J. Math. Kyoto Univ. (JMKYAZ)
39-3 (1999), 399–420

Well-posedness in C^∞ for some weakly hyperbolic equations

By

Ferruccio COLOMBINI and Nicola ORRÚ

1. Introduction

We consider a weakly hyperbolic equation:

$$L(t,x,\partial_t,\partial_x)u=\sum_{j+|\alpha|\le m} a_{j\alpha}(t,x)\partial_t^j\partial_x^\alpha u(t,x)=f, \tag{1}$$

that is, the characteristic polynomial

$$L_m(t,x,\tau,\xi)=\sum_{j+|\alpha|=m} a_{j\alpha}(t,x)\tau^j\xi^\alpha,$$

for all (t,x) and ξ real, has m real roots in τ

$$\tau_1(t,x,\xi)\le\tau_2(t,x,\xi)\le\cdots\le\tau_m(t,x,\xi).$$

Suppose that the coefficients $a_{j\alpha}(t,x)\in C^\infty$ and that $a_{m,0}(t,x)\equiv 1$. Then it was shown by Lax and Mizohata that for the Cauchy problem:

$$\begin{cases} Lu=f, \\ \partial_t^j u(0,x)=\phi_j(x), \quad (j=0,\cdots,m-1)\end{cases} \tag{2}$$

to be well-posed in C^∞ in a neighbourhood of the origin $(0,0)$ it is necessary that L is weakly hyperbolic. At the beginning of this century, Eugenio Elia Levi studied the case of one space variable x when the characteristic roots are of constant multiplicity, and proved that certain conditions are sufficient for well-posedness. These conditions are also necessary (see Mizohata-Ohya [MO] for the case of double characteristics). It is then natural to consider the problem that, when the roots have variable multiplicities and the coefficients depend only on time, what are the necessary or sufficient conditions for the well-posedness of the Cauchy problem.

We consider a homogeneous operator of arbitrary order. Even in this case, and even if the coefficients are analytic, one cannot deduce the C^∞ well-

Communicated by Prof. K. Ueno, August 10, 1998
Revised May 24, 1999

posedness. For example, the Cauchy problem for the equation

$$\partial_t^2 u - 2t\partial_t\partial_x u + t^2\partial_x^2 u = 0$$

is not well-posed in C^∞. Indeed with the change of variables

$$t' = t, \qquad x' = x + t^2/2$$

the equation becomes:

$$\partial_{t'}^2 u + \partial_{x'} u = 0.$$

Observe that in this case the characteristic roots are of constant multiplicity.

O. Oleinik in 1970 studied a hyperbolic equation of second order

$$\partial_t^2 u - \sum_{i,j} a_{ij}(t,x)\partial_{x_i}\partial_{x_j} u + \text{ lower order terms } = f \tag{3}$$

with characteristics of variable multiplicity, giving conditions which are sufficient for well-posedness. She supposed that the coefficients were in C^∞. Then Nishitani [Ni1] studied the case of one space variable and analytic coefficients, showing in particular that, if there are no lower order terms, the Cauchy problem is locally well-posed in C^∞. In [Ni2] he gave necessary and sufficient conditions for well-posedness in the presence of lower order terms. (D'Ancona [DA] examined the situation considered in [Ni1] and gave more precise energy estimates, following Oleinik). Colombini, Jannelli and Spagnolo in the meanwhile, studying equations of the form (3) with coefficients depending only on time, found a general conditions, sufficient for well-posedness, which they called logarithmic condition:

$$\int_0^T \frac{|a_t'(t,\xi)|}{a(t,\xi)+1}\,dt \le N\log(|\xi|+1) + C,$$

for any ξ, where $a(t,\xi) = \Sigma a_{ij}(t)\xi_i\xi_j$, and N, C are positive constants.

T. Yamazaki, studying equations of the form (3) in an abstract setting, obtained interesting results on the loss of regularity of the solution [Ya1], showing in particular that in an equation of the form

$$\partial_t^2 u - a(t)\partial_x^2 u = 0 \tag{4}$$

with $a(t) = t^{2k}$, k being an integer ≥ 1, there is no loss of regularity passing through the instant $t = 0$. That is, if $(u(t_0,\cdot),\ \partial_t u(t_0,\cdot)) \in H^1 \times H^0(R)$ for some $t_0 < 0$, then $(u(t,\cdot), \partial_t u(t,\cdot)) \in H^1 \times H^0(R)$, also when $t > 0$. For $t = 0$, $(u, \partial_t u) \in H^\sigma \times H^\gamma$, with $\sigma > 0$, $\gamma > 0$ and $\sigma + \gamma = 1$ (σ and γ depending on k). Using this method she gave an example [Ya2] of an equation of the form (4) with $a(t) \in C^\infty$, $a(t) \ge 0$, with infinitely many zeros near $t = 0$, for which the Cauchy problem is well-posed. In this case the logarithmic condition does not hold. We remark that an example of Colombini and Spagnolo [CS] shows that there exists a coefficient $a(t)\geq 0$, $a(t)\in C^\infty$, such that for (4) the Cauchy problem is ill-posed.

K. Kajitani studied the possibility of generalizing the logarithmic condition to the case of coefficients depending also on $x_1,\cdots,x_n$, that is to an equation of the form (3). As a special case he got some of the results obtained by Nishitani. He applied his method to general hyperbolic equations of second order and to equations of higher order. His results contain the logarithmic condition given in Remark 2.

Following his ideas, and those of Yamazaki, Orrú studied in [Or] some properties of the logarithmic condition.

We mention also the important works of Bronštein [Br1], [Br2] on the Lipschitz continuity of the characteristic roots of a hyperbolic polynomial (with sufficiently regular coefficients) and on the Cauchy problem in the Gevrey classes for hyperbolic equations of higher order. Ivrii and Petkov [IP] studied necessary conditions for C^∞ and Gevrey well-posedness of the Cauchy problem for hyperbolic equations showing, in special cases, that these conditions are also sufficient. Their results are used by Nishitani and are close to ours. We use a method of S. Mizohata ([Mi1], [Mi2]), using which he proved the well known Lax-Mizohata theorem.

If the coefficients of the equation are in C^∞ and if the differences of the characteristic roots vanish of finite order, then our theorems follow from some interesting results of Yamamoto and Mandai ([YaK] and [Ma]), on equations of order m with arbitrary lower order term. In difference from these, we will assume that the coefficients are not very regular.

We wish also to quote a paper of Ohya-Tarama on weakly hyperbolic equations in Gevrey spaces, some papers of D'Ancona, Ishida, Manfrin on weakly hyperbolic equations which are semilinear or quasilinear, a paper of D'Ancona-Spagnolo on the Cauchy problem for $N\times N$ hyperbolic homogeneous systems of first order and a paper of Nishitani in which he extends the results of [Ni2] to 2×2 systems of first order. In any way we observe that, for example, for the equation

$$\partial_t^2 u-3t^k\partial_t\partial_x u+2t^{2k}\partial_x^2 u=0$$

the Cauchy problem is well-posed (see the following theorem 1) but the associated 2×2 system doesn't satisfy the hypotheses of D'Ancona-Spagnolo.

2. Results

We give the following definitions (see [Mi2]) pag. 3 and pag. 7).

Definition 1. We say that (2) is H^∞-wellposed for $t\geq 0$, if for all $\phi_0,\cdots,\phi_{m-1}\in H^\infty$ and for $f\equiv 0$, there exists a unique solution $u(t,x)\in H^\infty(R_x^n)$ with $(\frac{\partial}{\partial t})^j u(t,x)\in H^\infty(R_x^n)$ $(0\leq j\leq m-1)$, $t\geq 0$.

Definition 2. We say that (2) is uniformly H^∞-wellposed in the positive direction, if for any $\phi_0,\cdots,\phi_{m-1}\in H^\infty$, $t_0\in[0,T]$ and any $f\in C([0,T];H^\infty)$, there exists a unique solution $u \in C^m([t_0, T]; H^\infty)$ of (1) such that $\partial_t^j u_{|t=t_0} = \phi_j$ $(0 \le j \le m-1)$.

We have the following theorems:

Theorem 1. *Consider the equation*

$$L(t, \partial_t, \partial_x)u = \partial_t^2 u + a(t)\partial_t \partial_x u + b(t)\partial_x^2 u = f,$$

with $a(t)$, $b(t) \in C^{2k+1}$ *in a neighbourhood of zero* (k *is a positive integer*), $a(0) = b(0) = 0$ *and such that they satisfy either the condition*

$$|b'(0)| + \cdots + |b^{(2k)}(0)| \neq 0,$$

or

$$|a'(0)| + \cdots + |a^{(k)}(0)| \neq 0.$$

Suppose that

$$L(t, \tau, 1) = \tau^2 + a(t)\tau + b(t)$$

has only real roots (*that is*, L *is hyperbolic*) *in* τ:

$$\tau_1(t) \le \tau_2(t).$$

Then the Cauchy problem

$$\begin{cases} Lu = f, \\ u|_{t=0} = \phi_0, \\ \partial_t u\,|_{t=0} = \phi_1 \end{cases}$$

is uniformly well-posed in H^∞ *near* 0 *if and only if*

$$\frac{|\tau_1(t)|^2 + |\tau_2(t)|^2}{|(\tau_1 - \tau_2)(t)|^2} \le C \tag{5}$$

near zero. (*see the book "On the Cauchy problem" of S. Mizohata* [*Mi*2]).

Remark 1. Condition (5) may also be written in terms of the coefficients of L in the following way:

$$\frac{a^2}{a^2 - 4b} \le C.$$

Remark 2. If $a, b \in C^\infty$ and vanish to infinite order at zero, condition (5) is neither necessary nor sufficient for the well-posedness of Cauchy problem.

For instance the equation

$$\partial_t^2 u - a(t)\partial_x^2 u = 0,$$

with $a(t)>0$ for $t\neq 0$, $\partial_t^j a(0)=0$ for any $j=0,1,\cdots$ obviously satisfies (5), but the Cauchy problem may be ill-posed ([CS]).

If $a,b\in C^1$ then the following condition is sufficient (logarithmic condition):

$$\int_0^T\left[\frac{|a'(t)|}{\sqrt{a^2-4b+\varepsilon^2}}+\frac{|(a^2-4b)'|}{a^2-4b+\varepsilon^2}\right]dt\leq C\log\frac{1}{\varepsilon}, \tag{6}$$

for any small $\varepsilon>0$ (C is a constant). See [CJS] and [Ka].

Put

$$(\tau_2-\tau_1)(t)=2^{-1/t}\quad \text{for}\quad t>0,\qquad =0\quad \text{for}\quad t\leq 0,$$

and

$$(\tau_1+\tau_2)(t)=-\sum_k 2^k 2^{-2^k}\chi\left(\frac{t-\frac{1}{2^{k+1}}}{\frac{1}{2^k}-\frac{1}{2^{k+1}}}\right),$$

where χ is a C^∞ function with support in $[0,1]$ and with $\chi(1/2)=1$. Then $a(t)$ and $b(t)$ are C^∞ function, vanish to infinite order at zero and satisfy (6) but not (5). The Cauchy problem is well-posed.

Theorem 2. *Consider the equation*

$$L(t,\partial_t,\partial_x)u=\partial_t^3u+a(t)\partial_t^2\partial_x u+b(t)\partial_t\partial_x^2u+c(t)\partial_x^3u=f. \tag{7}$$

Suppose that $a,b,c\in C^{l+1}$ *(l is an integer* ≥ 3*),* $a(0)=b(0)=c(0)=0$, *and satisfy either the condition*

$$|c'(0)|+\cdots+|c^{(l)}(0)|\neq 0 \tag{8}$$

or

$$|b'(0)|+\cdots+|b^{([l/2])}(0)|\neq 0. \tag{9}$$

Suppose that

$$L(t,\tau,1)=\tau^3+a(t)\tau^2+b(t)\tau+c(t)$$

has only real roots in τ:

$$\tau_1(t)\leq\tau_2(t)\leq\tau_3(t)$$

(*this implies that* $b'(0)=c'(0)=c''(0)=(0)$.

Then the Cauchy problem

$$\begin{cases} Lu=f,\\ u|_{t=0}=\phi_0,\\ \partial_t u|_{t=0}=\phi_1,\\ \partial_t^2 u|_{t=0}=\phi_2,\end{cases} \tag{10}$$

is uniformly well-posed in H^∞ near 0 *if and only if*

$$\frac{\tau_1^2+\tau_2^2}{(\tau_1-\tau_2)^2}+\frac{\tau_2^2+\tau_3^2}{(\tau_2-\tau_3)^2}+\frac{\tau_3^2+\tau_1^2}{(\tau_3-\tau_1)^2}\le C \tag{11}$$

near zero.

Remark 3. Again condition (11) may be expressed in terms of the coefficients of (7) in the following way:

$$\frac{-10a^3c+3a^2b^2+36abc-10b^3-27c^2}{-4a^3c+a^2b^2+18abc-4b^3-27c^2}\le C.$$

The denominator is the discriminant of the characteristic polynomial.

Remark 4. The following condition is equivalent to (11) when the hypotheses of theorem 2 are satisfied:

$$t^2\sum_{k\neq l}\frac{(\tau_k')^2+(\tau_l')^2}{(\tau_k-\tau_l)^2}\le C, \tag{12}$$

near zero. Observe that, as $l\ge 3$, the derivatives $\tau_j'(t)\in L^\infty$ (Bronštein).

If we remove the hypothesis that $a(0)=b(0)=c(0)=0$, and require that either

$$(\tau_1(t)-\tau_j(0))(\tau_2(t)-\tau_j(0))(\tau_3(t)-\tau_j(0))$$

vanishes to order $\le l$ for $t=0$, or

$$(\tau_1(t)-\tau_j(0))(\tau_2(t)-\tau_j(0))+(\tau_2(t)-\tau_j(0))(\tau_3(t)-\tau_j(0)) \\ +(\tau_3(t)-\tau_j(0))(\tau_1(t)-\tau_j(0))$$

vanishes to order $\le[l/2]$ for $t=0$, for $j=1,2,3$, then condition (12) is necessary and sufficient in order that (10) is well-posed in C^∞.

In particular, for example, the Cauchy problem for the operator

$$\partial_t^3-(1+3t^k)\partial_t^2\partial_x+(3t^k+2t^{2k})\partial_t\partial_x^2-2t^{2k}\partial_x^3$$

is well-posed.

Theorem 3. *Consider the equation*

$$L(t,\partial_t,\partial_x)u=\partial_t^m u+\sum_{j=0}^{m-1}a_{m-j}(t)\partial_t^j\partial_x^{m-j}u=f. \tag{13}$$

Suppose that the coefficients are in C^{l+1} (where l is an integer $\ge m$), that they vanish at zero, and that either

$$(14) \qquad |a'_m(0)|+\cdots+|a_m^{(l)}|\neq 0,$$

or

$$(15) \qquad |a'_{m-1}(0)|+\cdots+|a_{m-1}^{([l/2])}|\neq 0,$$

holds.

Suppose that the polynomial (in τ):

$$L(t,\tau,1)=\tau^m+\sum_{j=1}^{m-1} a_{m-j}(t)\tau^j$$

has only real roots:

$$\tau_1(t)\leq\tau_2(t)\leq\cdots\leq\tau_m(t).$$

Then the condition

$$(16) \qquad \sum_{j\neq k}\frac{\tau_j^2+\tau_k^2}{(\tau_j-\tau_k)^2}\leq C$$

near zero, is necessary and sufficient in order that the Cauchy problem for (13) *is uniformly well-posed in H^∞ near* 0.

Remark 5. We can see that for an equation like (13) if the Cauchy problem is H^∞ well-posed it is C^∞ well-posed, taking into account the finite speed of propagation.

In accordance with [IP] we give the following definition.

Definition 3. We say that the Cauchy problem for the equation (13) is C^∞ well-posed if

i) for any $f\in C^\infty$ and any $\phi_j\in C^\infty$ $(j=0,\cdots,m-1)$ there exists a solution $u\in C^m([0,+\infty);C^\infty)$.

ii) From the fact that $u\in C^m([0,+\infty)\times R)$ is a solution of the Cauchy problem with $\phi_0,\cdots,\phi_{m-1}=0$, $f=0$ in $[0,T_0]\times R_x$ it follows that $u=0$ in $[0,T_0]\times R_x$.

We give only the proof of theorem 2 and an outline of the proof of theorem 3. We wish to thank Domenico Luminati, Tatsuo Nishitani, Sergio Spagnolo and Jean Vaillant with whom we had useful discussions on the subject of this paper.

3. Proof of Theorem 2

We prove the sufficiency of condition (11). Let t be ≥ 0. Condition (11) implies that:

$$t^2 \sum_{i \neq j} \frac{(\tau_i')^2 + (\tau_j')^2}{(\tau_i - \tau_j)^2} \leq C_1$$

near zero. The following considerations are based on Newton's polygonal line method.

If $\tau_1(t) = d_1 t^j + \cdots$, $\tau_2(t) = d_2 t^j + \cdots$, $\tau_3(t) = d_3 t^j + \cdots$, with $j \geq 1$ an integer, d_1, d_2, $d_3 \in \boldsymbol{R}$, $d_1 \neq 0$, $d_1 \neq d_2$, $d_1 \neq d_3$ (d_2 or d_3 may vanish), then $|t\tau_1'(t)| \leq C|\tau_1|$ near zero. In fact

$$L_\tau(\tau_1)\tau_1' + L_t(\tau_1) = 0,$$

$$L_\tau(\tau_1) = (\tau_1 - \tau_2)(\tau_1 - \tau_3) \sim t^{2j}, \quad L_t(\tau_1) \precsim t^{3j-1}$$

(the symbol $\precsim$ means that $L_t(\tau_1)$ is an infinitesimal of order less or equal to t^{3j-1}.)

If $\tau_1(t) = d_1 t^j + \cdots$, $\tau_2(t) = d_2 t^k + \cdots$, $\tau_3(t) = d_3 t^k + \cdots$, with $1 \leq j < k$ integers, d_1, d_2, d_3 real, $d_1 \neq 0$, $d_2 \neq 0$, $d_3 \neq d_2$ (d_3 may be zero), then $|t\tau_2'(t)| \leq C|\tau_2|$ near zero.

$$L_\tau(\tau_2) \sim t^{j+k}, \quad L_t(\tau_2) = a'\tau_2^2 + b'\tau_2 + c' \precsim t^{j+2k-1}.$$

If $\tau_1(t) = d_1 t^j + \cdots$, $\tau_2(t) = d_2 t^k + \cdots$, $\tau_3(t) = o(t^k)$, with $1 \leq j \leq k$, d_1 and $d_2 \neq 0$, then $|t\tau_3'(t)| \leq C|\tau_2 - \tau_3| \sim t^k$ near zero.

$$L_\tau(\tau_3) \sim t^{j+k}, \quad L_t(\tau_3) = a'\tau_3^2 + b'\tau_3 + c' \precsim t^{j+2k-1}.$$

(this case occurs if $a \sim t^j$, $b \sim t^{j+k}$, $c = o(t^{j+2k})$, as one can see using Rouché's theorem).

We perform in (7) the Fourier transform with respect to x. Let $v = \mathscr{F}_x u$, take $f = 0$. Then

$$L(t, \partial_t, i\xi)v = \partial_t^3 v(t, \xi) + a(t)(i\xi)\partial_t^2 v + b(t)(i\xi)^2 \partial_t v + c(t)(i\xi)^3 v = 0. \tag{17}$$

Suppose that $u \in C^3([0, T], C^\infty)$ is a solution of (17), with compact, uniformly bounded support in x, then we have $v \in C^3([0, T], S_\xi)$.

Let us fix $\xi \geq 1$ and consider the energy

$$E(t) = |L_{12}(t, \partial_t, i\xi)v|^2 + |L_{23}(t, \partial_t, i\xi)v|^2 + |L_{31}(t, \partial_t, i\xi)v|^2,$$

with $L_1(t, \tau, \xi) = \tau - \tau_1(t)\xi$, $L_{12}(t, \tau, \xi) = (\tau - \tau_1(t)\xi)(\tau - \tau_2(t)\xi)$, and so on.

We have

$$\begin{aligned}
E'(t) &= 2Re(\partial_t L_{12} v)\overline{L_{12} v} + \text{ other terms} \\
&= 2Re(L_{12}\partial_t v + L'_{12,t} v)\overline{L_{12} v} + \text{ other terms} \\
&= 2Re(L_{12}\partial_t v - i\tau_3(t)\xi L_{12} v)\overline{L_{12} v} \\
&\quad + 2Re(-i\tau_1'(t)\xi L_2 v - i\tau_2'(t)\xi L_1 v)\overline{L_{12} v} + \text{ other terms} \\
&= 2Re(-i\tau_1'(t)\xi L_2 v - i\tau_2'(t)\xi L_1 v)\overline{L_{12} v} + \text{ other terms,}
\end{aligned}$$

where

$$L'_{12,t}(t,\tau,\xi)=\partial_t L_{12}(t,\tau,\xi)$$
$$=-\tau'_1(t)\xi L_2(t,\tau,\xi)-\tau'_2(t)\xi L_1(t,\tau,\xi),$$

which gives

$$L'_{12,t}(t,\partial_t,i\xi)v=-i\tau'_1\xi L_2 v-i\tau'_2\xi L_1 v.$$

We have

$$L_2=\tau-\tau_2(t)\xi=\frac{L_{12}-L_{23}}{(\tau_3(t)-\tau_1(t))\xi},$$

$$|\tau'_1(t)\xi L_2 v|\le\frac{C_1}{t}(\tau_3(t)-\tau_1(t))|\xi|\frac{|L_{12}v-L_{23}v|}{(\tau_3(t)-\tau_1(t))|\xi|}$$
$$\le\frac{C_1}{t}(|L_{12}v|+|L_{23}v|).$$

So

$$|E'(t)|\le\frac{C_1}{t}E(t).$$

Now we choose, for $0<t<\frac{1}{\xi}$, the energy

$$\tilde{E}(t)=|\partial_t^2 v|^2+\xi^2|\partial_t v|^2+\xi^4|v|^2.$$
$$|\tilde{E}'(t)|\le C_2\xi\tilde{E}(t).$$

If $0\le t_1\le t_2\le 1/\xi$, we obtain

$$\tilde{E}(t_2)\le e^{C_2}\tilde{E}(t_1).$$

If $1/\xi\le t_1\le t_2\le T$, we have:

$$E(t_2)\le\left(\frac{t_2}{t_1}\right)^{C_1}E(t_1)\le C_3\xi^{C_1}E(t_1).$$

Now $E(t)\le C\cdot\tilde{E}(t)$ near zero. Besides we have

$$\xi^2=\xi\frac{L_1-L_2}{\tau_2-\tau_1}=\frac{L_{12}-L_{31}}{(\tau_2-\tau_1)(\tau_3-\tau_2)}-\frac{L_{12}-L_{23}}{(\tau_2-\tau_1)(\tau_3-\tau_1)}.$$

By condition (11) there exists an integer $p>0$ such that $\tau_2-\tau_1\ge Ct^p>0$, $\tau_3-\tau_2\ge Ct^p$, so for $t\ge\frac{1}{\xi}$ we have

$$|\xi^2 v|\le C\xi^{2p}\sqrt{E(t)}.$$

The same estimate holds for $|\xi\partial_t v|$ and $|\partial_t^2 v|$. So

$$\tilde{E}(t)\leq C\xi^{4p}E(t), \qquad \text{if } t\geq 1/\xi.$$

If $0\leq t_1<\frac{1}{\xi}<t_2\leq T$, we estimate $\tilde{E}(t_2)$ with $E(t_2)$, $E(t_2)$ with $E(1/\xi)$, $E(1/\xi)$ with $\tilde{E}(1/\xi)$ and finally $\tilde{E}(1/\xi)$ with $\tilde{E}(t_1)$.

At last we have the estimate:

$$\tilde{E}(t_2)\leq C'(\xi^M+1)\tilde{E}(t_1), \qquad \text{if } 0\leq t_1\leq t_2\leq T, \tag{18}$$

where $M>0$ depends only on C_1 and p.

The inequality (18) holds, taking C' large, also for $0\leq\xi\leq 1$ and taking the absolute value, also for negative ξ.

Hence the Cauchy problem for (7) is well-posed in C^∞ (the uniqueness of the solution and the finite speed of propagation follow from the well-posedness in some Gevrey classes and from Holmgren's method; we can approximate the equation with strictly hyperbolic equations. Otherwise one can use the well-posedness of the Cauchy problem in the space of analytic functionals).

We prove now the necessity of condition (11). Suppose that (11) doesn't hold. We consider several different cases and we follow a method of Petrowsky and Mizohata.

CASE A. The three roots have the same principal part dt^k, with $k\geq 1$ an integer and $d\neq 0$. We can write the characteristic polynomial of (7) as follows:

$$\begin{gathered}(\tau-dt^k i\xi)^3+\mathcal{O}(t^{k+1})\xi\tau^2\\ +\mathcal{O}(t^{2k+1})\xi^2\tau+\mathcal{O}(t^{3k+1})\xi^3.\end{gathered}$$

Now

$$\begin{aligned}(\tau-dt^k i\xi)^3|_{\tau=\partial t}=(\partial_t&-dt^k i\xi)^3\\ &+3kdt^{k-1}i\xi(\partial_t-dt^k i\xi)\\ &+k(k-1)dt^{k-2}i\xi.\end{aligned}$$

If we replace $\partial_t-dt^k i\xi$ with σ, we have:

$$\sigma^3+3kdt^{k-1}i\xi\sigma+k(k-1)dt^{k-2}i\xi. \tag{19}$$

Suppose that $t^{k-1}\xi$ is small with respect to $t^{2k}\xi^2$. This is equivalent to $t^{k+1}\xi\gg 1$. Take

$$t=t'\xi^{-1/(k+1)+\varepsilon}, \qquad \text{with} \quad 0<\delta_1\leq t'\leq\delta_2, \quad \varepsilon>0. \tag{20}$$

Here δ_1, δ_2 are constants, $\varepsilon<1/(k+1)$ is a constant to be fixed.

Now, supposing $\xi>0$, we see that:

$$\begin{gathered}t^{k+1}\xi\gg 1,\\ t^{2k-4}\xi^2\ll t^{3k-3}\xi^3,\\ t^{k-2}\xi\ll(t^{k-1}\xi)^{\frac{3}{2}},\end{gathered}$$

hence the third term in (19) can be neglected.

The equation

$$\sigma^3+3kdt^{k-1}i\xi\sigma=0$$

has three roots

$$\sigma_{1,2}=\pm\sqrt{-3dkt^{k-1}i\xi}, \qquad \sigma^3=0.$$

Consider the polynomial $(\sigma-\sigma_1)(\sigma-\sigma_2)=\sigma^2-(\sigma_1+\sigma_2)\sigma+\sigma_1\sigma_2$ and associate to it the operator

$$L_{12}(t,\partial_t,i\xi)u=(\partial_t-dt^ki\xi)^2-(\sigma_1+\sigma_2)(\partial_t-dt^ki\xi)+\sigma_1\sigma_2.$$

L_{23}, L_{31} are defined similarly.

We now introduce

$$E(t)=|L_{23}v|^2-|L_{31}v|^2-|L_{12}v|^2.$$

We have

$$\begin{aligned}
\partial_t|L_{23}v|^2&=2Re(\partial_tL_{23}v)\overline{L_{23}v}\\
&=2Re((\partial_t-dt^ki\xi-\sigma_1)L_{23}v)\overline{L_{23}v}+2Re\sigma_1|L_{23}v|^2\\
&=2Re[(\partial_t-dt^ki\xi)^3v-(\sigma_2+\sigma_3)(\partial_t-dt^ki\xi)^2v\\
&\quad+\sigma_2\sigma_3(\partial_t-dt^ki\xi)v-\sigma_1(\partial_t-dt^ki\xi)^2v\\
&\quad+\sigma_1(\sigma_2+\sigma_3)(\partial_t-dt^ki\xi)v-\sigma_1\sigma_2\sigma_3v\\
&\quad-\sigma_2'(\partial_t-dt^ki\xi-\sigma_3)v-\sigma_3'(\partial_t-dt^ki\xi-\sigma_2)v]\overline{L_{23}v}\\
&\quad+2Re\sigma_1|L_{23}v|^2\\
&=2Re[(\partial_t-dt^ki\xi)^3v+3dkt^{k-1}i\xi(\partial_t-dt^ki\xi)v]\cdot\overline{L_{23}v}\\
&\quad+2Re(-\sigma_2'L_3v-\sigma_3'L_2v)\overline{L_{23}v}+2Re\sigma_1|L_{23}v|^2\\
&=2Re[(\tau-dt^ki\xi)^3|_{\tau=\partial_t}v-k(k-1)dt^{k-2}i\xi v]\overline{L_{23}v}\\
&\quad+2Re(-\sigma_2'L_3v-\sigma_3'L_2v)\overline{L_{23}v}+2Re\sigma_1|L_{23}v|^2.
\end{aligned}$$

($L_1=\partial_t-dt^ki\xi-\sigma_1$, and so on).

We can choose σ_1, σ_2 so that

$$Re\sigma_1\geq c_0\sqrt{t^{k-1}\xi}, \qquad Re\sigma_2\leq -c_0\sqrt{t^{k-1}\xi}, \tag{21}$$

with a positive constant c_0.

We have

$$\left|\frac{\sigma_2'}{\sigma_2-\sigma_1}\right|\leq\frac{C}{t}=o(\sqrt{t^{k-1}\xi}),$$

$$L_{23}-L_{31}=(\sigma_1-\sigma_2)L_3,$$

$$\sigma_3'=0,$$

$$\begin{aligned}t^{k-2}\xi v&=\frac{t^{k-2}\xi}{\sigma_2-\sigma_1}(L_1v-L_2v)\\&=\frac{t^{k-2}\xi}{(\sigma_2-\sigma_1)(\sigma_3-\sigma_2)}(L_{12}v-L_{13}v)-\frac{t^{k-2}\xi}{(\sigma_2-\sigma_1)(\sigma_3-\sigma_1)}(L_{12}v-L_{23}v)\end{aligned}$$

and $t^{k-2}\xi\ll(t^{k-1}\xi)^{3/2}$, as we have already seen.

Similarly

$$\begin{aligned}\mathcal{O}(t^{3k+1})\xi^3v&\sim t^{3k+1}\xi^3\frac{1}{|\sigma_2-\sigma_1|}(|L_1v|+|L_2v|)\\&\sim\frac{t^{3k+1}\xi^3}{t^{k-1}\xi}(|L_{12}v|+|L_{23}v|+|L_{31}v|).\end{aligned}$$

If the ε which appears in (20) is small enough, then

$$t^{k+5/3}\xi\ll1,\qquad t^{3k+5}\xi^3\ll1,$$

$$t^{6k+2}\xi^6\ll t^{3k-3}\xi^3,$$

$$t^{3k+1}\xi^3\ll(t^{k-1}\xi)^{3/2}.$$

Moreover

$$\begin{aligned}&\mathcal{O}(t^{2k+1})\xi^2(\partial_tv-dt^ki\xi v-\sigma_1v)\\&\sim t^{2k+1}\xi^2\frac{1}{\sqrt{t^{k-1}\xi}}(|L_{12}v|+|L_{31}v|).\end{aligned}$$

Now, since $t^{k+2}\xi\ll1$, we have

$$t^{2k+1}\xi^2\ll t^{k-1}\xi.$$

$\mathcal{O}(t^{k+1})\xi L_{12}v$ can be estimated because $t^{k+1}\xi\ll\sqrt{t^{k-1}\xi}$, since $t^{2k+2}\xi^2\ll t^{k-1}\xi$, $t^{k+3}\xi\ll1$.

So, deriving also $|L_{31}v|^2$ and $|L_{12}v|^2$, we get

$$\begin{aligned}E'(t)&\geq2c_0\sqrt{t^{k-1}\xi}(|L_{23}v|^2+|L_{31}v|^2)\\&\quad-o(\sqrt{t^{k-1}\xi})(|L_{23}v|^2+|L_{31}v|^2+|L_{12}v|^2)\\&\geq c_0\sqrt{t^{k-1}\xi}E(t)\end{aligned}$$

for large ξ (see the book of Mizohata, On the Cauchy problem [Mi2]). Put

$$t_{j,\xi}=\delta_j\xi^{-1/(k+1)+\varepsilon}, \quad \text{for} \quad j=1,2.$$

Let

$$(22) \qquad \begin{cases} L_{23}v=1 \\ L_{31}v=0 \\ L_{12}v=0 \end{cases} \quad \text{if} \quad t=t_{1,\xi}.$$

We have $E(t_{1,\xi})=1$. Integrating the energy inequality, and taking into account that

$$\int_{t_{1,\xi}}^{t_{2,\xi}}\sqrt{t^{k-1}\xi}\,dt=\frac{2}{k+1}t^{(k+1)/2}\Big|_{t_{1,\xi}}^{t_{2,\xi}}\xi^{1/2}$$

$$=\frac{2}{k+1}(\delta_2^{(k+1)/2}-\delta_1^{(k+1)/2})\xi^{\frac{k+1}{2}\varepsilon},$$

we get

$$E(t_{2,\xi})\geq\exp(c_1\xi^{(k+1)\varepsilon/2}),$$

with $c_1>0$.

We can find values for $v(t_{1,\xi})$, $\partial_t v(t_{1,\xi})$, $\partial_t^2 v(t_{1,\xi})$ with polynomial growth which verify (22). On the contrary $(v(t_{2,\xi}),\ \partial_t v(t_{2,\xi}),\ \partial_t^2 v(t_{2,\xi}))$ hasn't polynomial growth. Hence the Cauchy problem is not uniformly well-posed in C^∞ in any neighbourhood of zero.

Case B. Two roots have principal part dt^k, with $k\geq 1$ and $d\neq 0$, the other has p.p $d't^k$, with $d'\neq d$ (d' may be zero).

We can write the characteristic polynomial as follows:

$$(\tau-dt^k i\xi)^2(\tau-d't^k i\xi)+\mathcal{O}(t^{k+1})\xi\tau^2$$
$$+\mathcal{O}(t^{2k+1})\xi^2\tau+\mathcal{O}(t^{3k+1})\xi^3.$$

Consider the operator

$$\partial_t^2-2dt^k i\xi\partial_t+d^2t^{2k}(i\xi)^2=(\partial_t-dt^k i\xi)^2+kdt^{k-1}i\xi.$$

Let σ_1, σ_2 be the roots of

$$\sigma^2+kdt^{k-1}i\xi=0.$$

Take t as in (20). Consider the polynomials

$$L_{13}=(\tau-dt^k i\xi-\sigma_1)(\tau-d't^k i\xi),$$
$$L_{23}=(\tau-dt^k i\xi-\sigma_2)(\tau-d't^k i\xi),$$
$$L_{12}=(\tau-dt^k i\xi)^2$$

and the energy

$$E(t)=|L_{23}v|^2-|L_{31}|^2-|L_{12}v|^2.$$

We have

$$\begin{aligned}
\partial_t|L_{23}v|^2 &= 2Re(\partial_t L_{23}v)\overline{L_{23}v} \\
&= 2Re\sigma_1|L_{23}v|^2 \\
&\quad + 2Re((\tau-dt^k i\xi-\sigma_1)(\tau-dt^k i\xi-\sigma_2)(\tau-d't^k i\xi)|_{\tau=\partial_t}v)\overline{L_{23}v} \\
&\quad + 2Re(-kdt^{k-1}i\xi(\partial_t-d't^k i\xi)v \\
&\quad -\sigma'_2(\partial_t-d't^k i\xi)v-kd't^{k-1}i\xi(\partial_t-dt^k i\xi-\sigma_2)v)\overline{L_{23}v} \\
&= 2Re\sigma_1|L_{23}v|^2 \\
&\quad + 2Re((\tau-dt^k i\xi)^2(\tau-d't^k i\xi)|_{\tau=\partial_t}v \\
&\quad -(\sigma_1+\sigma_2)(\tau-dt^k i\xi)(\tau-d't^k i\xi)|_{\tau=\partial_t}v \\
&\quad +\sigma_1\sigma_2(\partial_t-d't^k i\xi)v)\overline{L_{23}v} \\
&\quad + 2Re(-kdt^{k-1}i\xi(\partial_t-d't^k i\xi)v \\
&\quad -\sigma'_2(\partial_t-d't^k i\xi)v-kd't^{k-1}i\xi(\partial_t-dt^k i\xi-\sigma_2)v)\overline{L_{23}v} \\
&= 2Re\sigma_1|L_{23}v|^2+2Re((\tau-dt^k i\xi)^2(\tau-d't^k i\xi)|_{\tau=\partial_t}v)\overline{L_{23}v} \\
&\quad + 2Re(-\sigma'_2(\partial_t-d't^k i\xi)v-kd't^{k-1}i\xi(\partial_t-dt^k i\xi-\sigma_2)v)\overline{L_{23}v}.
\end{aligned}$$

Moreover

$$Re\sigma_1\geq c_0\sqrt{t^{k-1}\xi}, \qquad (c_0>0 \text{ is a constant})$$

$$\left|\frac{\sigma'_2}{\sigma_2-\sigma_1}\right|\sim\frac{1}{t}=o(\sqrt{t^{k-1}\xi}),$$

$$\frac{t^{k-1}\xi}{t^k\xi}=\frac{1}{t}=o(\sqrt{t^{k-1}\xi}),$$

$$\sigma'_2(\partial_t-d't^k i\xi)v=\frac{\sigma'_2}{\sigma_2-\sigma_1}(L_{23}v-L_{12}v),$$

$$\begin{aligned}
t^{k-1}\xi(\partial_t-dt^k i\xi)v &\sim\frac{1}{t}[t^k\xi(\partial_t-dt^k i\xi)v] \\
&\sim\frac{1}{t}\left(\frac{1}{2}(L_{13}v+L_{23}v)-L_{12}v\right).
\end{aligned}$$

$$\begin{aligned}
&C(|L_{12}v|+|L_{23}v|+|L_{31}v|)\geq|\sigma_2-\sigma_1||\partial_t v-d't^k i\xi v| \\
&+t^k\xi|\partial_t v-dt^k i\xi v|\geq C'\sqrt{t^{k-1}\xi}t^k\xi|v| \\
&\sim|t^k\xi\sigma_2 v|. \qquad (C'>0).
\end{aligned}$$

$\partial_t|L_{13}v|^2$ can be estimated similarly (we have $Re\sigma_2 \le -c_0\sqrt{t^{k-1}\xi}$).

It can be seen that the quantity

$$\partial_t|L_{12}v|^2 = 2Re((\tau - dt^k i\xi)^2(\tau - d't^k i\xi)|_{\tau=\partial_t}v)\overline{L_{12}v}$$
$$+2Re(-2kdt^{k-1}i\xi(\partial_t - dt^k i\xi)v)\cdot\overline{L_{12}v}$$

yields a correct kind of estimate. The remainder of the proof is as in case A.

CASE C. Two roots have principal part dt^k, the other has p.p. $d't^j$, with $d\neq 0$, $d'\neq 0$, $1\le j<k$. We know that $\tau_3'(t)=\mathcal{O}(t^{j-1})$.

Consider the polynomials:

$$L_1 = \tau - dt^k i\xi - \sigma_1,$$
$$L_2 = \tau - dt^k i\xi - \sigma_2,$$
$$L_3 = \tau - \tau_3 i\xi, \quad L_{23} = L_2 L_3, \quad L_{31} = L_3 L_1, \quad L_{12} = (\tau - dt^k i\xi)^2.$$

(σ_1, σ_2 are the same as in case B).

Put

$$E(t) = |L_{23}v|^2 - |L_{31}v|^2 - |L_{12}v|^2.$$

We have, as in case B,

$$\partial_t|L_{23}v|^2 = 2Re\sigma_1|L_{23}v|^2$$
$$+2Re((\tau - dt^k i\xi)^2(\tau - \tau_3 i\xi)|_{\tau=\partial_t}v)\overline{L_{23}v}$$
$$+2Re(-\sigma_2' L_3 v - \tau_3' i\xi L_2 v)\overline{L_{23}v}.$$

$$Re\sigma_1 \ge c_0\sqrt{t^{k-1}\xi}, \quad (c_0>0),$$

$$\left|\frac{\sigma_2'}{\sigma_2-\sigma_1}\right| \sim \frac{1}{t} = o(\sqrt{t^{k-1}\xi}),$$

$$\frac{t^{j-1}\xi}{t^j\xi} = \frac{1}{t} = o(\sqrt{t^{k-1}\xi}).$$

$$(\partial_t - dt^k i\xi)v = \left(\frac{1}{2}(L_{13}v + L_{23}v) - L_{12}v\right)\cdot\frac{1}{(dt^k - \tau_3)i\xi},$$

$$\tau_3' i\xi(\partial_t - dt^k i\xi)v \le C\frac{t^{j-1}}{t^j}(|L_{13}v| + |L_{23}v| + |L_{12}v|).$$

$$|\tau_3'\xi\sigma_2 v| \le C\frac{t^{j-1}}{t^j}|\sigma_2|(|(\partial_t - dt^k i\xi)v| + |L_3 v|)$$
$$\le \tilde{C}\frac{1}{t}(|L_{13}v| + |L_{23}v| + |L_{12}v|).$$

Now we have

$$\begin{aligned}
&\tau^3+ai\xi\tau^2+b(i\xi)^2\tau+c(i\xi)^3-(\tau-dt^k i\xi)^2(\tau-\tau_3 i\xi)\\
&=(\tau-\tau_3 i\xi)[(\tau-\tau_1 i\xi)(\tau-\tau_2 i\xi)-(\tau-dt^k i\xi)^2]\\
&=(\tau-\tau_3 i\xi)[\mathcal{O}(t^{k+1/2})\xi(\tau-dt^k i\xi)+\mathcal{O}(t^{2k+1})\xi^2]
\end{aligned}$$

(τ_1, τ_2, τ_3 are the roots of the characteristic polynomial).
Let's prove this identity.

Lemma. *We have*

$$|\tau_j-dt^k|<\delta t^{k+1/2}, \quad \textit{for} \quad j=1,2$$

and for t small enough, with a suitable $\delta>0$.

Proof. $a(t)=a_j t^j+\cdots$, with $a_j\neq 0$. The quantity

$$|\tau^3+a\tau^2+b\tau+c-a_j t^j(\tau-dt^k)^2|,$$

if $|\tau-dt^k|=\delta t^{k+1/2}$, is smaller than $c_1 t^{2k+j+1}$, while

$$|a_j t^j(\tau-dt^k)|\geq|a_j|\delta^2 t^{2k+j+1}.$$

If δ is small enough we have

$$\begin{aligned}
&|\tau^3+a\tau^2+b\tau+c-a_j t^j(\tau-dt^k)^2|\\
&<|a_j t^j(\tau-dt^k)^2|
\end{aligned}$$

when $|\tau-dt^k|=\delta t^{k+1/2}$.
Applying Rouché's theorem we have the thesis.

It is simple to see that, choosing $\varepsilon>0$ small in (20), we have

$$\begin{aligned}
&t^{2k+1}\xi^2\ll t^{k-1}\xi,\\
&t^{k+1/2}\xi\ll\sqrt{t^{k-1}}\xi.
\end{aligned}$$

The derivative $\partial_t|L_{31}v|^2$ is handled similarly.
We have

$$\begin{aligned}
\partial_t|L_{12}v|^2&=2Re((\tau-dt^k i\xi)^2(\tau-\tau_3 i\xi)|_{\tau=\partial_t}v)\overline{L_{12}v}\\
&+2Re(-2kdt^{k-1}i\xi(\partial_t-dt^k i\xi)v)\cdot\overline{L_{12}v}
\end{aligned}$$

and

$$|(\partial_t - dt^k i\xi)v| \le \frac{C}{t^j \xi}(|L_{13}v| + |L_{23}v| + |L_{12}v|),$$

$$\frac{t^{k-1}}{t^j} \ll \sqrt{t^{k-1}\xi}.$$

We get the estimate

$$E'(t) \ge c_0\sqrt{t^{k-1}\xi}E(t),$$

for $t_{1,\xi} \le t \le t_{2,\xi}$ and ξ large, with a positive constant c_0. We conclude as in case A.

So, having considered all possible cases, we have that the Cauchy problem is well-posed in C^∞ if and only if

$$\sum_{i \ne j} \frac{\tau_i^2 + \tau_j^2}{(\tau_i - \tau_j)^2} \le C$$

near zero.

4. Outline of the proof of Theorem 3

(*Sufficiency*) By applying Newton's method we see that either (14) is valid and $\tau_1(t) = d_1 t^{k_1} +$ higher order terms, $\cdots$, $\tau_m(t) = d_m t^{k_m} + \cdots$, with $1 \le k_1 \le k_2 \le \cdots \le k_m$ and $d_j \ne 0$ for all j, or (14) is not valid but (15) is and $\tau_1(t) = d_1 t^{k_1} + \cdots, \cdots, \tau_{m-1}(t) = d_{m-1} t^{k_{m-1}} + \cdots$, $\tau_m(t) = o(t^{k_{m-1}})$, with $1 \le k_1 \le \cdots \le k_{m-1}$ and $d_j \ne 0$ for all j.

Inequality (16) holds if and only if the principal parts of the $\tau_j(t)$ are distinct in pairs, that is if $i \ne j$ and $k_i = k_j \Rightarrow d_i \ne d_j$. We make this hypothesis.

Suppose that $\tau_{p+1}(t), \cdots, \tau_q(t)$ $(p < q)$ are the roots which have p.p. $\sim t^k$ (p.p. means principal part), $\tau_{p+1}(t)$ has p.p. $d_{p+1}t^k$. Consider the terms

$$a_p \tau^{m-p}, \cdots, a_q \tau^{m-q}$$

in the polynomial $L(t, \tau, 1)$. We have

$$a_p(t) = \tilde{a}_p t^{k_1 + \cdots + k_p} + \cdots,$$

and so on.

Put

$$f(\tau) = \tilde{a}_p t^{k_1 + \cdots + k_p} \tau^{m-p} + \cdots + \tilde{a}_q t^{k_1 + \cdots + k_q} \tau^{m-q},$$
$$g(\tau) = L(t, \tau, 1).$$

We have

$$f_\tau(t, d_{p+1}t^k) k d_{p+1} t^{k-1} + f_t(t, d_{p+1}t^k) = 0,$$
$$g_\tau(t, \tau_{p+1})\tau'_{p+1} + g_t(t, \tau_{p+1}(t)) = 0.$$

But

$$g_\tau = f_\tau(t, d_{p+1}t^k) + o(t^{k_1+\cdots+k_p+k(m-p-1)})$$
$$g_t = f_t(t, d_{p+1}t^k) + o(t^{k_1+\cdots+k_p+k(m-p)-1})$$
$$f_\tau \sim t^{k_1+\cdots+k_p+k(m-p-1)}, \qquad f_t \sim t^{k_1+\cdots+k_p+k(m-p)-1}.$$

Hence

$$\tau'_{p+1}(t) = kd^{p+1}t^{k-1} + o(t^{k-1}).$$

(if (14) doesn't hold, then $\tau'_m(t) = o(t^{k_{m-1}-1})$).

Therefore we have

$$t^2 \sum_{i \neq j} \frac{(\tau'_i)^2 + (\tau'_j)^2}{(\tau_i - \tau_j)^2} \leq C$$

near zero.

The same argument adopted for a third order equation proves that Cauchy problem for the equation (13) is well-posed in C^∞ near zero.

(*necessity*). We sketch the proof. If all the roots of L have the same principal part dt^k, with $k \geq 1$ an integer and $d \neq 0$, we can write

$$L = (\tau - dt^k\xi)^m + \sum_{j=0}^{m-1} \mathcal{O}(t^{(m-j)k+1})\xi^{m-j}\tau^j.$$

We initially neglect the terms contained in the summation on the right hand side. We wish to study the operator

$$L = (\tau - dt^k\xi)^m|_{\tau=\partial_t},$$

which has one characteristic root of multiplicity m.

We need the following lemma

Lemma. *There exists a polynomial $p_m(\sigma)$, of degree m, which is even if m is even, odd if m is odd, such that*

$$(\tau - t^k\xi)^m|_{\tau=\partial_t} = (kt^{k-1}\xi)^{m/2} p_m\left(\frac{\sigma}{\sqrt{kt^{k-1}\xi}}\right)\Bigg|_{\sigma=\partial_t - t^k\xi}$$
$$+ \sum_{j=0}^{m-2} c_{m,j}(t, \xi)\sigma^j\Bigg|_{\sigma=\partial_t - t^k\xi}$$

Each $c_{m,j}$ is a polynomial in t, ξ. If $t^\alpha\xi^\beta$ is one of its terms, choosing t like in (20), *so that $t^{k+1}\xi \gg 1$, we have*

$$t^{\alpha}\xi^{\beta} \ll (kt^{k-1}\xi)^{(m-j)/2}.$$

The polynomials p_m are defined recursively as follows:

$$\begin{cases} p_0(\sigma) = 1, \\ p_{m+1}(\sigma) = p_m(\sigma)\sigma + p'_m(\sigma), \quad \text{for } m \geq 0. \end{cases}$$

Each p_m has distinct roots $c_1, \cdots, c_m$. Let

(23) $$\sigma_j = c_j(kdt^{k-1}i\xi)^{1/2}, \quad \text{for} \quad j = 1, \cdots, m.$$

They are the roots of

$$(kdt^{k-1}i\xi)^{m/2} p_m\left(\frac{\sigma}{\sqrt{kdt^{k-1}i\xi}}\right).$$

One can see that there are some σ_j with $Re\sigma_j > 0$, let them be $\sigma_1, \cdots, \sigma_p$. Let

$$L_{\hat{j}} = (\sigma - \sigma_1) \cdots (\widehat{\sigma - \sigma_j}) \cdots (\sigma - \sigma_m)|_{\sigma = \partial_t - t^k i\xi},$$

for $j = 1, \cdots, m$ (the term with the hat is omitted).

We put

(24) $$E(t) = |L_{\hat{1}}v|^2 + \cdots + |L_{\hat{p}}v|^2 - |L_{\widehat{p+1}}v|^2 - \cdots - |L_{\hat{m}}v|^2,$$

where $v = \mathscr{F}_x u$. $E(t)$ is the energy. Making some energy estimates we have:

$$E'(t) \geq c_0\sqrt{t^{k-1}\xi}E(t),$$

with $c_0 > 0$, if $t_{1,\xi} \leq t \leq t_{2,\xi}$.

The remaining part of the proof is as in the case of an operator of third order.

In general we have roots $\tau_1, \cdots, \tau_r$ with principal part of order less than k and distinct, we have roots $\tau_{r+1}, \cdots, \tau_{r+m_1}$ with p.p. d_1t^k $(d_1 \neq 0)$, $\tau_{r+m_1+1}, \cdots, \tau_{r+m_1+m_2}$ with p.p. d_2t^k $(d_2 \neq 0), \cdots, \tau_{r+m_1+\cdots+m_{n-1}+1}, \cdots, \tau_{r+m_1+\cdots+m_n}$ with p.p. d_nt^k $(d_n \neq 0)$, such that $d_1, d_2, \cdots, d_n$ are distinct real numbers. Finally we may have roots $\tau_{r+m_1+\cdots+m_n+1}, \cdots, \tau_m$ which are infinitesimals of order higher than k, when $t \to 0$. Let $\mu = m - (r + m_1 + \cdots + m_n)$ be their number. Suppose that $m_1 > 1$.

Put

$$B_0 = (\tau - \tau_1\xi) \cdots (\tau - \tau_r\xi),$$

$$\mathscr{B}_h = \tau - d_ht^k\xi, \quad B_h = \mathscr{B}_h^{m_h}, \quad \text{for} \quad h = 1, \cdots, n.$$

$$B_{n+1} = \tau^{\mu}.$$

Put

$$B_{h,j}=[(\sigma-\sigma_{h,1})\cdots(\widehat{\sigma-\sigma_{h,j}})\cdots(\sigma-\sigma_{h,m_h})]_{\sigma=\partial_t-d_ht^k\xi|\partial_t=\tau},$$

for $h=1,\cdots,n$ and for $j=1,\cdots,m_h$. The $\sigma_{h,j}$ are defined starting from B_h like the σ_j in (23), starting from L.

Put also

$$B_{n+1,j}=\left(\frac{c_0}{4}\sqrt{t^{k-1}\xi}\right)^{\mu-1-j}\tau^j, \quad \text{for } j=0,\cdots,\mu-1.$$

Here c_0 is a constant such that

$$Re\sigma_{h,j}>0 \Rightarrow Re\sigma_{h,j}\geq c_0\sqrt{t^{k-1}\xi}$$

(see (21)).

Let

$$L_{0,j}=\frac{B_0B_1\cdots B_{n+1}}{\tau-\tau_j\xi} \quad \text{if } j=1,\cdots,r,$$

$$L_{h,j}=B_0B_1\cdots B_{h,j}\cdots B_{n+1} \quad \text{if } h=1,\cdots,n, \text{ and if } j=1,\cdots,m_h,$$

$$L_{n+1,j}=B_0B_1\cdots B_nB_{n+1,j} \quad \text{if } j=0,\cdots,\mu-1.$$

We consider the energy

$$\begin{aligned}E(t)=&-|L_{0,1}v|^2-\cdots-|L_{0,r}v|^2\\&+|L_{1,1}v|^2+\cdots+|L_{1,p_1}v|^2-|L_{1,p_1+1}v|^2-\cdots-|L_{1,m_1}v|^2\\&+|L_{2,1}v|^2+\cdots+|L_{2,p_2}v|^2-|L_{2,p_2+1}v|^2-\cdots-|L_{2,m_2}v|^2\\&+\cdots\\&+|L_{n,1}v|^2+\cdots+|L_{n,p_n}v|^2-|L_{n,p_n+1}v|^2-\cdots-|L_{n,m_n}v|^2\\&-|L_{n+1,0}v|^2-\cdots-|L_{n+1,\mu-1}v|^2\end{aligned}$$

where the numbers $p_1,\cdots,p_n$ are chosen in the same way as in (24).

Deriving it and making some estimates (observe that

$$\tau_j(t)=c_jt^{k_j}+\cdots \Rightarrow \tau_j'(t)=k_jc_jt^{k_j-1}+\cdots,$$

for $j=1,\cdots,r$), we get

$$E'(t)\geq c_0\sqrt{t^{k-1}\xi}E(t),$$

if $t_{1,\xi}\leq t\leq t_{2,\xi}$ and ξ is large.

We conclude, as in the case of an operator of third order, that the Cauchy problem for (13) is not well-posed in C^∞ near zero.

Dipartimento di Matematica
Università di Pisa
Istituto di Matematica e Fisica
Università di Sassari

Bibliography

[Br1] M. D. Bronštein, The Cauchy problem for hyperbolic operators with characteristics of variable multiplicity, Trudy Moskov. Mat. Obšč., **41** (1980), 87–103.

[Br2] M. D. Bronštein, Smoothness of roots of polynomials depending on parameters, Sibirsk. Mat. Ž., **20** (1978), 493–501.

[CJS] F. Colombini, E. Jannelli and S. Spagnolo, Well-posedness in the Gevrey classes of the Cauchy problem for a nonstrictly hyperbolic equation with coefficients depending on time, Ann. Scuola Norm. Sup. Pisa, **10** (1983), 291–312.

[CS] F. Colombini and S. Spagnolo, An example of a weakly hyperbolic Cauchy problem not well-posed in C^∞, Acta Math., **148** (1982), 243–253.

[DA] P. D'Ancona, Well-posedness in C^∞ for a weakly hyperbolic second order equation, Rend. Sem. Mat. Univ. Padova, **91** (1994), 65–83.

[DM] P. D'Ancona and R. Manfrin, A class of locally solvable semilinear equations of weakly hyperbolic type, Ann. di Mat. Pura e Appl., (IV) **CLXVIII** (1995), 355–372.

[DS] P. D'Ancona and S. Spagnolo, On pseudosymmetric hyperbolic systems, Ann. Scuola Norm. Sup. Pisa, **25** (1997), 397–417.

[IS] H. Ishida, The Cauchy problem for weakly hyperbolic equations of second order, to appear.

[Iv] V. Ya. Ivrii, Cauchy problem conditions for hyperbolic operators with characteristics of variable multiplicity for Gevrey classes, Sibirski Mat. Ž., **17**–6 (1976), 1256–1270.

[IP] V. Ya. Ivrii and V. M. Petkov, Necessary conditions for the Cauchy problem for non strictly hyperbolic equations to be well-posed, Russian Math. Surveys, **29** (1974), 1–70.

[Ka] K. Kajitani, The well-posed Cauchy problem for hyperbolic operators, Exposé au Seminaire Vaillant, 1989.

[Le] E. E. Levi, Sul problema di CAUCHY per le equazioni lineari in due variabili a caratteristiche reali, Rendiconti dell'Istituto Lombardo, Nota I (§I-III), Nota II (§IV-VI) (1907–1908) (see also the collection of the works of E. E. Levi, under the care of U.M.I., published by Cremonese, Rome).

[Ma] T. Mandai, A necessary and sufficient condition for the well-posedness of some weakly hyperbolic Cauchy problems, Comm. in P.D.E. **8**–7 (1983), 735–771.

[Mi1] S. Mizohata, Some remarks on the Cauchy problem, J. Math. Kyoto Univ., **1** (1961), 109–127.

[Mi2] S. Mizohata, On the Cauchy problem, Academic Press, 1985.

[MO] S. Mizohata and Y. Ohya, Sur la condition de E. E. Levi concernant des équations hyperboliques, Publ. RIMS, **4** (1968), 511–526.

[Ni1] T. Nishitani, The Cauchy problem for weakly hyperbolic equations of second order, Comm. P.D.E., **5** (1980), 1273–1296.

[Ni2] T. Nishitani, A necessary and sufficient condition for the hyperbolicity of second order equations with two independent variables, J. Math. Kyoto University, **24** (1984), 91–104.

[Ni3] T. Nishitani, Hyperbolicity of two by two systems with two independent variables, Comm. P.D.E., **23** (1998), 1061–1110.

[Ol] O. A. Oleinik, On the Cauchy problem for weakly hyperbolic equations of second order, Comm. Pure Appl. Math., **23** (1970), 569–586.

[Or] N. Orrú On a weakly hyperbolic equation with a term of order zero, Annales de la Faculté des Sciences de Toulouse, **VI**-3 (1997), 525–534.

[OT] Y. Ohya and S. Tarama, Le problème de Cauchy à caracthéristiques multiples coefficients hölderiens en t, Proc. Taniguchi Intern. Sympos. on Hyperbolic Equations and Related Topics, 1984.

[P] I. G. Petrowsky, Über das Cauchysche Problem für ein System linearer partieller Differential gleichungen im Gebiete der nicht-analytischen Funktionen, Bull. de l'Univ. de l'Etat, Moscow (1938), 1–74.

[YaK] K. Yamamoto, The Cauchy problem for some class of hyperbolic differential operators with variable multiple characteristics, J. Math. Soc. Japan, **31** (1979), 481–502.

[Ya1] T. Yamazaki, Linear evolution equations and a mixed problem for singular or degenerate wave equations, Comm. P.D.E., **12**-7 (1987), 701–776.

[Ya2] T. Yamazaki, Unique existence of evolution equations of hyperbolic type with countably many singular or degenerate points, Journal of Differential Equations, **77** (1989), 38–72.