J. Math. Kyoto Univ.
4-1 (1964) 77-97

Contributions to the theory of differentials on open Riemann surfaces

Dedicated to Professor A. Kobori on his 60th birthday

By

Mineko MORI

(Received Sept. 10, 1964)

Introduction

Ahlfors [2] and Ahlfors-Sario [3] have extended the theory of differentials on open Riemann surfaces, and Kusunoki [6] have developed the theory of Abelian differentials on open Riemann surfaces. Accola [1] has established some results on bilinear relations with respect to those differentials. We shall here use the same notations for the classes of differentials as in Ahlfors-Sario [3], and discuss relations between those classes of differentials, including the bilinear relations.

In § 1 we establish a relation between the class of canonical differentials and the class of distinguished differentials (Theorem 1), which asserts that a meromorphic differential φ is a canonical semiexact differential if and only if the real part of φ is distinguished. An essential tool used there is a method of principal operator investigated by Sario [10]. In § 2 we generalize the notion of the finite bilinear relation, which was defined on Riemann surfaces of class O_{HD} by Accola [1], to arbitrary open Riemann surfaces, and extend the results obtained by Accola [1]. Analogous results have been obtained by Oikawa, but he does not yet publish them. Section 3 deals with three classes of Riemann surfaces; the class O_{KD}, the class of surfaces on which $\Gamma_{he} \cap \Gamma_{hse}^{*} \subset \Gamma_{he}^{*}$ is valid and the class of surfaces on which $\Gamma_{hm} = \Gamma_{he} \cap \Gamma_{h0}$ holds. We give there equivalent conditions for each of these classes. Finally in § 4, a necessary and sufficient condition for the validity of the generalized bilinear relation is given (Theorem 8), which shows that the differentials of class $\Gamma_{h0}\cap\Gamma_{hse}^*$ play an essential role in the bilinear relation. Moreover a few related results are proved under the condition $\Gamma_{he}\cap\Gamma_{hse}^*\subset\Gamma_{he}^*$.

§ 1. Canonical differentials and distinguished differentials

1. On an open Riemann surface R of genus g $(0\leqq g\leqq\infty)$ we take a canonical homology basis $\{A_k, B_k\}_{k=1,\cdots,g}$ and $\{C_\nu\}_{\nu=1,\cdots,p}$ $(0\leqq p\leqq\infty)$ such that 1) any cycle in R is homologous to a finite linear combination $\sum(p_kA_k+q_kB_k)+\sum r_\nu C_\nu$, where p_k, q_k and r_ν are integers, 2) the intersection numbers are characterized by $A_h\times B_k=\delta_{hk}$, $A_h\times A_k=B_h\times B_k=0$ for $h, k=1, 2, \cdots, g$, and 3) any dividing cycle in R is homologous to a finite linear combination $\sum r_\nu C_\nu$.

Then, there exist *canonical semiexact differentials* φ_{A_k}, φ_{B_k} on R which are uniquely determined by the conditions

$$Re\iint_{B_h}\varphi_{A_k}=-Re\int_{A_h}\varphi_{B_k}=\delta_{hk}\text{ and }Re\int_{A_h}\varphi_{A_k}=Re\int_{B_h}\varphi_{B_k}=0,$$

and *canonical differentials* φ_{C_ν}, whose real parts are generalized harmonic measures associated with C_ν (Kusunoki [6]).

On the other hand, to any cycle c in R there corresponds a *unique real harmonic differential* $\sigma(c)$ of class Γ_{h0} such that

$$(\omega, \sigma(c)^*)=\int_c\omega$$

for all closed differential ω, and to any non-dividing cycle c' in R there corresponds a *unique real harmonic differential* $\tilde{\sigma}(c')$ of class $\Gamma_{h0}\cap\Gamma_{hse}^*$ such that

$$(\omega, \tilde{\sigma}(c')^*)=\int_{c'}\omega$$

for all $\omega\in\Gamma_{hse}$.

It is known that

$$\begin{aligned}\Gamma_{h0}&=[\sigma(A_k), \sigma(B_k), \sigma(C_\nu)]\\ \Gamma_{h0}\cap\Gamma_{hse}^*&=[\tilde{\sigma}(A_k), \tilde{\sigma}(B_k)],\end{aligned}\tag{1}$$

where the brackets denote the closed linear subspace spanned by the indicated differentials (Ahlfors-Sario [3]).

At first we shall show the following relations between these differentials.

Proposition 1. (Kusunoki [7]) *It holds that*

$$\begin{aligned}\varphi_{A_k} &= \tilde{\sigma}(A_k)+i\tilde{\sigma}(A_k)^* \\ \varphi_{B_k} &= \tilde{\sigma}(B_k)+i\tilde{\sigma}(B_k)^* \\ \varphi_{C_\nu} &= \sigma(C_\nu)+i\sigma(C_\nu)^* .\end{aligned}$$

Proof. Let $\{R_n\}$ be an exhaustion of R with regular regions, and we denote by $\varphi^{(n)}_{A_k}$, $\tilde{\sigma}^{(n)}(A_k)$ etc. the corresponding differentials on R_n. Then we have

$$\varphi^{(n)}_{\substack{A_k\\B_k}} \longrightarrow \varphi_{\substack{A_k\\B_k}} \qquad (n\to\infty). \tag{2}$$

On the interiors R_n of compact bordered surfaces we can easily see that the relations

$$\begin{aligned}\varphi^{(n)}_{A_k} &= \tilde{\sigma}^{(n)}(A_k)+i\tilde{\sigma}^{(n)}(A_k)^* \\ \varphi^{(n)}_{B_k} &= \tilde{\sigma}^{(n)}(B_k)+i\tilde{\sigma}^{(n)}(B_k)^* \\ \varphi^{(n)}_{C_k} &= \sigma^{(n)}(C_\nu)+i\sigma^{(n)}(C_\nu)^*\end{aligned} \tag{3}$$

hold. By (2) and (3) we see that $\lim_{n\to\infty}\tilde{\sigma}^{(n)}(A_k)$ exists for each k and

$$\lim_{n\to\infty}(\omega, (Re\,\varphi_{A_k})^*-\tilde{\sigma}^{(n)}(A_k)^*) = 0$$

for any $\omega\in\Gamma_{hse}$. Therefore we have

$$(\omega, (Re\,\varphi_{A_k})^*) = \lim_{n\to\infty}(\omega, \tilde{\sigma}^{(n)}(A_k)^*) = \int_{A_k}\omega . \tag{4}$$

Since $(Re\,\varphi_{A_k})^*-\tilde{\sigma}(A_k)^*$ belongs to Γ_{hse}, we get

$$\|(Re\varphi_{A_k})^*-\tilde{\sigma}(A_k)^*\|^2 = 0$$

by (4) and the reproducing property of $\tilde{\sigma}(A_k)^*$ for Γ_{hse}.

If $\{R_n\}$ is a canonical exhaustion, we have, by the definitions

$$\begin{aligned}\varphi^{(n)}_{C_\nu} &\longrightarrow \varphi_{C_\nu} \qquad (n\to\infty)\\ \sigma^{(n)}(C_\nu) &\longrightarrow \sigma(C_\nu) \qquad (n\to\infty).\end{aligned}$$

Therefore we get

$$\varphi_{C_\nu} = \sigma(C_\nu)+i\sigma(C_\nu)^* .$$

2. Now we treat the bilinear relation for analytic differentials in the following form.

Proposition 2. *Let φ be a canonical semiexact differential of the first kind, then the bilinear relation*

$$(\varphi \cdot \psi^*) = 2\sum_{k=1}^{\infty}\int_{A_k} Re\,\varphi \int_{B_k}\bar{\psi} - \int_{A_k}\bar{\psi}\int_{B_k} Re\,\varphi \qquad \text{(a finite sum)}$$

holds for any $\psi \in \Gamma_{ase}$.

Proof. Any canonical differential has at most a finite number of non-vanishing real periods. If

$$\int_{A_k} Re\,\varphi = x_k \quad \text{and} \quad \int_{B_k} Re\,\varphi = y_k ,$$

φ can be expressed as

$$(5) \qquad \varphi = \sum_{k=1}^{\infty}(-x_k\varphi_{B_k}+y_k\varphi_{A_k})$$

by the uniquenes theorem for canonical differentials (Kusunoki [6]). For any $\psi \in \Gamma_{ase}$ we have, by Proposition 1 and (5),

$$\begin{aligned}(\varphi, \psi^*) &= \sum_{k=1}^{\infty}\{-x_k(\bar{\sigma}(B_k)+i\bar{\sigma}(B_k)^*, \psi^*)+y_k(\bar{\sigma}(A_k)+i\bar{\sigma}(A_k)^*, \psi^*)\} \\ &= 2\sum_{k=1}^{\infty}\int_{A_k} Re\,\varphi \int_{B_k}\bar{\psi} - \int_{A_k}\bar{\psi}\int_{B_k} Re\,\varphi ,\end{aligned}$$

because

$$\begin{aligned}(\bar{\sigma}(B_k)+i\bar{\sigma}(B_k)^*, \psi^*) &= -\overline{(\psi, \bar{\sigma}(B_k)^*)}+i\overline{(\psi^*, \bar{\sigma}(B_k)^*)} \\ &= -\int_{B_k}\bar{\psi}+i\int_{B_k}\bar{\psi}^* \\ &= -2\int_{B_k}\bar{\psi} .\end{aligned}$$

By making use of this bilinear relation we get easily

Corollary 1. *Any canonical semiexact differential of the first kind without A periods vanishes identically.*

3. We denote by ψ_{Pr} and $\tilde{\psi}_{Pr}$ canonical semiexact differentials with singularities $1/z^r$ and i/z^r $(r \geqq 2)$ at P whose real parts are exact; and by ϕ_{PQ} and $\tilde{\phi}_{PQ}$ canonical semiexact differentials with

singularities $-1/z$ or $-i/z$ at P and $+1/z$ or $+i/z$ at Q whose real parts have single-valued integrals. Let $\{R_n\}$ be an exhaustion of R with regular regions and let us denote by $\psi_{Pr}^{(n)}$, $\phi_{PQ}^{(n)}$ etc. the corresponding differentials on R_n. We have already known that

$$(6) \qquad \psi_{Pr}^{(n)} \longrightarrow \psi_{Pr},\ \tilde{\psi}_{Pr}^{(n)} \longrightarrow \tilde{\psi}_{Pr} \qquad (n \to \infty).$$

We shall now establish the analogous characterization for ϕ_{PQ} and $\tilde{\phi}_{PQ}$, using the following result obtained by Kusunoki [6].

Let du be a real part of a canonical differential and dU a harmonic differential square integrable outside of a compact set K and $\int_\gamma dU^* = 0$ for every dividing cycle $\gamma \subset R-K$, then for any exhaustion $\{R_n\}$ of R we have

$$(7) \qquad \lim_{n\to\infty} \int_{\partial R_n} u\, dU^* = 0.$$

Lemma 1. *For any exhaustion $\{R_n\}$ of R with regular regions, we have*

$$\phi_{PQ}^{(n)} \longrightarrow \phi_{PQ},\ \tilde{\phi}_{PQ}^{(n)} \longrightarrow \tilde{\phi}_{PQ} \qquad (n \to \infty).$$

Proof. We consider the class $\{S\}$ of analytic functions with logarithmic singularities $-\log z_1$ at P and $+\log z_2$ at Q, where z_1 and z_2 are local parameters at P and Q respectively, and satisfy the following conditions:

1) real part u is single-valued and has finite Dirichlet integral over a boundary neighborhood of R,

2) imaginary part v satisfies $\int_\gamma dv = 0$ for every dividing curve γ, and

3) $\int_\beta u dv = 0$, where β denotes the ideal boundary of R.

Let

$$\int \phi_{PQ} = u_0 + iv_0 = \begin{cases} -\log z_1 + a_0^{(1)} + \sum_{j=1}^{\infty} a_{0j}^{(1)} z_1^j & \text{at } P \\ +\log z_2 + a_0^{(2)} + \sum_{j=1}^{\infty} a_{0j}^{(2)} z_2^j & \text{at } Q. \end{cases}$$

Then $\int \phi_{PQ}$ belongs to $\{S\}$ because of (7). Suppose that f is a function of class $\{S\}$ and

$$f = u+iv = \begin{cases} -\log z_1 + a^{(1)} + \sum_j a_j^{(1)} z_1^j & \text{at} \quad P \\ +\log z_2 + a^{(2)} + \sum_j a_j^{(2)} z_2^j & \text{at} \quad Q\,. \end{cases}$$

Taking (7) into consideration, we get by direct calculation

$$\begin{aligned} 0 &\leqq D_R(u-u_0) = \lim_{\rho_1,\rho_2\to 0} D_{R-(\Delta_1+\Delta_2)}(u-u_0) \\ &= \lim_{\rho_1,\rho_2\to 0} \{D_{R-(\Delta_1+\Delta_2)}(u) - 2D_{R-(\Delta_1+\Delta_2)}(u, u_0) + D_{R-(\Delta_1+\Delta_2)}(u_0)\} \\ &\leqq \lim_{\rho_1,\rho_2\to 0} \left\{ \int_{\partial(\Delta_1+\Delta_2)} (u-u_0)dv - \int_{\partial(\Delta_1+\Delta_2)} (u-u_0)dv_0 \right\} \\ &= 2\pi\, Re\, \{(a^{(1)}-a^{(2)}) - (a_0^{(1)} - a_0^{(2)})\}\,, \end{aligned} \tag{8}$$

where Δ_1 and Δ_2 are disks about P and Q with radii ρ_1 and ρ_2 respectively. Hence we conclude that the minimum of $Re\,(a^{(1)}-a^{(2)})$ is attained for the function $\int \phi_{PQ}$ in the class $\{S\}$.

Now let

$$\int \phi_{PQ}^{(n)} = u_n + iv_n = \begin{cases} -\log z_1 + a_n^{(1)} + \sum_j a_{nj}^{(1)} z_1^j & \text{at} \quad P \\ +\log z_2 + a_n^{(2)} + \sum_j a_{nj}^{(2)} z_2^j & \text{at} \quad Q\,, \end{cases}$$

then we have

$$Re\,(a_n^{(1)} - a_n^{(2)}) \leqq Re\,(a_{n+1}^{(1)} - a_{n+1}^{(2)}) \leqq \cdots \leqq Re\,(a_0^{(1)} - a_0^{(2)})\,,$$

because

$$\int_{\partial R_n} u_m dv_m = -D_{R_m - R_n}(u_m) \leqq 0$$

for $m > n$. Hence $\lim_{n\to\infty} Re(a_n^{(1)} - a_n^{(2)})$ exists, and since

$$\begin{aligned} 0 &\leqq D_K(u_m - u_n) \leqq D_{R_n}(u_m - u_n) \\ &\leqq 2\pi\, Re\, \{(a_m^{(1)} - a_m^{(2)}) - (a_n^{(1)} - a_n^{(2)})\} \end{aligned}$$

for any compact set K by (8), there exists a subsequence $\{u_{n_\nu}\}$ which converges uniformly to a function u' harmonic on $R-(P+Q)$ and $D_{R-(\Delta_1+\Delta_2)}(u') < \infty$. We can show that $u' = u_0$ immediately.

4. Let us recall that a differential ω on R, harmonic except for harmonic poles, is called *distinguished* if

1) ω^* is semiexact outside of some compact subset of R, and

2) there exist differentials $\omega_{hm} \in \Gamma_{hm}$ and $\omega_{e0} \in \Gamma_{e0} \cap \Gamma^1$ such that $\omega = \omega_{hm} + \omega_{e0}$ in a boundary neighborhood of R.

There exists a unique distinguished differential with a finite number of given harmonic poles, the sum of their residues being zero, and a finite number of given periods (Ahlfors-Sario [3]).

Let u be a harmonic function on a boundary neighborhood with zero flux over the ideal boundary of R and let Ω be a canonical subregion of R. The *principal operator* $\tilde{L}_{1\Omega}$ on Ω associates to u_Ω, which is a restriction of u to Ω, a function $\tilde{p}_{1\Omega}$ harmonic on a boundary neighborhood of Ω so that $\tilde{p}_{1\Omega}-u_\Omega$ is constant on each boundary component of Ω and has zero flux over each dividing cycle of Ω. The *principal operator* $\tilde{L}_1$ on R associates to u a function $\tilde{p}_1$ which is a limit of $\tilde{p}_{1\Omega}$ as Ω tends to R (Ahlfors-Sario [3]). We say that $\tilde{p}_1-u$ has $\tilde{L}_1$*-behavior* at the ideal boundary of R. *To each harmonic semiexact differential* ω *with a finite number of singularities and periods, there corresponds a distinguished differential* $\lambda(\omega)$ *with the singularities and periods of* ω *and which, in a boundary neighborhood of* R, *is the differential of a function whose real and imaginary parts have* $\tilde{L}_1$*-behavior. Therefore we have*

$$\lambda(\omega)\equiv\omega \tag{9}$$

if and only if ω *is distinguished* (Rodin [8]).

Lemma 2. *Let* φ *be a canonical semiexact differential, then*

$$\lambda(Re\,\varphi)\equiv Re\,\varphi\,.$$

Proof. Suppose that

$$\int_{A_k} Re\,\varphi = x_k\,,\quad \int_{B_k} Re\,\varphi = y_k$$

and φ has singularities

$$\sum_{r=1}^{m_j}\frac{a_{jr}}{z^r}\quad\text{at}\quad P_j\ (=1,2,\cdots,s)\,.$$

Then we have

$$\varphi=\sum_{j=1}^{s}\{Re\,a_{j1}\phi_{QPj}+Im\,a_{j1}\tilde{\phi}_{QPj}+\sum_{r=2}^{m_j}(Re\,a_{jr}\psi_{Pjr}+Im\,a_{jr}\tilde{\psi}_{Pjr})\}$$
$$+\sum_{k=1}^{\infty}(-x_k\varphi_{Bk}+y_k\varphi_{Ak})$$

where Q is a point different from all the P_j. For a canonical exhaustion $\{R_n\}$, we consider differentials

$$\varphi^{(n)} = \sum_{j=1}^{s} \{Re\, a_{j1}\phi_{QPj}^{(n)} + Im\, a_{j1}\tilde{\phi}_{QPj}^{(n)} + \sum_{r=2}^{m_j} (Re\, a_{jr}\psi_{Pjr}^{(n)} + Im\, a_{jr}\tilde{\psi}_{Pjr}^{(n)})\} + \sum_{k=1}^{\infty} (-x_k\varphi_{Bk}^{(n)} + y_k\varphi_{Ak}^{(n)})$$

on R_n, then

$$\varphi^{(n)} \longrightarrow \varphi \qquad (n \to \infty). \tag{10}$$

We form $\lambda(Re\,\varphi)$ which is, by definition,

$$\lambda(Re\,\varphi) = \lim_{n\to\infty} \lambda_{R_n}(Re\,\varphi) \tag{11}$$

where $\lambda_{R_n}(Re\,\varphi)$ denote the differentials corresponding to $Re\,\varphi$ by the same kind of operators as λ on R_n. The differentials $Re\,\varphi^{(n)} - \lambda_{R_n}(Re\,\varphi)$ have no singularities and no periods, and they belong to $\Gamma_{he} \cap \Gamma_{hse}^*$ on R_n. Because R_n are canonical regions, the functions

$$f_n = \int \{Re\,\varphi^{(n)} - \lambda_{R_n}(Re\,\varphi)\}$$

are constant on each boundary component of R_n, and we have solutions $H_{f_n}^{R_n}$ of Dirichlet problems on R_n with boundary values f_n. Then $dH_{f_n}^{R_n}$ belong to Γ_{hm} on R_n and since

$$Re\,\varphi^{(n)} - \lambda_{R_n}(Re\,\varphi) = dH_{f_n}^{R_n},$$

it must be identically zero. Therefore we get the conclusion by (10) and (11).

5. Now we consider the completions of classes of canonical semiexact differentials and canonical differentials of the first kind, that is the spaces spanned by $\{\varphi_{Ak}, \varphi_{Bk}\}_{k=1,\cdots,g}$ and $\{\varphi_{Ak}, \varphi_{Bk}, \varphi_{C\nu}\}_{k=1,\cdots,g;\,\nu=1,\cdots,p}$ over the real numbers respectively. We shall denote them as

$$\Gamma_{kse} = [\varphi_{Ak}, \varphi_{Bk}]_{\text{real}}, \quad \Gamma_k = [\varphi_{Ak}, \varphi_{Bk}, \varphi_{C\nu}]_{\text{real}}. \tag{12}$$

Then it can be readily seen that

$$\Gamma_{aS} \cap \Gamma_{ase} = Cl(\Gamma_{kse} + \Gamma_{kse}^*), \quad \Gamma_{aS} = Cl(\Gamma_k + \Gamma_k^*). \tag{13}$$

These expressions are the counterparts for analytic differentials of the expression

$$\Gamma_S = Cl(\Gamma_{he} + \Gamma^*_{h0})$$

for harmonic differentials.

Theorem 1. 1) *A meromorphic differential φ is a canonical semiexact differential if and only if $Re\,\varphi$ is a distinguished differential,* 2) *φ belongs to Γ_{kse} if and only if $Re\,\varphi$ belongs to $\Gamma_{h0} \cap \Gamma^*_{hse}$, and* 3) *$\varphi$ belongs to Γ_k if and only if $Re\,\varphi$ belongs to Γ_{h0}.*

If φ is a canonical differential, then $Re\,\varphi$ is distinguished by Lemma 2 and (9). If ω is a real distinguished differential, there is a canonical semiexact differential φ whose real part has the same singularities and periods as ω, which is seen by the uniqueness theorem for canonical differentials. Then we have $Re\,\varphi = \lambda(Re\,\varphi)$ by Lemma 2, and therefore $Re\,\varphi$ is a distinguished differential. Hence it must be equal to ω by the uniqueness theorem for distinguished differentials. 2) and 3) are seen by (1), Proposition 1 and (13).

Corollary 2. 1) *R is a Riemann surface of class O_{HD} if and only if $\Gamma_k = \Gamma_k^* = \Gamma_a$,* 2) *$R$ is a Riemann surface of class O_{KD} if and only if $\Gamma^*_{kse} = \Gamma_{kse} = \Gamma_{ase}$, and* 3) *$\Gamma_{he} \cap \Gamma^*_{hse} \subset \Gamma^*_{he}$ holds if and only if $\Gamma_{kse} = \Gamma^*_{kse} = \Gamma_{aS} \cap \Gamma_{ase}$.*

Proof. R belongs to O_{HD} if and only if $\Gamma_h = \Gamma_{h0}$, and we get 1) by 3) of Theorem 1. R belongs to O_{KD} if and only if $\Gamma_{hse} = \Gamma_{h0}$ because $\Gamma_{hse} = \Gamma_{h0} \oplus \Gamma_{hse} \cap \Gamma^*_{he}$, here $\oplus$ means orthogonal direct sum, and $\Gamma_{hse} = \Gamma_{h0}$ if and only if $\Gamma_{ase} = \Gamma_{kse} = \Gamma^*_{kse}$. From 2) of Theorem 1 it follows that $\Gamma_{aS} \cap \Gamma_{ase} = \Gamma_{kse} = \Gamma^*_{kse}$ if and only if $\Gamma_{h0} \cap \Gamma^*_{h0} = \Gamma_{h0} \cap \Gamma^*_{hse}$. We have $\Gamma_{h0} \cap \Gamma^*_{hse} = \Gamma_{h0} \cap \Gamma^*_{h0}$ if and only if $\Gamma_{he} \cap \Gamma^*_{hse} \subset \Gamma^*_{h0}$. Indeed, taking orthogonal complements of the both sides of the latter relation, we get $\Gamma^*_{h0} \oplus \Gamma_{hm} \supset \Gamma_{h0}$. Hence $\Gamma_{h0} = \Gamma_{h0} \cap (\Gamma_{hm} \oplus \Gamma^*_{h0}) = \Gamma_{hm} \oplus \Gamma_{h0} \cap \Gamma^*_{h0}$, and it can be readily seen that $[\sigma(A_k), \sigma(B_k)] = \Gamma_{h0} \cap \Gamma^*_{h0}$. The converse is almost trivial.

By 2) and 3) of this Corollary we see

Corollary 3. (Oikawa) *In order that R be a Riemann surface of class O_{KD}, it is necessary and sufficient that $R \in O_{AD}$ and $\Gamma_{he} \cap \Gamma^*_{hse} \subset \Gamma^*_{he}$.*

§ 2. Generalization of finite bilinear relation

6. Accola [1] defined the finite bilinear relation on open Riemann surfaces of class O_{HD}. We shall here generalize the notion to arbitrary open Riemann surfaces and define it as follows. Let R be an arbitrary open Riemann surface and let ω and σ be elements of Γ_{hse} on R which have only a finite number of non-vanishing A periods. Then we say that the *finite bilinear relation* holds for ω and σ, if the following relation holds:

$$(\omega, \sigma^*) = \sum_{k=1}^{\infty} \int_{A_k} \omega \int_{B_k} \bar{\sigma} - \int_{A_k} \bar{\sigma} \int_{B_k} \omega \qquad \text{(a finite sum).} \tag{14}$$

We denote by Γ_{hA} and Γ_{hB} the spaces spanned by $\{\sigma(A_k)\}_{k=1,\cdots,g}$ and $\{\sigma(B_k)\}_{k=1,\cdots,g}$ respectively, where g is the genus of R, and by $\hat{\Gamma}_{hA}$ and $\hat{\Gamma}_{hB}$ the spaces spanned by the $\hat{\sigma}(A_k)$ and $\hat{\sigma}(B_k)$ respectively, where the $\hat{\sigma}(A_k)^*$ and $\hat{\sigma}(B_k)^*$ are the period reproducers in Γ_{h0} (Rodin [8]).

Theorem 2. (Oikawa) *The following three conditions are equivalent:*

1) *The finite bilinear relation holds for any $\omega \in \Gamma_{h0}$ and $\sigma \in \Gamma_{hse}$ with a finite number of non-vanishing A periods.*

2) *Any $\varphi \in \Gamma_{kse}$ without A periods vanishes identically.*

3) $\Gamma_{h0} \cap \Gamma_{hse}^* = \Gamma_{hA} \oplus \hat{\Gamma}_{hA}^*$.

Proof. 1) implies 2): If φ is an elment of Γ_{kse} with a finite number of non-vanishing A periods, we have

$$\begin{aligned} \|\varphi\|^2 &= -i(\varphi, \varphi^*) \\ &= -2(Re\, \varphi, Im\, \varphi^*) - 2i(Re\, \varphi, Re\, \varphi^*) \\ &= -2i(Re\, \varphi, \varphi^*) \\ &= -2i \sum_k \left(\int_{A_k} Re\, \varphi \int_{B_k} \bar{\varphi} - \int_{A_k} \bar{\varphi} \int_{B_k} Re\, \varphi \right) \end{aligned}$$

because $Re\, \varphi \in \Gamma_{h0} \cap \Gamma_{hse}^*$ and $\varphi \in \Gamma_{kse} \subset \Gamma_{hse}$. Therefore $\|\varphi\|^2 = 0$ if φ has no A periods. If φ of class Γ_{kse}^* has no A periods, we can show analogously that $\varphi \equiv 0$.

2) implies 3): We always have $\Gamma_{h0} \cap \Gamma_{hse}^* \supset \Gamma_{hA} \oplus \hat{\Gamma}_{hA}^*$. Suppose that ω belongs to $\Gamma_{h0} \cap \Gamma_{hse}^*$ and is real. If $\omega \perp \Gamma_{hA} \oplus \hat{\Gamma}_{hA}^*$, ω has

no A periods and $\omega^* \perp \Gamma_{hA}^*$. Therefore $\omega + i\omega^*$ has no A periods, and it must be identically zero because $\omega + i\omega^* \in \Gamma_{kse}$. For a complex ω of class $\Gamma_{h0} \cap \Gamma_{hse}^*$, we can show that the real and imaginary parts of ω are zero respectively, if ω is orthogonal to $\Gamma_{hA} \oplus \hat{\Gamma}_{hA}^*$.

3) implies 1): Suppose that $\omega \in \Gamma_{h0}$ and $\sigma \in \Gamma_{hse}$, and both have only a finite number of non-vanishing A periods. We decompose ω so that

$$\omega = \omega_1 + \omega_2, \quad \text{where} \quad \omega_1 \in \Gamma_{h0} \cap \Gamma_{hse}^*, \quad \text{and} \quad \omega_2 \in \Gamma_{hm}.$$

Then we have

$$\int_{A_k} \omega = \int_{A_k} \omega_1 = \alpha_k(\omega)$$

and

$$(\omega, \sigma^*) = (\omega_1, \sigma^*). \tag{15}$$

We see that $\omega_1 + \sum_k \alpha_k(\omega)\tilde{\sigma}(B_k)$ belongs to Γ_{hA} by 3), because it belongs to $\Gamma_{h0} \cap \Gamma_{hse}^*$ and has no A periods. Moreover, $\sigma + \sum_k \alpha_k(\sigma)\tilde{\sigma}(B_k)$ $\in \Gamma_{hse}$ and this has no A periods, where $\alpha_k(\sigma)$ denote the periods of σ along A_k. Therefore we have

$$\begin{aligned}
&-\overline{(\sigma + \sum_k \alpha_k(\sigma)\tilde{\sigma}(B_k),\ \omega_1^* + \sum_k \alpha_k(\omega)\tilde{\sigma}(B_k)^*)} \\
&= (\omega_1 + \sum_k \alpha_k(\omega)\tilde{\sigma}(B_k),\ \sigma^* + \sum_k \alpha_k(\sigma)\tilde{\sigma}(B_k)^*) \\
&= 0.
\end{aligned}$$

Expanding this result and using the relation (15), we get

$$(\omega, \sigma^*) = \sum_k \int_{A_k} \omega \int_{B_k} \bar{\sigma} - \int_{A_k} \bar{\sigma} \int_{B_k} \omega.$$

Corollary 1 shows that *the conditions of Theorem 2 are satisfied if the genus of R is finite.*

Using the decomposition

$$\begin{aligned}
\Gamma_{hse} &= \Gamma_{hm} \oplus \Gamma_{hse} \cap \Gamma_{hse}^* \\
&= \Gamma_{hm} \oplus \Gamma_{he} \cap \Gamma_{he}^* \oplus \Gamma_S \cap \Gamma_{hse} \cap \Gamma_{hse}^*
\end{aligned}$$

it can be proved in the analogous way as in the previous Theorem:

Theorem 3. (Oikawa) *The following three conditions are equivalent:*

1) *The finite bilinear relation holds for any* $\omega \in \Gamma_S \cap \Gamma_{hse}$ *and* $\sigma \in \Gamma_{hse}$ *with a finite number of non-vanishing A periods.*

2) $\Gamma_{aS} \cap \Gamma_{ase}$ *are spanned by the* φ_{A_k}.

3) $\Gamma_S \cap \Gamma_{hse} \cap \Gamma_{hse}^* = \Gamma_{hA} \oplus \Gamma_{hA}^*$.

A Riemann surface of finite genus satisfies the three conditions of this Theorem if and only if $\Gamma_{he} \cap \Gamma_{hse}^* \subset \Gamma_{he}^*$ *holds.* Indeed if $\Gamma_{he} \cap \Gamma_{hse}^* \subset \Gamma_{he}^*$ holds, then $\Gamma_{aS} \cap \Gamma_{ase} = \Gamma_{kse}$ and it is spanned by the φ_{A_k}. Conversely, for any $\omega \in \Gamma_S \cap \Gamma_{he} \cap \Gamma_{hse}^*$ we have

$$||\omega||^2 = -(\omega, \omega^{**}) = 0$$

by 1) of the Theorem, which implies $\Gamma_{he} \cap \Gamma_{hse}^* \subset \Gamma_{he}^*$.

7. Now we construct normal differentials after Accola [1], using the $\tilde{\sigma}(B_k)$ instead of the $\sigma(B_k)$. Let

$$\tilde{\sigma}(B_k) = \theta_k + \tau_k \quad \text{where} \quad \theta_k \in \Gamma_{hA}^* \quad \text{and} \quad \tau_k \in (\Gamma_{hA}^*)^{\perp}.$$

Set

$$\phi_k = -\theta_k - i\theta_k^*, \tag{16}$$

then $\phi_k \in \Gamma_{kse}^*$, and

$$\int_{A_h} \phi_k = (\phi_k, \tilde{\sigma}(A_h)^*) = (-\tilde{\sigma}(B_k), \tilde{\sigma}(A_h)^*) = \delta_{hk}.$$

We can prove the following Lemma completely in the same way as Lemma 6 in Accola [1].

Lemma 3. *If* $\Gamma_{hm} = \Gamma_{he} \cap \Gamma_{h0}$, *then the* θ_k *are complete in* Γ_{hA}^*.

Using this Lemma we establish the following Theorem quite analogously as Accola [1] did for Riemann surfaces of class O_{HD}.

Theorem 4. *If the* φ_k *span* $\Gamma_{aS} \cap \Gamma_{ase}$, *then the* φ_{A_k} *span* $\Gamma_{aS} \cap \Gamma_{ase}$. *Conversely, if* $\Gamma_{hm} = \Gamma_{he} \cap \Gamma_{h0}$ *holds and the* φ_{A_k} *span* $\Gamma_{aS} \cap \Gamma_{ase}$, *then the* φ_k *span* $\Gamma_{aS} \cap \Gamma_{ase}$.

For completeness we sketch the outline of the proof. Suppose that $\sigma \in \Gamma_{hA}^* \subset \Gamma_{h0}^* \cap \Gamma_{hse}$ and $\sigma \perp \theta_k$ for any k. Then we have $\sigma^* \in \Gamma_{hA}$, that is, $\sigma^* \perp \Gamma_{hA}^*$. Since $\tau_k \perp \Gamma_{hA}^*$, we have $\sigma \perp \theta_k + \tau_k$ for all k, that is, $\sigma^* \perp \Gamma_{hB}^*$. Hence

$$\sigma \in \Gamma_{he}^* \cap \Gamma_{h0}^* \cap \Gamma_{hse},$$

and σ must be identically zero by the assumption. Which asserts the Lemma.

To prove the Theorem we suppose that φ of class $\Gamma_{aS} \cap \Gamma_{ase}$ has no A periods. Then

$$\begin{aligned}(\varphi, \phi_k) &= -(\varphi, \theta_k)+i(\varphi^*, \theta_k{}^*)\\ &= -2(Re\,\varphi, \theta_k)-2i(Im\,\varphi, \theta_k^*)\\ &= 0\,.\end{aligned}$$

By the assumption we conclude that $\varphi\equiv 0$, which means that the φ_{A_k} span $\Gamma_{aS}\cap\Gamma_{ase}$. If φ of class $\Gamma_{aS}\cap\Gamma_{ase}$ is perpendicular to all the ϕ_k, we have

$$\begin{aligned}2(\varphi, \theta_k) &= (\varphi, \theta_k)+(\varphi^*, \theta_k^*)\\ &= -(\varphi, \phi_k)\\ &= 0\,.\end{aligned}$$

Hence we see that φ nas no A periods by the above Lemma, and the conclusion yields.

8. As sufficient conditions in order that the conditions of Theorems 2 and 3 be satisfied, the following results are obtained.

Theorem 5 (Oikawa) 1) *If* $\Gamma_{he}\cap\Gamma_{hse}^*\subset\Gamma_{he}^*$ *holds and the vector sum* $\Gamma_{hA}+\Gamma_{hB}$ *is closed, then the three conditions of Theorem* 3 *are satisfied, and therefore* $\Gamma_{aS}\cap\Gamma_{ase}$ *is spanned by the* φ_k. 2) *If* $\Gamma_{hm}=\Gamma_{he}\cap\Gamma_{h0}$ *holds and the vector sum* $\Gamma_{hA}+\Gamma_{hB}$ *is closed, the three conditions of Theorem 2 are satisfied.*

Proof. To prove 1), we show that 1) of Theorem 3 holds. Suppose that $\omega\in\Gamma_S\cap\Gamma_{hse}$ and $\sigma\in\Gamma_{hse}$ have only a finite number of non-vanishing A periods. By the assumption we have

$$\begin{aligned}\Gamma_{hse}\cap\Gamma_S &= \Gamma_{h0}\oplus\Gamma_{he}^*\cap\Gamma_{hse}\cap\Gamma_S\\ &= \Gamma_{h0}\\ &= \Gamma_{hm}\oplus\Gamma_{h0}\cap\Gamma_{h0}^*\end{aligned}$$

and

$$\begin{aligned}\Gamma_{hse} &= \Gamma_{hm}\oplus\Gamma_{hse}\cap\Gamma_{hse}^*\\ &= \Gamma_{hm}\oplus\Gamma_{he}\cap\Gamma_{he}^*\oplus\Gamma_S\cap\Gamma_{hse}\cap\Gamma_{hse}^*\\ &= \Gamma_{hm}\oplus\Gamma_{he}\cap\Gamma_{he}^*\oplus\Gamma_{h0}\cap\Gamma_{h0}^*.\end{aligned}$$

Let

$$\omega=\omega_1+\omega_2 \quad \text{where} \quad \omega_1\in\Gamma_{h0}\cap\Gamma_{h0}^*,\ \omega_2\in\Gamma_{hm};$$

$$\sigma=\sigma_1+\sigma_2 \quad \text{where} \quad \sigma_1\in\Gamma_{h0}\cap\Gamma_{h0}^*,\ \sigma_2\in\Gamma_{hm}\oplus\Gamma_{he}\cap\Gamma_{he}^*$$

and

$$\omega'=\omega_1+\sum_k \alpha_k(\omega)\bar{\sigma}(B_k) \quad \text{where} \quad \alpha_k(\omega)=\int_{A_k}\omega;$$

$$\sigma'=\sigma_1+\sum_k \alpha_k(\sigma)\bar{\sigma}(B_k) \quad \text{where} \quad \alpha_k(\sigma)=\int_{A_k}\sigma,$$

then ω' and σ' have no A periods and

$$\omega',\ \sigma'\in\Gamma_{h0}\cap\Gamma_{h0}^*.$$

On the other hand, because $\Gamma_{hA}+\Gamma_{hB}$ is closed and $\Gamma_{he}\cap\Gamma_{h0}\cap\Gamma_{h0}^*$ has no non-zero elements, it is easily seen that $\omega',\sigma'\in\Gamma_{hA}$. Therefore it holds that

$$\begin{aligned}(\omega,\sigma^*)&=(\omega_1,\sigma_1^*)\\&=\sum_k\int_{A_k}\omega\int_{B_k}\bar{\sigma}-\int_{A_k}\bar{\sigma}\int_{B_k}\omega.\end{aligned}$$

To prove 2), we suppose that $\omega\in\Gamma_{h0}$ and $\sigma\in\Gamma_{hse}$ have only a finite number of non-vanishing A periods. Let

$$\omega=\omega_1+\omega_2 \quad \text{where} \quad \omega_1\in\Gamma_{h0}\cap\Gamma_{hse}^*,\ \omega_2\in\Gamma_{hm}$$

and

$$\omega'=\omega_1+\sum_k\alpha_k(\omega)\bar{\sigma}(B_k)$$

$$\sigma'=\sigma+\sum_k\alpha_k(\sigma)\bar{\sigma}(B_k).$$

Then ω' belongs to Γ_{hA} by the assumption that $\Gamma_{hA}+\Gamma_{hB}$ is closed, and σ' is semiexact and has no A periods, that is, σ' is orthogonal to Γ_{hA}^*. Therefore we have

$$(\omega',\sigma'^*)=0,$$

and expanding this we get the finite bilinear relation for ω and σ.

§ 3. Some classes of Riemann surfaces

9. We have already treated the classes of Riemann surfaces on which $\Gamma_{he}\cap\Gamma_{hse}^*\subset\Gamma_{he}^*$ or $\Gamma_{hm}=\Gamma_{he}\cap\Gamma_{h0}$ holds. The surfaces on which $\Gamma_{hm}=\Gamma_{he}\cap\Gamma_{h0}$ holds were discussed by Accola [1] and

$\Gamma_{he}\cap\Gamma_{hse}^*\subset\Gamma_{he}^*$ holds were taken up by Oikawa in his unpublished study. A Riemann surface of class O_{KD} is a surface on which $\Gamma_{he}\cap\Gamma_{hse}^*=\{0\}$ holds. If $\Gamma_{he}\cap\Gamma_{hse}^*=\{0\}$, then $\Gamma_{he}\cap\Gamma_{hse}^*\subset\Gamma_{he}^*$, and if $\Gamma_{he}\cap\Gamma_{hse}^*\subset\Gamma_{he}^*$ then $\Gamma_{hm}=\Gamma_{he}\cap\Gamma_{h0}$. But the converses are not true. These three classes of Riemann surfaces seem to be important when we discuss on differentials of some classes, because we get rid of some complexities. We give here the summaries of equivalent conditions.

Proposition 3. *A Riemann surface R belongs to O_{KD} if and only if one of the following conditions is fulfilled:*

1) $\Gamma_{hm}=\Gamma_{he}$.
2) $\Gamma_{hse}=\Gamma_{h0}$.
3) *Any element of Γ_{ase} with exact real part is identically zero.*

By $\Gamma_{he}=\Gamma_{hm}\oplus\Gamma_{he}\cap\Gamma_{hse}^*$ we get 1), and because $\Gamma_{hse}=\Gamma_{h0}\oplus\Gamma_{hse}\cap\Gamma_{he}^*$, we get the condition 2). It is obvious that $\Gamma_{he}\cap\Gamma_{hse}^*=\{0\}$ is equivalent to 3).

Proposition 4. *The following conditions are equivalent:*

1) $\Gamma_{hm}=\Gamma_{he}\cap\Gamma_{S}$.
2) $\Gamma_{he}\cap\Gamma_{hse}^*\cap\Gamma_{S}=\{0\}$.
3) $\Gamma_{he}\cap\Gamma_{hse}^*\subset\Gamma_{he}^*$.
4) $\Gamma_{hse}=\Gamma_{h0}\oplus\Gamma_{he}\cap\Gamma_{he}^*$.
5) $\Gamma_{h0}\cap\Gamma_{hse}^*=\Gamma_{h0}\cap\Gamma_{h0}^*$.
6) *Any element of $\Gamma_{aS}\cap\Gamma_{ase}$ with exact real part is identically zero.*

We have already shown that $\Gamma_{he}\cap\Gamma_{hse}^*\subset\Gamma_{he}^*$ is equivalent to the condition 5). We have orthogonal decompositions

$$\Gamma_{he}\cap\Gamma_{hse}^* = \Gamma_{he}\cap\Gamma_{he}^*\oplus\Gamma_{he}\cap\Gamma_{hse}^*\cap\Gamma_{S},$$
$$\Gamma_{he}\cap\Gamma_{S} = \Gamma_{hm}\oplus\Gamma_{he}\cap\Gamma_{S}\cap\Gamma_{hse}^*.$$

By the former of which we get the condition 2), and the latter of which shows the equivalency of the conditions 1) and 2). Taking the orthogonal complements of the equation 1) we get 4), and 6) is immediately seen by 2).

Proposition 5. *The following conditions are equivalent:*

1) $\Gamma_{hm}=\Gamma_{he}\cap\Gamma_{h0}$.

2) $\Gamma_{he}\cap\Gamma_{hse}^{*}\cap\Gamma_{h0}=\{0\}$.

3) $\Gamma_{he}\cap\Gamma_{hse}^{*}\subset Cl(\Gamma_{h0}^{*}+\Gamma_{he}^{*})$.

4) $\Gamma_{hse}=Cl(\Gamma_{he}+\Gamma_{h0})$.

5) *Any element of class* Γ_{kse} *with exact real part is identically zero.*

By taking orthogonal complements of $\Gamma_{hm}^{*}=\Gamma_{he}^{*}\cap\Gamma_{h0}^{*}$, we get 4), and by taking orthogonal complements of the relation 3) we get $\Gamma_{h0}^{*}\oplus\Gamma_{hm}\supset\Gamma_{he}\cap\Gamma_{h0}$, which means 1).

10. Accola [1] showed that $\Gamma_{hm}=\Gamma_{he}\cap\Gamma_{h0}$ is equivalent to that if $\omega\in\Gamma_{h0}$, $\sigma\in\Gamma_{hse}$ and ω has a finite number of non-vanishing A and B periods, then

$$(\omega,\sigma^{*})=\sum_{k=1}^{\infty}\int_{A_k}\omega\int_{B_k}\bar{\sigma}-\int_{A_k}\bar{\sigma}\int_{B_k}\omega \qquad \text{(a finite sum)} \tag{17}$$

ohlds. By which we know that the validity of the relation (17) is independent of homology basis, though the validity of the generalized bilinear relation depends on, and even the validity of the finite bilinear relation seems to depend on homology basis.

We get analogous equivalent conditions for the classes O_{KD} and of surfaces on which $\Gamma_{he}\cap\Gamma_{hse}^{*}\subset\Gamma_{he}^{*}$ holds.

Theorem 6. *A Riemann surface R is of class O_{KD} if and only if the relation* (17) *holds for $\omega\in\Gamma_{hse}$ with a finite number of non-zero periods and for any $\sigma\in\Gamma_{hse}$.*

Proof. Suppose that $R\in O_{KD}$. If $\omega\in\Gamma_{hse}$ has only a finite number of non-zero periods, set

$$\omega'=\omega+\sum_{k}\{\alpha_k\bar{\sigma}(B_k)-\beta_k\bar{\sigma}(A_k)\}$$

where $\alpha_k=\int_{A_k}\omega$ and $\beta_k=\int_{B_k}\omega$. Then $\omega'\in\Gamma_{he}$, and because $\sigma\in\Gamma_{h0}$ by Proposition 3, we have $(\omega',\sigma^{*})=0$. Expanding this yields the result.

Conversely, if the condition holds, any element σ of class $\Gamma_{he}\cap\Gamma_{hse}^{*}$ must be identically zero, because $\|\sigma\|^2=-(\sigma,\sigma^{**})=0$.

Theorem 7. *A necessary and sufficient condition for $\Gamma_{he}\cap\Gamma_{hse}^{*}\subset\Gamma_{he}^{*}$ is that the relation* (17) *holds for $\omega\in\Gamma_{S}\cap\Gamma_{hse}$ with a finite* *number of non-zero periods and for any* $\sigma \in \Gamma_{hse}$.

Proof. Suppose that $\Gamma_{he} \cap \Gamma_{hse}^* \subset \Gamma_{he}^*$ and set

$$\omega' = \omega + \sum_k \{\alpha_k \tilde{\sigma}((B_k) - \beta_k(A_k)\}$$

where $\alpha_k = \int_{A_k} \omega$ and $\beta_k = \int_{B_k} \omega$. Then $\omega' \in \Gamma_S \cap \Gamma_{he} = \Gamma_{hm}$ by Proposition 4, and $(\omega', \sigma^*) = 0$. Conversely, for any $\sigma \in \Gamma_{he} \cap \Gamma_{hse}^* \cap \Gamma_S$ we get $\|\sigma\|^2 = 0$ by the condition.

§ 4. Generalized bilinear relation

11. Let R be an open Riemann surface, $\{R_n\}$ an exhaustion of R, and let $A_1, B_1, \cdots, A_{p(n)}, B_{p(n)}, \cdots$ be a corresponding canonical homology basis such that $A_1, B_1, \cdots, A_{p(n)}, B_{p(n)}$ is a basis modulo ∂R_n on R_n. For a fixed $\sigma \in \Gamma_{hse}$, the *generalized bilinear relation* is said to hold if for all $\omega \in \Gamma_{h0}$, we have

$$(\omega, \sigma^*) = \lim_{n\to\infty} \sum_{k=1}^{p(n)} \int_{A_k} \omega \int_{B_k} \bar{\sigma} - \int_{A_k} \bar{\sigma} \int_{B_k} \omega \tag{18}$$

(Accola [1]).

We define linear operators $\tilde{T}_n$ on Γ_{hse} as follows. For any $\sigma \in \Gamma_{hse}$

$$\tilde{T}_n \sigma = \sum_{k=1}^{p(n)} \{\beta_k \tilde{\sigma}(A_k) - \alpha_k \tilde{\sigma}(B_k)\} \tag{19}$$

where $\alpha_k = \int_{A_k} \sigma$ and $\beta_k = \int_{B_k} \sigma$. Accola [1] defined linear operators T_n using the $\sigma(A_k)$ and $\sigma(B_k)$ in place of our $\tilde{\sigma}(A_k)$ and $\tilde{\sigma}(B_k)$, and gave a necessary and sufficient condition that the relation (18) holds for a fixed $\sigma \in \Gamma_{hse}$.

Theorem 8. *The generalized bilinear relation* (18) *holds for all* $\omega \in \Gamma_{h0}$ *and* $\sigma \in \Gamma_{hse}$ *if and only if, for any* $\tau \in \Gamma_{h0} \cap \Gamma_{hse}^*$, *we have*

$$\|\tilde{T}_n \tau - \tau\| \longrightarrow 0 \qquad (n \to \infty).$$

In other words, the norms $\|\tilde{T}_n\|$ *are bounded as n tends to* ∞.

Proof. Sufficiency. We have orthogonal decompositions

$$\Gamma_{h0} = \Gamma_{hm} \oplus \Gamma_{h0} \cap \Gamma_{hse}^*$$
$$\Gamma_{hse} = \Gamma_{he} \oplus \Gamma_{hse} \cap \Gamma_{h0}^*.$$

Let

$$\omega = \omega_1 + \omega_2 \quad \text{where} \quad \omega_1 \in \Gamma_{h0} \cap \Gamma_{hse}^*, \ \omega_2 \in \Gamma_{hm} ;$$
$$\sigma = \sigma_1 + \sigma_2 \quad \text{where} \quad \sigma_1 \in \Gamma_{hse} \cap \Gamma_{h0}^*, \ \sigma_2 \in \Gamma_{he} .$$

Then we have

$$\begin{aligned}(\omega, \sigma^*) &= (\omega_1, \sigma_1^*) = \lim_{n\to\infty} (\tilde{T}_n\omega_1, \sigma_1^*) \\ &= \lim_{n\to\infty} \sum_{k=1}^{p(n)} (\beta_k\tilde{\sigma}(A_k) - \alpha_k\tilde{\sigma}(B_k), \sigma_1^*) \\ &= \lim_{n\to\infty} \sum_{k=1}^{p(n)} \{\alpha_k\overline{(\sigma_1, \tilde{\sigma}(B_k)^*)} - \beta_k\overline{(\sigma_1, \tilde{\sigma}(A_k)^*)}\} \\ &= \lim_{n\to\infty} \sum_{k=1}^{p(n)} \int_{A_k} \omega \int_{Bk} \bar{\sigma} - \int_{A_k} \bar{\sigma} \int_{Bk} \omega .\end{aligned}$$

The necessity can be readily seen because any $\tau \in \Gamma_{h0} \cap \Gamma_{hse}^*$ can be approximated arbitrarily closely by finite combinations of the $\tilde{\sigma}(A_k)$ and $\tilde{\sigma}(B_k)$, and for any $\tilde{\sigma}(A_k)$ $(\tilde{\sigma}(B_k))$ it holds that

$$\|\tilde{\sigma}(A_k) - \tilde{T}_n\tilde{\sigma}(A_k)\|^2 = 0$$

for sufficiently large n by (18).

If R is of class O_{HD}, then we have

$$\Gamma_{he} \cap \Gamma_{hse}^* = \Gamma_h \quad \text{and} \quad \tilde{T}_n = T_n .$$

Hence our Theorem reduces to Theorem 10 and Corollary 11 in Accola [1].

Corollary 4. *If* $\Gamma_{hm} = \Gamma_{he} \cap \Gamma_{h0}$, *it is necessary and sufficient for the validity of the generalized bilinear relation that* $\|\tilde{T}_n\tau\|$ *are bounded as n tends to* ∞ *for any* $\tau \in \Gamma_{h0} \cap \Gamma_{hse}^*$.

12. Concerning a canonical homology basis with which the generalized bilinear relation is valid for any $\omega \in \Gamma_{h0}$ and $\sigma \in \Gamma_{hse}$, we obtain

Theorem 9. *Suppose that* $\Gamma_{he} \cap \Gamma_{hse}^* \subset \Gamma_{he}^*$ *holds on a Riemann surface R. Let* $\{R_n\}$ *be an exhaustion of R and* $A_1, B_1, \cdots, A_{p(n)}, B_{p(n)}, \cdots$ *a corresponding canonical homology basis with respect to which the generalized bilinear relation holds. Then for any* $\varphi \in \Gamma_{aS} \cap \Gamma_{ase}$, *we have*

$$\left.\begin{array}{l}\dfrac{1}{2}\sum\limits_{k=1}^{p(n)}(\varphi,\ \varphi_{A_k}^*)\phi_k \longrightarrow \varphi \\[2ex] \dfrac{1}{2}\sum\limits_{k=1}^{p(n)}(\varphi,\ \phi_k)\varphi_{A_k}^* \longrightarrow \varphi\end{array}\right. \qquad (n\to\infty).$$

Proof. By the assumption, we have $\Gamma_{aS}\cap\Gamma_{ase}=\Gamma_{kse}=\Gamma_{kse}^*\subset \Gamma_{h0}\cap\Gamma_{h0}^*$, and

$$\tilde{T}_n\varphi = \sum_{k=1}^{p(n)}\{\beta_k\tilde{\sigma}(A_k)-\alpha_k\tilde{\sigma}(B_k)\}$$

where $\alpha_k=\int_{A_k}\varphi$ and $\beta_k=\int_{B_k}\varphi$. Let

$$\tilde{T}_n\varphi = \theta_n(\varphi)+\tau_n(\varphi)$$

where $\theta_n(\varphi)\in\Gamma_{hA}^*$ and $\tau_n(\varphi)\in(\Gamma_{hA}^*)^{\perp}$, then

$$\theta_n(\varphi) = -\sum_{k=1}^{p(n)}\alpha_k\theta_k\,,$$

here θ_k are the differentials defined in (16), and

$$\theta_n(\varphi)+i\theta_n(\varphi)^* = -\sum_{k=1}^{p(n)}\alpha_k(\theta_k+i\theta_k^*) = \sum_{k=1}^{p(n)}\alpha_k\phi_k\,.$$

By the validity of the generalized bilinear relation, $||\tilde{T}_n\tau||$ are bounded as $n\to\infty$, and we have

$$\lim_{n\to\infty}\sum_{k=1}^{p(n)}\alpha_k\phi_k\in\Gamma_{aS}\cap\Gamma_{ase}\,.$$

Again by the generalized bilinear relation, an element of $\Gamma_{aS}\cap\Gamma_{ase}$ without any A periods is identically zero, and we have

$$\lim_{n\to\infty}\sum_{k=1}^{p(n)}\alpha_k\phi_k = \frac{1}{2}\lim_{n\to\infty}\sum_{k=1}^{p(n)}(\varphi,\ \varphi_{A_k}^*)\phi_k = \varphi\,.$$

For any $\varphi,\ \psi\in\Gamma_{aS}\cap\Gamma_{ase}$, we get

$$\lim_{n\to\infty}\sum_{k=1}^{p(n)}\frac{1}{2}(\varphi,\ \varphi_{A_k}^*)(\phi_k,\ \psi) = (\varphi,\ \psi)\,.$$

This shows that $\frac{1}{2}\sum_k(\psi,\ \phi_k)\varphi_{A_k}^*$ converges weakly to ψ. If the generalized bilinear relation is valid, the $\varphi_{A_k}^*$ span $\Gamma_{aS}\cap\Gamma_{ase}$, which is seen by Theorem 2 and Corollary 2, and this weak convergence is, in fact, strong convergence (Karlin [4]).

13. Finally the following Theorem gives another equivalent condition for the validity of the generalized bilinear relation on the surfaces on which $\Gamma_{he}\cap\Gamma_{hse}^{*}\subset\Gamma_{he}^{*}$ holds.

Theorem 10. *Under the condition* $\Gamma_{he}\cap\Gamma_{hse}^{*}\subset\Gamma_{he}^{*}$, *the generalized bilinear relation is valid for any* $\omega\in\Gamma_{h0}$ *and* $\sigma\in\Gamma_{hse}$, *if and only if*

$$\frac{1}{4}\sum_{k=1}^{p(n)}\{(\varphi,\varphi_{B_k}^{*})\varphi_{A_k}-(\varphi,\varphi_{A_k}^{*})\varphi_{B_k}\}\longrightarrow\varphi\qquad(n\to\infty)\tag{20}$$

holds for any $\varphi\in\Gamma_{as}\cap\Gamma_{ase}$.

Proof. By the assumption, φ, $Re\,\varphi$ and $Im\,\varphi$ belong to $\Gamma_{h0}\cap\Gamma_{h0}^{*}$, and the norms of

$$\begin{aligned}\tilde{T}_n\varphi &= \sum_{k=1}^{p(n)}\left\{\left(\int_{B_k}\varphi\right)\bar{\sigma}(A_k)-\left(\int_{A_k}\varphi\right)\bar{\sigma}(B_k)\right\}\\ &=\frac{1}{2}\sum_{k=1}^{p(n)}\{(\varphi,\varphi_{B_k}^{*})\bar{\sigma}(A_k)-(\varphi,\varphi_{A_k}^{*})\bar{\sigma}(B_k)\},\\ \tilde{T}_n(Re\,\varphi) &= \sum_{k=1}^{p(n)}\left\{\left(\int_{B_k}Re\,\varphi\right)\bar{\sigma}(A_k)-\left(\int_{A_k}Re\,\varphi\right)\bar{\sigma}(B_k)\right\}\end{aligned}$$

and

$$\tilde{T}_n(Im\,\varphi) = \sum_{k=1}^{p(n)}\left\{\left(\int_{B_k}Im\,\varphi\right)\bar{\sigma}(A_k)-\left(\int_{A_k}Im\,\varphi\right)\bar{\sigma}(B_k)\right\}$$

are bounded as $n\to\infty$. Therefore

$$\lim_{n\to\infty}\sum_{k=1}^{p(n)}\left\{\left(\int_{B_k}Re\,\varphi\right)\varphi_{A_k}-\left(\int_{A_k}Re\,\varphi\right)\varphi_{B_k}\right\}\in\Gamma_{kse}$$

and this has the same real periods as φ. Hence it is equal to φ because of Proposition 4. Further, $-i\varphi\in\Gamma_{kse}^{*}=\Gamma_{kse}$ and

$$\lim_{n\to\infty}\sum_{k=1}^{p(n)}\left\{\left(\int_{B_k}Im\,\varphi\right)\varphi_{A_k}-\left(\int_{A_k}Im\,\varphi\right)\varphi_{B_k}\right\}\in\Gamma_{kse},$$

and this has the same real periods as $-i\varphi$. Thus we get the necessity.

Suppose next that $\Gamma_{he}\cap\Gamma_{hse}^{*}\subset\Gamma_{he}^{*}$, but the generalized bilinear relation does not holds for some $\sigma\in\Gamma_{hse}$. Then there exists a τ of class $\Gamma_{h0}\cap\Gamma_{hse}^{*}=\Gamma_{h0}\cap\Gamma_{h0}^{*}$ such that $\lim_{n\to\infty}\|\tilde{T}_n\tau\|=\infty$ by Corollary 4. Let $\varphi=\tau+i\tau^{*}$, then $\varphi\in\Gamma_{kse}$ and $\lim_{n\to\infty}\|\tilde{T}_n\varphi\|=\infty$. Hence

$$\lim_{n\to\infty}\left\| \frac{1}{2}\sum_{k=1}^{p(n)} \{(\varphi, \varphi_{B_k^*})\varphi_{A_k} - (\varphi, \varphi_{A_k^*})\varphi_{B_k}\} \right\| = \infty .$$

The assertion has been completely proved.

KYOTO UNIVERSITY

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