho}\int_{D_\rho} p(s, x, y)\, m(dy)=\int_{\partial D} p(s, x, y)\,\tilde{m}(dy),$$

uniformly in $s\in[t_0, T]$ and $x\in\bar{D}$ by Lemma 3.3. Thus we can take $\rho_1>0$ such that $0<\rho<\rho_1$ implies

$$\left|\frac{1}{\rho}\int_{D_\rho} p(s, x, y)\, m(dy)-\int_{\partial D} p(s, x, y)\,\tilde{m}(dy)\right|<\frac{\varepsilon}{T}$$

uniformly in $s\in[t_0, T]$ and $x\in\bar{D}$. Hence we have

$$\begin{aligned}
&\left|\frac{1}{\rho}\int_0^t ds\int_{D_\rho} p(s, x, y)\, m(dy)-\int_0^t ds\int_{\partial D} p(s, x, y)\,\tilde{m}(dy)\right|\\
&\leqslant\frac{1}{\rho}\int_0^{t_0} ds\int_{D_\rho} p(s, x, y)\, m(dy)+\int_0^{t_0} ds\int_{\partial D} p(s, x, y)\,\tilde{m}(dy)\\
&\quad+\int_{t_0}^{\max(t, t_0)} ds\left|\frac{1}{\rho}\int_{D_\rho} p(s, x, y)\, m(dy)-\int_{\partial D} p(s, x, y)\,\tilde{m}(dy)\right|\\
&<2\varepsilon+\{\max(t, t_0)-t_0\}\frac{\varepsilon}{T}\leqslant 3\varepsilon,
\end{aligned}$$

uniformly in $0<t\leqslant T$, $x\in\bar{D}$ and $0<\rho\leqslant\min(\rho_0, \rho_1)$, completing the proof.

Chapter II. Analytical construction of the diffusion

To find the diffusion determined by

$$\frac{\partial u}{\partial t}=Au \text{ and } Lu(x)=0,\ x\in\partial D,$$

it is sufficient to construct the semigroup on $C(\overline{D})$ with the Green operators $\{G_\alpha\}$ such that

$$(\alpha-\overline{A})G_\alpha u=u \text{ and } LG_\alpha u(x)=0,\ x\in\partial D.$$

Since $(\alpha-\overline{A})(G_\alpha u-G_\alpha^{\min} u)=0$, $G_\alpha u$ is written as

$$G_\alpha u=G_\alpha^{\min} u+H_\alpha\varphi,\ \varphi=[G_\alpha u]_{\partial D}.$$

In order that $LG_\alpha u=0$ is satified, the following equation should hold.

$$LG_\alpha^{\min} u+LH_\alpha\varphi=0.$$

And hence, φ is obtained by

$$\varphi=-(LH_\alpha)^{-1}LG_\alpha^{\min} u.$$

Thus, G_α should be given by

$$G_\alpha u=G_\alpha^{\min}-H_\alpha(LH_\alpha)^{-1}LG_\alpha^{\min} u$$

by a purely formal computation, which will be rigorously justified in §4–§5. LH_α, considered as an operator, has a closed extension $\overline{LH_\alpha}$, which is the generator of a semigroup on $C(\partial D)$, if an equation of type

$$(\lambda-LH_\alpha)\psi=\varphi$$

can be solved for sufficiently many φ. $\overline{LH_\alpha}$ is the generator of a Markov process, which will be called the Markov process on the boundary of order α. The equation $(\lambda-LH_\alpha)\psi=\varphi$ will be reduced to an integro-differential equation and will be solved in some special cases in §6.

§4. Operators induced by Wentzell's boundary conditions

Let A and D satisfy the conditions at the beginning of §2.

For each x in ∂D we assign a class of functions $\{\xi_x^i(y), i=1, \cdots, N\}$ in $C^3(\bar{D})$ satisfying:

i) there is a neighbourhood U_x of x such that the restriction of $\{\xi_x^i(y)\}$ to $U_x \cap \bar{D}$ is a canonical coordinate system.

ii) $\xi_x^N(y) \geqslant 0$ for each $y \in \bar{D}$; $\xi_x^i(y)=0$ $(1 \leqslant i \leqslant N)$ if and only if $y=x$.

Consider

$$
\begin{aligned}
Lu(x) = & \sum_{i,j=1}^{N-1} \alpha^{ij}(x) \frac{\partial^2 u}{\partial \xi_x^i \partial \xi_x^j}(x) + \sum_{i=1}^{N-1} \beta^i(x) \frac{\partial u}{\partial \xi_x^i}(x) \\
& + \gamma(x) u(x) + \delta(x) \lim_{y \in D, y \to x} Au(y) \\
& + \mu(x) \frac{\partial u}{\partial n}(x) + \int_{\bar{D}} \Big\{ u(y) - u(x) \\
& - \sum_{i=1}^{N-1} \frac{\partial u}{\partial \xi_x^i}(x) \xi_x^i(y) \Big\} \nu_x(dy),
\end{aligned} \tag{4.1}
$$

where $(\alpha^{ij}(x))$ is symmetric and non-negative definite, $\gamma(x)$, $\delta(x)$ and $-\mu(x)$ are non-positive, and ν_x is a σ-finite measure on $\bar{D}$ satisfying

$$
\begin{aligned}
& \nu_x(\{x\}) = 0 \\
& \nu_x(\bar{D} - U_x) < \infty \\
& \int_{U_x} \Big\{ \xi_x^N(y) + \sum_{i=1}^{N-1} (\xi_x^i(y))^2 \Big\} \nu_x(dy) < \infty.
\end{aligned} \tag{4.2}
$$

$Lu(x)$ exists at any point x in ∂D, if u is in $C^2(\bar{D})$. In fact, the integrand of the last term being $O(\xi_x^N(y)) + \sum_{i=1}^{N-1} O(\xi_x^i(y))^2$ for $u \in C^2(\bar{D})$, the integral exists by (4.2). The other terms in (4.1) clearly exist for $u \in C^2(\bar{D})$.

Now, we assume, throughout this paper, condition

(L. 1) $Lu(x)$ is continuous in $x \in \partial D$, if u is in $C^2(\bar{D})$.

Sometimes, we also assume one of the following conditions:

(L. 2) $\nu_x(D) = \infty$ for each $x \in \partial D$ such that $-\delta(x) + \mu(x) = 0$.

(L. 2′) $-\gamma(x) - \delta(x) + \mu(x) + \nu_x(D) > 0$, for any $x \in \partial D$.

Let $\mathfrak{D}(L)$ be a linear subspace of $C(\bar{D})$ satisfying

(4. 3) $Lu(x)$ exists for each x and is continuous in $x\in\partial D$, if u is in $\mathfrak{D}(L)$;[21]

(4. 4) $C^2(\bar{D})\subset\mathfrak{D}(L)\subset\bigcup_{\kappa>0}C^{0,\kappa}(\bar{D})$.

Let L be the operator defined on $\mathfrak{D}(L)$ by

$$u\to Lu(x),\quad x\in\partial D.$$

For $\alpha\geqslant 0$, let $\mathfrak{D}(LH_\alpha)$ be the set of functions $\{\varphi\in C(\partial D)\,|\,H_\alpha\varphi\in\mathfrak{D}(L)\}$ and let LH_α be the operator defined on $\mathfrak{D}(LH_\alpha)$ by

$$\varphi\to(LH_\alpha)\varphi=L(H_\alpha\varphi).$$

Clearly, $C^{2,\kappa}(\partial D)$ is contained in $\mathfrak{D}(LH_\alpha)$ by Theorem 2. 5, v) and (4. 4). We note that $\mathfrak{D}(L)$ can be chosen in different ways, as long as it satisfies (4. 3) and (4. 4).

Lemma 4. 1. *If φ in $\mathfrak{D}(LH_\alpha)$ takes a positive maximum at $x_0\in\partial D$.*

$$LH_\alpha\varphi(x_0)\leqslant 0.$$

If, moreover, L satisfies (L. 2′) *and if $\alpha>0$, we have*

$$LH_\alpha\varphi(x_0)<0.$$

Especially, $LH_\alpha 1(x)<0$ for each $x\in\partial D$, if $\alpha>0$ and (L. 2′) *holds.*

Proof. Let $u=H_\alpha\varphi$. Since u coincides with φ on ∂D, we have

$$\sum_{i,j=1}^{N-1}\alpha^{ij}(x_0)\frac{\partial^2 u}{\partial\xi^i_{x_0}\partial\xi^j_{x_0}}(x_0)\leqslant 0,\quad\frac{\partial u}{\partial\xi^i_{x_0}}(x_0)=0\quad(i=1,\cdots,N-1)$$

and $\gamma(x)u(x)\leqslant 0$. Since $(\alpha-A)u(x)=0$ for each $x\in D$,

$$\delta(x_0)\lim_{y\in D,\,y\to x_0}Au(y)=\delta(x_0)\lim_{y\in D,\,y\to x_0}\alpha u(y)=\alpha\delta(x_0)u(x_0)\leqslant 0.$$

Since u takes a positive maximum at x_0 as a function on $\bar{D}$ by virtue of Theorem 2. 5, iii), we have $\mu(x_0)(\partial u/\partial n)(x_0)\leqslant 0$ and

$$u(y)-u(x_0)-\sum_{i=1}^{N-1}\frac{\partial u}{\partial\xi^i_{x_0}}(x_0)\xi^i_{x_0}(y)=u(y)-u(x_0)\leqslant 0.$$

Thus $LH_\alpha\varphi(x_0)=Lu(x_0)\leqslant 0$. If $\alpha>0$, then $H_\alpha\varphi$ is not a constant

21) We neglect the term whose coefficient is zero. For instance, if $\alpha^{ij}(x)=\beta^i(x)=0$ and $\nu_x(\bar{D})<\infty$, then $Lu(x)$ exists for all u in $C^1(\bar{D})$.

function, and hence does not take a positive maximum in D, $(\partial u/\partial n)(x_0)<0$ and $u(y)-u(x_0)<0$ for each $y\in D$ by Theorem 2.5, iii) and iv). Thus,

$$Lu(x_0)\leqslant(\gamma(x_0)+\alpha\delta(x_0))u(x_0)+\mu(x_0)\frac{\partial u}{\partial n}(x_0)$$
$$+\int_D(u(y)-u(x_0))\nu_{x_0}(dy)<0,$$

if one of $\mu(x_0)$, $-\gamma(x_0)$, $-\delta(x_0)$ and $\nu_{x_0}(D)$ is positive by (L. 2′)

Corollary. *LH_α has the smallest closed extension $\overline{LH_\alpha}$. If φ in $\mathfrak{D}(\overline{LH_\alpha})$ takes a positive maximum at $x\in\partial D$, $\overline{LH_\alpha}\varphi(x)\leqslant 0$.*

This is clear by Lemma 4.1, Theorem 1.2 and the remark to Theorem 1.2.

Now, we consider the set of functions $\mathfrak{D}(LG_\alpha^{\min})=\{f\in C(\overline{D})\mid G_\alpha^{\min}f\in\mathfrak{D}(L)\}$ and define $LG_\alpha^{\min}$ for $f\in\mathfrak{D}(LG_\alpha^{\min})$ by

$$f\to(LG_\alpha^{\min})f=L(G_\alpha^{\min}f).$$

$\mathfrak{D}(LG_\alpha^{\min})$ contains $C^{0,\kappa}(\overline{D})$ and hence it is dense in $C(\overline{D})$ by virtue of Theorem 2.4 and (L. 1).

Lemma 4.2. *$LG_\alpha^{\min}$ can be uniquely extended to a non-negative, bounded linear operator on $C(\overline{D})$ taking values in $C(\partial D)$ for each $\alpha\geqslant 0$.*

We write $\overline{LG_\alpha^{\min}}$ for the extension.

Proof. Let u be non-negative and contained in $\mathfrak{D}(LG_\alpha^{\min})$. If $G_\alpha^{\min}u$ is of class C^2 in a neighbourhood of $y\in D$, then $AG_\alpha^{\min}u(y)=\bar{A}G_\alpha^{\min}u(y)=\alpha G_\alpha^{\min}u(y)-u(y)$ by Lemmas 2.3 and 2.4. Noting that $G_\alpha^{\min}u$ vanishes on ∂D and is non-negative, we have

$$LG_\alpha^{\min}u(x)=-\delta(x)u(x)+\mu(x)\frac{\partial}{\partial n}G_\alpha^{\min}u(x)+\int_D G_\alpha^{\min}(y)\nu_x(dy),$$

which is non-negative at each $x\in\partial D$. Thus $LG_\alpha^{\min}$ is nonnegative, and hence $-LG_\alpha^{\min}\|u\|\leqslant LG_\alpha^{\min}u\leqslant LG_\alpha^{\min}\|u\|$ for each u in $\mathfrak{D}(LG_\alpha^{\min})$, implying boundedness of $LG_\alpha^{\min}$ with norm $\|LG_\alpha^{\min}\|=\|LG_\alpha^{\min}1\|$. $\mathfrak{D}(LG_\alpha^{\min})$ being dense in $C(\overline{D})$, $LG_\alpha^{\min}$ can be extended uniquely to a non-negative bounded linear operator $\overline{LG_\alpha^{\min}}$ on $C(\overline{D})$.

Corollary. *The range of* $\overline{LG_\alpha^{\min}}$ *does not depend on* $\alpha\geqslant 0$. *Moreover we have*

(4. 5) $\overline{LG_\alpha^{\min}}f-\overline{LG_\beta^{\min}}f+(\alpha-\beta)\overline{LG_\alpha^{\min}}G_\beta^{\min}f=0, \quad f\in C(\overline{D})$.

This is clear from the resolvent equation for $\{G_\alpha^{\min}\}$ and the definition of $\overline{LG_\alpha^{\min}}$.

Lemma 4. 3. *For any* $\varphi\in C(\partial D)$ *and* α, $\beta\geqslant 0$, *we have*

(4. 6) $H_\alpha\varphi-H_\beta\varphi+(\alpha-\beta)G_\alpha^{\min}H_\beta\varphi=0.$

Proof. Let us denote the lefthand side of (4. 6) by u_φ. The mapping $\varphi\rightarrow u_\varphi$ is clearly bounded and linear from $C(\partial D)$ into $C(\overline{D})$. If φ is in $C^{2,\kappa}(\partial D)$, u_φ belongs to $C^2(\overline{D})$, because $H_\alpha\varphi$ and $H_\beta\varphi$ is in $C^2(\overline{D})$ by Theorem 2. 5, v) and $G_\alpha^{\min}H_\beta\varphi$ is in $C^2(\overline{D})$ by Theorem 2. 4, i). Hence, for $\varphi\in C^{2,\kappa}(\partial D)$, we have $(\alpha-A)u_\varphi=0$ by an easy calculation and it results from $[u_\varphi]_{\partial D}=0$ that $u_\varphi\equiv 0$ by Theorem 2. 5, i). Thus, $u_\varphi=0$ for each $\varphi\in C(\partial D)$, since $C^{2,\kappa}(\partial D)$ is dense in $C(\partial D)$, completing the proof.

Lemma 4. 4. $\mathfrak{D}(\overline{LH_\alpha})$ *does not depend on* $\alpha\geqslant 0$. *If we denote the common domain by* $\widetilde{\mathfrak{D}}$, *we have*

(4. 7) $\overline{LH_\alpha}\varphi-\overline{LH_\beta}\varphi+(\alpha-\beta)\overline{LG_\alpha^{\min}}H_\beta\varphi=0,$

for any α, $\beta\geqslant 0$ and $\varphi\in\widetilde{\mathfrak{D}}$.

Proof. Let φ be in $\mathfrak{D}(LH_\beta)$. By definition of $\mathfrak{D}(LH_\beta)$ $H_\beta\varphi$ belongs to $C^{0,\kappa}(\partial D)$, and hence $G_\alpha^{\min}H_\beta\varphi$ is in $C^2(D)\subset\mathfrak{D}(L)$. Thus, we can apply L on the both sides of $H_\alpha\varphi=H_\beta\varphi-(\alpha-\beta)G_\alpha^{\min}H_\beta\varphi$, and obtain

(4. 8) $LH_\alpha\varphi=LH_\beta\varphi-(\alpha-\beta)LG_\alpha^{\min}H_\beta\varphi, \quad \varphi\in\mathfrak{D}(LH_\beta)$.

For any $\varphi\in\mathfrak{D}(\overline{LH_\beta})$ there is a sequence $\{\varphi_n\in\mathfrak{D}(LH_\beta)\}$ such that $\varphi_n\rightarrow\varphi$ and $LH_\beta\varphi_n\rightarrow\overline{LH_\beta}\varphi$. But, $\varphi_n\rightarrow\varphi$ implies $H_\beta\varphi_n\rightarrow H_\beta\varphi$ and hence $LG_\alpha^{\min}H_\beta\varphi_n=\overline{LG_\alpha^{\min}}H_\beta\varphi_n\rightarrow\overline{LG_\alpha^{\min}}H_\beta\varphi$. Thus, by (4. 8), $LH_\alpha\varphi_n$ converges to $\overline{LH_\beta}\varphi-(\alpha-\beta)\overline{LG_\alpha^{\min}}H_\beta\varphi$ as $n\rightarrow\infty$. This means that φ in $\mathfrak{D}(\overline{LH_\beta})$ belongs to $\mathfrak{D}(\overline{LH_\alpha})$ and that (4. 7) holds. Interchanging α and β, we have $\mathfrak{D}(\overline{LH_{\alpha}})\subset\mathfrak{D}(\overline{LH_{\beta}})$, completing the proof.

Let $\mathfrak{D}(\widehat{L})$ be the set of all functions which can be written in the form

$$\sum_{i=1}^{m}G_{\alpha_i}^{\min}f_i+\sum_{j=1}^{n}H_{\beta_j}\varphi_j,\quad f_i\in C(\overline{D}),\ \varphi_j\in\widetilde{\mathfrak{D}}, \tag{4.9}$$

where m and n are non-negative integers and $\alpha_i\geqslant 0$, $\beta_j\geqslant 0$ $(1\leqslant i\leqslant m,$ $1\leqslant j\leqslant n)$.

Lemma 4.5. *$C^{2,\kappa}(\overline{D})$ is contained in $\mathfrak{D}(\widehat{L})$. There is a linear operator $\widehat{L}$ defined on $\mathfrak{D}(\widehat{L})$ satisfying*

$$\widehat{L}u=Lu,\quad \textit{for } u\in C^{2,\kappa}(\overline{D})$$
$$\widehat{L}G_{\alpha}^{\min}f=\overline{LG_{\alpha}^{\min}}f,\quad \textit{for } f\in C(\overline{D}),$$
$$\widehat{L}H_{\alpha}\varphi=\overline{LH_{\alpha}}\varphi,\quad \textit{for } \varphi\in\widetilde{\mathfrak{D}}.$$

Such an operator $\widehat{L}$ is unique.

Proof. For any u with expression (4.9) we assign

$$\widehat{L}u=\sum_{i=1}^{m}\overline{LG_{\alpha_i}^{\min}}f_i+\sum_{j=1}^{n}\overline{LH_{\beta_j}}\varphi_j. \tag{4.10}$$

In order that $\widehat{L}u$ depends only on u, not on the choice of the expression, it is sufficient to prove that

$$u=\sum_{i=1}^{m}G_{\alpha_i}^{\min}f_i+\sum_{j=1}^{n}H_{\beta_j}\varphi_j=0 \tag{4.11}$$

implies $\sum_{i=1}^{m}\overline{LG_{\alpha_i}^{\min}}f_i+\sum_{j=1}^{n}\overline{LH_{\beta_j}}\varphi_j=0$. Since $\sum_{j=1}^{n}\varphi_j$ is the boundary value of u and $u=0$, we have

$$\sum_{j=1}^{n}\varphi_j=0. \tag{4.12}$$

Since $\quad(\alpha-\overline{A})G_{\alpha_i}^{\min}f_i=f_i+(\alpha-\alpha_i)G_{\alpha_i}^{\min}f_i$

and $\quad(\alpha-\overline{A})H_{\beta_j}\varphi_j=(\alpha-\beta_j)H_{\beta_j}\varphi_j$

follow from Lemma 2.4, we have, using (4.5) and (4.7),

$$\sum_{i=1}^{m}\overline{LG_{\alpha_i}^{\min}}f_i+\sum_{j=1}^{n}\overline{LH_{\beta_j}}\varphi_j=\sum_{i=1}^{m}(\overline{LG_{\alpha}^{\min}}f_i+(\alpha-\alpha_i)\overline{LG_{\alpha}^{\min}}G_{\alpha_i}^{\min}f_i)$$
$$+\sum_{j=1}^{n}(\overline{LH_{\alpha}}\varphi_j+(\alpha-\beta_j)\overline{LG_{\alpha}^{\min}}H_{\beta_j}\varphi_j)$$

$$=\overline{LG_\alpha^{\min}}\left\{\sum_{i=1}^m f_i+\sum_{i=1}^m(\alpha-\alpha_i)G_{\alpha_i}^{\min}f_i+\sum_{j=1}^n(\alpha-\beta_j)H_{\beta_j}\varphi_j\right\}+\overline{LH_\alpha}\left(\sum_{j=1}^n\varphi_j\right)$$

$$=\overline{LG_\alpha^{\min}}(\alpha-\bar{A})\left\{\sum_{i=1}^m G_{\alpha_i}^{\min}f_i+\sum_{j=1}^n H_{\beta_j}\varphi_j\right\}+\overline{LH_\alpha}\left(\sum_{j=1}^n\varphi_j\right)=0,$$

by (4.11) and (4.12). $\hat{L}G_\alpha^{\min}f=\overline{LG_\alpha^{\min}}f$, $\hat{L}H_\alpha\varphi=\overline{LH_\alpha}\varphi$ and the uniqueness of $\hat{L}$ are clear by definition of $\hat{L}$.

If u is in $C^{2,\kappa}(\bar{D})$, $[u]_{\partial D}$ is in $C^{2,\kappa}(\partial D)\subset\tilde{\mathfrak{D}}$, and hence $H_\alpha[u]_{\partial D}$ belongs to $C^2(\bar{D})$ by Theorem 2.5, v). Since $u-H_\alpha[u]_{\partial D}$ vanishes on ∂D and is in $C^2(\bar{D})$, there is a $v\in C(\bar{D})$ such that $u-H_\alpha[u]_{\partial D}=G_\alpha^{\min}v$ by Proposition 2.1. Thus, we have $\hat{L}u=\hat{L}(H_\alpha[u]_{\partial D}+G_\alpha^{\min}v)=\overline{LH_\alpha}[u]_{\partial D}+\overline{LG_\alpha^{\min}}v=LH_\alpha[u]_{\partial D}+LG_\alpha^{\min}v=Lu$, completing the proof.

Remark 4.1. *u belongs to $\mathfrak{D}(\hat{L})$ if and only if $u\in\mathfrak{D}(\bar{A})$ and $[u]_{\partial D}\in\tilde{\mathfrak{D}}$.* In fact, if $u\in\mathfrak{D}(\hat{L})$, then u is in $\mathfrak{D}(\bar{A})$ by (4.9) and Lemma 2.4, and we have $[u]_{\partial D}=\sum_{j=1}^n\varphi_j\in\tilde{\mathfrak{D}}$. Conversely, if $u\in\mathfrak{D}(\bar{A})$ and $[u]_{\partial D}\in\tilde{\mathfrak{D}}$, then, we have $u=G_\alpha^{\min}f+H_\alpha[u]_{\partial D}$ by some $f\in C(\bar{D})$ similarly to the last paragraph of the proof of Lemma 4.5, and hence $u\in\mathfrak{D}(\hat{L})$. We note that u in $\mathfrak{D}(\hat{L})$ is expressed by

$$u=G_\alpha^{\min}(\alpha-\bar{A})u+H_\alpha[u]_{\partial D},\quad \alpha>0,$$

which we can prove by applying $\alpha-\bar{A}$ to the both sides of $u=G_\alpha^{\min}f+H_\alpha[u]_{\partial D}$ by Lemma 2.4.

§5. Semigroups on $C(\partial D)$ and construction of the diffusion

Theorem 5.1. i) *Let $\alpha>0$. $\overline{LH_\alpha}$ is a generator of a semigroup on $C(\partial D)$, if and only if there exists a number $\lambda>0$ such that*

$$(5.1)\qquad \begin{aligned}&(\alpha-A)u(x)=0 && x\in D,\\ &(\lambda-L)u(x)=\varphi(x), && x\in\partial D\end{aligned}$$

has a solution $u\in\mathfrak{D}(L)\cap C^2(D)$ for each φ in a dense subset of $C(\partial D)$.

ii) *If $\overline{LH_\alpha}$ generates a semigroup on $C(\partial D)$ for some $\alpha>0$,* *then* $\overline{LH_\beta}$ *is also the generator of a semigroup on* $C(\partial D)$ *for each* $\beta > 0$.

We call the semigroup on $C(\partial D)$ with generator $\overline{LH_\alpha}$ *the semigroup on* $C(\partial D)$ *of order* α, and denote it by $\{S_t^\alpha, t \geqq 0\}$. The Green operator of $\{S_t^\alpha\}$ is denoted by

$$K_\lambda^\alpha \varphi = \int_0^\infty e^{-\lambda t} S_t^\alpha \varphi dt, \quad \lambda > 0.$$

Proof. By Theorem 2.5, i) and the definition of $\mathfrak{D}(\overline{LH_\alpha})$, the equation (5.1) is equivalent to

$$(5.1') \qquad (\lambda - LH_\alpha)\psi = \varphi, \quad \psi \in \mathfrak{D}(LH_\alpha).$$

The solution u of (5.1) is given by $u = H_\alpha \psi$. If (5.1) has a solution for each φ in a dense subset of $C(\partial D)$, or equivalently, $\mathfrak{R}(\lambda - LH_\alpha)$ is dense in $C(\partial D)$, then LH_α satisfies (1.5), (1.7) and (1.8) of Theorem 1.2 by Lemma 4.1, and hence $\overline{LH_\alpha}$ generates a semigroup on $C(\partial D)$. Conversely, if $\overline{LH_\alpha}$ is the generator, $\mathfrak{R}(\lambda - \overline{LH_\alpha})$ is $C(\partial D)$ for all positive λ, and hence, $\mathfrak{R}(\lambda - LH_\alpha)$ is dense in $C(\partial D)$, completing the proof of i). For any α and β, $\overline{LH_\beta}$ is $\overline{LH_\alpha}$ plus a bounded operator by Lemmas 4.2 and 4.4. Noting Corollary to Lemma 4.1, we see that Corollary to Theorem 1.2 is applicable to our case. Hence, we have ii), and the proof is complete.

Corollary. $\overline{LH_\alpha}$ *is the generator of a semigroup on* $C(\partial D)$, *if and only if there is a number* $\lambda > 0$ *such that the equation*

$$(5.2) \qquad \begin{aligned} &(\alpha - \bar{A})u(x) = 0, && x \in \bar{D}, \\ &(\lambda - \hat{L})u(x) = \varphi(x), && x \in \partial D, \end{aligned}$$

has a solution $u \in \mathfrak{D}(\hat{L})$[22] *for each* φ *in a dense set of* $C(\partial D)$, *or, equivalently, the equation*

$$(5.2') \qquad (\lambda - \overline{LH_\alpha})\psi = \varphi$$

has a solution $\psi \in \mathfrak{D}(\overline{LH_\alpha})$ *for each* φ *in a dense set.*

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22) Note that $\mathfrak{D}(\hat{L}) \subset \mathfrak{D}(\bar{A})$.

Proof. Suppose that u belongs to $\mathfrak{D}(\hat{L})$ and satisfies (5.2). Then, $u=H_\alpha[u]_{\partial D}$ and $[u]_{\partial D}\in\tilde{\mathfrak{D}}$ by Remark 4.1, and hence, $[u]_{\partial D}$ satisfies (5.2′). Conversely, if ψ is the solution of (5.2′), we have $H_\alpha\psi\in\mathfrak{D}(\hat{L})$, and $H_\alpha\psi$ is the solution of (5.2). (5.2′) has a solution for all φ in a dense set, if and only if (5.1′) does so. Thus the proof is complete.

Lemma 5.1. *Let L satisfy* (L.2′) *and $\alpha>0$.*[23] *If $\overline{LH_\alpha}$ generates a semigroup on $C(\partial D)$, then*

$$(5\cdot 3) \qquad \overline{LH_\alpha}\psi=\varphi$$

has a unique solution for each $\varphi\in C(\partial D)$, and hence $\overline{LH}_\alpha^{-1}$ is defined on $C(\partial D)$. $-\overline{LH}_\alpha^{-1}$ is non-negative and bounded.

Proof. Since constant 1 belongs to $\mathfrak{D}(LH_\alpha)$ and $LH_\alpha 1(x)<0$ at each $x\in\partial D$ for positive α by (L.2′) and Lemma 4.1, $\overline{LH_\alpha}+k$ is the generator of a semigroup on $C(\partial D)$ by the corollary to Theorem 1.1, where $-k=\sup_{x\in\partial D}\overline{LH_\alpha}1(x)$. Thus, $-\overline{LH_\alpha}\psi=\{k-(\overline{LH_\alpha}+k)\}\psi=\varphi$ has a unique solution ψ for each $\varphi\in C(\partial D)$ by Theorem 1.1. $-\overline{LH}_\alpha^{-1}$ is clearly non-negative and $\|-\overline{LH}_\alpha^{-1}\|\leqslant k^{-1}$, completing the proof.

We sometimes write K_0^α for $-\overline{LH}_\alpha^{-1}$, if it exists.

Lemma 5.2. *If u in $\mathfrak{D}(\hat{L})$ satisfies*

$$(\alpha-\bar{A})u=0, \qquad \text{on } \bar{D},$$
$$(\lambda-\hat{L})u=0, \qquad \text{on } \partial D,$$

for some $\alpha\geqslant 0$ and $\lambda>0$, then $u=0$. Moreover, if $\alpha>0$ and if L satisfies (L.2′)[23] *and $\overline{LH_\alpha}$ is a generator, then*

$$(\alpha-\bar{A})u=0. \qquad \text{on } \bar{D},$$
$$\hat{L}u=0, \qquad \text{on } \partial D,$$

imply $u=0$.

---

23) The condition "(L.2′) and $\alpha>0$" can be replaced by any one of conditions "$-\gamma(x)>0$ for each $x\in\partial D$" and "$-\gamma(x)+\mu(x)+\nu_x(D)>0$ for each $x\in\partial D$ and c is not identically zero." In fact, all we have to use is $LH_\alpha 1<0$.

Proof. Let $(\alpha-\bar{A})u=0$ and $(\lambda-\widehat{L})u=0$. Then $u=H_\alpha[u]_{\partial D}$ and $[u]_{\partial D}\in\widetilde{\mathfrak{D}}$ by Remark 4.1. Thus, $(\lambda-\overline{LH_\alpha})[u]_{\partial D}=(\lambda-\widehat{L})u=0$. Since $\lambda>0$, this implies $[u]_{\partial D}=0$ by the property of $\overline{LH_\alpha}$ stated in the corollary to Lemma 4.1. Now, let $\alpha>0$ and let L satisfy (L.2′). If $(\alpha-\bar{A})u=0$ and $\widehat{L}u=0$, we have $\overline{LH_\alpha}[u]_{\partial D}=0$, and hence $[u]_{\partial D}=0$ by the uniqueness of the solution of (5.3) in Lemma 5.1. In both cases, we have $u=H_\alpha[u]_{\partial D}=0$.

Lemma 5.3. *For any* $u\in C(\bar{D})$, *we have*

$$\lim_{\alpha\to\infty}(\alpha G_\alpha^{\min}u+H_\alpha[u]_{\partial D})=u \tag{5.4}$$

Proof. Fix $\beta\geqq 0$, and put $v=u-H_\beta[u]_{\partial D}$. By (4.6), we have

$$\begin{aligned}\alpha G_\alpha^{\min}v-v&=\alpha G_\alpha^{\min}u-\alpha G_\alpha^{\min}H_\beta[u]_{\partial D}-u+H_\beta[u]_{\partial D}\\&=(\alpha G_\alpha^{\min}u+H_\alpha[u]_{\partial D}-u)-\beta G_\alpha^{\min}H_\beta[u]_{\partial D}.\end{aligned}$$

Since v vanishes on ∂D, $\alpha G_\alpha^{\min}v-v$ converges uniformly to 0 by Theorem 2.4, v). Moreover, $G_\alpha^{\min}H_\beta[u]_{\partial D}$ also converges to 0, and hence $\alpha G_\alpha^{\min}u+H_\alpha[u]_{\partial D}$ converges to u.

Lemma 5.4. $(\partial/\partial n)H_\alpha 1$ *diverges uniformly and monotonically to* $-\infty$, *when* $\alpha\to\infty$.

Proof. Since we have $H_\alpha 1=H_\beta 1-(\alpha-\beta)G_\alpha^{\min}H_\beta 1$, by (4.6), $\alpha>\beta$ implies $H_\alpha 1\leqq H_\beta 1$, and hence $(\partial/\partial n)H_\alpha 1$ is monotone non-increasing in α. Moreover, Theorem 2.4, v) implies $H_\alpha 1(x)\downarrow 0$ for $x\in D$ when $\alpha\to\infty$, with β fixed in the above equality, Now, we note that there is a function $u\in C^2(\bar{D})$ satisfying

$$[u]_{\partial D}=1 \text{ and } \frac{\partial u}{\partial n}u(x)\leqq -K,\quad x\in\partial D \tag{5.5}$$

for any fixed $K>0$. In fact, $(H_{\alpha_0}1)^{n_0}$ belongs to $C^2(\bar{D})$ and satisfies

$$\begin{aligned}\frac{\partial}{\partial n}(H_{\alpha_0}1)^{n_0}(x)&=n_0\cdot(H_{\alpha_0}1)^{n_0-1}(x)\cdot\frac{\partial}{\partial n}H_{\alpha_0}1(x)\\&=n_0\frac{\partial}{\partial n}H_{\alpha_0}1(x)\leqq -n_0\inf_{x\in\partial D}\left|\frac{\partial}{\partial n}H_{\alpha_0}1(x)\right|,\quad x\in\partial D.\end{aligned}$$

Since $\inf_{x\in\partial D}|(\partial/\partial n)H_{\alpha_0}1(x)|>0$ by Theorem 2.5, iv), it satisfies (5.5) for sufficiently lagre n_0 and fixed α_0. Take such a neighbourhood U of ∂D relative to $\overline{D}$ that $u(x)\geqslant 1/2$ holds on U and U has a smooth boundary. Since the convergence $H_\alpha 1\rightarrow 0$ on D is monotone, it is uniform on $\partial U-\partial D$ by Dini's theorem, and hence $H_\alpha 1(x)\leqslant u(x)$ on ∂U and $\alpha>2\|Au\|$ for sufficnently large α. Thus, by

$$(A-\alpha)(H_\alpha 1-u)(x)=\alpha u(x)-Au(x)\geqslant\frac{\alpha}{2}-\|Au\|>0,\ x\in U,$$

$H_\alpha 1-u$ never takes a positive maximum in U, and hence $H_\alpha 1(x)\leqslant u(x)$ in U, implying $(\partial/\partial n)H_\alpha 1(x)\leqslant(\partial/\partial n)u(x)\leqslant -K$, $x\in\partial D$, foɹ sufficiently large $\alpha>0$, completing the proof.

Corollary. *If* $\overline{LH_\alpha}$ *is a generator and* (L. 2) *holds, then*

$$\lim_{\alpha\rightarrow\infty}\|K_\lambda^\alpha\|=0,\quad \text{for } \lambda\geqslant 0.$$

Proof. If $\delta(x)$ or $\mu(x)$ is positive at $x\in\partial D$,

$$LH_\alpha 1(x)=\gamma(x)+\alpha\delta(x)+\mu(x)\frac{\partial}{\partial n}H_\alpha 1(x)+\int_D\{H_\alpha 1(y)-1\}\nu_x(dy)$$

diverges monotonically to $-\infty$ be Lemma 5. 4. But, condition (L. 2) implies $\nu_x(D)=\infty$ and hence $\int_D\{H_\alpha 1(y)-1\}\nu_x(dy)$ decreases to $-\infty$, if $\delta(x)=\mu(x)=0$. Thus, $(LH_\alpha 1(x))^{-1}$ converges monotonically to 0, and hence uniformly on ∂D, if $\alpha\rightarrow\infty$, by Dini's theorem. Thus, we have $1=-|(LH_\alpha 1)^{-1}|\cdot LH_\alpha 1\leqslant\|(LH_\alpha 1)^{-1}\|(\lambda-LH_\alpha 1)$, which implies

$$K_\lambda^\alpha 1\leqslant\|(LH_\alpha 1)^{-1}\|K_\lambda^\alpha(\lambda-LH_\alpha 1)=\|(LH_\alpha 1)^{-1}\|\rightarrow 0,\ \text{as } \alpha\rightarrow\infty,$$

completing the proof.

Remark 5. 1. *Under the assumption that* $\overline{LH_\alpha}$ *is a generator,*

$$\lim_{\alpha\rightarrow\infty}\|K_\lambda^\alpha\|>0$$

if $\delta(x)=\mu(x)=0$ *for all* $x\in\partial D$ *and* $\nu_x(D)$ *is bounded in* $x\in\partial D$.

Proof. We have $L1(x)=\gamma(x)$ by $\delta(x)=0$, and hence γ is bounded. By assumption $LH_\alpha 1(x)$ is bounded both in α and x, since

$$|LH_\alpha 1(x)| = |\gamma(x) + \int_D \{H_\alpha 1(y) - 1\} \nu_x(dy)|$$
$$\leqslant \sup_{x\in\partial D} |\gamma(x)| + \sup_{x\in\partial D} \nu_x(D) = K < \infty.$$

Hence, $1 \geqslant \|\lambda - LH_\alpha 1\|^{-1}(\lambda - LH_\alpha)1$, which implies $K_\lambda^\alpha 1 \geqslant \|\lambda - LH_\alpha 1\|^{-1} \geqslant (\lambda + K)^{-1} > 0$, completing the proof.

Theorem 5.2. *Let* $\bar{A}_{\hat{L}}$ *be the restriction of* $\bar{A}$ *to the subset* $\mathfrak{D}(\bar{A}_{\hat{L}}) = \{u \mid u \in \mathfrak{D}(\hat{L})$ *and* $Lu = 0\}$ *of* $\mathfrak{D}(\bar{A})$. *If* $\overline{LH_\alpha}$ *is a generator and* (L. 2) *holds, then* $\bar{A}_{\hat{L}}$ *is the generator of a semigroup on* $C(\bar{D})$. *The Green operator* G_α *of the semigroup is given by*

$$(5.6) \qquad G_\alpha u = G_\alpha^{\min} u + H_\alpha K_0^\alpha \overline{LG_\alpha^{\min}} u, \quad u \in C(\bar{D}).$$

Thus, we have obtained the semigroup on $C(\bar{D})$ determined by A and L. This is a special case of a little more general

Theorem 5.2′. *Let* $\bar{A}_{\hat{L}-\lambda}$ *be the restriction of* $\bar{A}$ *to the set* $\mathfrak{D}(\bar{A}_{\hat{L}-\lambda}) = \{u \mid u \in \mathfrak{D}(\hat{L})$ and $\hat{L}u = \lambda u\}$, $\lambda \geqslant 0$. *If* $\overline{LH_\alpha}$ *is a generator and* (L. 2) *holds, then* $\bar{A}_{\hat{L}-\lambda}$ *is the generator of a semigroup on* $C(\bar{D})$. *The Green operator* G_α^λ *of the semigroup is given by*

$$(5.7) \qquad G_\alpha^\lambda u = G_\alpha^{\min} u + H_\alpha K_\lambda^\alpha \overline{LG_\alpha^{\min}} u, \quad u \in C(\bar{D}).$$

Proof. Since $K_\lambda^\alpha = (\lambda - \overline{LH_\alpha})^{-1}$ exists for each $\lambda \geqslant 0$ and $\alpha > 0$ by Lemma 5.1, we can define G_α^λ on $C(\bar{D})$ by (5.7). $G_\alpha^\lambda u$ clearly belongs to $\mathfrak{D}(\hat{L})$, and moreover, to $\mathfrak{D}(\bar{A}_{\hat{L}-\lambda})$, since

$$(\lambda - \hat{L}) G_\alpha^\lambda u = (\lambda - \hat{L}) G_\alpha^{\min} u + (\lambda - \hat{L}) H_\alpha K_\lambda^\alpha \overline{LG_\alpha^{\min}} u$$
$$= -\overline{LG_\alpha^{\min}} u + (\lambda - \overline{LH_\alpha}) K_\lambda^\alpha \overline{LG_\alpha^{\min}} u = -\overline{LG_\alpha^{\min}} u + \overline{LG_\alpha^{\min}} u = 0 \quad \text{on } \partial D.$$

Now, we verify the conditions (1.1)-(1.4) in Theorem 1.1 for $\bar{A}_{\hat{L}-\lambda}$. For any $\alpha > 0$ and $u \in C(\bar{D})$, $G_\alpha^\lambda u$ is the unique solution of $(\alpha - \bar{A})v = u$ contained in $\mathfrak{D}(\bar{A}_{\hat{L}-\lambda})$, since

$$(\alpha - \bar{A}) G_\alpha^\lambda u = (\alpha - \bar{A}) G_\alpha^{\min} u + (\alpha - \bar{A}) H_\alpha K_\lambda^\alpha \overline{LG_\alpha^{\min}} u = u$$

and the uniqueness follows from Lemma 5.2. Thus, $G_\alpha^\lambda = (\alpha - \bar{A}_{\hat{L}-\lambda})^{-1}$, implying (1.2). $G_\alpha^\lambda \geqslant 0$ in (1.4) follows from evident inequalities $G_\alpha^{\min} \geqslant 0$, $H_\alpha \geqslant 0$, $K_\lambda^\alpha \geqslant 0$ and $\overline{LG_\alpha^{\min}} \geqslant 0$ and (5.7). To prove $\|G_\alpha^\lambda\| \leqslant 1/\alpha$

in (1. 3), we note that the following equality holds:

$$\alpha G_\alpha^{\min} 1 + H_\alpha 1 = 1 + G_\alpha^{\min} c,$$

since the boundary values of the both sides coincide and we have the same value when we apply $\alpha - A$ on the both sides. Hence, we have

$$(\lambda - \overline{LH_\alpha})1 = \lambda - LH_\alpha 1 = \lambda + \alpha LG_\alpha^{\min} 1 - L1 - LG_\alpha^{\min} c$$
$$= \lambda + \alpha LG_\alpha^{\min} 1 - \gamma - \mu \frac{\partial}{\partial n} G_\alpha^{\min} c - \int_D G_\alpha^{\min} c(y) \nu_x(dy) \geqslant \alpha \overline{LG_\alpha^{\min}} 1.$$

Then, applying K_λ^α on the both ends, we have

$$1 = K_\lambda^\alpha(\lambda - \overline{LH_\alpha})1 \geqslant \alpha K_\lambda^\alpha \overline{LG_\alpha^{\min}} 1$$

This, combined with $\alpha G_\alpha^{\min} + H_\alpha 1 \leqslant 1$, imply $\alpha G_\alpha^\lambda 1 = \alpha G_\alpha^{\min} 1 + \alpha H_\alpha K_\lambda^\alpha \overline{LG_\alpha^{\min}} 1 \leqslant 1$, and hence (1. 3).

To prove (1. 1), it is sufficient to verify $\alpha G_\alpha^\lambda u \to u$ when $\alpha \to \infty$ for each u in some dense subset of $C(\overline{D})$, for instance, $\mathfrak{D}(\widehat{L})$. Since the first summand of the rigththand side of

$$\|\alpha G_\alpha^\lambda u - u\| = \|\alpha G_\alpha^{\min} u + \alpha H_\alpha K_\lambda^\alpha \overline{LG_\alpha^{\min}} u - u\|$$
$$\leqslant \|\alpha G_\alpha^{\min} u + H_\alpha [u]_{\partial D} - u\| + \|\alpha K_\lambda^\alpha \overline{LG_\alpha^{\min}} u - [u]_{\partial D}\|$$

converges to 0 by Lemma 5. 3, we have only to prove $\alpha K_\lambda^\alpha \overline{LG_\alpha^{\min}} u \to [u]_{\partial D}$ for $u \in \mathfrak{D}(\widehat{L})$. Writing $u = G_\beta^{\min} v + H_\beta \varphi$ for fixed β, where $v \in C(\overline{D})$ and $\varphi = [u]_{\partial D} \in \widetilde{\mathfrak{D}}$, and noting that

$$G_\alpha^{\min} u = G_\alpha^{\min} G_\beta^{\min} v + G_\alpha^{\min} H_\beta \varphi = \frac{1}{\alpha - \beta}(G_\beta^{\min} v - G_\alpha^{\min} v + H_\beta \varphi - H_\alpha \varphi),$$

we have

$$\|\alpha K_\lambda^\alpha \widehat{L} G_\alpha^{\min} u - [u]_{\partial D}\| = \left\| \frac{\alpha}{\alpha - \beta} K_\lambda^\alpha \widehat{L}(G_\beta^{\min} v - G_\alpha^{\min} v + H_\beta \varphi - H_\alpha \varphi) - \varphi \right\|$$
$$= \left\| \frac{\alpha}{\alpha - \beta} K_\lambda^\alpha \widehat{L}(G_\beta^{\min} v - G_\alpha^{\min} v + H_\beta \varphi) + \frac{\alpha}{\alpha - \beta}(\varphi - \lambda K_\lambda^\alpha \varphi) - \varphi \right\|$$
$$\leqslant \frac{\alpha}{\alpha - \beta} \|K_\lambda^\alpha\| (\|\widehat{L} G_\beta^{\min} v + \widehat{L} H_\beta \varphi - \lambda \varphi\| + \|\widehat{L} G_\alpha^{\min}\| \cdot \|v\|) + \left\| \frac{\alpha}{\alpha - \beta} \varphi - \varphi \right\|,$$

where we have used $K_\lambda^\alpha(\lambda - \overline{LH_\alpha})\varphi = \varphi$ for $\varphi \in \widetilde{\mathfrak{D}}$. Since $\|K_\lambda^\alpha\| \to 0$ by the corollary to Lemma 5.4 and $\|\widehat{L}G_\alpha^{\min}\|=\|\widehat{L}G_\alpha^{\min}1\|=\|\widehat{L}G_\beta^{\min}1-(\alpha-\beta)\widehat{L}G_\alpha^{\min}G_\beta^{\min}1\|\leqq\|\widehat{L}G_\beta^{\min}1\|$ for $\alpha>\beta$, $\|\alpha K_\lambda^\alpha \widehat{L}G_\alpha^{\min}u-[u]_{\partial D}\|$ converges to 0, completing the proof.

Remark 5.2. *Suppose that* $\overline{LH_\alpha}$ *is a generator and* (L. 2′) *is satisfied. In order that* $\overline{A}_{\hat{L}-\lambda}$ *be a generator it is necessary that* $\lim_{\alpha\to\infty}\|K_\lambda^\alpha\|=0$ *hold.*

Proof. By assumption, G_α^λ in (5.7) can be defined and satisfies $G_\alpha^\lambda=(\alpha-\overline{A}_{\hat{L}-\lambda})^{-1}$ as in the proof of Theorem 5.2′. If $\overline{A}_{\hat{L}-\lambda}$ is a generator, $\alpha G_\alpha^\lambda u\to u$ $(\alpha\to\infty)$ for $u\in C(\overline{D})$, and hence we have

$$\alpha K_\lambda^\alpha \overline{LG_\alpha^{\min}}u\to[u]_{\partial D},\quad \alpha\to\infty,$$

noting that $G_\alpha^\lambda u$ is reduced to $K_\lambda^\alpha\overline{LG_\alpha^{\min}}u$ on ∂D. Put $u=H_\beta\varphi$ and $\varphi=K_\lambda^\beta 1$. We have

$$\begin{aligned}&\alpha K_\lambda^\alpha\overline{LG_\alpha^{\min}}u=\alpha K_\lambda^\alpha\widehat{L}(G_\alpha^{\min}H_\beta\varphi)\\&=\frac{\alpha}{\alpha-\beta}K_\lambda^\alpha(\overline{LH_\beta}\varphi-\overline{LH_\alpha}\varphi)=\frac{\alpha}{\alpha-\beta}K_\lambda^\alpha\{(\lambda-\overline{LH_\alpha})\varphi-(\lambda-\overline{LH_\beta})\varphi\}\\&=\frac{\alpha}{\alpha-\beta}\varphi-\frac{\alpha}{\alpha-\beta}K_\lambda^\alpha(\lambda-\overline{LH_\beta})K_\lambda^\beta 1=\frac{\alpha}{\alpha-\beta}\varphi-\frac{\alpha}{\alpha-\beta}\cdot K_\lambda^\alpha 1.\end{aligned}$$

Since $\alpha K_\lambda^\alpha\overline{LG_\alpha^{\min}}u\to[u]_{\partial D}=\varphi$ and $\frac{\alpha}{\alpha-\beta}\to 1$, we have $K_\lambda^\alpha 1\to 0$, which implies $\|K_\lambda^\alpha\|=\|K_\lambda^\alpha 1\|\to 0$.

Remark 5.3. There is a freedom of choice in defining $\mathfrak{D}(L)$ and hence $\mathfrak{D}(\widehat{L})$ as we have noted in §4. But, the following assertion holds. *Let* L_1 *be also an operator of type* (4.1) *and satisfy* (4.3) *and* (4.4). *Suppose that* L_1 *is an extension of* L. *Then, if* $\overline{LH_\alpha}$ *is a generator, we have* $\mathfrak{D}(\widehat{L})=\mathfrak{D}(\widehat{L}_1)$ *and* $\widehat{L}=\widehat{L}_1$.

Proof. Since $\overline{LG_\alpha^{\min}}=\overline{L_1G_\alpha^{\min}}$ is obvious, we have only to prove $\overline{LH_\alpha}=\overline{L_1H_\alpha}$. The range of $\lambda-LH_\alpha$ being dense in $C(\partial D)$ for $\lambda>0$, $\overline{L_1H_\alpha}$ is also a generator. $(\lambda-\overline{L_1H_\alpha})^{-1}$ is an extension of $(\lambda-\overline{LH_\alpha})^{-1}$, because $\overline{L_1H_\alpha}$ is an extension of $\overline{LH_\alpha}$. But, since $(\lambda-\overline{LH_\alpha})^{-1}$ has domain $C(\partial D)$, we have $(\lambda-\overline{L_1H_\alpha})^{-1}=(\lambda-\overline{LH_\alpha})^{-1}$ and hence, $\overline{L_1H_\alpha}=\overline{LH_\alpha}$.

We have obtained a class of semigroups on $C(\bar{D})$ with generators $\bar{A}_{\hat{L}-\lambda}$ ($\lambda\geqslant 0$), if there is a semigroup on $C(\partial D)$ with generator $\overline{LH_\alpha}$ for some $\alpha\geqslant 0$. A converse problem may be of some interest. Here, we formulate a result.

Theorem 5.3. *Suppose that, for each $\lambda\geqslant 0$, $\bar{A}_{\hat{L}-\lambda}$ is the generator of a semigroup on $C(\bar{D})$. Then, $\overline{LH_\alpha}$ is the generator of a semigroup on $C(\partial D)$ for each $\alpha\geqslant 0$.*

Proof. By the corollary to Theorem 5.1, it is sufficient to prove that $(\alpha-\bar{A})u=0$ and $(\lambda-\hat{L})u=\varphi$ has a solution u for each φ in a dense subset of $C(\partial D)$ for some fixed $\alpha\geqslant 0$ and $\lambda\geqslant 0$. Let $\lambda'\neq\lambda$. Take any f in $\mathfrak{D}(\hat{L})$ satisfying $(\lambda'-\hat{L})f=0$ and write $v=\frac{1}{\lambda-\lambda'}f$. Since $\bar{A}_{\hat{L}-\lambda}$ is a generator, there is a w such that $(\alpha-\bar{A})w=-(\alpha-\bar{A})v$ and $(\lambda-\hat{L})w=0$. Then, $u=v+w$ satisfies $(\alpha-\bar{A})u=(\alpha-\bar{A})v+(\alpha-\bar{A})w=0$ and $(\lambda-\hat{L})u=(\lambda-\hat{L})v+(\lambda-\hat{L})w=(\lambda-\hat{L})v=(\lambda-\lambda')[v]_{\partial D}+(\lambda'-\hat{L})v=[f]_{\partial D}$. Since $\bar{A}_{\hat{L}-\lambda'}$ is a generator $\{u\in\mathfrak{D}(\hat{L})\,|\,(\lambda'-\hat{L})u=0\}$ is dense in $C(\bar{D})$, implying that $\overline{LH_\alpha}$ is a generator.

Remark 5.4. In the above proof, we have really proved that if $\bar{A}_{\hat{L}-\lambda}$ generates a semigroup on $C(\bar{D})$ for some $\lambda\geqslant 0$, and if $\{u\,|\,u\in\mathfrak{D}(\hat{L})$ and $(\lambda'-\hat{L})u=0\}$ is dense in $C(\bar{D})$ for some $\lambda'\neq\lambda$, then $\overline{LH_\alpha}$ is a generator. Probably the second assumption can be dropped, but it is not yet proved.

Now, we prove some equalities connecting $\{G_\alpha^\lambda\}$ and $\{K_\lambda^\alpha\}$. The resolvent equations for $\{G_\alpha^\lambda\}$ and $\{K_\lambda^\alpha\}$ are obtained by putting $\lambda=\mu$ or $\alpha=\beta$ in the following.

Proposition 5.1.[24] *If $\overline{LH_\alpha}$ and $\overline{LH_\beta}$ are generators of semigroups on $C(\partial D)$, then we have, for any λ and $\mu>0$,*

$$G_\alpha^\lambda u-G_\beta^\mu u+(\alpha-\beta)G_\alpha^\lambda G_\beta^\mu u+(\lambda-\mu)\hat{K}_\lambda^\alpha[G_\beta^\mu u]_{\partial D}=0,\quad u\in C(\bar{D}),\tag{5.8}$$

$$\hat{K}_\lambda^\alpha\varphi-\hat{K}_\mu^\beta\varphi+(\lambda-\mu)\hat{K}_\lambda^\alpha K_\mu^\beta\varphi+(\alpha-\beta)G_\alpha^\lambda\hat{K}_\mu^\beta\varphi=0,\quad \varphi\in C(\partial D),\tag{5.9}$$

24) Some of these relations are obtained also in [12, 25] under a different set-up.

(5. 10) $K_{\lambda}^{\alpha}\varphi-K_{\mu}^{\beta}\varphi+(\lambda-\mu)K_{\lambda}^{\alpha}K_{\mu}^{\beta}\varphi+K_{\lambda}^{\alpha}(\overline{LH_{\beta}}-\overline{LH_{\alpha}})K_{\mu}^{\beta}\varphi=0,$
$\varphi\in C(\partial D),$

where G_{α}^{λ} *is defined in* (5. 7) *and* $\widehat{K_{\lambda}^{\alpha}}=H_{\alpha}K_{\lambda}^{\alpha}$. *Above equalities hold even when* λ *or* $\mu=0$ *if* K_{0}^{α} *or* K_{0}^{β} *exists.*

Proof. If we apply $\alpha-\overline{A}$ and $\lambda-\widehat{L}$ to the lefthand sides of (5. 8) and (5. 9), then the result is 0. But the solution $u\in C(\overline{D})$ of $(\alpha-\overline{A})u=0$ and $(\lambda-\widehat{L})u=0$ is unique and is constant 0 for $\alpha\geqslant 0$ and $\lambda>0$ (also for $\lambda=0$ if K_{0}^{α} exists) by Lemma 5. 2, and hence (5. 8) and (5. 9) are proved. As for (5. 10), the boundary value of the left hand side of (5. 9) being $K_{\lambda}^{\alpha}\varphi-K_{\mu}^{\beta}\varphi+(\lambda-\mu)K_{\lambda}^{\alpha}K_{\mu}^{\beta}\varphi+(\alpha-\beta)K_{\lambda}^{\alpha}\overline{LG_{\alpha}^{\min}}H_{\beta}K_{\mu}^{\beta}\varphi$, it is sufficient to verify $(\alpha-\beta)\overline{LG_{\alpha}^{\min}}H_{\beta}K_{\mu}^{\beta}\varphi=(\overline{LH_{\beta}}-\overline{LH_{\alpha}})K_{\mu}^{\beta}\varphi$. But, this is clear by (4. 7) with φ replaced by $K_{\mu}^{\beta}\varphi$.

§6. A reduction to an integro-differential equation

The problem of constructing the diffusion determined by A and L is reduced to solve the equation of type $(\lambda-\overline{LH_{\alpha}})\psi=\varphi$. But this equation is essentially an integro-differential equation given on the boundary ∂D. To show this we prepare

Lemma 6. 1.[25] *Let* K *be a* P*-dimensional compact manifold of class* C^{2} *and let* $\{T_{t}, t\geqslant 0\}$ *be a semigroup on* $C(K)$ *with generator* $\mathfrak{G}$. *Let* $\{\eta^{i}(y)\in C^{2}(K), 1\leqslant i\leqslant P\}$ *be extensions of local coordinates in a neighbourhood of a point* $x\in K$ *such that* 1, $\eta^{i}(y)$ $(1\leqslant i\leqslant P)$ *and* $\sum_{i=1}^{P}(\eta^{i}(y))^{2}$ *belong to* $\mathfrak{D}(\mathfrak{G})$. *Moreover, let* $\eta^{i}(y)=0$, $1\leqslant i\leqslant P$, *if and only if* $y=x$. *Then, we have*

$$(6.1)\quad \begin{aligned}\mathfrak{G}f(x)&=\sum_{i,j=1}^{P}\overline{\alpha}^{ij}(x)\frac{\partial^{2}f}{\partial\eta^{i}\partial\eta^{j}}(x)+\sum_{i=1}^{P}\overline{\beta}^{i}(x)\frac{\partial f}{\partial\eta^{i}}(x)+\overline{\gamma}(x)f(x)\\&\quad+\int_{K}\left\{f(y)-f(x)-\sum_{i=1}^{P}\frac{\partial f}{\partial\eta^{i}}(x)\eta^{i}(y)\right\}\overline{\nu}_{x}(dy),\end{aligned}$$
$$f\in C^{2}(K)\cap\mathfrak{D}(\mathfrak{G}),$$

25) A similar result is obtained by K. Yosida [38]. The proof of this lemma is a modification of Wentzell's method of seeking the boundary condition.

where $(\bar{a}^{ij}(x))$ *is non-negative definite,* $\bar{\gamma}(x)$ *is non-positive, and* $\bar{\nu}_x$ *is a measure on* K *such that*

$$\bar{\nu}_x(K-U)<\infty,\quad \int_U \sum_{i=1}^{p}(\eta^i(y))^2\bar{\nu}_x(dy)<\infty$$

for any neighbourhood U *of* x.

Proof. For f in $\mathfrak{D}(\mathfrak{G})\cap C^2(K)$,

$$\begin{aligned}\mathfrak{G}f(x)&=\lim_{t\to 0}\frac{1}{t}(T_t f(x)-f(x))\\&=\lim_{t\to 0}\frac{1}{t}\Big(\int_K f(y)P(t,x,dy)-f(x)\Big)\\&=\lim_{t\to 0}\Big[\gamma(t)f(x)+\sum_{i=1}^{p}\beta^i(t)\frac{\partial f}{\partial\eta^i}(x)\\&\quad+\int_K\Big\{f(y)-f(x)-\sum_{i=1}^{p}\frac{\partial f}{\partial\eta^i}(x)\eta^i(y)\Big\}\Big\{\sum_{i=1}^{p}\eta^i(y)^2\Big\}^{-1}\nu(t,dy)\Big]\end{aligned}$$

where

$$\begin{aligned}\gamma(t)&=t^{-1}(P(t,x,K)-1)=t^{-1}(T_t1(x)-1)\leqslant 0\\ \beta^i(t)&=t^{-1}\int_K\eta^i(y)P(t,x,dy)=t^{-1}(T_t\eta^i(x)-\eta^i(x)),\\ \nu(t,E)&=t^{-1}\int_E\sum_{i=1}^{p}\eta^i(y)^2P(t,x,dy),\quad E\in\boldsymbol{B}(K).\end{aligned}$$ [26]

By assumption $\lim_{t\to 0}\gamma(t)=\mathfrak{G}1(x)=\gamma(x)$ and $\lim_{t\to 0}\beta^i(t)=\mathfrak{G}\eta^i(x)=\bar{\beta}^i(x)$ exist. Moreover, there is $\varepsilon_0>0$ such that

$$\begin{aligned}\nu(t,K)&\leqslant\lim_{t\to 0}\nu(t,K)+1\\&=\lim_{t\to 0}t^{-1}\Big\{T_t\Big(\sum_{i=1}^{p}(\eta^i)^2\Big)(x)-\Big(\sum_{i=1}^{p}(\eta^i)^2\Big)(x)\Big\}+1\\&\leqslant\mathfrak{G}\Big(\sum_{i=1}^{p}(\eta^i)^2\Big)(x)+1,\quad\text{for } t\leqslant\varepsilon_0.\end{aligned}$$

Putting

$$g(y)=\Big\{f(y)-f(x)-\sum_{i=1}^{p}\frac{\partial f}{\partial\eta^i}(x)\eta^i(y)\Big\}\Big\{\sum_{i=1}^{p}\eta^i(y)^2\Big\}^{-1},$$

$$z^{ij}(y)=\eta^i(y)\eta^j(y)\Big\{\sum_{i=1}^{p}\eta^i(y)^2\Big\}^{-1},$$

we have

26) $\boldsymbol{B}(K)$ is the topological Borel field of K.

$$g(y)=\frac{1}{2}\sum_{i,j=1}^{P}\frac{\partial^2 f}{\partial\eta^i\partial\eta^j}(x)z^{ij}(y)+r(f,y)\left\{\sum_{i=1}^{P}\eta^i(y)^2\right\}^{-1}.$$

$r(f,y)$ being continuous in $K-\{x\}$ and $o\left(\sum_{i=1}^{P}\eta^i(y)^2\right)$ around x, the second summand of $g(y)$ can be continuously extended on K. Let φ be a mapping from $K-\{x\}$ into $K\times R^{P2}$ defined by

$$\varphi: y\to\varphi(y)=(y, z^{ij}(y))\in K\times R^{P2},$$

and let M be the closure of the image $\varphi(K-\{x\})$ in $K\times R^{P2}$. M is compact for $-1\leqslant z^{ij}(y)\leqslant 1$. Define

$$G(y, Z^{ij})=\frac{1}{2}\sum_{i,j=1}^{P}\frac{\partial^2 f}{\partial\eta^i\partial\eta^j}(x)Z^{ij}+r(f,y)\left\{\sum_{i=1}^{P}\eta^i(y)^2\right\}^{-1}$$

on $\varphi(K-\{x\})$, and extend it continuously on M, and denote it by the same notation $G(\zeta)$, $\zeta\in M$. This is possible by what we have noted about $g(y)$. Clearly, $g(y)=G(\varphi(y))$, $y\in K-\{x\}$. Define $N(t, E)=\nu(t, \varphi^{-1}(E))$, $E\subset M$. Since $N(t, M)=\nu(t, K-\{x\})\leqslant \mathfrak{G}\left(\sum_{i=1}^{P}(\eta^i)^2\right)(x)+1$ for sufficiently small t, there is a sequence $\{t_n\searrow 0\}$ such that $\{N(t_n,\cdot)\}$ tends to $N(\cdot)$ in weak star. Thus, we have

$$\begin{aligned}\mathfrak{G}u(x)&=\lim_{n\to\infty}\left\{\gamma(t_n)f(x)+\sum_{i=1}^{P}\beta^i(t_n)\frac{\partial f}{\partial\eta^i}(x)+\int_M G(\zeta)N(t_n, d\zeta)\right\}\\ &=\bar{\gamma}(x)f(x)+\sum_{i=1}^{P}\bar{\beta}^i(x)\frac{\partial f}{\partial\eta^i}(x)+\int_M G(\zeta)N(d\zeta).\end{aligned}\tag{6.2}$$

Writing $\nu(E)=N(\varphi(E))$, $E\subset K-\{x\}$, we have

$$\begin{aligned}\int_M G(\zeta)N(d\zeta)&=\int_{M\cap\{y=x\}}G(\zeta)N(d\zeta)+\int_{M\cap\{y\neq x\}}G(\zeta)N(d\zeta)\\ &=\int_{M\cap\{y=x\}}\left\{\frac{1}{2}\sum_{i,j=1}^{P}\frac{\partial^2 f}{\partial\eta^i\partial\eta^j}(x)Z^{ij}\right\}N(d\zeta)+\int_{M\cap\{y\neq x\}}G(\zeta)N(d\zeta)\\ &=\sum_{i,j=1}^{P}\frac{\partial^2 f}{\partial\eta^i\partial\eta^j}(x)\int_{M\cap\{y=x\}}\frac{1}{2}Z^{ij}N(d\zeta)+\int_{K-\{x\}}G(\varphi(y))\nu(dy)\\ &=\sum_{i,j=1}^{P}\bar{\alpha}^{ij}(x)\frac{\partial^2 f}{\partial\eta^i\partial\eta^j}(x)+\int_{K-\{x\}}g(y)\nu(dy)\\ &=\sum_{i,j=1}^{P}\bar{\alpha}^{ij}(x)\frac{\partial^2 f}{\partial\eta^i\partial\eta^j}(x)+\int_{K-\{x\}}\left\{f(y)-f(x)\right.\\ &\qquad\left.-\sum_{i=1}^{P}\frac{\partial f}{\partial\eta^i}(x)\eta^i(y)\right\}\left\{\sum_{i=1}^{P}\eta^i(y)^2\right\}^{-1}\nu(dy),\end{aligned}$$

where $$\bar{\alpha}^{ij}(x)=\frac{1}{2}\int_{M\cap\{y=x\}}Z^{ij}N(d\zeta).$$

Putting $$\bar{\nu}_x(E)=\int_E\left\{\sum_{i=1}^{P}\gamma^i(y)^2\right\}^{-1}\nu(dy)$$

and noting (6.2) and (6.3) we have (6.1). It is clear by definitions that $\bar{\gamma}(x)\leqslant 0$, $(\bar{\alpha}^{ij}(x))$ is non-negative definite and $\bar{\nu}_x$ satisfies the conditions stated in the lemma.

Lemma 6.2. *Let L be $\partial/\partial n$.*[27] *Then, there is a semigroup on $C(\partial D)$ with generator $\overline{LH_\alpha}$ for any $\alpha\geqslant 0$. If φ belongs to $C^{2,\kappa}(\partial D)$, then*

$$\overline{\frac{\partial}{\partial n}H_\alpha}\varphi(x)=\frac{\partial}{\partial n}H_\alpha\varphi(x)$$

$$(6.4)\qquad =\sum_{i,j=1}^{N-1}\bar{\alpha}^{ij}(\alpha,x)\frac{\partial^2\varphi}{\partial\xi_x^i\partial\xi_x^j}(x)+\sum_{i=1}^{N-1}\bar{\beta}^i(\alpha,x)\frac{\partial\varphi}{\partial\xi_x^i}(x)$$
$$+\bar{\gamma}(\alpha,x)\varphi(x)+\int_{\partial D}\left\{\varphi(y)-\varphi(x)-\sum_{i=1}^{N-1}\frac{\partial\varphi}{\partial\xi_x^i}(x)\xi_x^i(y)\right\}$$
$$\bar{\nu}_x(\alpha,dy),\text{[28]}\quad x\in\partial D,$$

where $(\bar{\alpha}^{ij}(\alpha,x))$ is non-negative definite, $\bar{\gamma}(\alpha,x)\leqslant 0$ and $\bar{\nu}_x(\alpha,dy)$ is a σ-finite measure on ∂D satisfying

$$(6.5)\qquad \bar{\nu}_x(\alpha,\partial D-U_x)<\infty,\quad \int_{U_x\cap\partial D}\sum_{i=1}^{N-1}(\xi_x^i(y))^2\bar{\nu}_x(\alpha,dy)<\infty.$$

for any neighbourhood U_x of x.

Proof. Since there is a unique solution of the equation $(\alpha-A)u=0$ on D, $\left(\lambda-\frac{\partial}{\partial n}\right)u=\varphi$ on ∂D, for $\varphi\in C^{0,\kappa}(\partial D)$ by Theorem 2.3, there is a semigroup on $C(\partial D)$ with generator $\overline{\frac{\partial}{\partial n}H_\alpha}$ by Theorem 5.1. Since $\xi_x^i(y)$ satisfies the conditions in Lemma 6.1, $\overline{\frac{\partial}{\partial n}H_\alpha}$ can be represented by (6.1) for φ in $C^2(\partial D)\cap\mathfrak{D}(\overline{LH_\alpha})$. But $C^{2,\kappa}(\partial D)$ being contained in $\mathfrak{D}(LH_\alpha)\subset\mathfrak{D}(\overline{LH_\alpha})$ as we have noted just before Lemma 4.1, (6.4) holds for $C^{2,\kappa}(\partial D)$.

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27) We put $\mathfrak{D}(\partial/\partial n)=C^1(\bar{D})$.

28) Recently, S. A. Molčanov [23] proved that the first term in the right side of (6.4) can be omitted, and he represented $\bar{\nu}_x(\alpha,\cdot)$ in a more concrete form.

Lemma 6. 3. *For any given* $\nu_x(dy)$ *satisfying* (4. 2) *and any* $\alpha\geqslant 0$, *there is a* σ*-finite measur* $\bar{\nu}_x(\alpha, dy)$ *satisfying the condition* (6. 5) *with* $\nu_x(\alpha, \cdot)$ *replaced by* $\bar{\nu}_x(\alpha, \cdot)$ *such that*

$$(6.6)\qquad \int_{\partial D}\varphi(y)\bar{\nu}_x(\alpha, dy)=\int_D H_\alpha\varphi(y)\nu_x(dy)$$

if $\varphi\in C(\partial D)$ *and* $\varphi(y)=O\left(\sum_{i=1}^{N-1}(\xi_x^i(y))^2\right)$, $y\to x$.

Proof. Fix an $x\in\partial D$ and put $\eta(y)=\left[\sum_{i=1}^{N-1}(\xi_x^i(y))^2\right]_{\partial D}$. Each ξ_x^i belonging to $C^3(\partial D)$, $\eta(y)\in C^3(\partial D)$ and hence $\varphi\cdot\eta$ belonge to $C^3(\partial D)\subset C^{2,\kappa}(\partial D)$ for any $\varphi\in C^3(\partial D)$, implying $H_\alpha(\varphi\cdot\eta)\in C^2(\bar{D})$. Thus, noting that $\varphi\cdot\eta(x)=0$ and $(\partial/\partial\xi_x^i)(\varphi\cdot\eta)(x)=0$ $(1\leqslant i\leqslant N-1)$, functional

$$\begin{aligned}\Phi(\varphi)&=\int_D H_\alpha(\varphi\cdot\eta)(y)\nu_x(dy)\\ &=\int_D\left\{H_\alpha(\varphi\cdot\eta)(y)-H_\alpha(\varphi\cdot\eta)(x)-\sum_{i=1}^{N-1}\frac{\partial}{\partial\xi_x^i}H_\alpha(\varphi\cdot\eta)(x)\xi_x^i(y)\right\}\nu_x(dy)\end{aligned}$$

can be defined on $C^3(\partial D)$. $\Phi(\varphi)$ is clearly linear, non-negative and bounded, because $\Phi(1)$ is finite. $C^3(\partial D)$ being dense in $C(\partial D)$, $\Phi(\varphi)$ can be extended to a bounded, linear and non-negative functional on $C(\partial D)$ uniquely, and hence represented as an integral

$$\Phi(\varphi)=\int_{\partial D}\varphi(y)\mu(dy),\quad \varphi\in C(\partial D)$$

by a bounded measure μ on ∂D in virtue of Riesz' theorem.

Now, define $\bar{\nu}_x(\alpha, dy)$ by

$$\bar{\nu}_x(\alpha, E)=\int_{E-\{x\}}\eta(y)^{-1}\mu(dy),$$

which is clearly a σ-finite measure on ∂D and satisfies (6. 5) with $\nu_x(\alpha, \cdot)$ replaced by $\bar{\nu}_x(\alpha, \cdot)$. If $\varphi\in C^3(\partial D)$ and φ vanishes in a neighbourhood of x, then $\varphi\eta^{-1}$ can be considered as in $C^3(\partial D)$, and hence

$$\begin{aligned}\int_{\partial D}\varphi(y)\bar{\nu}_x(\alpha, dy)&=\int_{\partial D-\{x\}}\varphi\eta^{-1}(y)\mu(dy)=\Phi(\varphi\eta^{-1})\\ &=\int_D H_\alpha\varphi(y)\nu_x(dy).\end{aligned}$$

Let φ be in $C(\partial D)$ and satisfy $\varphi(y)=O\left(\sum_{i=1}^{N-1}\xi_x^i(y)^2\right)$, $y\to x$. Then, we can find a sequence $\{\varphi_n\}\subset C^3(\partial D)$ tending to φ such that the support of φ_n does not contain x and $|\varphi_n(y)|\leqslant K\eta(y)$. We have

$$\int_{\partial D}\varphi_n(y)\bar{\nu}_x(\alpha, dy)=\int_D H_\alpha\varphi_n(y)\nu_x(dy),$$

and this formula becomes (6.6) as $n\to\infty$ by the dominated convergence theorem, since $|H_\alpha\varphi_n(y)|\leqslant H_\alpha|\varphi_n|(y)\leqslant KH_\alpha\eta(y)=O(\eta_x^N(y))+O\left(\sum_{i=1}^{N-1}\eta_x^i(y)^2\right)$.

Theorem 6.1. *For any given L, we have*

$$(6.7)\quad \begin{aligned}\overline{LH_\alpha}\varphi=LH_\alpha\varphi=&\sum_{i,j=1}^{N-1}\alpha^{ij}(\alpha, x)\frac{\partial^2\varphi}{\partial\xi_x^i\partial\xi_x^j}(x)\\&+\sum_{i=1}^{N-1}\beta^i(\alpha, x)\frac{\partial\varphi}{\partial\xi_x^i}(x)+\gamma(\alpha, x)\varphi(x)\\&+\int_{\partial D}\left\{\varphi(y)-\varphi(x)-\sum_{i=1}^{N-1}\frac{\partial\varphi}{\partial\xi_x^i}(x)\xi_x^i(y)\right\}\nu_x(\alpha, dy),\end{aligned}$$

for $\varphi\in C^{2,\kappa}(\partial D)$ *and* $x\in\partial D$,

where $(\alpha^{ij}(\alpha, x))$ *is non-negative definite,* $\gamma(\alpha, x)\leqslant 0$, *and* $\nu_x(\alpha,\cdot)$ *is a σ-finite measure on* ∂D *satisfying*

$$(6.8)\quad \nu_x(\alpha, \partial D-U_x)<\infty,\ \int_{\partial D}\sum_{i=1}^{N-1}(\xi_x^i(y))^2\nu_x(\alpha, dy)<\infty,$$

for any neighbourhood U_x *of* x.

Proof. First, we note that the following quantities are finite.

$$(6.9)\quad \begin{aligned}\bar{\bar{\gamma}}(\alpha, x)&=\int_D(H_\alpha 1(y)-1)\nu_x(dy),\\ \bar{\bar{\beta}}^i(\alpha, x)&=\int_D(H_\alpha\xi_x^i(y)-\xi_x^i(y))\nu_x(dy).\end{aligned}$$

In fact, the integrands of (6.9) and their derivatives with respect to ξ_x^i $(1\leqslant i\leqslant N-1)$ vanish at x and the integrands belong to $C^2(\bar{D})$, implying that the integrals are finite by assumptions on ν_x.

Now, we compute $\overline{LH_\alpha}\varphi$ for $\varphi\in C^{2,\kappa}(\partial D)$ by definitions

$$\overline{LH_\alpha\varphi(x)}=LH_\alpha\varphi(x)=\sum_{i,j=1}^{N-1}\alpha^{ij}(x)\frac{\partial^2\varphi}{\partial\xi_x^i\partial\xi_x^j}(x)+\sum_{i=1}^{N-1}\beta^i(x)\frac{\partial\varphi}{\partial\xi_x^i}(x)$$

$$(6.10)\qquad +\gamma(x)\varphi(x)+\delta(x)\lim_{y\in D,y\to x}A(H_\alpha\varphi)(y)+\mu(x)\frac{\partial}{\partial n}H_\alpha\varphi(x)$$

$$+\int_{\bar{D}}\Big(H_\alpha\varphi(y)-\varphi(x)-\sum_{i=1}^{N-1}\frac{\partial\varphi}{\partial\xi_x^i}(x)\xi_x^i(y)\Big)\nu_x(dy).$$

Since $(\alpha-A)H_\alpha\varphi(x)=0$ for $x\in D$, the forth summand is given by

$$(6.11)\qquad \delta(x)\lim_{y\in D,y\to x}A(H_\alpha\varphi)(y)=\alpha\varphi(x).$$

The fifth summand can be rewritten making use of (6. 4) by Lemma 6. 2. As for the last summand, the integral restricted to D is given in the following, making use of Lemma 6. 3 and (6. 9).

$$\int_D\Big(H_\alpha\varphi(y)-\varphi(x)-\sum_{i=1}^{N-1}\frac{\partial\varphi}{\partial\xi_x^i}(x)\xi_x^i(y)\Big)\nu_x(dy)$$

$$=\int_D H_\alpha\Big\{\varphi-\varphi(x)-\sum_{i=1}^{N-1}\frac{\partial\varphi}{\partial\xi_x^i}(x)\xi_x^i\Big\}(y)\nu_x(dy)$$

$$(6.12)\qquad +\int_D\{H_\alpha 1(y)-1\}\varphi(x)\nu_x(dy)$$

$$+\int_D\sum_{i=1}^{N-1}\frac{\partial\varphi}{\partial\xi_x^i}(x)\Big\{H_\alpha\xi_x^i(y)-\xi_x^i(y)\Big\}\nu_x(dy)$$

$$=\int_{\partial D}\Big\{\varphi(y)-\varphi(x)-\sum_{i=1}^{N-1}\frac{\partial\varphi}{\partial\xi_x^i}(x)\xi_x^i(y)\Big\}\bar{\nu}_x(\alpha,dy)$$

$$+\bar{\bar{\gamma}}(\alpha,x)\varphi(x)+\sum_{i=1}^{N-1}\bar{\bar{\beta}}^i(\alpha,x)\frac{\partial\varphi}{\partial\xi_x^i}(x).$$

Denote the retriction on ∂D of $\nu_x(\cdot)$ by $[\nu_x]_{\partial D}(\cdot)$ and put

$$(6.13)\qquad \begin{aligned}&\alpha^{ij}(\alpha,x)=\alpha^{ij}(x)+\mu(x)\bar{\alpha}^{ij}(\alpha,x)\\&\beta^i(\alpha,x)=\beta^i(x)+\mu(x)\bar{\beta}^i(\alpha,x)+\bar{\bar{\beta}}^i(\alpha,x)\\&\gamma(\alpha,x)=\gamma(x)+\alpha\delta(x)+\mu(x)\bar{\gamma}(\alpha,x)+\bar{\bar{\gamma}}(\alpha,x)\\&\nu_x(\alpha,\cdot)=[\nu_x]_{\partial D}(\cdot)+\mu(x)\bar{\nu}_x(\alpha,\cdot)+\bar{\bar{\nu}}_x(\alpha,\cdot).\end{aligned}$$

Thus, (6. 4) and (6. 9)–(6. 12) imply the representation (6. 7). By definition (6. 13) the properties of $(\alpha^{ij}(\alpha,x))$ and $\nu_x(\alpha,\cdot)$ mentioned in the theorem and that $\gamma(\alpha,x)\leqslant 0$ are clear, completing the proof.

By the above theorem, to solve the equation

$$(\lambda-\overline{LH_\alpha})\psi=\varphi$$

for φ in a set dense in $C(\partial D)$ is reduced to solve an integro-differential equation

$$\begin{aligned}(6.14)\qquad &\lambda\psi(x)-\Big(\sum_{i,j=1}^{N-1}\alpha^{ij}(\alpha,x)\frac{\partial^2\psi}{\partial\xi_x^i\partial\xi_x^j}(x)\\ &+\sum_{i=1}^{N-1}\beta^i(\alpha,x)\frac{\partial\psi}{\partial\xi_x^i}(x)+\gamma(\alpha,x)\psi(x)\\ &+\int_{\partial D}\Big\{\psi(y)-\psi(x)-\sum_{i=1}^{N-1}\frac{\partial\psi}{\partial\xi_x^i}(x)\xi_x^i(y)\Big\}\nu_x(\alpha,dy)\\ &=\varphi(x),\quad x\in\partial D,\end{aligned}$$

for so many φ in $C(\partial D)$.

To consider some examples, we prepare

Lemma 6.4. *If $\overline{LH_\alpha}$ is a generator of a semigroup on $C(\partial D)$, then $\overline{L'H_\alpha}$ is also a generator, where*

$$L'u(x)=Lu(x)+\gamma'(x)u(x)+\delta'(x)\lim_{y\in D,y\to x}Au(y),\quad x\in\partial D,$$

and $\gamma'(x)$ and $\delta'(x)$ are non-positive continuous functions of $x\in\partial D$,

Proof. Since γ' and δ' are continuous, L' satisfies (L. 1) and we can choose $\mathfrak{D}(L')=\mathfrak{D}(L)\cap\{u\mid u\in C^2(D)$ and Au can be extended to a function in $C(\overline{D})\}$. Thus we have $\mathfrak{D}(LH_\alpha)=\mathfrak{D}(L'H_\alpha)$ and

$$\begin{aligned}L'(H_\alpha\varphi)(x)&=LH_\alpha\varphi(x)+\gamma'(x)H_\alpha\varphi(x)+\delta'(x)\lim_{y\to x}AH_\alpha\varphi(y)\\&=LH_\alpha\varphi(x)+\gamma'(x)\varphi(x)+\alpha\delta'(x)\varphi(x),\quad\text{for }\varphi\in\mathfrak{D}(LH_\alpha),\end{aligned}$$

and hence, $\overline{L'H_\alpha}=\overline{LH_\alpha}+\gamma'+\alpha\delta'$. By applying the corollary to Theorem 1.2 for $\mathfrak{D}=\overline{LH_\alpha}$ and $M=\gamma'+\alpha\delta'$, we complete the proof.

Example 1. If L is given by

$$Lu(x)=\frac{\partial}{\partial n}u(x)+\gamma(x)u(x)+\delta(x)\lim_{y\in D,y\to x}Au(y),$$

where γ and δ are non-positive and continuous and if we choose $\mathfrak{D}(L)=\{u\mid u\in C^1(\overline{D})\cap C^2(D)$ and Au can be extended to a function in $C(\overline{D})\}$, then $\overline{A}_{\hat{L}-\lambda}$ is the generator of a semigroup on $C(\overline{D})$ for each $\lambda\geqslant 0$.

In fact, by Lemma 6.2 $(\partial/\partial n)H_\alpha$ is a generator, and hence $\overline{LH_\alpha}$ is also a generator by Lemma 6.4 for any $\alpha > 0$. Since L satisfies (L. 2), $\bar{A}_{\hat{L}-\lambda}$ is a generator by Theorem 5.2′ for each $\lambda > 0$.

Example 2. Consider the case where ∂D is a compact Lie group and each translation of ∂D can be extended to an isometric transformation of $\bar{D}$. Let A be invariant under the transformations induced by translations of ∂D. Let $\alpha^{ij}(x)$, $\beta^i(x)$, and $\mu(x)$ in L be constants and let $\gamma(x)$ and $\delta(x)$ be continuous in $x \in \partial D$. Let $\nu_x(\cdot)$ and $\{\xi_x^i(y)\}$ satisfy $\nu_x(E) = \nu_e(x^{-1}(E))$ and $\xi_x^i(y) = \xi_e^i(x^{-1}(y))$, $x \in \partial D$, $y \in \bar{D}$, where e is the neutral element of ∂D and $x^{-1}(\cdot)$ is the transformation of $\bar{D}$ induced by the translation of ∂D determined by $x^{-1} \in \partial D : y \in \partial D \to x^{-1} \cdot y \in \partial D$. Then, $\bar{A}_{\hat{L}-\lambda}$ is the generator of a semigroup on $C(\bar{D})$ for each $\lambda > 0$, if L satisfies (L. 2). Moreover, if we assume only (L. 2′) for L and require δ to be a constant, then (L. 2) is necessary and sufficient in order that $\bar{A}_{\hat{L}-\lambda}$ be a generator.

In fact, if $\gamma = \delta = 0$, then $\alpha^{ij}(\alpha, x)$, $\beta^i(\alpha, x)$, and $\gamma(\alpha, x)$ in (6.7) are constants, and $\nu_x(\alpha, \cdot)$ is translation invariant, and hence, by a theorem of G. Hunt [11, p. 279] $\overline{LH_\alpha}$ is the generator of a semigroup on $C(\partial D)$. Therefore, $\overline{LH_\alpha}$ is a generator even if γ and δ do not vanish, by Lemma 6.4. If L satisfies (L. 2), $\bar{A}_{\hat{L}-\lambda}$ is a generator by Theorem 5.2′. Let L satisfy (L. 2′) and let δ be a constant and $\bar{A}_{\hat{L}-\lambda}$ be a generator. If $\mu - \delta > 0$, (L. 2) is satisfied. We have $\|K_\lambda^\alpha\| \to 0$ as $\alpha \to \infty$ by Remark 5.2. Thus, if $\mu = \delta = 0$, then by Remark 5.1 $\nu_x(D)$ is not bounded in x, and hence $\nu_x(D)$ is always ∞, completing the proof.

As concrete examples, consider a circular disc in R^2 or a solid sphere in R^4 with rotation invariant A and L. As another example, let $\bar{D}$ be a set $\{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 \leqslant 1,\ 0 \leqslant x_3 \leqslant 2\pi\}$ in R^3 where $(x_1, x_2, 0)$ is identified with $(x_1, x_2, 2\pi)$. ∂D is a 2-dimensional torus. Let A and L be invariant under the rotations around x_3-axis and translations along x_3-axis.

We note a little more general result is obtained in case ∂D is a torus [33, p. 537] by making use of [27].

Example 2 can be extended to the case where ∂D is a homogeneous space, since Hunt [11, pp. 286–293] has obtained a corresponding result about homogeneous space. In this case, the conditions on A and L are more restrictive to a certain extent.

A similar result is obtained in the case of the Brownian motion in the half space of R^n in [32].

Chapter III. Local time and the Markov process on the boundary

To inquire the probabilistic meaning of the Markov process on the boundary, we consider in this chapter the reflecting diffusion, that is, the diffusion determined by

$$\frac{\partial u}{\partial t}=Au \quad \text{and} \quad \frac{\partial u}{\partial n}(x)=0, \quad x\in\partial D.$$

In this case, local time on the boundary $\mathfrak{t}(t, w)$ will be defined by

$$\mathfrak{t}(t, w)=\lim_{\rho\to 0}\frac{1}{\rho}\int_0^t \chi_{D_\rho}(x_s(w))\,ds,^{29)}$$

which has some properties similar to the case of one dimension, as will be proved in §7. Then, $x_{\mathfrak{t}^{-1}(t,w)}(w)$ will be proved in §9 to be a Markov process on the boundary with the generator $\overline{\frac{\partial}{\partial n}H_0}$, where $\mathfrak{t}^{-1}(t, w)$ is the right continuous inverse of $\mathfrak{t}(t, w)$. This means that the Markov process on the boundary of order 0 is the trace on the boundary of the trajectory of the reflecting diffusion, and that $\mathfrak{t}^{-1}(t, w)$ is a time scale suitable to describe this motion. Moreover, the diffusion determined by

$$\frac{\partial u}{\partial t}=Au \text{ and } \gamma(x)u(x)+\delta(x)\lim_{y\in D,\, y\to x}Au(y)+\frac{\partial u}{\partial n}(x)=0,\ x\in\partial D$$

will be constructed from the reflecting diffusion using $\mathfrak{t}(t, w)$ in §9,

29) χ_E is the indicator function of a set E.

Some lemmas related to certain differential equations and $\mathfrak{t}(t, w)$ will be proved in §8 as a preliminary to §9.

§7. Local time on the boundary of the reflecting diffusion

Let $p^+(t, x, y)$ be the fundamental solution of the Cauchy problem for the parabolic equation

$$\frac{\partial u}{\partial t}(t, x) = Au(t, x), \qquad t>0, \ x\in D$$

with *reflecting barrier condition*

$$(7.1) \qquad \frac{\partial u}{\partial n}(t, x) = 0, \qquad t>0, \ x\in\partial D,$$

where A contains *no constant term*:

$$(7.2) \qquad Au(t, x) = \sum_{i,j=1}^{N} \frac{1}{\sqrt{a(x)}} \frac{\partial}{\partial x^i}\left(a^{ij}(x)\sqrt{a(x)}\,\frac{\partial u}{\partial x^j}(t, x)\right) + \sum_{i=1}^{N} b^i(x)\frac{\partial u}{\partial x^i}(t, x).$$

$\{T_t^+, t\geqslant 0\}$ defined by

$$(7.3) \qquad T_t^+ f(x) = \int_{\bar D} p^+(t, x, y) f(y)\, m(dy) \quad \text{for } t>0 \text{ and } T_0^+ f(x) = f(x)$$

are non-negative linear operators on $C(\bar D)$, and form a semigroup on $C(\bar D)$ in the sense of §1 by virtue of Theorem 2. 1.

Let W be the set of all functions w defined on $[0, +\infty]$, taking values in $\bar D\cup\{\Delta\}$, where Δ is adjoined to $\bar D$ as an isolated point, and satisfying following conditions:

1. There is such a $\zeta(w)\in[0, +\infty]$ that $w(t)\in\bar D$ for $0\leqslant t<\zeta(w)$, and $w(t)=\Delta$ for $\zeta(w)\leqslant t\leqslant+\infty$.

2. $w(t)$ is continuous in t for $0\leqslant t<\zeta(w)$. We sometimes write $x_t(w)=w(t)$. Let $\boldsymbol{B}_t$ and $\boldsymbol{B}$ be the smallest Borel fields of subsets of W, which make $\{x_s, s\in[0, t]\}$ and $\{x_s, s\in[0, +\infty)\}$ measurable, respectively. For each $w\in W$ and $t>0$, we have a *shifted path*

$w_t^+ \in W$ defined by $w_t^+(s)=w(t+s)$, $s\in[0, +\infty]$. A mapping $w\to w_t^+$ is clearly $\boldsymbol{B}$-measurable.

Theorem 7.1. *There is a Markov process* $\mathbf{X}=(x_t, W, \boldsymbol{B}_t, P_x, x\in\overline{D}\cup\{\Delta\})$ *with* $\{p^+(t,x,y)m(dy)\}$ *as a system of transition probabilities, that is, there is a system of measures* $\{P_x(\cdot), x\in\overline{D}\cup\{\Delta\}\}$ *on* $(W, \boldsymbol{B})$ *such that* $P_x(B)$ *is* $\boldsymbol{B}(\overline{D}\cup\{\Delta\})$*-measurable in* x *for each* B, $P_x(x_0(w)=x)=1$ *for each* $x\in\overline{D}$, $E_x(f(w_t^+)\,|\,\boldsymbol{B}_t)=E_{x_t}(f)$[30] *holds for each* $\boldsymbol{B}$*-measurable bounded function* f *with* P_x*-probability one, and*

$$E_x(f(x_t))=\int_{\overline{D}}p^+(t,x,y)f(y)m(dy), \quad f\in C(\overline{D}).$$

Such a Markov process is unique.[31] *Moreover,* $\mathbf{X}$ *is conservative, that is,*

$$(7.4)\qquad P_x(\zeta(w)=+\infty)=1, \quad \text{for } x\in\overline{D}.$$

The transition operators of $\mathbf{X}$ *form a semigroup on* $C(\overline{D})$ *and its generator is* $\overline{A}_{\mathfrak{L}}$, *where* $L=\partial/\partial n$ *and* $\mathfrak{D}(L)=C^1(\overline{D})$.

We call $\mathbf{X}$ *the reflecting diffusion on* $\overline{D}$ *determined by* A.

Proof. First, let us prove that the generator of T_t^+ is $\overline{A}_{\mathfrak{L}}$. Since $(\alpha-A)u=0$ and $Lu=\varphi$ can be solved for dense φ in $C(\partial D)$ by Theorem 2.3, $\overline{A}_{\mathfrak{L}}$ generates a semigroup on $C(\overline{D})$ by Theorems 5.1 and 5.2. Let G_α^+ be its Green operator. Since $\varphi\in C^{0,\kappa}(\partial D)$ implies $u\in C^1(\overline{D})$ and $u=H_\alpha K_0^\alpha\varphi$, $f\in C^{0,\kappa}(\overline{D})$ implies $H_\alpha K_0^\alpha\frac{\partial}{\partial n}G_\alpha^{\min}f\in C^1(\overline{D})$ by Theorem 2.4, (i). Hence, G_α^+f belongs to $C^2(D)\cap C^1(\overline{D})$ and satisfies $(\alpha-A)G_\alpha^+f=f$ in D and $\frac{\partial}{\partial n}G_\alpha^+f=0$ on ∂D for each $f\in C^{0,\kappa}(\overline{D})$. Therefore, we have

$$G_\alpha^+f(x)=\int_{\overline{D}}\left(\int_0^\infty e^{-\alpha t}p^+(t,x,y)\,dt\right)f(y)m(dy)$$

by Theorem 2.3, since $e^{-\alpha t}p^+(t,x,y)$ is the fundamental solution of

30) $E_x(\cdot)$ is the integration by measure $P_x(\cdot)$.

31) More precisely, P_x is unique if W, $\boldsymbol{B}_t$ and $\boldsymbol{B}$ are fixed.

the Cauchy problem of $\partial u/\partial t=(A-\alpha)u$ with boundary condition $\partial u/\partial n=0$. This implies that G_{α}^{+} is the Green operator of T_t^{+}, and hence the generator of T_t^{+} is $\bar{A}_{\hat{L}}$.

For the existence of $P_x(\cdot)$ on the space of continuous path functions W, it is sufficient to prove

$$(7.5)\qquad \lim_{t\downarrow 0}\sup_{x\in D}\frac{1}{t}\left(1-\int_{U_\varepsilon(x)}p^{+}(t,x,y)m(dy)\right)=0 \quad \text{for } \varepsilon>0,$$

by virtue of a result of Dynkin [4] and Kinney [19], where $U_\varepsilon(x) =\{y\in\bar{D}\,|\,d(x,y)<\varepsilon\}$. Now, fix an $x_0\in\bar{D}$ and find such a function $f\in C^{2,\kappa}(\bar{D})$ that satisfies $\partial f/\partial n=0$, $0\leqslant f\leqslant 1$, vanishes everywhere in $D-U_{\varepsilon/2}(x_0)$, and is constantly 1 in some neighbourhood V of x_0. It is easy to find such an f if x_0 is in D. For $x_0\in\partial D$, choosing sufficiently small a and $b>0$ and $\theta\in C^{2,\kappa}([0,+\infty))$ which satisfies $0\leqslant\theta\leqslant 1$, $\theta(z)=1$ for $0\leqslant z\leqslant a$, and $\theta(z)=0$ for $z\geqslant b$, we obtain such an f by putting $f(x)=\theta\left(\sum_{i=1}^{N-1}(\bar{x}^i-\bar{x}_0^i)^2\right)\cdot\theta(\bar{x}^N)$, where $(\bar{x}^i)$ and $(\bar{x}_0^i)$ are canonical coordinate systems of x and x_0, respectively. Noting that f belongs to the domain of the generator of $\{T_t^{+}\}$ since $f\in\mathfrak{D}(\hat{L})$ and $\hat{L}f=\partial f/\partial n=0$ by Lemma 4.5, that $\bar{A}f(x)=Af(x)=0$ for $x\in V$, and that $\chi_{U_\varepsilon(x)}\geqslant f$ for each $x\in V$, we have

$$\begin{aligned}
&\sup_{x\in V}\frac{1}{t}\left\{1-\int_{U_\varepsilon(x)}p^{+}(t,x,y)m(dy)\right\}\\
&\qquad\leqslant\sup_{x\in V}\frac{1}{t}\left\{1-\int_{\bar{D}}p^{+}(t,x,y)f(y)m(dy)\right\}\\
&\qquad=\sup_{x\in V}\left[\frac{1}{t}\left\{1-\int_{\bar{D}}p^{+}(t,x,y)f(y)m(dy)\right\}+Af(x)\right]\\
&\qquad\leqslant\left\|\frac{1}{t}(f-T_t^{+}f)+Af\right\|,
\end{aligned}$$

which converges to 0 when t tends to 0. Since $\bar{D}$ is compact, (7.5) has been proved. The uniqueness of $\boldsymbol{X}$ is assured by K. Ito [15, p. 35] and (7.4) follows from $\int_{\bar{D}}p^{+}(t,x,y)m(dy)=1$, which is obtained by (7.1), (7.2) and Theorem 2.1, ii).

Theorem 7.2. *There is a sequence of positive numbers* $\{\rho_n\downarrow 0\}$

such that $\mathfrak{t}_{\rho_n}(t, w)$ *converges to a continuous, non-negative additive functional* $\mathfrak{t}(t, w)$ *of* $\boldsymbol{X}$ *uniformly on any compact time interval with* P_x*-probability* 1 *for any* $x \in \bar{D}$, *where* $\{\rho_n\}$ *does not depend on the choice of* $x \in \bar{D}$, *and*

$$(7.6) \qquad \mathfrak{t}_\rho(t, w) = \frac{1}{\rho} \int_0^t \chi_{D_\rho}(x_s(w))\, ds, \quad \rho > 0.$$

$\mathfrak{t}(t, w)$ *satisfies*

$$(7.7) \qquad E_x(\mathfrak{t}(t, w)) = \int_0^t ds \int_{\partial D} p^+(s, x, y) \widetilde{m}(dy), \quad x \in \bar{D}.$$

Such an additive functional $\mathfrak{t}(t, w)$ *is unique up to* P_x*-probability* 0 *for all* x. *Moreover,*

$$(7.8) \qquad P_x(\lim_{t \to \infty} \mathfrak{t}(t, w) = \infty) = 1, \quad \text{if } x \in \bar{D},$$

$$(7.9) \qquad P_x(\mathfrak{t}(t, w) > 0 \textit{ for all } t > 0) = 1, \textit{ if and only if } x \in \partial D,$$

and $\mathfrak{t}(t, w)$ *increases at* t *only when* $x_t(w)$ *is on the boundary.*

$\mathfrak{t}(t, w)$ is called the *local time on the boundary* for the reflecting diffusion $\boldsymbol{X}$.

By an additive functional of $\boldsymbol{X}$, we mean that $\mathfrak{t}(t, w)$ is $\boldsymbol{B}_t$-measurable and that

$$P_x(\mathfrak{t}(t+s, w) = \mathfrak{t}(t, w) + \mathfrak{t}(s, w_t^+) \text{ for all } t, s \geqq 0) = 1, \quad x \in \bar{D},$$

holds.

Proof.[32] Put $e_\rho(t, x) = E_x(\mathfrak{t}_\rho(t, w))$. Then, we have

$$(7.10) \qquad \lim_{\rho, \rho' \downarrow 0} \sup_{x \in \bar{D}} E_x(|\mathfrak{t}_\rho(t, w) - \mathfrak{t}_{\rho'}(t, w)|^2) = 0.$$

In fact, noting that $e_\rho(t, x)$ converges to $\int_0^t ds \int_{\partial D} p^+(s, x, y) \widetilde{m}(dy)$ uniformly in $x \in \bar{D}$ and $0 \leqq t \leqq T$ for any fixed $T > 0$ by Lemma 3. 6, and putting $f(x) = \frac{1}{\rho} \chi_{D_\rho}(x) - \frac{1}{\rho'} \chi_{D_{\rho'}}(x)$, we have

$$E_x(|\mathfrak{t}_\rho(t, w) - \mathfrak{t}_{\rho'}(t, w)|^2) = E_x\left\{\left(\int_0^t f(x_s) ds\right)^2\right\}$$

32) The method is suggested by McKean-Tanaka [21].

$$=2E_x\Big(\int_0^t f(x_s)ds\int_s^t f(x_r)dr\Big)=2E_x\Big\{\int_0^t f(x_s)dsE_{x_s}\Big(\int_0^{t-s} f(x_r)dr\Big)\Big\}$$

$$\leqslant 2E_x\Big(\int_0^t |f(x_s)|ds\Big)\sup_{0\leq s\leq t, y\in\bar{D}}\Big|E_y\Big(\int_0^s f(x_r)dr\Big)\Big|$$

$$\leqslant 4\sup_{\rho>0} e_\rho(t,x)\cdot\sup_{0\leq s\leq t, y\in\bar{D}}|e_\rho(s,y)-e_{\rho'}(s,y)|\to 0,$$

uniformly in $x\in\bar{D}$ as ρ and $\rho'\to 0$. Now, define

$$\mathfrak{l}_\rho(t,w)=E_x(\mathfrak{t}_\rho(T)\,|\,\boldsymbol{B}_t)\quad\text{for } t\leqslant T.$$

Then, we have

$$\text{(7. 11)}\qquad \mathfrak{l}_\rho(t,w)=\mathfrak{t}_\rho(t)+e_\rho(T-t,x_t).$$

Since $(\mathfrak{l}_\rho(t)-\mathfrak{l}_{\rho'}(t),\ \boldsymbol{B}_t,\ 0\leqslant t\leqslant T,\ P_x)$, is a separable martingale for any $x\in\bar{D}$, by an extension of Kolmogorov's inequality due to Doob [3, p. 353] we have

$$\text{(7. 12)}\qquad P_x(\max_{0\leq t\leq T}|\mathfrak{l}_\rho(t)-\mathfrak{l}_{\rho'}(t)|>\varepsilon)\leqslant\varepsilon^{-2}E_x(|\mathfrak{l}_\rho(T)-\mathfrak{l}_{\rho'}(T)|^2)$$
$$=\varepsilon^{-2}E_x(|\mathfrak{t}_\rho(T)-\mathfrak{t}_{\rho'}(T)|^2).$$

Thus, by (7. 10) and (7. 12) and by making use of the Borel-Cantelli Lemma we can find such a subsequence $\{\rho_{n'}\}$ for any sequence $\{\rho_n\downarrow 0\}$ that

$$P_x(\mathfrak{l}_{\rho_{n'}}(t,w)\text{ converges uniformly in } 0\leqslant t\leqslant T,\text{ when } n'\to\infty)=1,\quad x\in\bar{D}.$$

Hence, by (7. 11) and Lemma 3. 6 we have

$$P_x(\mathfrak{t}_{\rho_{n'}}(t,w)\text{ converges uniformly in } 0\leqslant t\leqslant T,\text{ when } n'\to\infty)=1,\quad x\in\bar{D}.$$

Since the convergence in (7. 10) is uniform in x, $\{\rho_{n'}\}$ can be taken independently of $x\in\bar{D}$. Then, by making use of the diagonal method we can prove that there is a sequence $\rho_n\downarrow 0$ such that $P_x(W')=1$ for each $x\in\bar{D}$, where

$$W'=\{w\,|\,\mathfrak{t}_{\rho_n}(t,w)\text{ converges uniformly on any compact time interval, when } n\to\infty\}.$$

Now, define $\mathfrak{t}(t, w)$ by $\mathfrak{t}(t, w)=\lim_{n\to\infty}\mathfrak{t}_{\rho_n}(t, w)$ for $w\in W'$ and $\mathfrak{t}(t, w)\equiv+\infty$ for $w\notin W'$. That $\mathfrak{t}(t, w)$ is a non-negative additive functional is clear by the construction. Continuity follows from the uniform convegence of $\mathfrak{t}_{\rho_n}(t, w)$ to $\mathfrak{t}(t, w)$. By (7. 10) $\mathfrak{t}(t, w)$ is also a limit of $\mathfrak{t}_\rho(t, w)$ in the sense of $L^2(W, P_x)$ and hence in the sense of $L^1(W, P_x)$. This, combined with $e_\rho(t, x)=E_x(\mathfrak{t}_\rho(t, x))\rightarrow\int_0^t ds\int_{\partial D}p^+(s, x, y)\widetilde{m}(dy)$, implies (7. 7). Now, let $x_t(w)$ be in D and $t>0$. Then, by the continuity of path functions there are t_1 and t_2 such that $t_1<t<t_2$ and $\mathfrak{t}_\rho(t_1, w)=\mathfrak{t}_\rho(t_2, w)$ for sufficiently small ρ, and hence $\mathfrak{t}(t_1, w)=\mathfrak{t}(t_2, w)$, implying that $\mathfrak{t}(t, w)$ increases at t only when $x_t(w)$ is on ∂D. (In case $x_t(w)\in D$ and $t=0$, we have only to consider $0=t<t_2$ with $\mathfrak{t}(t_2, w)=0$.) (7. 8) and (7. 9) will be proved later as a corollary to Theorem 9. 1.

Let us prove the uniqueness. Suppose that $\mathfrak{t}(t, w)$ and $\mathfrak{t}'(t, w)$ are both non-negative additive functionals of $\boldsymbol{X}$ satisfying (7. 7). Put $\mathfrak{s}(t, w)=\mathfrak{t}(t, w)-\mathfrak{t}'(t, w)$. Since T_t^+ forms a semigroup on $C(\overline{D})$, the reflecting diffusion has the strong Markov property by [14, p. 60], and the right continuous inverse $\mathfrak{t}^{-1}(t, w)$ of $\mathfrak{t}(t, w)$ is a Markov time. Hence we have

$$E_x\left(\int_0^t\mathfrak{t}(ds)\int_s^t\mathfrak{s}(dr)\right)=E_x\left(\int_0^t\mathfrak{t}(ds)\int_0^{t-s}\mathfrak{s}(dr, w_s^+)\right)$$

$$=E_x\left(\int_0^{\mathfrak{t}(t)}ds\int_0^{t-\mathfrak{t}^{-1}(s)}\mathfrak{s}(dr, w^+_{\mathfrak{t}^{-1}(s)})\right)=E_x\left(\int_0^{\mathfrak{t}(t)}\left[E_{x_{t'}}\left(\int_0^{t-t'}\mathfrak{s}(dr)\right)\right]_{t'=\mathfrak{t}^{-1}(s)}ds\right)=0,$$

by the assumption $E_x(\mathfrak{s}(t, w))=0$. Similarly we have

$$E_x\left(\int_0^t\mathfrak{t}'(ds)\int_s^t\mathfrak{s}(dr)\right)=0,$$

and hence

$$E_x(\mathfrak{s}(t, w)^2)=E_x\left(\int_0^t\int_0^t\mathfrak{s}(ds)\mathfrak{s}(dr)\right)=2E_x\left(\int_0^t\mathfrak{s}(ds)\int_s^t\mathfrak{s}(dr)\right)=0,$$

which implies $P_x(\mathfrak{t}(t, w)=\mathfrak{t}'(t, w)$ for all $t)=1$. The proof of Theorem 7. 2 is complete.

Remark. *For any* t, *we have*

(7.12) $\lim_{\rho\downarrow 0} E_x(|\mathfrak{t}_\rho(t,w)-\mathfrak{t}(t,w)|^2)=0$, *uniformly in* x.

§8. Solutions for some parabolic equations

Lemma 8.1.[33] *If f and b belong to $C^{0,\kappa}(\bar{D})$, then*

$$v(t,x)=E_x\left\{\int_0^t f(x_s)\exp\left(-\int_0^s b(x_r)\,dr\right)ds\right\}$$

is continuously differentiable in $t>0$, belongs to $C^1(\bar{D})\cap C^2(D)$ as a function of x, and satisfies

(8.1) $\left(\frac{\partial}{\partial t}+b(x)-A\right)v(t,x)=f(x),\quad t>0,\ x\in D,$

(8.2) $\frac{\partial v}{\partial n}(t,x)=0,\quad t>0,\ x\ni\partial D,$

(8.3) $\lim_{t\downarrow 0} v(t,x)=0,\quad x\in\bar{D}.$

Proof.

$$\begin{aligned}
v(t,x)-E_x\left(\int_0^t f(x_s)\,ds\right)&=E_x\left(\int_0^t\left\{\exp\left(-\int_0^s b(x_r)\,dr\right)-1\right\}f(x_s)\,ds\right)\\
&=-E_x\left(\int_0^t f(x_s)\,ds\int_0^s b(x_u)\exp\left(-\int_u^s b(x_r)\,dr\right)du\right)\\
&=-E_x\left(\int_0^t b(x_u)\,du\int_u^t f(x_s)\exp\left(-\int_u^s b(x_r)\,dr\right)ds\right)\\
&=-E_x\left\{\int_0^t b(x_u)\,du\,E_{x_u}\left(\int_0^{t-u} f(x_s)\exp\left(-\int_0^s b(x_r)\,dr\right)ds\right)\right\}\\
&=-E_x\left(\int_0^t b(x_s)v(t-s,x_s)\,ds\right).
\end{aligned}$$

Thus, we have

$$v(t,x)=\int_0^t ds\int_{\bar{D}} p^+(s,x,y)\,(f(y)-b(y)v(t-s,y))\,m(dy).$$

Hence, $v(t,x)$ is continuously differentiable in $x\in\bar{D}$ and satisfies (2.9) with u replaced by v, by Lemma 2.2. Therefore,

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33) This is a parabolic version of the theorem of Kac [14, p. 50]. A similar result is obtained by Has'minskii [9, p. 8] independently.

$$|v(t, x)-v(t', x')| \leqslant |v(t, x)-v(t', x)| + |v(t', x)-v(t', x')|$$
$$\leqslant e^{T\|b\|}\cdot \|f\| \cdot |t-t'| + K\sum_{i=1}^{N}|x^i - x'^i|,$$

implying $v \in C^{0,1}([0, T] \times \overline{D})$. Thus, $v(t, x)$ satisfies (8.1)-(8.3) by Theorem 2.2.

Lemma 8.2. *Let f and b be in $C^{0,\kappa}(\overline{D})$ and let φ and β be in $C^{0,\kappa}(\partial D)$ and $\beta \geqslant 0$. Then,*

$$u(t, x) = E_x\left(\int_0^t f(x_s)\exp\left(-\int_0^s b(x_r)\,dr - \int_0^s \beta(x_r)\mathfrak{t}(dr)\right)ds\right) \tag{8.4}$$

is continuously differentiable in $t>0$, belongs to $C^1(\overline{D}) \cap C^2(D)$ as a function of x, and satisfies

$$\left(\frac{\partial}{\partial t} + b(x) - A\right)u(t, x) = f(x), \quad t>0,\ x \in D, \tag{8.5}$$

$$\left(\beta(x) - \frac{\partial}{\partial n}\right)u(t, x) = 0, \quad t>0,\ x \in \partial D, \tag{8.6}$$

$$\lim_{t \downarrow 0} u(t, x) = 0, \quad x \in \overline{D}. \tag{8.7}$$

$$v(t, x) = E_x\left(\int_0^t \varphi(x_s)\exp\left(-\int_0^s b(x_r)\,dr - \int_0^s \beta(x_r)\mathfrak{t}(dr)\right)\mathfrak{t}(ds)\right) \tag{8.4'}$$

is continuously differentiable in $t>0$, belongs to $C^1(\overline{D}) \cap C^2(D)$ as a function of x, and satisfies

$$\left(\frac{\partial}{\partial t} + b(x) - A\right)v(t, x) = 0, \quad t>0,\ x \in D, \tag{8.5'}$$

$$\left(\beta(x) - \frac{\partial}{\partial n}\right)v(t, x) = \varphi(x), \quad t>0,\ x \in \partial D, \tag{8.6'}$$

$$\lim_{t \downarrow 0} v(t, x) = 0, \quad x \in \overline{D}. \tag{8.7'}$$

Proof. Here, we prove the propositions for v alone, since the proof for u is almost the same. Take continuous extensions on $\overline{D}$ of φ and β denoted by $\hat{\varphi}$ and $\hat{\beta}$, respectively. $\hat{\beta}$ is taken to be non-negative. Put

$$(8.8)\qquad v_\rho(t,x)=E_x\left\{\int_0^t\hat{\varphi}(x_s)\exp\left(-\int_0^s b(x_r)\,dr-\int_0^s\hat{\beta}(x_r)\mathfrak{t}(dr)\right)\mathfrak{t}_\rho(ds)\right\},$$

$$(8.9)\qquad v_{\rho,\rho'}(t,x)=E_x\left\{\int_0^t\hat{\varphi}(x_s)\exp\left(-\int_0^s b(x_r)\,dr-\int_0^s\hat{\beta}(x_r)\mathfrak{t}_{\rho'}(dr)\right)\mathfrak{t}_\rho(ds)\right\}.$$

We note that

$$(8.10)\qquad \lim_{n\to\infty}v_{\rho,\rho_n}(t,x)=v_\rho(t,x)\quad(\text{boundedly in } 0\leqslant t\leqslant T,\ x\in\overline{D}),$$

$$(8.11)\qquad \lim_{n\to\infty}v_{\rho_n}(t,x)=v(t,x)\quad(\text{boundedly in } 0\leqslant t\leqslant T,\ x\in\overline{D}),$$

where $\{\rho_n\}$ is the sequence in Theorem 7.2, and T is arbitrary number. In fact, $\int_0^s\hat{\beta}(x_r)\mathfrak{t}_{\rho_n}(dr)\to\int_0^s\hat{\beta}(x_r)\mathfrak{t}(dr)$ and the dominated convergence theorem imply (8.10). Moreover, we have

$$(8.12)\qquad \begin{aligned}&\int_0^t\hat{\varphi}(x_s)\exp\left(-\int_0^s b(x_r)\,dr-\int_0^s\hat{\beta}(x_r)\mathfrak{t}(dr)\right)\mathfrak{t}_{\rho_n}(ds)\\ &\to\int_0^t\hat{\varphi}(x_s)\exp\left(-\int_0^s b(x_r)\,dr-\int_0^s\hat{\beta}(x_r)\mathfrak{t}(dr)\right)\mathfrak{t}(ds),\end{aligned}$$

when $n\to\infty$. Since $\mathfrak{t}_{\rho_n}(t)\to\mathfrak{t}(t)$ in $L_1(W,P_x)$, $\{\mathfrak{t}_{\rho_n}\}$ are uniformly integrable in the sense of Doob [3]. The left hand side of (8.12) is bounded by $\|\hat{\varphi}\|e^{t\|b\|}\mathfrak{t}_{\rho_n}(t)$, and hence, uniformly integrable in n. Therefore, (8.11) follows from (8.12). The convergences in (8.10) and (8.11) are bounded, because both $v_{\rho,\rho'}(t,x)$ and $v_\rho(t,x)$ are bounded by $\|\hat{\varphi}\|e^{T\|b\|}\cdot E_x(\mathfrak{t}_\rho(t,w))=\|\hat{\varphi}\|e^{T\|b\|}e_\rho(T,x)$ for any $x\in\overline{D}$ and $0\leqslant t\leqslant T$.

Now, fix ρ and ρ', and take sequences $\{g_m\}$ and $\{h_m\}$ in $\bigcup_{\kappa>0}C^{0,\kappa}(\overline{D})$ converging boundedly to $\hat{\varphi}(x)\frac{1}{\rho}\chi_{D_\rho}(x)$ and $b(x)+\hat{\beta}(x)\frac{1}{\rho'}\chi_{D_{\rho'}}(x)$, respectively. By Lemma 8.1

$$u_m(t,x)=E_x\left\{\int_0^t g_m(x_s)\exp\left(-\int_0^s h_m(x_r)\,dr\right)ds\right\}$$

satisfies $\left(\frac{\partial}{\partial t}+h_m-A\right)u_m=g_m$ on $\overline{D}$, $\frac{\partial u_m}{\partial n}=0$ on ∂D, and $\lim_{t\downarrow 0}u_m=0$.

Hence, making use of the fundamental solution $p(t, x, y)$ of the equation $\frac{\partial u}{\partial t}=Au$ with boundary condition $\left(\beta-\frac{\partial}{\partial n}\right)u=0$, we have, by Theorem 2.2,

$$u_m(t, x)=\int_0^t ds\int_{\bar{D}} p(s, x, y)\{g_m(y)-h_m(y)u_m(t-s, y)\}m(dy)$$
$$+\int_0^t ds\int_{\partial D} p(s, x, y)\beta(y)u_m(t-s, y)\tilde{m}(dy).$$

Since u_m converges boundedly to $v_{\rho,\rho'}$ when $m\to\infty$, it follows that

$$v_{\rho,\rho'}(t, x)=\int_0^t ds\int_{\bar{D}} p(s, x, y)\Big\{\frac{1}{\rho}\chi_{D_\rho}(y)\hat{\varphi}(y)$$
$$(8.13)\quad -b(y)v_{\rho,\rho'}(t-s, y)-\frac{1}{\rho'}\chi_{D_{\rho'}}(y)\hat{\beta}(y)v_{\rho,\rho'}(t-s, y)\Big\}m(dy)$$
$$+\int_0^t ds\int_{\partial D} p(s, x, y)\beta(y)v_{\rho,\rho'}(t-s, y)\tilde{m}(dy).$$

To apply Lemma 3.4 to $\{f_{\rho'}(x)=\hat{\beta}(x)v_{\rho,\rho'}(s, x)\}$ we have, by (8.13) above,

$$\sup_{x\in V(\rho', r_j, x_j; \bar{\sigma}_j)}|f_{\rho'}(x)-f_{\rho'}(P_j(x))|\leqq \text{const.}\sup_{x\in V(\rho', r_j, x_j; \bar{\sigma}_j)}$$
$$\Big\{\int_0^t ds\int_{\bar{D}}|p(s, x, y)-p(s, P_j(x), y)|m(dy)$$
$$+\frac{1}{\rho'}\int_0^t ds\int_{D_{\rho'}}|p(s, x, y)-p(s, P_j(x), y)|m(dy)$$
$$+\int_0^t ds\int_{\partial D}|p(s, x, y)-p(s, P_j(x), y)|\tilde{m}(dy)\Big\},$$

which converges to 0 when $\rho'\downarrow 0$ by Lemmas 2.1, 3.3 and 3.5. Hence, letting $\rho'=\rho_n\downarrow 0$ in (8.13), we obtain

$$v_\rho(t, x)=\int_0^t ds\int_{\bar{D}} p(s, x, y)\Big(\frac{1}{\rho}\chi_{D_\rho}(y)\hat{\varphi}(y)-b(y)v_\rho(t-s, y)\Big)m(dy)$$

in virtue of Lemma 3.4. Thus, we have, by letting $\rho=\rho_n\downarrow 0$,

$$v(t, x)=\int_0^t ds\int_{\partial D} p(s, x, y)\varphi(y)\tilde{m}(dy)$$
$$-\int_0^t ds\int_{\bar{D}} p(s, x, y)b(y)v(t-s, y)m(dy),$$

implying that $v \in C^{0,1}((0, T) \times \bar{D})$ by Theorem 2.2 and Lemma 2.2, and that v satisfies (8.5')-(8.7') by Theorem 2.2.

§9. Markov processes on the boundary and some modifications of the reflecting diffusion

Now, we construct Markov processes on the boundary corresponding to the semigroups on $C(\partial D)$ in §6 in some special cases. In a similar way we can obtain some modifications of the reflecting diffusion.

Let $\mathfrak{a}(t, w)$ be a non-negative continuous additive functional of $\boldsymbol{X}$, and let $\mathfrak{a}^{-1}(t, w)$ be the right continuous inverse of $\mathfrak{a}(t, w)$, that is,

$$\mathfrak{a}^{-1}(t, w) = \sup\{s \mid \mathfrak{a}(s, w) \leqslant t\}.$$

Noting that $P_x(\mathfrak{a}(t, w) > 0 \text{ for all } t > 0) = 0$ or 1 for each $x \in \bar{D}$, we denote by S^* the set of all such x that the above probability is 1. Then, it can be proved that

$$P_x(x_{\mathfrak{a}^{-1}(t,w)}(w) \text{ takes values in } S^* \cup \{\Delta\}, \text{ is right continuous and has left limits as a function of } t \in [0, \mathfrak{a}(\infty, w))) = 1, \quad x \in \bar{D}.$$

Suppose that S^* is measurable. We define W^* to be the set of all functions w^* of $[0, +\infty]$, taking values in $S^* \cup \{\Delta\}$ and satisfying the following conditions:

1. There is such a $\zeta^*(w^*) \in [0, +\infty]$ that $w^*(t) \in S^*$ for $0 \leqslant t < \zeta^*(w^*)$ and $w^*(t) = \Delta$ for $\zeta^*(w^*) \leqslant t \leqslant +\infty$.
2. $w^*(t)$ is right continuous and has left limits for each $t \in [0, \zeta^*(w^*))$.

We write $x_t^*(w^*) = w^*(t)$. Let $\boldsymbol{B}_t^*$ and $\boldsymbol{B}^*$ be the smallest Borel fields of subsets of W^*, which make $\{x_s^*; s \in [0, t]\}$ and $\{x_s^* : s \in [0, +\infty]\}$ measurable, respectively. We define P_x^* on $(W^*, \boldsymbol{B}^*)$ by

$$P_x^*(B) = P_x(x_{\mathfrak{a}^{-1}(t,w)}(w) \text{ belongs to } B, \text{ as a function of } t)$$

for $B \in \boldsymbol{B}^*$. Then, the system $\boldsymbol{X}^* = (x_t^*, W^*, \boldsymbol{B}_t^*, P_x^*; x \in S^* \cup \{\Delta\})$ is a Markov process and has the strong Markov property. The transition operator of $\boldsymbol{X}^*$ and the Green operator are determined by

$$(9.1)\quad \begin{aligned} T_t^* f(x) &= E_x^*(f(x_t^*)) = E_x(f(x_{\mathfrak{a}^{-1}(t)})\chi_{\{t<\mathfrak{a}(\infty)\}}), \\ K_\lambda^* f(x) &= \int_0^\infty e^{-\lambda t} T_t^* f(x)\,dt = E_x\Big(\int_0^\infty e^{-\lambda \mathfrak{a}(s)} f(x_s)\mathfrak{a}(ds)\Big), \end{aligned}$$

where f is bounded measurable on S^*. The righthand sides of the above equalities have meaning, because for any measurable extension on $\bar{D}$ of f the righthand sides exist and they depend only on f, not on the choice of the extension. We call $\boldsymbol{X}^*$ *the Markov process obtained from* $\boldsymbol{X}$ *through time change by* $\mathfrak{a}(t)$.

Suppose that another non-negative continuons additive functional $\mathfrak{b}(t, w)$ of $\boldsymbol{X}$ is given. Let $P(\cdot)$ be a probability measure on $[0, +\infty]$ with density e^{-t}, and let $\bar{P}_x$ be the product measure of P_x and P on the space $\Omega = W\times[0, +\infty]$. Define $\bar{x}_t(\omega)$ on Ω for $t\in[0\ +\infty]$ by

$$\begin{aligned} \bar{x}_t(\omega) &= x_{\mathfrak{a}^{-1}(t,w)}(w), && \text{if } \mathfrak{b}(\mathfrak{a}^{-1}(t, w), w) < T, \\ &= \Delta, && \text{if otherwise} \\ \bar{\zeta}(\omega) &= \inf\{t\,|\,\bar{x}_t(\omega) = \Delta\}, && \text{where } \omega = (w, T)\in\Omega. \end{aligned}$$

We have

$$\bar{P}_x(\bar{x}_t(\omega) \text{ takes values in } S^*\cup\{\Delta\},\text{ is right continuous and has left limits as a function of } t\in[0,\bar{\zeta}(\omega))) = 1,\ x\in\bar{D},$$

and define a measure $P_x^\sharp$ on $(W^*, \boldsymbol{B}^*)$ by

$$P_x^\sharp(B) = \bar{P}_x(\bar{x}_t(\omega) \text{ belongs to } B \text{ as a function of } t)$$

for $B\in\boldsymbol{B}^*$. Then, the system $X^\sharp = (x_t^*, W^*, \boldsymbol{B}_t^*, P_x^\sharp, x\in S^*\cup\{\Delta\})$ is also a Markov process with the strong Markov property, called *the Markov process obtained from* $\boldsymbol{X}$ *through time change by* $\mathfrak{a}(t)$ *and killing by* $\mathfrak{b}(t)$. Its transition operator and Green operator are

$$(9.2)\quad \begin{aligned} T_t^\sharp f(x) &= E_x^\sharp(f(x_t^*)) = E_x(f(x_{\mathfrak{a}^{-1}(t)})e^{-\mathfrak{b}(\mathfrak{a}^{-1}(t))}\chi_{\{t<\mathfrak{a}(\infty)\}}) \\ K_\lambda^\sharp f(x) &= \int_0^\infty e^{-\lambda t} T_t^\sharp f(x)\,dt = E_x\Big(\int_0^\infty e^{-\lambda\mathfrak{a}(s)-\mathfrak{b}(s)} f(x_s)\mathfrak{a}(ds)\Big). \end{aligned}$$

The proof of the above results is referred mainly to Volkonskiĭ [34, §1].

Theorem 9.1. *Let* $\mathbf{X}^*$ *be the Markov process obtained from the reflecting diffusion* $\mathbf{X}$ *through time change by the local time on the boundary* $\mathfrak{t}(t, w)$. *Then,* $\mathbf{X}^*$ *is conservative, has state space* $S^* = \partial D$, *and* T_t^* *is a semigroup on* $C(\partial D)$ *with generator* $\overline{\frac{\partial}{\partial n} H_0}$.[34]

More generally, we can prove the following.

Theorem 9.1′. *For* β *and* γ *in* $C^{2,\kappa}(\partial D)$ *satisfying* $\beta > 0$ *and* $\gamma \leqslant 0$, *put*

$$\begin{aligned} \mathfrak{a}(t, w) &= \int_0^t \beta(x_s(w))\mathfrak{t}(ds, w), \\ \text{(9.3)} \qquad \mathfrak{b}(t, w) &= \alpha t + \int_0^t |\gamma(x_s(w))| \beta(x_s(w))\mathfrak{t}(ds, w), \end{aligned}$$

where α *is a non-negative number. Then, the Markov process* $\mathbf{X}^\sharp$ *obtained from* $\mathbf{X}$ *through time change by* $\mathfrak{a}(t)$ *and killing by* $\mathfrak{b}(t)$ *has the state space* $S^* = \partial D$ *and* $T_t^\sharp$ *forms a semigroup on* $C(\partial D)$ *with generator* $\overline{\frac{1}{\beta}\frac{\partial}{\partial n} H_\alpha} + \gamma$.

Proof. Since $\mathfrak{t}(t, w)$ and, consequently, $\mathfrak{a}(t, w)$ increase at t only when $x_t(w)$ is on ∂D, S^* is contained in ∂D. Hence we can define, for $\varphi \in C^{0,\kappa}(\partial D)$,

$$\widetilde{K}_\lambda \varphi(x) = E_x\left(\int_0^\infty e^{-\lambda \mathfrak{a}(s) - \mathfrak{b}(s)} \varphi(x_s) \mathfrak{a}(ds)\right).$$

Since $v(t, x) = E_x\left(\int_0^t e^{-\lambda \mathfrak{a}(s) - \mathfrak{b}(s)} \varphi(x_s) \mathfrak{a}(ds)\right)$ satisfies

$$\left(\frac{\partial}{\partial t} - A + \alpha\right) v(t, x) = 0,$$

$$\left(\lambda - \frac{1}{\beta}\frac{\partial}{\partial n} - \gamma\right) v(t, x) = \varphi(x),$$

$$\lim_{t \downarrow 0} v(t, x) = 0$$

by Lemma 8.2, we can see, by Corollary to Theorem 2.3, that

34) $\partial/\partial n$ is understood to be an operator with domain $C^1(\overline{D})$.

$\widetilde{K}_\lambda\varphi(x)=\lim_{t\to\infty}v(t,x)=u(x)$ belongs to $C^1(\overline{D})\cap C^2(D)$ and satisfies

$$(A-\alpha)u(x)=0, \qquad x\in D,$$
$$\left(\lambda-\frac{1}{\beta}\frac{\partial}{\partial n}-\gamma\right)u(x)=\varphi(x), \qquad x\in\partial D.$$

Thus, by Theorem 5.1, $\overline{\left(\frac{1}{\beta}\frac{\partial}{\partial n}+\gamma\right)H_\alpha}=\frac{1}{\beta}\overline{\frac{\partial}{\partial n}H_\alpha}+\gamma$ generates a semi-group $\widetilde{T}_t$ on $C(\partial D)$. Since $\int_0^\infty e^{-\alpha t}\widetilde{T}_t\varphi(x)dt=\widetilde{K}_\lambda f(x)=\int_0^\infty e^{-\alpha t}$ $T_t^\sharp\varphi(x)dt$, we have $\widetilde{T}_t\varphi(x)=T_t^\sharp\varphi(x)$ for all $\varphi\in C^{0,\kappa}(\partial D)$. Since $\widetilde{T}_t$ is strongly continuous in $t\geqslant 0$, we have $P_x(x_{\mathfrak{a}^{-1}(0)}=x)=1$ for each $x\in\partial D$. Therefore, we have, by the strong Markov property, $P_x(\mathfrak{a}^{-1}(0)=0)=E_x(P_{x_{\mathfrak{a}^{-1}(0)}}(\mathfrak{a}^{-1}(0)=0))=P_x(\mathfrak{a}^{-1}(0,w^+_{\mathfrak{a}^{-1}(0,w)})=0)=P_x(\mathfrak{a}^{-1}$ $(0,w)<\infty)=1$ for each $x\in\partial D$. Thus, $S^*=\partial D$ and $\widetilde{T}_t=T_t^\sharp$, completing the proof of Theorem 9.1′. Conservativity of $\boldsymbol{X}^*$ in Theorem 9.1 follows from that the domain of $\overline{\frac{\partial}{\partial n}H_0}$ contains the constant function 1 and $\overline{\frac{\partial}{\partial n}H_0}1=0$, and hence $T_t^*1(x)-1=\int_0^t T_s^*\overline{\frac{\partial}{\partial n}H_0}1(x)ds$ $=0$.

As a special case where $\beta=1$ and $\gamma=0$, we have obtained a system of *Markov processes on the boundary ∂D with generator* $\overline{\frac{\partial}{\partial n}H_\alpha}$. Thus we have established a justification of the interpretation of $\overline{LH_\alpha}$ in the introduction in the case of the reflecting diffusion.

Corollary.[35] $P_x(\lim_{t\to\infty}\mathfrak{t}(t,w)=\infty)=1$ *if* $x\in\overline{D}$.

$P_x(\mathfrak{t}(t,w)>0$ *for all* $t>0)=1$ *if and only if* $x\in\partial D$.

Proof. From conservativity of $\boldsymbol{X}^*$ we have $P_x(\lim_{t\to\infty}\mathfrak{t}(t,w)=\infty)$ $=1$ for each $x\in\partial D$. The strong Markov property implies this for $x\in\overline{D}$, since $P_x(\lim_{t\to\infty}\mathfrak{t}(t,w)=\infty)=E_x(P_{x_\sigma}(\lim_{t\to\infty}\mathfrak{t}(t,w)=\infty))=1$ where σ is the first hitting time to ∂D. The second assertion has already been proved.

Note. Replacing $(\alpha-A)u(x)=0$ by $(A+c(x))u(x)=0$ in

35) Cf. Theorem 7.2, (7.8) and (7.9).

Theorem 2.5, we obtain a non-negative bounded linear operator $H_{|c|}$ instead of H_α, where $0 \geqslant c \in C^{0,\kappa}(\bar{D})$. We can define $\frac{\partial}{\partial n} H_{|c|}$ and prove properties of $\frac{\partial}{\partial n} H_{|c|}$ similar to those of $\frac{\partial}{\partial n} H_\alpha$. For instance, Markov process on the boundary corresponding to $\overline{\frac{1}{\beta} \frac{\partial}{\partial n} H_{|c|}} + \gamma$ is obtained by putting, in (9.3),

$$\mathfrak{b}(t, w) = \int_0^t |c(x_s(w))| ds + \int_0^t |\gamma(x_s(w))| \beta(x_s(w)) \mathfrak{t}(ds, w).$$

An alternative proof of Theorem 9.1. H. Tanaka (private communication) suggested that Theorem 9.1 can be proved essentially by Theorem 2.3 and a property of additive functionals [25, Theorem 4.1]. Here is a proof. First, we have for $\alpha > 0$,

$$E_x\left(\int_0^\infty e^{-\alpha t} \mathfrak{t}(dt)\right) = \lim_{n \to \infty} E_x\left(\int_0^\infty e^{-\alpha([nt]+1)/n} \mathfrak{t}(dt)\right)$$
$$= \lim_{n \to \infty} \sum_{k=1}^\infty E_x\left(\int_{(k-1)/n}^{k/n} e^{-\alpha k/n} \mathfrak{t}(dt)\right)$$
$$= \lim_{n \to \infty} \sum_{k=1}^\infty e^{-\alpha k/n} \int_{(k-1)/n}^{k/n} dt \int_{\partial D} p^+(t, x, y) \tilde{m}(dy),$$
$$= \int_0^\infty e^{-\alpha t} dt \int_{\partial D} p^+(t, x, y) \tilde{m}(dy) = \int_{\partial D} g_\alpha^+(x, y) \tilde{m}(dy),$$

where
$$g_\alpha^+(x, y) = \int_0^\infty e^{-\alpha t} p^+(t, x, y) dt.$$

Hence, by [25, Theorem 4.1],

$$E_x\left(\int_0^\infty e^{-\alpha t} \varphi(x_t) \mathfrak{t}(dt)\right) = \int_{\partial D} g_\alpha^+(x, y) \varphi(y) \tilde{m}(dy), \quad x \in \bar{D}, \tag{9.4}$$

holds for all $\varphi \in C(\partial D)$. The condition on $g_\alpha^+(x, y)$ for the use of [25, Theorem 4.1] is easily checked, for $g_\alpha^+(x, y)$ is continuous in $\bar{D} \times \bar{D} - \{(x, y) \mid x = y\}$. Let $\{K_\lambda^\alpha, \lambda \geqslant 0\}$ be the Green operator for the semigroup on $C(\partial D)$ generated by $\overline{\frac{\partial}{\partial n} H_\alpha}$, and let $p_\lambda^+(t, x, y)$ be the fundamental solution of the Cauchy problem for the equation $\frac{\partial u}{\partial t} = Au$ with boundary condition $\left(\lambda - \frac{\partial}{\partial n}\right) u = 0$. Then, $e^{-\alpha t} p_\lambda^+(t, x, y)$

is the fundamental solution of the Cauchy problem for $\partial u/\partial t=(A-\alpha)u$ and $\int_0^\infty e^{-\alpha t}p_\lambda^\dagger(t, x, y)dt$ is the Green functon for $(\alpha-A)u=f$ with boundary condition unchanged. Hence,

$$K_\lambda^\alpha\varphi(x)=\int_0^\infty e^{-\alpha t}dt\int p_\lambda^\dagger(t, x, y)\varphi(y)\widetilde{m}(dy)$$

by virtue of Theorem 2. 3. The right side of (9. 4) is just $K_0^\alpha\varphi(x)$, if $x\in\partial D$. Put

$$\widetilde{K}_\lambda^\alpha\varphi(x)=E_x\left(\int_0^\infty e^{-\alpha t-\lambda\mathfrak{t}(t)}\varphi(x_t)\mathfrak{t}(dt)\right), \qquad x\in\overline{D}.$$

Then, (9. 4) means $\widetilde{K}_0^\alpha\varphi=K_0^\alpha\varphi$ on ∂D, and hence, $\widetilde{K}_\lambda^\alpha\varphi=K_\lambda^\alpha\varphi$ on ∂D for all $\lambda\geqslant 0$ by the force of the resolvent equations:

$$\widetilde{K}_\lambda^\alpha-\widetilde{K}_\mu^\alpha+(\lambda-\mu)\widetilde{K}_\lambda^\alpha\widetilde{K}_\mu^\alpha=0 \quad \text{and} \quad K_\lambda^\alpha-K_\mu^\alpha+(\lambda-\mu)K_\lambda^\alpha K_\mu^\alpha=0.$$

Letting $\alpha\to 0$, we obtain $\widetilde{K}_\lambda^0\varphi=K_\lambda^0\varphi$ on ∂D, from which Theorem 9. 1 follows.

Now, as in the case of one dimensional diffusion [16], we construct some diffusions modifying the reflecting diffusion $\boldsymbol{X}$ by making use of the local time on the boundary.

Theorem 9. 2. *For b in $C^{2,\kappa}(\overline{D})$ and c in $C^{0,\kappa}(\overline{D})$ and γ and δ in $C^{2,\kappa}(\partial D)$ satisfying $b>0$, $c\leqslant 0$, $\gamma\leqslant 0$, and $\delta\leqslant 0$, put*

$$\mathfrak{a}(t, w)=\int_0^t b(x_s(w))ds+\int_0^t|\delta(x_s(w))|\mathfrak{t}(ds, w),$$

$$\mathfrak{b}(t, w)=\int_0^t|c(x_s(w))|b(x_s(w))ds+\int_0^t|\gamma(x_s(w))|\mathfrak{t}(ds, w).$$

Then, the Markov process $\boldsymbol{X}^\sharp$ obtained from $\boldsymbol{X}$ through time change by $\mathfrak{a}$ and killing by $\mathfrak{b}$ has the state space $\overline{D}$ and $\{T_t^\sharp\}$ is a semi-group on $C(\overline{D})$, whose generator is the contraction of $\overline{A}_1$ by the lateral condition $\widehat{L}_1u=0$, where $A_1=\frac{1}{b}A+c$ and

$$L_1u(x)=\gamma(x)u(x)+\delta(x)\lim_{y\in D,\, y\to x}A_1u(y)+\frac{\partial}{\partial n}u(x).^{36)}$$

36) $\partial/\partial n$ is the normal derivative associated with A (not with A_1). We put $\mathfrak{D}(L_1)=\{u\mid u\in C^1(\overline{D})\cap C^2(D)$ and $A_1u(x)$ is extended to be continuous on $\overline{D}\}$.

Proof. Since $\mathfrak{a}(t, w)$ is strictly increasing, the state space S^* of $X^\sharp$ is clearly $\overline{D}$. Let $f \in C^{0,\kappa}(\overline{D})$. Since

$$v(t, x) = E_x\left(\int_0^\infty f(x_s)\exp(-\lambda\mathfrak{a}(s) - \mathfrak{b}(s))\mathfrak{a}(ds)\right)$$

satisfies

$$\left(\frac{\partial}{\partial t} + \lambda b - A - cb\right)v = fb \qquad \text{on } D,$$

$$\left(\gamma + \lambda\delta + \frac{\partial}{\partial n}\right)v = f\delta \qquad \text{on } \partial D$$

by Lemma 8.2, $\lim_{t\to\infty} v(t, x) = u(x)$ exists, belongs to $\mathfrak{D}(L_1)$ and satisfies $(\lambda - A_1)u = f$ on D, $L_1 u = 0$ on ∂D, by the corollary to Theorem 2.3. On the other hand, $u = K_\lambda^\sharp f$ in view of (9.2). Moreover, u belongs to $\mathfrak{D}(\widehat{L}_1) \subset \mathfrak{D}(\overline{A}_1)$ and satisfies $(\lambda - \overline{A}_1)u = f$ and $\widehat{L}_1 u = 0$. In fact, $u = G_{\alpha,1}^{\min} f + H_{\alpha,1}[u]_{\partial D}$, where $G_{\alpha,1}^{\min}$ and $H_{\alpha,1}$ correspond to A_1. Since $G_{\alpha,1}^{\min} f \in C^2(\overline{D}) \subset \mathfrak{D}(L_1)$ by Theorem 2.4, $H_{\alpha,1}[u]_{\partial D}$ belongs to $\mathfrak{D}(L_1)$, and hence $u \in \mathfrak{D}(\widehat{L}_1)$ and $\widehat{L}_1 u = L_1 G_{\alpha,1}^{\min} f + L_1 H_{\alpha,1}[u]_{\partial D} = L_1 u$. Next, note that there is a semigroup on $C(\overline{D})$ with generator $\overline{A}_1$ restricted to $\{u \in \mathfrak{D}(\widehat{L}_1) \mid \widehat{L}_1 u = 0\}$ by Theorems 5.1 and 5.2, since the equation $(\alpha - A_1)u = 0$, $L_1 u = \varphi$ is equivalent to $(A + cb - \alpha b)u = 0$, $\left(\gamma + \alpha\delta + \frac{\partial}{\partial n}\right)u = \varphi$. Then, the Green operator of this semigroup coincides with $K_\lambda^\sharp$, and the proof is complete.

Comments on the general case

The definition of the local time on the boundary in §7 is based essentially on the special case, while the following method will be useful in general.

Consider the diffusion $x_t(w)$ on $\overline{D}$ corresponding to a semigroup $\{T_t, t \geqslant 0\}$ on $C(\overline{D})$ with Green operator $\{G_\alpha, \alpha > 0\}$. Let u be a function in $C(\overline{D})$ non-negative and not identically zero. Then,

$$u_\alpha = G_\alpha u - G_\alpha^{\min} u = H_\alpha[G_\alpha u]_{\partial D}$$

is uniformly α-excessive relative to x_t, that is, non-negative, $e^{-\alpha t} T_t u_\alpha$

$\leqslant u_\alpha$, and $e^{-\alpha t}T_t u_\alpha(x)$ converges uniformly to $u_\alpha(x)$ when $t\to 0$. Hence, there is a unique non-negative, continuous additive functional $\mathfrak{t}_u^\alpha(t, w)$ of x_t such that

$$E_x\left(\int_0^\infty e^{-\alpha t}\mathfrak{t}_u^\alpha(dt, w)\right)=u_\alpha(x).$$

It can be proved that $\mathfrak{t}_u^\alpha(t, w)$ increases only when $x_t(w)$ is on the boundary. The Markov process $y_t(w)$ obtained from $x_t(w)$ through time change by $\mathfrak{t}_u^\alpha(t, w)$, that is, $y_t(w)=x_{(\mathfrak{t}_u^\alpha)^{-1}(t,w)}(w)$, would share the essential character with the Markov process x_t^* on the boundary having generator $\overline{LH_0}$, where L is the operator of boundary condition. Namely, x_t^* is expected to be obtained from y_t through time change by an additive functional of the type $\mathfrak{a}(t)=\int_0^t \varphi(y_s)ds$, where φ is a function on ∂D, and conversely.

In the case of the reflecting diffusion, the local time on the boundary $\mathfrak{t}(t, w)$ defined in §7 is connected with $\mathfrak{t}_u^\alpha(t, w)$ by the relation

$$\mathfrak{t}(t, w)=\int_0^t \frac{1}{\psi(x_s(w))}\mathfrak{t}_u^\alpha(ds, w),$$

if $u\in C^{0,\kappa}(\overline{D})$. Here, $\psi=\frac{\partial}{\partial n}G_\alpha^{\min}u>0$. The proof is as follows. Put $\psi'=\frac{\partial}{\partial n}G_\alpha^{\min}u$, noting that $G_\alpha^{\min}u\in C^1(\overline{D})$ by Theorem 2.4. ψ' is positive everywhere on ∂D by the footnote 16) to Theorem 2.5, iv). Since u_α belongs to $C^1(\overline{D})\cap C^2(D)$ and satisfies $(\alpha-A)u_\alpha=0$ on D and $-\frac{\partial}{\partial n}u_\alpha=\psi'$, we have, by Theorem 2.3, $u_\alpha(x)=\int_{\partial D}g_\alpha^+(x, y)$ $\psi'(y)\tilde{m}(dy)$, from which

$$u_\alpha(x)=E_x\left(\int_0^\infty e^{-\alpha t}\psi'(x_t)\mathfrak{t}(dt)\right)$$

follows (see the second proof of Theorem 9.1). This proves the connection between $\mathfrak{t}(t)$ and $\mathfrak{t}_u^\alpha(t)$ stated above, and $\psi=\psi'$.

Returning to the general case, $\mathfrak{t}_u^\alpha(t, w)$ depends on the choice of α and u. But, if we take another pair α' and u' and suppose that u and u' are positive, then, there is a function φ on ∂D such that

$$\mathfrak{t}_u^{\alpha}(t, w) = \int_0^t \varphi(x_s(w))\mathfrak{t}_{u'}^{\alpha'}(ds, w)$$

holds, and conversely. φ is a positive function on ∂D, bounded and boundedly away from zero. For the proof, note a result of Motoo [24, Proposition 6. 17.] and the fact that, if we have $0 < c_1 \leqslant u \leqslant c_2$ by constant c_1 and c_2, then $\mathfrak{t}_{c_1}^{\alpha}(t) \leqslant \mathfrak{t}_u^{\alpha}(t) \leqslant \mathfrak{t}_{c_2}^{\alpha}(t) = \frac{c_2}{c_1}\mathfrak{t}_{c_1}^{\alpha}(t)$. Dependence of the process $y_t(w)$ above on the choice of α and u corresponds to the situation that L is not uniquely determined by the diffusion. In fact, the boundary conditions $Lu=0$ and $L'u=0$ coincides if $L'=\varepsilon(x)L$, where $\varepsilon(x)$ is a positive function on ∂D.

Motoo has proved in [24] that (7. 9) holds if u is positive, and that (7. 8) holds if u is positive and x_t is conservative, where $\mathfrak{t}(t, w)$ in (7. 8) and (7. 9) is replaced by $\mathfrak{t}_u^{\alpha}(t, w)$.

Appendix. Proof of lemmas concerning fundamental solutions

In order to prove Lemmas 2. 1, 2. 2, and 3. 5, we state the method of construction of the fundamental solution $p(t, x, y)$ of the Cauchy problem for parabolic equation (2. 2) with boundary condition (2. 3) according to S. Ito [17, 18] with slight modifications. We have introduced in §3 the definition of canonical coordinate neighbourhood U of a point on the boundary ∂D and canonical coordinate system in U. For a point x_0 in D, any coordinate neighbourhood U not reaching ∂D and any coordinate system in U are called canonical. We can choose a finite number of canonical coordinate neighbourhoods U_i, $1 \leqslant i \leqslant M$, open[37] subsets B_{ij}, $1 \leqslant j \leqslant M_i$, of U_i and non-negative functions λ_{ij} in $C^{2,\kappa}(\overline{D})$ with supports contained in B_{ij}, satisfying the following conditions: $\{B_{ij}, 1 \leqslant j \leqslant M_i, 1 \leqslant i \leqslant M\}$ is a covering of $\overline{D}$; if B_{ij} intersects $B_{i'j'}$, then $\overline{B}_{i'j'} \subset U_i$; $\sum_{i,j}\lambda_{ij}(x)^2$

37) In the topology on $\overline{D}$.

$=1$ for $x\in\overline{D}$; $\frac{\partial\lambda_{ij}}{\partial n}(x)=0$ for $x\in\partial D$. Suppose that U_i contains boundary points for $1\leqslant i\leqslant M'$, while U_i $(M'+1\leqslant i\leqslant M)$ does not. Let $\sigma_i(x)=(x_{(i)}^1,\cdots,x_{(i)}^N)$ be canonical coordinate system in U_i, and $a^{(i)}$ and $a_{kl}^{(i)}$ be the values in σ_i of a and a_{kl}, respectively. For $1\leqslant i\leqslant M'$, define

$$r_i(t,x,y)=\frac{1}{\sqrt{a^{(i)}(y)}}\left(\frac{a^{(i)}(y)}{4\pi t}\right)^{N/2}\exp\Big(-(4t)^{-1}\sum_{k,l=1}^{N}a_{k,l}^{(i)}(y)$$
$$\times(y_{(i)}^k-x_{(i)}^k)(y_{(i)}^l-x_{(i)}^l)\Big)$$

for $x,y\in U_i$ and $r_i^*(t,x,y)$ by replacing $x_{(i)}^k$ and $x_{(i)}^l$ by $x_{(i)}^{k*}$ and $x_{(i)}^{l*}$, respectively, where $(x_{(i)}^{1*},\cdots,x_{(i)}^{N-1*},x_{(i)}^{*})=(x_{(i)}^1,\cdots,x_{(i)}^{N-1},-x_{(i)}^N)$, and,

$$\alpha_i(t,x,y)=\frac{2|\gamma(P_i(x))|}{1+|\gamma(P_i(x))|}t$$

$$\beta_i(t,x,y)=1+\frac{|\gamma(P_i(x))|}{1+|\gamma(P_i(x))|}t^{1/2}[1-\exp(-t^{-1/2}x_{(i)}^N)]$$
$$\times\left[1+|\gamma(P_i(x))|-\sum_{k=1}^{N}a_{1k}^{(i)}(y)(y_{(i)}^k-x_{(i)}^k)\right]$$

$$q_i(t,x,y)=\alpha_i(t,x,y)(r_i(t,x,y)-r_i^*(t,x,y))$$
$$+\beta_i(t,x,y)(r_i(t,x,y)+r_i^*(t,x,y))^{38)}$$

where $P_i(x)=\sigma_i^{-1}(x_{(i)}^1,\cdots,x_{(i)}^{N-1},0)$. For $M'+1\leqslant i\leqslant M$ we define $r_i(t,x,y)$ by the same formula and put

$$q_i(t,x,y)=r_i(t,x,y).$$

Further, we define

$$q(t,x,y)=\sum_{i=1}^{M}\sum_{j=1}^{M_i}\lambda_{ij}(x)q_i(t,x,y)\lambda_{ij}(y).$$

The fundamental solution $p(t,x,y)$ is constructed from $q(t,x,y)$ by

38) The construction in [18] contains a fault, which is corrected by S. Ito in his lectures at University of Tokyo and Rikkyo University. The definition of $q_i(t,x,y)$ here is slightly modified further.

(a. 1) $p(t, x, y)=q(t, x, y)+\int_0^t ds\int_{\bar{D}} q(t-s, x, z)f(s, z, y)m(dz),$

where f is the solution of the integral equation

$$f(t, x, y)=\left(A_x-\frac{\partial}{\partial t}\right)q(t, x, y)$$
$$+\int_0^t ds\int_{\bar{D}}\left(A_x-\frac{\partial}{\partial t}\right)q(t-s, x, z)f(s, z, y)m(dz)^{39)}$$

and is obtained by successive approximation as follows:

(a. 2) $e_0(t, x, y)=\left(A_x-\frac{\partial}{\partial t}\right)q(t, x, y)$

(a. 3) $e_{n+1}(t, x, y)=\int_0^t ds\int_{\bar{D}} e_0(t-s, x, z)e_n(s, z, y)m(dz)$

(a. 4) $f(t, x, y)=\sum_{n=0}^{\infty} e_n(t, x, y).$

Let us fix T arbitrarily. We prove necessary estimates in the following. Note that constants $K_1, K_2, \cdots, K_{13}$ in this section are independent of $t\in(0, T]$ and $x, y\in\bar{D}$. First, we have

(a. 5) $|q(t, x, y)|\leqslant K_1\sum_{i,j}\lambda_{ij}(x)\lambda_{ij}(y)t^{-N/2}$
$\times\exp\left(-K_2t^{-1}\sum_{k=1}^{N}|y_{(i)}^k-x_{(i)}^k|^2\right)$

(a. 6) $|e_0(t, x, y)|\leqslant K_1\sum_{i,j}\chi_{B_{ij}}(x)\lambda_{ij}(y)t^{(-N-1)/2}$
$\times\exp\left(-K_2t^{-1}\sum_{k=1}^{N}|y_{(i)}^k-x_{(i)}^k|^2\right)$

(a. 7) $|e_n(t, x, y)|\leqslant K_1^{n+1}\Gamma\left(\frac{n+1}{2}\right)^{-1}t^{(n-N-1)/2},\quad n\geqslant 0.$

The proof of (a·5)—(a·7) is as follows.[40] First, we have (a. 5) with K_1 and K_2 replaced by some K_1' and K_2', respectively, from the definition of q and

$$\exp\left(-Kt^{-1}\sum_{k=1}^{N}|y_{(i)}^k-x_{(i)}^{k*}|^2\right)\leqslant\exp\left(-Kt^{-1}\sum_{k=1}^{N}|y_{(i)}^k-x_{(i)}^k|^2\right).$$

Calculating $e_0(t, x, y)$ from (a. 2) and using $(a_{kl}^{(i)})=(a_{(i)}^{kl})^{-1}$, $a_{(i)}^{kl}(x)$

39) $A_x q(t, x, y)$ denotes the operation of A to $q(t, x, y)$ with respect to x.
40) The proof of (a. 7) is communicated from S. Ito.

$=a^{kl}_{(i)}(y)+O\left(\sum_{i=1}^{N}|x^{k}_{(i)}-y^{k}_{(i)}|\right)$, $a^{(i)}_{Nk}(y)=O(y^{N}_{(i)})$, and $\theta^{n}\exp(-C_1\theta^2)\leqslant C_2\exp(-C_3\theta^2)$, we obtain (a. 6) with K_1 and K_2 replaced by some K_1'' and K_2'', respectively. (a. 7) for $n=0$ with K_1 replaced by some K_1''' is an obvious consequence. Before the proof of (a. 7) for $n\geqslant 1$, let us prove, for some K_3,

$$\text{(a. 8)}\quad \int_{\bar D}|e_n(t,x,y)|\,m(dx)\leqslant K_3^{n+1}\Gamma\left(\frac{n+1}{2}\right)^{-1}t^{(n-1)/2}$$

$$\text{(a. 9)}\quad \int_{\bar D}|e_n(t,x,y)|\,m(dy)\leqslant K_3^{n+1}\Gamma\left(\frac{n+1}{2}\right)^{-1}t^{(n-1)/2}.$$

We can choose such a constant K_3 that (a. 8) and (a. 9) hold for $n=0$, by virtue of (a. 6) with K_1 and K_2 replaced by K_1'' and K_2'' above. Moreover, we take $K_3\geqslant K_1'''$. If (a. 8) holds for some n, then

$$\begin{aligned}&\int|e_{n+1}(t,x,y)|\,m(dx)\\&\leqslant\int_0^t ds\iint|e_0(t-s,x,z)|\,|e_n(s,z,y)|\,m(dx)m(dz)\\&\leqslant K_3^{n+2}\Gamma\left(\frac{1}{2}\right)^{-1}\Gamma\left(\frac{n+1}{2}\right)^{-1}\int_0^t(t-s)^{-1/2}s^{(n-1)/2}ds\\&=K_3^{n+2}\Gamma\left(\frac{n+2}{2}\right)^{-1}t^{n/2},\end{aligned}$$

which proves (a. 8) for $n+1$. Thus (a. 8) is verified for all $n\geqslant 0$. (a. 9) is proved in the same way. Using (a. 8) and (a. 9), we have

$$\text{(a. 10)}\quad |e_n(t,x,y)|\leqslant 2^{Nn/1}K_3^{n+1}\Gamma\left(\frac{n+1}{2}\right)^{-1}t^{(n-N-1)/2}.$$

In fact, (a. 10) holds for $n=0$ by $K_3\geqslant K_1'''$, and if (a. 10) is true for some n, then, separating the integral in s in the right side of (a. 3) into $[0,t/2]$ and $[t/2,t]$, we have

$$\begin{aligned}|e_{n+1}(t,x,y)|&\leqslant 2^{Nn/2}K_3^{n+2}\Gamma\left(\frac{1}{2}\right)^{-1}\Gamma\left(\frac{n+1}{2}\right)^{-1}\\&\times\left(\int_0^{t/2}(t-s)^{(-N-1)/2}s^{(n-1)/2}ds+\int_{t/2}^{t}(t-s)^{(-N-1)/2}s^{(n-1)/2}ds\right)\end{aligned}$$

$$=2^{Nn/2}K_3^{n+2}\Gamma\left(\frac{1}{2}\right)^{-1}\Gamma\left(\frac{n+1}{2}\right)^{-1}t^{(n-N)/2}\left(\int_0^{1/2}(1-s)^{(-N-1)/2}s^{(n-1)/2}ds\right.$$

$$\left.+\int_{1/2}^{1}(1-s)^{-1/2}s^{(n-N-1)/2}ds\right)$$

$$\leqslant 2^{\frac{N(n+1)}{2}}K_3^{n+2}\Gamma\left(\frac{1}{2}\right)^{-1}\Gamma\left(\frac{n+1}{2}\right)^{-1}B\left(\frac{1}{2},\ \frac{n+1}{2}\right)t^{(n-N)/2}.$$

Thus, we have (a. 5)-(a. 7) if we put $K_1=\max(K_1', K_1'', 2^{N/2}K_3)$ and $K_2=\min(K_2', K_2'')$.

The estimate (a. 7) implies the convergence of the series in the right side of (a. 4). The existence of the integral in (a. 1) is verified in the proof of Lemma 2. 1 below.

Proof of Lemma 2. 1. Obviously, $q(t, x, y)$ is a continuous function of three variables. So, by (a. 1), we have only to prove the continuity of

$$u(t, x, y)=\int_0^t ds\int_{\bar{D}} q(t-s, x, z)f(s, z, y)m(dz).$$

First, note that there is a K_4 such that

$$\text{(a. 11)} \qquad \int_{\bar{D}}|q(t, x, y)|m(dy)\leqslant K_4,$$

which follows from (a. 5). Fix $0<t_1<t_2<\infty$ and let $t_1<t<t_2$. Since $f(t, x, y)$ is bounded on $[t_1/2, t_2]\times\bar{D}\times\bar{D}$ by (a. 7), we have

$$\text{(a. 12)} \qquad \left|\int_{t-\delta}^t ds\int q(t-s, x, z)f(s, z, y)m(dz)\right|\leqslant K\delta$$

for sufficiently small $\delta>0$, where K is a constant independent of t, δ, x and y. Fix δ and put

$$v(s;\ t, x, y)=\chi_{(0, t-\delta)}(s)\int_{\bar{D}} q(t-s, x, z)f(s, z, y)m(dz).$$

If we fix $s\neq 0$, $t-\delta$ and let $t'\to t$, $x'\to x$, and $y'\to y$, then $v(s; t', x', y')\to v(s;\ t, x, y)$. In fact, $e_n(t, x, y)$ defined by (a. 2) and (a. 3) are continuous in y, which is proved by induction, and hence, $f(t, x, y)$ is continuous in y, and we have, using the dominated convergence theorem, the continuity of $v(s;\ t, x, y)$ with respect to

(t, x, y). The estimates (a. 5) and (a. 8) imply

$$|v(s;\, t, x, y)| \leqslant K_5 \delta^{-N/2} \int |f(s, z, y)|\, m(dz)$$
$$\leqslant K_5 \delta^{-N/2} \sum_{n=0}^{\infty} K_3^{n+1} \Gamma\left(\frac{n+1}{2}\right)^{-1} s^{(n-1)/2},$$

for a suitable constant K_5. Hence, using the dominated convergence theorem again, we have

$$\int_0^{t_2} v(s;\, t', x', y')\, ds \to \int_0^{t_2} v(s;\, t, x, y)\, ds,$$

that is,

$$\int_0^{t'-\delta} ds \int q(t'-s, x', z) f(s, z, y')\, m(dz)$$
$$\to \int_0^{t-\delta} ds \int q(t-s, x, z) f(s, z, y)\, m(dz).$$

This combined with (a. 12) prove that $u(t', x', y') \to u(t, x, y)$, and the proof of Lemma 2. 1 is complete.

Proof of Lemma 2. 2. Let us prove, for a constant K_6,

(a. 13) $$\left| \frac{\partial p}{\partial x_{(i)}^l}(t, x, y) \right| \leqslant K_6 t^{(-N-1)/2}$$

and

(a. 14) $$\int_{\bar{D}} \left| \frac{\partial p}{\partial x_{(i)}^l}(t, x, y) \right| m(dy) \leqslant K_6 t^{-1/2}.$$

First, by a simple calculation, we have, for some K_7 and K_8,

(a. 15) $$\left| \frac{\partial q}{\partial x_{(i)}^l}(t, x, y) \right| \leqslant K_7 \sum_{i,j} \chi_{B_{ij}}(x) \lambda_{ij}(y) t^{(-N-1)/2} \times \exp\left(-K_8 t^{-1} \sum_{k=1}^{N} |y_{(i)}^k - x_{(i)}^k|^2 \right),$$

(a. 16) $$\int_{\bar{D}} \left| \frac{\partial q}{\partial x_{(i)}^l}(t, x, y) \right| m(dy) \leqslant K_7 t^{-1/2}.$$

Since we have, by (a. 5), (a. 7), (a. 8) and (a. 11),

$$\int_0^t ds \int_{\bar{D}} |q(t-s, x, z)|\, |e_n(s, z, y)|\, m(dz)$$
$$\leqslant K_1^{n+1} K_9 \Gamma\left(\frac{n+1}{2}\right)^{-1} \left(\int_0^{t/2} (t-s)^{-N/2} s^{(n-1)/2} ds + \int_{t/2}^{t} s^{(n-N-1)/2} ds \right)$$

$$=K_1^{n+1}K_9\Gamma\left(\frac{n+1}{2}\right)^{-1}t^{(n-N+1)/2}\left(\int_0^{1/2}(1-s)^{-N/2}s^{(n-1)/2}ds+\int_{1/2}^1 s^{(n-N-1)/2}ds\right)$$

$$\leqslant 2^{N/2}K_1^{n+1}K_9\Gamma\left(\frac{n+1}{2}\right)^{-1}t^{(n-N+1)/2}\int_0^1 s^{(n-1)/2}ds$$

$$=2^{N/2}K_1^{n+1}K_9\Gamma\left(\frac{n+3}{2}\right)^{-1}t^{(n-N+1)/2},$$

$$\bar{e}_{n+1}(t,x,y)=\int_0^t ds\int_{\bar{D}}q(t-s,x,z)e_n(s,z,y)m(dz)$$

exists and

$$p(t,x,y)=q(t,x,y)+\sum_{n=1}^{\infty}\bar{e}_n(t,x,y) \tag{a. 17}$$

holds. By

$$\frac{\partial}{\partial x'_{(i)}}\int_{\bar{D}}q(t-s,x,z)e_n(s,z,y)m(dz)$$

$$=\int_{\bar{D}}\frac{\partial q}{\partial x'_{(i)}}(t-s,x,z)e_n(s,z,y)m(dz)$$

and by an estimate similar to the proof of (a. 10), we have

$$\frac{\partial\bar{e}_{n+1}}{\partial x'_{(i)}}(t,x,y)=\int_0^t ds\int_{\bar{D}}\frac{\partial q}{\partial x'_{(i)}}(t-s,x,z)e_n(s,z,y)m(dz)$$

and

$$\left|\frac{\partial\bar{e}_{n+1}}{\partial x'_{(i)}}(t,x,y)\right|\leqslant 2^{N/2}K_1^{n+1}K_7\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n+2}{2}\right)^{-1}t^{(n-N)/2}. \tag{a. 18}$$

Here, we have used (a. 7), (a. 8), (a. 15) and (a. 16). Thus $p(t,x,y)$ is of class C^1 as a function of x, and we can differentiate (a. 17) with respect to $x'_{(i)}$ term by term. (a. 13) follows from (a. 15) and (a. 18), and (a. 9) combined with (a. 16) implv

$$\int_{\bar{D}}\left|\frac{\partial\bar{e}_{n+1}}{\partial x'_{(i)}}(t,x,y)\right|m(dy)\leqslant K_3^{n+1}K_7\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n+2}{2}\right)^{-1}t^{n/2}.$$

Hence, we have (a. 14).

To conclude the proof, put

$$v(s,t,x)=\int_{\bar{D}}p(s,x,y)h(t-s,y)m(dy).$$

Then, by (a. 13), (a. 14) and boundedness of h, $v(s,t,x)$ and $u(t,x)$ are of class C^1 as function of x and

$$\frac{\partial v}{\partial x_{(i)}^{l}}(s, t, x)=\int_{\bar{D}} \frac{\partial p}{\partial x_{(i)}^{l}}(s, x, y)h(t-s, y)m(dy)$$

$$\frac{\partial v}{\partial x_{(i)}^{l}}(t, x)=\int_{0}^{t} \frac{\partial v}{\partial x_{(i)}^{l}}(s, t, x)ds.$$

(2.9) follows from the estimate (a. 14).

Proof of Lemma 3.5. It is sufficient to prove

$$\text{(a. 19)} \qquad \frac{1}{\rho}\int_{D_\rho} p(t, x, y)m(dy)\leqslant Kt^{-1/2},$$

from which $\int_{\partial D} p(t, x, y)\widetilde{m}(dy)\leqslant Kt^{-1/2}$ follows by Lemma 3.3 if we let $\rho\to 0$. Let us give the estimates

$$\text{(a. 20)} \qquad \frac{1}{\rho}\int_{D_\rho} |q(t, x, y)|m(dy)\leqslant K_{10}t^{-1/2}$$

$$\text{(a. 21)} \quad \frac{1}{\rho}\int_{D_\rho} m(dy)\int_0^t ds\int_{\bar{D}} |q(t-s, x, z)|\,|e_0(s, z, y)|m(dz)\leqslant K_{10}$$

$$\text{(a. 22)} \quad \frac{1}{\rho}\int_{D_\rho} m(dy)\int_0^t ds\int_{\bar{D}} |e_0(t-s, x, z)|\,|e_0(s, z, y)|m(dz)\leqslant K_{10}t^{-1/2}$$

where K_{10} is independent not only of $t\in(0, T]$ and $x, y\in\bar{D}$, but also of $\rho\in(0, \rho_0]$. Choose sets $V_{ijk}=V(\rho_{ijk}, r_{ijk}, x_{ijk}; \sigma_i)$ $1\leqslant i\leqslant M$, $1\leqslant j\leqslant M_i$, $1\leqslant k\leqslant M_{ij}$, of the type (3.2) such that $D_\rho\subset\bigcup_{i,j,k} V_{ijk}$ and $D_\rho\cap B_{ijk}\subset\bigcup_k V_{ijk}$ for any sufficiently small ρ, and put $\rho''=\rho''(\rho)=\max_{i,j,k}\rho''_{ijk}(\rho)$ where $\rho''_{ijk}(\rho)$ is defined by using Corollary to Lemma 3.2 in V_{ijk}. Then, we have, by (a. 5),

$$\frac{1}{\rho}\int_{D_\rho}|q(t, x, y)|m(dy)\leqslant\sum_{i,j,k}\frac{1}{\rho}\int_{D_\rho\cap V_{ijk}}\lambda_{ij}(x)r_i'(t, x, y)\lambda_{ij}(y)m(dy)$$

$$\leqslant\frac{\rho''}{\rho}\sum_{i,j,k}\frac{1}{\rho''}\int_0^{\rho''}dy_{(i)}^N\int\cdots\int_{R_{ijk}}\lambda_{ij}(x)r_i'(t, x, y)\lambda_{ij}(y)\sqrt{a^{(i)}(y)}\,dy_{(i)}^1\cdots dy_{(i)}^{N-1},$$

where

$$r_i'(t, x, y)=K_1t^{-N/2}\exp\left(-K_2t^{-1}\sum_{l=1}^{N}|y_{(i)}^l-x_{(i)}^l|^2\right)$$

and

$$R_{ijk}=\left\{(y_{(i)}^1, \cdots, y_{(i)}^{N-1})\,\Big|\,\sum_{l=1}^{N-1}|y_{(i)}^l-(x_{ijk})_{(i)}^l|^2<r_{ijk}^2\right\}.$$

From this follows (a. 20), since $\rho''/\rho\to 1$. If we denote by $\sum_{i',j'}^{(i,j)}$ the summation for all i', j' such that $B_{ij}\cap B_{i'j'}\neq\phi$ and recall that $B_{i'j'}\subset U_i$ for such i', j', then, we have, by (a. 5) and (a. 6),

$$\frac{1}{\rho}\int_{D_\rho}m(dy)\int_0^t ds\int_{\bar D}|q(t-s,x,z)|\,|e_0(s,z,y)|\,m(dz)$$

$$\leqslant\frac{1}{\rho}\int_{D_\rho}m(dy)\int_0^t ds\int_{\bar D}m(dz)\sum_{i,j}\sum_{i',j'}^{(i,j)}K_1^2\lambda_{ij}(x)\lambda_{ij}(z)\chi_{B_{i'j'}}(z)\lambda_{i'j'}(y)$$
$$\times(t-s)^{-N/2}s^{(-N-1)/2}$$
$$\times\exp\Big(-K_2(t-s)^{-1}\sum_{l=1}^N|z_{(i)}^l-x_{(i)}^l|^2\Big)\exp\Big(-K_2 s^{-1}\sum_{l=1}^N|y_{(i')}^l-z_{(i')}^l|^2\Big)$$

$$\leqslant\frac{1}{\rho}\int_{D_\rho}m(dy)\int_0^t ds\sum_{i,j}\sum_{i',j'}^{(i,j)}K_{11}\lambda_{ij}(x)\lambda_{i'j'}(y)s^{-1/2}t^{-N/2}$$
$$\times\exp\Big(-K_{12}t^{-1}\sum_{l=1}^N|y_{(i)}^l-x_{(i)}^l|^2\Big)$$

$$\leqslant 2K_{11}t^{1/2}\frac{\rho''}{\rho}\sum_{i,j}\sum_{i',j'}^{(i,j)}\sum_{k'}\frac{1}{\rho''}\int_0^{\rho''}dy_{(i)}^N\int\cdots\int_{R_{i',j',k'}}\lambda_{ij}(x)\lambda_{i'j'}(y)t^{-N/2}$$
$$\times\exp\Big(-K_{12}t^{-1}\sum_{l=1}^N|y_{(i)}^l-x_{(i)}^l|^2\Big)\sqrt{a^{(i)}(y)}\,dy_{(i)}^1\cdots dy_{(i)}^{N-1},$$

for some K_{11} and K_{12}, from which (a. 21) follows. Here we have chosen K_{12} satisfying $K_{12}\leqslant K_2$ and

$$\exp\Big(-K_2 s^{-1}\sum_{l=1}^N|y_{(i')}^l-z_{(i')}^l|^2\Big)\leqslant\exp\Big(-K_{12}s^{-1}\sum_{l=1}^N|y_{(i)}^l-z_{(i)}^l|^2\Big).$$

Similarly we can prove the estimate (a. 22).

Now, we have

$$\text{(a. 23)}\quad \frac{1}{\rho}\int_{D_\rho}|e_n(t,x,y)|\,m(dy)\leqslant K_{13}^{n+1}\Gamma\Big(\frac{n}{2}\Big)^{-1}t^{(n-2)/2},\quad n\geqslant 1,$$

where $K_{13}=\max(\Gamma(1/2)K_{10},K_3)$. In fact, (a. 23) holds for $n=1$ by virtue of (a. 22). If $e_n(t,x,y)$ satisfies (a. 23), we have, using (a. 9),

$$\frac{1}{\rho}\int_{D_\rho}|e_{n+1}(t,x,y)|\,m(dy)$$
$$\leqslant\frac{1}{\rho}\int_{D_\rho}m(dy)\int_0^t ds\int_{\bar D}|e_0(t-s,x,z)|\,|e_n(s,z,y)|\,m(dz)$$
$$\leqslant K_{13}K_{13}^{n+1}\Gamma\Big(\frac{1}{2}\Big)^{-1}\Gamma\Big(\frac{n}{2}\Big)^{-1}\int_0^t(t-s)^{-1/2}s^{(n-2)/2}ds\leqslant K_{13}^{n+2}\Gamma\Big(\frac{n+1}{2}\Big)^{-1}t^{(n-1)/2}.$$

Thus, (a. 23) holds for all $n\geqslant 1$. From (a. 11) and (a. 23) we obtain

$$\frac{1}{\rho}\int_{D_\rho} m(dy)\int_0^t ds\int_{\bar{D}} |q(t-s,x,z)|\,|e_n(s,z,y)|\,m(dz)$$
$$\leqslant K_4 K_{13}^{n+1}\Gamma\left(\frac{n+2}{2}\right)^{-1} t^{n/2}, \qquad n\geqslant 1.$$

This combined with (a. 20) and (a. 21) prove (a. 19), and the proof of Lemma 3. 5 is complete.

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Tokyo University of Education and
The Collage of General Education,
University of Tokyo