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On the mod 3 homotopy type of the classifying space of a central product of $SU(3)$'s

By

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1. Introduction

Let $SU(3)$ be the compact Lie group of special unitary complex matrices of order 3. It is well known that the center of $SU(3)$, namely Γ, is isomorphic to $\mathbf{Z}/3$ and it is generated by the matrix (ω, ω, ω) where $\omega \in \mathbf{C}$ such that $\omega^3 = 1$ and $\omega \neq 1$. The compact Lie group $SU(3,3)$ is defined as the central product $SU(3) \times_{\mathbf{Z}/3} SU(3)$, i.e., as the quotient

$$SU(3,3) = SU(3) \times SU(3)/\Delta$$

where Δ is the subgroup of $SU(3) \times SU(3)$ generated by the elements (A, A) such that $A \in \Gamma$.

The group $SU(3,3)$ plays an important role when studying the homotopy type of the classifying space of the exceptional compact Lie group of rank 4, F_4, at primes greater than 3 (see [17] and [6]), and specially at the prime 3 (see [21]). This justify a deep study of the structure of $SU(3,3)$, as well as those of its classifying space $BSU(3,3)$, at the prime 3.

Our first result describes the mod 3 cohomology of $SU(3,3)$ as Hopf algebra.

Theorem 1.1. *$H^*SU(3,3) = \mathbf{F}_3[y_2]/y_2^3 \otimes \Lambda_{\mathbf{F}_3}(x_1, x_3, x_3', x_5)$, where subindex indicates degree. Moreover, the Hopf algebra structure is given by the reduced diagonal map*

a	x_1	x_3	x_3'	x_5	y_2
$\bar{\phi}(a)$	0	$y_2 \otimes x_1$	$y_2 \otimes x_1$	$y_2 \otimes (x_3' - x_3)$	0

Proof. See Section 3.

Then we calculate the mod 3 cohomology of the classifying space of $SU(3,3), BSU(3,3)$.

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Theorem 1.2. *There is an algebra isomorphism:*

$$H^*(BSU(3,3);\mathbf{F}_3) \cong \mathbf{F}_3[y_2, y_4, y_8, w_8, y_{12}, y_{18}] \otimes \Lambda_{\mathbf{F}_3}(y_3, y_7, y_9)/R$$

where $|y_i| = i, |w_8| = 8$ *and* R *is the ideal generated by*

$$\begin{array}{cccc} y_2y_3, & y_3y_4, & y_2y_7, & y_2w_8 + y_3y_7, \\ y_7y_4 - y_3y_8, & y_3y_8 + y_2y_9, & w_8y_4 + y_3y_9, & y_4y_9, \\ y_7y_8, & y_8y_9, & w_8y_8 - y_7y_9, & y_2^3y_{18} - y_8^3 + y_2^2y_4y_8^2 - y_4^3y_{12}. \end{array}$$

We also know that $\beta y_2 = y_3$, $\beta y_7 = w_8$, $\beta y_8 = y_9$, $\mathscr{P}^1 y_3 = y_7$, $\mathscr{P}^1 y_4 = -y_8 + y_4^2$, $\mathscr{P}^1 y_{12} = w_8^2 + y_{12}y_2^2$, $\mathscr{P}^1 y_{18} = y_2^2y_{18} - y_2y_4y_8^2$, $\mathscr{P}^3 y_7 = y_7y_{12} + y_3w_8^2$, $\mathscr{P}^3 w_8 = w_8y_{12}$ *and* $\mathscr{P}^3 y_{12} = y_{12}(y_2^6 - y_{12})$.

Proof. Consider the universal fibration

$$SU(3,3) \to * \to BSU(3,3)$$

and let $(E_*^{*,*}, d_*)$ be the mod 3 Eilenberg-Moore spectral sequence converging to $H^*BSU(3,3)$. This spectral sequence starts as

$$E_2^* = \mathrm{Cotor}^*_{H^*SU(3,3)}(\mathbf{F}_3, \mathbf{F}_3).$$

In Section 4, Theorem 4.9, we calculate this E_2 term. In Section 5, we prove that all the possible differentials vanish and therefore the spectral sequence collapses at the 2-stage.

To finish, we should calculate the algebra structure. In order to do that, we first calculate the mod 3 invariants by the action of the Weyl group on the cohomology of the maximal tori (Section 6). Using that information we finally calculate the algebra structure as well as part of the Steenrod algebra structure in Section 7.

According to [17], the homotopy type of $BSU(3,3)$ is determined, up to completion, by its mod p cohomology at primes different from 3, for in that case

$$BSU(3,3)_p^{\wedge} \simeq (BSU(3)^2)_p^{\wedge}, \qquad p \neq 3.$$

In this note, we use strongly Theorem 1.2 to prove that $BSU(3,3)$ is determined up to completion by its cohomology at the torsion prime 3, as well.

Theorem 1.3. *Let* X *be a* 3*-complete space such that*

$$H^*(X;\mathbf{F}_3) \cong H^*(BSU(3,3);\mathbf{F}_3)$$

as algebras over the mod 3 *Steenrod algebra. Then* X *is homotopy equivalent to* $BSU(3,3)$ *up to* 3*-completion.*

Proof. See Section 10.

A different question is whether or not the homotopy type of a compact Lie group or p-compact group is determined by the Weyl group representation. The concept of p-compact group, a homotopy theoretic generalization of compact Lie group, was introduced by Dwyer and Wilkerson in [4]. A loop space X is said to be a p-compact group if X is $\mathbf{F}_p$ finite, p-complete, and $\pi_0 X$ is a finite p-group. Then p-compact groups are shown to admit maximal tori in the sense of Rector ([19], [4]) and the Weyl group of X, W_X, is defined as $\pi_0 \mathbf{W}_X$, where $\mathbf{W}_X$ is the space of self-maps of BT_X over BX, if $BT_X \to BX$ is the maximal torus of X. The p-adic representation of the Weyl group as a pseudo reflection group is therefore obtained as $W_X \to \mathrm{Aut}(H_2(BT_X; \mathbf{Z}_p^\wedge))$.

Møller and Notbohm have considered the torsion free case: [15]. Here we have considered again the case of $SU(3,3)$.

Theorem 1.4. *Let X be a connected 3-compact group with the same Weyl group type as $SU(3,3)$. Then BX and $BSU(3,3)$ are homotopy equivalent up to 3-completion.*

Proof. See Section 10.

This result allows us to determine the integral homotopy type of $BSU(3,3)$ in the following sense:

Corollary 1.5. *Let L be a connected finite loop space with maximal torus and Weyl group W_L such that the integral representation of W_L is conjugate to that of $W_{SU(3,3)}$. Then BL is homotopy equivalent to $BSU(3,3)$.*

Proof. See Section 10.

Organization of the paper. The paper is organized as follows. In Section 2 we describe the mod 3 cohomology of $PU(3^n)$ and $BPU(3)$ as it will be usefull for following calculations. In Section 3, we prove Theorem 1.1. In Section 4 we calculate the E_2-term of the Eilenberg-Moore spectral sequence associated to the universal fibration of $BSU(3,3)$. In Section 5 we prove that the Eilenberg-Moore spectral sequence cited above collapses at the E_2-term. In Section 6 we deal with the action of the Weyl group of $SU(3,3)$ on a maximal torus. In Section 7 we determine the algebra structure of $H^*(BSU(3,3); \mathbf{F}_3)$. In the following sections we follow the ideas in [3] to prove Theorems 1.3 and 1.4. In Section 8, given a 3-complete space X, with the same cohomology as $BSU(3,3)$, we construct a couple of principal fibrations that allow us to compute the cohomology of $X\{3\}$, the 3-connected cover of X, additively. In Section 9 we compute the algebra structure of $H^*(X\{3\}; \mathbf{F}_3)$ so obtaining $X\{3\} \simeq (BSU(3)^2)_3^\wedge$. In the last section we prove Theorems 1.3 and 1.4, as well as Corollary 1.5.

Notation. Here $\mathscr{A}_3$ is the mod 3 Steenrod algebra, all spaces are assumed to have the homotopy type of CW-complexes, and completion means Bousfield-Kan completion. Given a space Y, we write H^*Y for $H^*(Y; \mathbf{F}_3)$, $H^*_{\mathbf{Q}_3}(Y)$ for $H^*(Y, \mathbf{Z}_3^\wedge) \otimes \mathbf{Q}$ and $Y_p^\wedge$ for the Bousfield-Kan p-completion of the space Y. We write "Bss" for the Bockstein spectral sequence, "EMss" for the Eilenberg-Moore spectral sequence and "Sss" for the Serre spectral sequence. The symbol

$\Lambda_{\mathbf{F}}$ is used to denote an exterior algebra over the coefficient field $\mathbf{F}$. Given a group A and an A-module M, we denote by $\mathscr{H}^*(A;M)$ the cohomology with twisted coefficients.

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2. The groups $PU(3)$ and $PU(9)$

The compact Lie group $PU(3^n)$ is defined as the quotient $SU(3^n)/\Gamma$, being Γ the center of $SU(3^n)$. The Hopf algebra structure of the cohomology of $PU(3^n)$ appears in [2].

Theorem 2.1. *There is an algebra isomorphism:*

$$H^*PU(3^n) \cong \mathbf{F}_3[\hat{y}]/\hat{y}^{3^n} \otimes \Lambda_{\mathbf{F}_3}(\hat{x}_1, \hat{x}_3, \dots, \hat{x}_{2\cdot 3^n-2}),$$

where $|\hat{x}_i| = i$ *and* $|\hat{y}| = 2$. *The Hopf algebra structure is given by the reduced diagonal map*

a	$\hat{x}_1$	$\hat{x}_{2i-1}$	$\hat{y}_2$
$\bar{\phi}(a)$	0	$\sum_{j=1}^{i-1} \binom{i-1}{j-1} \hat{y}_2^{i-j} \otimes \hat{x}_{2j-1}$	0

The cohomology ring of $BPU(3)$ is given by Kono, Mimura and Shimada in [7]:

Theorem 2.2. *There is an algebra isomorphism:*

$$H^*BPU(3) \cong \mathbf{F}_3[t_2, t_8, t_{12}] \otimes \Lambda_{\mathbf{F}_3}(t_3, t_7)/J$$

where $|t_i| = i$ *and* J *is the ideal generated by* t_2t_3, t_2t_7 *and* $t_3t_7 + t_2t_8$.

The Steenrod algebra structure was calculated by Kono and Yagita in [8]:

Theorem 2.3. *The Steenrod algebra action is determined by the following relations*

$$\beta t_2 = t_3, \qquad \beta t_7 = t_8,$$

$$\mathscr{P}^1 t_3 = t_7, \qquad \mathscr{P}^1 t_{12} = t_8^2 + t_{12}t_2^2,$$

$$\mathscr{P}^3 t_7 = t_7 t_{12} + t_3 t_8^2, \qquad \mathscr{P}^3 t_8 = t_8 t_{12}, \qquad \mathscr{P}^3 t_{12} = t_{12}(t_2^6 - t_{12}).$$

Then the groups $PU(3)$ and $SU(3,3)$ are related by the following commutative diagram

$$\begin{array}{ccccc} & & \mathbf{Z}/3 & = & \mathbf{Z}/3 \\ & & \downarrow & & \downarrow \\ SU(3) & \longrightarrow & SU(3)^2 & \xrightarrow{\pi_i} & SU(3) \\ \| & & \downarrow p & & \downarrow \\ SU(3) & \longrightarrow & SU(3,3) & \xrightarrow{\tilde{\pi}_i} & PU(3) \end{array} \tag{1}$$

where all the rows and columns are exact sequences and π_i is the projection of the i-th factor. Moreover, the short exact sequence

$$SU(3) \longrightarrow SU(3,3) \xrightarrow{\tilde{\pi}_i} PU(3)$$

has a section $s : PU(3) \to SU(3,3)$ that maps the class $[A]$ to the class $[A, A]$. Hence the induced fibration

$$BSU(3) \longrightarrow BSU(3,3) \xrightarrow{B\tilde{\pi}_i} BPU(3)$$

has also a section and therefore it proves

Lemma 2.4. *The maps $B\tilde{\pi}_i^*$ and $\tilde{\pi}_i^*$ induce a monomorphism in cohomology.*

This lemma is quite useful as we know $H^*PU(3)$ as well as $H^*BPU(3)$, and we want to calculate $H^*SU(3,3)$ and $H^*BSU(3,3)$.

We are also interested in the relation between $PU(9)$ and $SU(3,3)$. Let g denote the composition

$$SU(3) \times SU(3) \xrightarrow{1\times\Delta} SU(3) \times SU(3) \times SU(3) \hookrightarrow SU(9),$$

then g induces the following commutative diagram

$$\begin{array}{ccccc} \mathbf{Z}/(3) & \longrightarrow & SU(3) \times SU(3) & \longrightarrow & SU(3,3) \\ \downarrow & & \downarrow g & & \downarrow \hat{g} \\ \mathbf{Z}/9 & \longrightarrow & SU(9) & \longrightarrow & PU(9) \end{array} \tag{2}$$

which will allow us to calculate the structure of $H^*SU(3,3)$.

3. The Hopf algebra structure of $H^*SU(3,3)$

In this section we prove Theorem 1.1. In order to do it, we use the information about $PU(3)$ and $PU(9)$ stated in the previous section as well as the diagrams relating those spaces with $SU(3,3)$. Now, the proof of Theorem 1.1 follows as

Proof of Theorem 1.1. By Lemma 2.4, we know that the fibration

$$SU(3) \longrightarrow SU(3,3) \xrightarrow{\tilde{\pi}_i} PU(3) \tag{3}$$

has a section and $\tilde{\pi}_i^*$ is injective. An easy analysis of the mod 3 Sss associated to fibration (3) shows that this spectral sequence collapses at the 2-stage and therefore

$$H^*SU(3,3) = \mathbf{F}_3[y_2]/y_2^3 \otimes \Lambda_{\mathbf{F}_3}(x_1, x_3, x_3', x_5),$$

where subindex indicates degree. Moreover, the generators can be chosen such that $\tilde{\pi}_i(\hat{x}_1) = x_1, \tilde{\pi}_i(\hat{y}) = y, \tilde{\pi}_1(\hat{x}_3) = x_3$ and $\tilde{\pi}_2(\hat{x}_3) = x_3'$. Therefore we know that

a	x_1	x_3	x_3'	y_2
$\bar{\phi}(a)$	0	$y_2 \otimes x_1$	$y_2 \otimes x_1$	0

and $\beta x_1 = y_2$, and $\beta x_3 = \beta x_3' = y_2^2$.

To calculate the reduced diagonal of the element x_5, we use the information about $PU(9)$ as follows: diagram (2) induces a diagram of principal fibrations

$$\begin{array}{ccccc} SU(3) \times SU(3) & \longrightarrow & SU(3,3) & \longrightarrow & B\mathbf{Z}/3 \\ {\scriptstyle g}\downarrow & & {\scriptstyle \hat{g}}\downarrow & & \downarrow \\ SU(9) & \longrightarrow & PU(9) & \longrightarrow & B\mathbf{Z}/9. \end{array}$$

Comparing the mod 3 Sss sequences associated to both fibrations, we can easily deduce that $\hat{g}^* : H^*PU(9) \to H^*SU(3,3)$ is trivial on all the generators of $H^*PU(9)$ but in the cases $\hat{g}^*(\hat{y}) = y_2, \hat{g}^*(\hat{x}_3) = x_3 - x_3'$ and $\hat{g}^*(\hat{x}_5) = \pm x_5$. We can assume that $\hat{g}^*(\hat{x}_5) = x_5$, therefore

$$\begin{aligned} \bar{\phi}(x_5) &= \bar{\phi}(\hat{g}^*(\hat{x}_5)) \\ &= (\hat{g}^* \otimes \hat{g}^*)(\bar{\phi}(\hat{x}_5)) \\ &= (\hat{g}^* \otimes \hat{g}^*)(\hat{y}^2 \otimes \hat{x}_1 - \hat{y} \otimes \hat{x}_3) \\ &= y_2 \otimes (x_3' - x_3) \end{aligned}$$

which finishes the proof.

4. The Cotor* calculation

Let $\mathscr{A} \cong H^*SU(3,3)$ as coalgebras. In this section we calculate $\mathrm{Cotor}^*_{\mathscr{A}}(\mathbf{F}_3, \mathbf{F}_3)$, and to do that, we construct an injective resolution of $\mathbf{F}_3$ over $\mathscr{A}$ using the same construction and the same notation as those in §4 of [12].

Consider $L \subset \mathscr{A}$ the submodule generated by $\{y_2, y_2^2, x_1, x_3, z_3, x_5\}$ where $z_3 = -x_3'$. Let $\theta : \mathscr{A} \to L$ and $\iota : L \to \mathscr{A}$ be the natural projection and the injection respectively, such that $\iota \circ \theta = 1_{\mathscr{A}}$.

Consider $sL = \{c_3, c_5, a_2, a_4, b_4, b_6\}$ the suspension of L, and define $\bar{\theta} : \mathscr{A} \to sL$ by $\bar{\theta} = s \circ \theta$ and, $\bar{\imath} : sL \to \mathscr{A}$ by $\bar{\imath} = \imath \circ s^{-1}$.

Let $T(sL)$ be the free tensor algebra over sL with the (natural) product ψ. Consider the two sided ideal I of $T(sL)$ generated by $\operatorname{Im} \psi \circ (\bar{\theta} \otimes \bar{\theta}) \circ \phi(\operatorname{Ker} \bar{\theta})$. Then I is generated by:

$$[a_i, a_j],\ [a_i, b_j],\ [a_i, c_5],\ [b_i, c_5],\ [a_2, c_3],$$

$$[a_4, c_3] - c_5 a_2, \qquad [b_4, c_3] + c_5 a_2, \quad \text{and} \quad [b_6, c_3] + c_5(a_4 + b_4),$$

where $[\alpha, \beta] = \alpha\beta - (-1)^n \beta\alpha$ with $n = \deg(\alpha)\deg(\beta)$.

Put $C = T(sL)/I$ and we can now construct the twisted tensor product $W = \mathscr{A} \otimes C$ with respect to $\bar{\theta}$. That is, W is an $\mathscr{A}$-comodule with a differential operator

$$d_W = 1 \otimes d_C + ((1 \otimes \psi) \circ (1 \otimes \bar{\theta} \otimes 1) \circ (\phi \otimes 1)),$$

where d_C is defined as

$$d_C = -\psi \circ (\bar{\theta} \otimes \bar{\theta}) \circ \phi \circ \bar{\imath}.$$

In our case, d_W and d_C are given by:

$$d_W(x_1) = 1 \otimes a_2,$$

$$d_W(x_3) = 1 \otimes a_4 + y_2 \otimes a_2, \qquad d_W(z_3) = 1 \otimes b_4 + y_2 \otimes a_2,$$

$$d_W(x_5) = 1 \otimes b_6 - y_2 \otimes (a_4 + b_4),$$

$$d_W(y_2) = 1 \otimes c_3, \qquad d_W(y_2^2) = 1 \otimes c_5 - y_2 \otimes c_3,$$

$$d_C(a_2) = 0,$$

$$d_C(a_4) = -c_3 a_2, \qquad d_C(b_4) = c_3 a_2,$$

$$d_C(b_6) = c_3(a_4 + b_4),$$

$$d_C(c_3) = 0 \qquad \text{and} \qquad d_C(c_5) = c_3^2.$$

Now we define weight in W as follows:

Weight	0	2	2	4	1	2
$\mathscr{A}$	x_1	x_3	z_3	x_5	y_2	y_2^2
C	a_2	a_4	b_4	b_6	c_3	c_5

and the weight of a monomial is the sum of weights of each element.

Define a filtration $F_r = \{x \mid \text{weight}(x) \leq r\}$ and put $E_0 W = \sum F_i / F_{i-1}$. Then it is easy to see that:

$$E_0 W \cong \Lambda_{\mathbf{F}_3}(x_1, x_3, z_3, x_5) \otimes \mathbf{F}_3[a_2, a_4, b_4, b_6] \otimes C(Q(y_2)),$$

where $C(Q(y_2))$ is the cobar construction of $\mathbf{F}_3[y_2]/y_2^3$. Now the differential formulae imply that E_0W is acyclic, and hence W is acyclic. So W is an injective $\mathbf{F}_3$-resolution over $\mathscr{A}$. Consequently we have:

$$H(C:d_C) = \mathrm{Cotor}^*_{\mathscr{A}}(\mathbf{F}_3, \mathbf{F}_3).$$

In what follows, we denote d_C simply by d.

In order to calculate $H(C:d)$, we follow the ideas in [13]. Call $y_4 = a_4 + b_4$, then we have an additive isomorphism between C and $T(c_3, c_5) \otimes \mathbf{F}_3[a_2, a_4, y_4, b_6]$. Thus $\{c_3^{l_1} c_5^{l_2} a_2^{l_3} a_4^{l_4} y_4^{l_5} b_6^{l_6} \mid l_i \in \mathbf{N}\}$ is a basis of C.

Define a new weight in C by

$\omega(\alpha)$	1	1	1	1	1	2
α	a_2	a_4	y_4	b_6	c_3	c_5

and for a monomial $\alpha_1 \dots \alpha_n$,

$$\omega(\alpha_1 \dots \alpha_n) = \omega(\alpha_1) + \cdots + \omega(\alpha_n).$$

For an element $x = \sum \lambda_i x_i$, where x_i is a monomial and $\lambda_i \in \mathbf{F}_3$, we define the weight of x as the infimum of the weights of the x_i's. Then the filtration $F^r = \{x \mid \omega(x) \geq r\}$ gives rise to a spectral sequence $\{E_r, d_r\}$ such that $E_0 = C$ and $E_\infty = H(C:d)$, namely converging to $\mathrm{Cotor}^*_{\mathscr{A}}(\mathbf{F}_3, \mathbf{F}_3)$.

Notice that the only non trivial d_0 is $d_0(c_5) = c_3^2$, therefore

Lemma 4.1. *We have*

$$E_1 \cong \mathbf{F}_3[a_2, a_4, y_4, b_6, w_8] \otimes \Lambda_{\mathbf{F}_3}(c_3)$$

Now, the only non zero d_1 are $d_1(a_4) = -c_3 a_2$ and $d_1(b_6) = c_3 y_4$, thus

Lemma 4.2. $E_2 \cong \mathbf{F}_3[a_4^3, b_6^3, w_8] \otimes A$, *where*

$$\begin{aligned} A = {} & \{1, y_8, a_2 b_6^2, c_3 a_4^2 b_6, c_3 a_4^2 b_6^2\} \mathbf{F}_3[a_2, y_4] \\ & \oplus \{y_7, c_3 a_4 b_6, c_3 b_6^2, c_3 a_4, b_6^2\} \mathbf{F}_3[y_4] \\ & \oplus \{c_3, y_4\} \mathbf{F}_3, \end{aligned}$$

being $y_7 = c_3 a_4 - c_5 a_2$ *and* $y_8 = a_2 b_6 + y_4 a_4$.

To handle d_2, we introduce an auxiliary derivation δ on the subalgebra $\mathbf{F}_3[a_2, a_4, y_4, b_6]$ of C as

$\delta(\alpha)$	0	$-a_2$	0	y_4
α	a_2	a_4	y_4	b_6

Then, the derivation δ has the following properties:

Lemma 4.3. *For a polynomial* $P \in \mathbf{F}_3[a_2, a_4, y_4, b_6]$ *we have:*
1) $\delta^3 P = 0$,
2) $[c_3, P] = -c_5 \delta P$,
3) $dP = c_3 \delta P + c_5 \delta^2 P$.

Proof. (By induction.) Suppose that $\delta^3 P = 0$ holds for any polynomial P of degree up to l. Then:

$$\delta^3(xP) = \delta^3 x \cdot P + x \cdot \delta^3 P = 0.$$

Thus $\delta^3 P = 0$ holds for a polynomial of degree $l+1$.

Suppose that $[c_3, P] = -c_5 \delta P$ holds for any polynomial P of degree up to l. Then:

$$[c_3, xP] = [c_3, x]P + x[c_3, P] = -c_5 \delta x \cdot P - x c_5 \delta P = -c_5 \delta(xP).$$

Thus the relation holds for a polynomial of degree $l+1$.

Suppose that $dP = c_3 \delta P + c_5 \delta^2 P$ holds for any polynomial of degree up to l. Then:

$$\begin{aligned} d(xP) &= dx \cdot P + x \cdot dP \\ &= (c_3 \delta x + c_5 \delta^2 x)P + x(c_3 \delta P + c_5 \delta^2 P) \\ &= c_3 \delta x \cdot P + c_5 \delta^2 x \cdot P + (c_3 x - c_5 \delta x)\delta P - c_5 x \delta^2 P \\ &= c_3 \delta(xP) + c_5 \delta^2 (xP). \end{aligned}$$

Thus the differential formula holds for a polynomial of degree $l+1$.

This derivation has "enough" information about the differential d as the following shows.

Lemma 4.4. *Let* P *be non trivial in* $\mathbf{F}_3[a_2, a_4, y_4, b_6]$. *Then* P *is a non trivial cocycle if and only if* $\delta P = 0$.

Proof. If P is a cocycle then $dP = 0$. Then by the differential formula, we have $\delta P = 0$.

Conversely, if $\delta P = 0$, so does $\delta^2 P$, therefore we have $dP = 0$ by the differential formula. Since P contains neither c_3 nor c_5, it is not in the d-image, hence it is a non trivial cocycle.

Now, Lemma 4.3 gives rise to the formulae

$$y_7 \delta^2 P = d(a_2 P - a_4 \delta P), \qquad w_8 \delta^2 P = d(-c_3 P - c_5 \delta P),$$

which together with the δ^2-image:

$$\begin{aligned} a_2^2 &= \delta^2(-a_4^2), & a_2 y_4 &= \delta^2(a_4 b_6), \\ y_4^2 &= \delta^2(-b_6^2), & a_2 y_8 &= \delta^2(-a_4^2 b_6), \\ y_4 y_8 &= \delta^2(-a_4 b_6^2), & y_8^2 &= \delta^2(a_4^2 b_6^2), \end{aligned}$$

proves the following lemma:

Lemma 4.5. *The only non trivial d_2 are:*

$$d_2(a_2b_6^2) = y_7y_4^2, \qquad d_2(c_3a_4^2) = w_8a_2^2, \qquad d_2(c_3a_4^2b_6^2) = w_8y_8^2,$$

$$d_2(c_3a_4^2b_6) = w_8y_8a_2, \qquad d_2(-c_3a_4b_6) = w_8a_2y_4,$$

$$d_2(c_3b_6^2) = w_8y_4^2, \qquad \textit{and} \qquad d_2(c_3a_4b_6^2) = w_8y_4y_8.$$

Thus,

Lemma 4.6.

$$E_3 \cong \mathbf{F}_3[a_4^3, b_6^3] \otimes (\{1, y_8, y_8^2\}\mathbf{F}_3[a_2, y_4]$$
$$\oplus \{c_3, y_7, w_8, y_9, a_2w_8, y_4y_7, y_4w_8, w_8y_8\}\mathbf{F}_3[w_8]).$$

When we put $y_{12} = a_4^3$ and $y_{18} = b_6^3$, all the generators are permanent, hence $E_3 = E_\infty$ and we have proved

Proposition 4.7. *We have an additive isomorphism*

$$\mathrm{Cotor}^*_{\mathscr{A}}(\mathbf{F}_3, \mathbf{F}_3) \cong (\{1, y_8, y_8^2\}\mathbf{F}_3[a_2, y_4]$$
$$\oplus \{c_3, y_7, w_8, y_9, a_2w_8, y_4y_7, y_4w_8, w_8y_8\}\mathbf{F}_3[w_8]) \otimes \mathbf{F}_3[y_{12}, y_{18}].$$

Now, it is easy to get the algebra structure of $\mathrm{Cotor}^*_{\mathscr{A}}(\mathbf{F}_3, \mathbf{F}_3)$ as

Proposition 4.8. $\mathrm{Cotor}^*_{\mathscr{A}}(\mathbf{F}_3, \mathbf{F}_3)$ *is commutative.*

Proof. To begin with, we have the following d-images:

$$c_3^2 = dc_5, \qquad y_7^2 = d(c_5a_4^2), \qquad y_9^2 = d(c_5b_6^2),$$

$$[c_3, y_7] = d(c_5a_4), \qquad [c_3, w_8] = d(c_5^2), \qquad [c_3, y_9] = d(-c_5b_6),$$

$$[y_7, w_8] = d(c_5^2a_4), \qquad [y_7, y_9] = d(-c_5a_4b_6), \qquad [y_9, w_8] = d(c_5^2b_6).$$

In $C, [c_3, P] = c_5\delta P$ holds for $P \in \mathbf{F}_3[a_2, a_4, y_4, b_6]$, hence if P is a cocycle we have $[c_3, P] = 0$. Therefore commutativity holds in $\mathrm{Cotor}^*_{\mathscr{A}}(\mathbf{F}_3, \mathbf{F}_3)$.

And we have the following d-images:

$$c_3^2 = dc_5, \qquad y_7^2 = d(c_5a_4^2),$$

$$y_9^2 = d(c_5b_6^2), \qquad c_3\delta^2Q = d(\delta Q),$$

$$y_7\delta^2Q = d(a_2Q - a_4\delta Q), \qquad w_8\delta^2Q = d(-c_3Q - c_5\delta Q),$$

$$y_9\delta^2Q = d(b_6\delta Q - y_4Q), \qquad c_3a_2 = d(b_4),$$

$$c_3y_4 = d(b_6), \qquad y_7a_2 = d(a_4^2),$$

$$y_7y_8 = d(-a_4^2b_6), \qquad y_9y_4 = d(b_6^2),$$

$$y_9y_8 = d(-a_4b_6^2), \qquad c_3y_7 + a_2w_8 = d(c_5a_4),$$
$$c_3y_8 + y_9y_2 = d(a_4b_6), \qquad y_7y_4 + y_9y_2 = d(a_4b_6),$$
$$w_8y_8 - y_7y_9 = d(-c_5a_4b_6), \qquad w_8y_4 + y_9c_3 = d(c_5b_6).$$

Finally, note that the elements which are 0 as polynomial in $\mathbf{F}_3[a_2, a_4, y_4, b_6]$ are generated by $y_8^3 - a_2^3y_{18} - y_4^3y_{12}$, hence we have proved:

Theorem 4.9. *For $\mathscr{A} = H^*SU(3,3)$, we have as algebra:*

$$\mathrm{Cotor}^*_{\mathscr{A}}(\mathbf{F}_3, \mathbf{F}_3) \cong \mathbf{F}_3[a_2, y_4, y_8, w_8, y_{12}, y_{18}] \otimes \Lambda_{\mathbf{F}_3}(c_3, y_7, y_9)/R,$$

where R is the ideal generated by:

$$\begin{array}{cccc} c_3a_2, & c_3y_4, & y_7a_2, & y_7y_8 \\ y_9y_4, & y_9y_8, & c_3y_7 + a_2w_8, & c_3y_8 + y_9y_2 \\ y_7y_4 + y_9y_2, & w_8y_8 - y_7y_9, & w_8y_4 + y_9c_3, & y_8^3 - a_2^3y_{18} - y_4^3y_{12}. \end{array}$$

Corollary 4.10. *We have*

$$\mathrm{Cotor}^*_{\mathscr{A}}(\mathbf{F}_3, \mathbf{F}_3) \cong (A + B) \otimes \mathbf{F}_3[y_{12}, y_{18}]$$

where $A = \{1, y_8, y_8^2\}\mathbf{F}_3[a_2, y_4]$ and $B = \{1, a_2, c_3, y_4, y_7, y_8, y_9, c_3y_8\}\mathbf{F}_3[w_8]$. Moreover, $A \cap B = \{1, a_2, y_4, y_8\}\mathbf{F}_3$.

5. The spectral sequence collapses

In this section we prove that the EMss associated to the fibration

$$SU(3,3) \to * \to BSU(3,3),$$

namely $(E_*^{*,*}, d_*)$, collapses at the 2-stage. We prove that all the possible differentials vanish and therefore the spectral sequence collapses at the 2-stage.

The description of the E_2-term, $\mathrm{Cotor}^*_{H^*SU(3,3)}(\mathbf{F}_3, \mathbf{F}_3)$, appears in Theorem 4.9.

The classes a_2 and c_3 of $\mathrm{Cotor}^*_{H^*SU(3,3)}(\mathbf{F}_3, \mathbf{F}_3)$ are the transgressions of the classes x_1 and y_2 of $H^*SU(3,3)$, so they are permanent cycles.

Since d_ry_4 is of degree 5 and $\mathrm{Cotor}^5_{H^*SU(3,3)}(\mathbf{F}_3, \mathbf{F}_3) = 0$, y_4 is a permanent cycle.

In the mod 3 Sss associated to the fibration

$$SU(3,3) \to * \to BSU(3,3),$$

we use Kudo's Theorem and obtain a non trivial element $\beta\mathscr{P}^1c_3 \in H^8BSU(3,3)$, hence $H^7BSU(3,3) \neq 0$. The only element of degree 7 in $\mathrm{Cotor}^*_{H^*SU(3,3)}(\mathbf{F}_3, \mathbf{F}_3)$ is y_7, so it is a permanent cycle and represents $\mathscr{P}^1c_3$.

The elements d_ry_8 and d_rw_8 are of degree 9. The only element of degree 9 in $\mathrm{Cotor}^*_{H^*SU(3,3)}(\mathbf{F}_3, \mathbf{F}_3)$ is y_9 and it is a permanent cycle because

$$1 = \dim_{\mathbf{F}_3}\mathrm{Cotor}^9_{H^*SU(3,3)}(\mathbf{F}_3, \mathbf{F}_3) \geq \dim_{\mathbf{F}_3}H^9BSU(3,3) \geq \dim_{\mathbf{F}_3}H^9BF_4 = 1.$$

So w_8 and y_8 are permanent cycles too.

Since $d_r y_{12}$ is of degree 13 and $\mathrm{Cotor}^{13}_{H^*SU(3,3)}(\mathbf{F}_3, \mathbf{F}_3) = 0$, y_{12} is a permanent cycle.

Finally, we know that

$$12 = \dim_{\mathbf{Q}} H^{18}(BSU(3,3); \mathbf{Q}) \le \dim_{\mathbf{F}_3} H^{18} BSU(3,3)$$

$$\dim_{\mathbf{F}_3} H^{18} BSU(3,3) \le \dim_{\mathbf{F}_3} \mathrm{Cotor}^{18}_{H^*SU(3,3)}(\mathbf{F}_3, \mathbf{F}_3) = 12,$$

hence y_{18} has to be a permanent cycle too.

6. Mod 3 invariants forms

In this section, we calculate the invariants under the action of the Weyl group of $SU(3,3)$ on H^*BT. In what follows in this section, the lattice L_G of a compact Lie group G means the lattice associated to the 3-adic representation of W_G, the Weyl group of G, on $GL(H^*(BT_G, \mathbf{Z}_3^\wedge))$.

The first step is to calculate $L_{SU(3,3)}$, the lattice of $SU(3,3)$. According to Notbohm [18], the lattice of $SU(3) \times SU(3)$ is projective and that one of $PU(3)\times PU(3)$ is simply connected, therefore the lattice L of any quotient of $SU(3)\times SU(3)$, with Weyl group W, fits in the following diagram:

$$\begin{array}{ccccc}
L_{PU(3)\times PU(3)} & \longrightarrow & L & \longrightarrow & L_W \\
\| & & \downarrow & & \downarrow \\
L_{PU(3)\times PU(3)} & \longrightarrow & L_{SU(3)\times SU(3)} & \longrightarrow & \mathbf{Z}/3 \oplus \mathbf{Z}/3 \\
 & & {\scriptstyle\sigma}\downarrow & & \downarrow \\
 & & Z(L) & = & Z(L)
\end{array}$$

where either rows and columns are short exact sequences, and $Z(L)$ is the center of the lattice. In our case $Z(L_{SU(3,3)}) = \mathbf{Z}/3$.

Let $\{\tilde{e}_1, \dots, \tilde{e}_4\}$ a $\mathbf{Z}_3^\wedge$-base of $L_{SU(3)\times SU(3)}$, then the action of $W_{SU(3)\times SU(3)} \cong \Sigma_3 \times \Sigma_3$ is given by the matrices

$$\widetilde{TA} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \widetilde{TB} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\widetilde{CA} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad \widetilde{CB} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix},$$

where $\widetilde{TA}$ and $\widetilde{CA}$, and $\widetilde{TB}$ and $\widetilde{CB}$ represent the permutations $(1,2)$ and $(1,2,3)$ in the first and second copy of Σ_3 respectively. This implies that $\sigma(\tilde{e}_1) = \sigma(\tilde{e}_2)$ and $\sigma(\tilde{e}_3) = \sigma(\tilde{e}_4)$.

Because σ is surjective and the inclusion of the center of $SU(3,3)$ is diagonal, we can assume that $\sigma(\tilde{e}_1) = \sigma(\tilde{e}_3) = 1$ without lost of generality. Hence, a $\mathbf{Z}_3^\wedge$-base of $L_{SU(3,3)} \cong \operatorname{Ker}\sigma$ is

$$e_1 = \tilde{e}_1 - \tilde{e}_2, \qquad e_2 = \tilde{e}_1 - \tilde{e}_3$$

$$e_3 = \tilde{e}_1 - \tilde{e}_4, \qquad e_4 = 2\tilde{e}_1 + \tilde{e}_2$$

and therefore the action of $W_{SU(3,3)}$ is given by the matrices

$$TA = \begin{pmatrix} -1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad TB = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$CA = \begin{pmatrix} -1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad CB = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Now, we have to calculate the action of $W_{SU(3,3)}$ on H^*BT, that is, on $L_{SU(3,3)}/3 \cong L_{SU(3,3)} \otimes \mathbf{F}_3$.

Let us consider the $\mathbf{F}_3$-base of $H^*BT \cong H^*(BT; \mathbf{Z}_3^\wedge) \otimes \mathbf{Z}/3$:

$$t_1 = e_1 \otimes 1 + e_2 \otimes 1 + e_3 \otimes 1$$

$$t_2 = e_2 \otimes 1 + e_3 \otimes 1$$

$$t_3 = e_2 \otimes 1 + e_3 \otimes 1 + e_4 \otimes 1$$

$$t_4 = e_2 \otimes 1 - e_3 \otimes 1,$$

then the action of $W_{SU(3,3)}$ on $L_{SU(3,3)}/3 \cong H^*BT$ is given by the matrices,

$$TA = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad TB = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$CA = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad CB = \begin{pmatrix} -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

The calculation of the mod 3 invariant forms is now done in several steps, following the chain of subgroups

$$\langle TA, CA \rangle \subset \langle TA, CA, CB \cdot TB \rangle \subset W_{SU(3,3)}.$$

Easily we can see that,

Lemma 6.1. $\mathbf{F}_3[t_1,t_2,t_3,t_4]^{\langle TA,CA\rangle} \cong \mathbf{F}_3[z_2,s_4,s_6,t_4]$ *where* $z_2 = t_1+t_2+t_3$, $s_4 = t_1t_2+t_1t_3+t_2t_3$ *and* $s_6 = t_1t_2t_3$.

Let R denote the element $CB\cdot TB \in W_{SU(3,3)}$, the next step is,

Lemma 6.2. $\mathbf{F}_3[z_2,s_4,s_6,t_4]^R \cong \mathbf{F}_3[z_2,x_4,y_4,x_6]$ *where subindex indicates degree and*,

$$x_4 = s_4 + z_2t_4,$$

$$y_4 = t_4^2 - t_4z_2,$$

$$x_6 = s_6 - t_4(s_4 + t_4^2 - z_2^2).$$

Proof. It can easily checked that $\mathbf{F}_3[z_2,x_4,y_4,x_6] \subset \mathbf{F}_3[z_2,s_4,s_6,t_4]^R$. Now, an arbitrary element $f \in \mathbf{F}_3[z_2,s_4,s_6,t_4]$ can be written in an unique manner as $f = g + t_4h$ where $g,h \in \mathbf{F}_3[z_2,x_4,y_4,x_6]$. If f is invariant under the action of R, we have that

$$g + (z_2 - t_4)h = g + t_4h,$$

what implies that $h \equiv 0$, and the lemma have been proved.

Finally,

Theorem 6.3. $\mathbf{F}_3[t_1,t_2,t_3,t_4]^{W_{SU(3,3)}} \cong \mathbf{F}_3[z_2,z_4,z_8,z_{12},\hat{z}_{18}]/(r_{24})$ *where the elements* z_j *and* r_j *have degree* j *and can be expressed as elements of* $\mathbf{F}_3[z_2,x_4,y_4,x_6]$ *in the following manner*,

$$z_2 = z_2$$

$$z_4 = z_2^2 + x_4 + y_4$$

$$z_8 = z_2(x_6 + z_2x_4) - y_4(z_2^2 + x_4 + y_4)$$

$$z_{12} = y_4^3 + z_2^2y_4^2$$

$$\hat{z}_{18} = x_6^3 - x_6^2(x_4 + y_4 - z_2^2)z_2 - x_6(x_4 + y_4 - z_2^2)^2y_4,$$

$$r_{24} = z_2^3(\hat{z}_{18} + z_2z_8^2 - z_2^3z_{12} - z_2^3z_4z_8 - z_2^7z_4 + z_2z_4^2z_8 - z_2^3z_4^3 - z_2^9)$$
$$- z_8^3 + z_2^2z_4z_8^2 - z_4^3z_{12}.$$

Proof. We have that $\mathbf{F}_3[t_1,t_2,t_3,t_4]^{W_{SU(3,3)}} \cong ((\mathbf{F}_3[t_1,t_2,t_3,t_4]^{\langle TA,CA\rangle})^R)^{TB}$ hence by the lemmas above, all we have to do is to calculate

$$\mathbf{F}_3[z_2, s_4 + z_2t_4, s_6 - t_4(s_4 + t_4^2 - z_2^2), t_4^2 - t_4z_2]^{TB}.$$

An element $f \in \mathbf{F}_3[z_2, s_4 + z_2t_4, s_6 - t_4(s_4 + t_4^2 - z_2^2), t_4^2 - t_4z_2]$ is written (no necessarily uniquely) in a form

$$f = g + (t_4^2 - t_4z_2)h + (s_6 - t_4(s_4 + t_4^2 - z_2^2))i + (s_6 - t_4(s_4 + t_4^2 - z_2^2))^2 j$$

where $g, h, i, j \in \mathbf{F}_3[z_2, z_4, z_8, z_{12}, \hat{z}_{18}]$.

As f is invariant,

$$0 = f - TB(f) = t_4z_2h + t_4(s_4 + t_4^2 - z_2^2)i - s_6t_4(s_4 + t_4^2 - z_2^2)j$$

$$0 = z_2h + (s_4 + t_4^2 - z_2^2)i - s_6(s_4 + t_4^2 - z_2^2)j.$$

Therefore $z_2h = (s_4 + t_4^2 - z_2^2)(s_6j - i)$ and applying CB we get that $j \equiv 0$ because z_2, h, i, j and $s_4 + t_4^2 - z_2^2$ are invariant by the action of BC. Hence $z_2h = -(s_4 + t_4^2 - z_2^2)i$ and there exists $\hat{h} \in \mathbf{F}_3[z_2, z_4, z_8, z_{12}, \hat{z}_{18}]$ such that $h = -(s_4 + t_4^2 - z_2^2)\hat{h}$ and $i = z_2\hat{h}$.

We have proved that if f is invariant then

$$f = g - (t_4^2 - t_4z_2)(s_4 + t_4^2 - z_2^2)\hat{h} + z_2(s_6 - t_4(s_4 + t_4^2 - z_2^2))\hat{h}$$

$$= g + (z_8 - z_2^2 z_4)\hat{h}$$

that is $f \in \mathbf{F}_3[z_2, z_4, z_8, z_{12}, \hat{z}_{18}]/(r_{24})$. Clearly,

$$\mathbf{F}_3[z_2, z_4, z_8, z_{12}, \hat{z}_{18}]/(r_{24}) \subset \mathbf{F}_3[t_1, t_2, t_3, t_4]^{W_{SU(3,3)}}$$

and the theorem is proved.

Remark 6.4. An easier expression of r_{24} can be obtained if we consider the class $z_{18} = \hat{z}_{18} + z_2z_8^2 - z_2^3z_{12} - z_2^3z_4z_8 - z_2^7z_4 + z_2z_4^2z_8 - z_2^3z_4^3 - z_2^9$. In this case,

$$\mathbf{F}_3[t_1, t_2, t_3, t_4]^{W_{SU(3,3)}} \cong \mathbf{F}_3[z_2, z_4, z_8, z_{12}, z_{18}]/(z_2^3z_{18} - z_8^3 + z_2^2z_4z_8^2 - z_4^3z_{12}).$$

Remark 6.5. Note that the classes z_2 and z_{12} have been chosen such that they are the images of the generators of $H^*BT_{PU(3)}^{W_{PU(3)}}$ described in [7] by the map $H^*BT_{PU(3)} \to H^*BT_{SU(3,3)}$ induced by the projection $SU(3,3) \to PU(3)$.

The classes z_4 and z_8 have been chosen such that the natural inclusion $H^*BT_{F_4}^{W_{F_4}} \subset H^*BT_{SU(3,3)}^{W_{SU(3,3)}}$ maps the classes p_1 and $\bar{p}_2$ described in [20] to z_4 and z_8 respectively.

By means of the Cartan formula as well as the information given in the remarks above, we can get some information about the action of the Steenrod algebra.

Proposition 6.6. *The action of* $\mathscr{A}_3$ *on* $\mathbf{F}_3[t_1, t_2, t_3, t_4]^{W_{SU(3,3)}}$ *is given by,*

i) $\mathscr{P}^1z_2 = z_2^3, \mathscr{P}^1z_4 = -z_8 + z_4^2$, $\mathscr{P}^1z_8 = z_4z_8, \mathscr{P}^1z_{12} = z_2^2z_{12}$, *and* $\mathscr{P}^1z_{18} = z_2^3z_{18} - z_2z_4z_8^2$

ii) $\mathscr{P}^3z_2 = 0, \mathscr{P}^3z_4 = 0, \mathscr{P}^3z_8 = z_2z_{18} + f$, *and* $\mathscr{P}^3z_{12} = z_{12}(z_2^6 - z_{12})$, *where* $f \in \mathbf{F}_3[z_2, z_4, z_8, z_{12}]$

iii) $\mathscr{P}^9z_2 = 0, \mathscr{P}^9z_4 = 0$, $\mathscr{P}^9z_8 = 0, \mathscr{P}^9z_{12} = 0$, *and* $\mathscr{P}^9z_{18} = z_{18}^3$

7. The algebra structure of $H^*BSU(3,3)$

In this section we calculate the algebra structure of $H^*BSU(3,3)$ as well as the action of some Steenrod operations.

Remember we have a fibration (see Section 2)

$$BSU(3) \longrightarrow BSU(3,3) \xrightarrow{B\tilde{\pi}} BPU(3)$$

such that $B\tilde{\pi}^* : H^*BPU(3) \to H^*BSU(3,3)$ is injective (Lemma 2.4). Call $y_i = B\tilde{\pi}^*(t_i)$ for $i = 2, 3, 7, 12$ and $w_8 = B\tilde{\pi}^*(t_8)$ where $t_i \in H^*BPU(3)$ are the classes described in Theorem 2.2. Then trivially we get the algebra relations

$$y_2y_3 = 0, \qquad y_2y_7 = 0, \qquad y_2w_8 + y_3y_7 = 0$$

also some information about the Steenrod algebra action

$$\beta y_2 = y_3, \qquad \beta y_7 = w_8,$$

$$\mathscr{P}^1 y_3 = y_7, \qquad \mathscr{P}^1 y_{12} = w_8^2 + y_{12}y_2^2,$$

$$\mathscr{P}^3 y_7 = y_7y_{12} + y_3w_8^2, \qquad \mathscr{P}^3 w_8 = w_8y_{12}, \qquad \mathscr{P}^3 y_{12} = y_{12}(y_2^6 - y_{12}).$$

Moreover, we also have the inclusion $SU(3,3) \hookrightarrow F_4$ that induces an injection $H^*BF_4 \hookrightarrow H^*BSU(3,3)$. Call y_4, y_8 and y_9 the images of $z_4, z_8, z_9 \in H^*BF_4$ respectively, therefore we have the algebra relations

$$y_4y_9 = 0, \qquad y_8y_9 = 0$$

and

$$\mathscr{P}^1 y_4 = -y_8 + y_4^2, \qquad \beta y_8 = y_9, \qquad \mathscr{P}^1 y_9 = 0.$$

Now by dimensional reasons we get that

$$y_3y_4 = 0,$$

and applying $\mathscr{P}^1$ to this equality and $\beta\mathscr{P}^1$ we also get the algebra relations

$$0 = \mathscr{P}^1(y_3y_4) = y_7y_4 - y_3y_8$$

$$0 = \beta\mathscr{P}^1(y_3y_4) = w_8y_4 + y_3y_9.$$

As $\dim_{\mathbf{Q}_3^\wedge} H^{10}_{\mathbf{Q}_3} BSU(3,3) = \dim_{\mathbf{Q}_3^\wedge} H^{10}_{\mathbf{Q}_3} BSU(3)^2 = 4$ and $y_3w_8 \neq 0$ we have the following new algebra relation

$$0 = \beta(y_2y_8) = y_3y_8 + y_2y_9.$$

Now, note that $\mathscr{P}^1 y_8 = y_2^2 f + \lambda y_3y_9$ where $f \in \mathbf{F}_3[y_2, y_4, y_8, w_8]$ and $\lambda \in \mathbf{F}_3$, therefore $y_3\mathscr{P}^1 y_8 = 0$ and applying $\mathscr{P}^1$ and $\beta\mathscr{P}^1$ to the relation above we get,

$$0 = \mathscr{P}^1(y_3y_8 + y_2y_9) = y_7y_8$$

$$0 = \beta\mathscr{P}^1(y_3y_8 + y_2y_9) = w_8y_9 - y_7y_9.$$

At this point, to have completely determined the algebra structure of $H^*BSU(3,3)$, we only need to find a nice generator in dimension 18 and to describe the algebra relation that appears in dimension 24. In order to do that we use the information of Section 6.

Lemma 7.1. *Let $T \overset{\iota}{\hookrightarrow} SU(3,3)$ be the standard inclusion of the maximal torus of $SU(3,3)$. Then $H^*BSU(3,3) \xrightarrow{B\iota^*} (H^*BT)^{W_{SU(3,3)}}$ is surjective.*

Proof. By Remark 6.4 we know that

$$\mathbf{F}_3[t_1,t_2,t_3,t_4]^{W_{SU(3,3)}} \cong \mathbf{F}_3[z_2,z_4,z_8,z_{12},z_{18}]/(z_2^3z_{18} - z_8^3 + z_2^2z_4z_8^2 - z_4^3z_{12}).$$

We already know that $H^*BSU(3,3)$ is generated (as algebra) by $y_2, y_3, y_4, y_7, y_8, w_8, y_{12}$ and a generator in dimension 18, and because the way they have been chosen and Remark 6.5, we also know that

$$B\iota^*(y_2) = z_2, \quad B\iota^*(y_3) = 0, \quad B\iota^*(y_4) = z_4, \quad B\iota^*(y_7) = 0,$$
$$B\iota^*(w_8) = 0, \quad B\iota^*(y_8) = z_8, \quad B\iota^*(y_{12}) = z_{12}.$$

Therefore the image of this new generator in dimension 18 has to be 0 or z_{18} probably plus decomposables. If it is 0, then $\operatorname{Im} B\iota^* = \mathbf{F}_3[z_2,z_4,z_8,z_{12}]$, but by Proposition 6.6 we know that $\mathscr{P}^3z_8 = z_2z_{18} + f$ where $f \in \mathbf{F}_3[z_2,z_4,z_8,z_{12}]$, which means that $\mathscr{P}^3z_8 \notin \operatorname{Im} B\iota^*$ what is impossible.

Hence z_{18} is in $\operatorname{Im} B\iota^*$ and the lemma is proved.

By the last lemma we know that there exist at least one element y_{18} such that $B\iota^*(y_{18}) = z_{18}$. It is clear that y_{18} is a new generator and, because

$$\dim_{\mathbf{Q}} H^{18}(BSU(3,3);\mathbf{Q}) = \dim_{\mathbf{F}_3} H^{18}BSU(3,3),$$

we also have that $\beta y_{18} = 0$.

Now, by the last lemma we know that the algebra relation in dimension 24 has to be of the form

$$0 = y_2^3y_{18} - y_8^3 + y_2^2y_4y_8^2 - y_4^3y_{12} + ay_4w_8y_{12} + bw_8^2y_8 + cw_8^3, \tag{4}$$

where $a,b,c \in \mathbf{F}_3$. Applying β to (4) we get that $0 = bw_8^2y_9$, hence $b = 0$ and the relation is reduced to

$$0 = y_2^3y_{18} - y_8^3 + y_2^2y_4y_8^2 - y_4^3y_{12} + ay_4w_8y_{12} + cw_8^3, \tag{5}$$

where $a,c \in \mathbf{F}_3$. Applying β to $y_2(5)$ we get that $0 = cy_3w_8^3$, hence $c = 0$ and the relation is reduced to

$$0 = y_2^3y_{18} - y_8^3 + y_2^2y_4y_8^2 - y_4^3y_{12} + ay_4w_8y_{12}, \tag{6}$$

where $a \in \mathbf{F}_3$. Applying β to $y_7(6)$ we get that $0 = ay_4w_8^2y_{12}$, hence $a = 0$ and finally we get the algebra relation

$$0 = y_2^3y_{18} - y_8^3 + y_2^2y_4y_8^2 - y_4^3y_{12}$$

which finishes the calculations.

The action of $\mathcal{P}^1$ on y_{18} can be easily deduced of that relation and we get

$$\mathcal{P}^1 y_{18} = y_2^2 y_{18} - y_2 y_4 y_8^2.$$

8. Two principal fibrations

Let X be a 3-complete space with $H^*X \cong_{\mathcal{A}_3} H^*BSU(3,3)$. First of all, we obtain some information about the low dimensional 3-adic cohomology of X.

Lemma 8.1. *Let X be a 3-complete space with $H^*X \cong_{\mathcal{A}_3} H^*BSU(3,3)$, then the low dimensional 3-adic cohomology of X is*

n	0	1	2	3	4	5	6	7
$H^*(X;\mathbf{Z}_3^\wedge)$	$\mathbf{Z}_3^\wedge$	0	0	$\bar{y}_3\mathbf{Z}/3$	$\{\overline{y_2^2}, \bar{y}_4\}\mathbf{Z}_3^\wedge$	0	$\{\overline{y_2^3}, \overline{y_4 y_2}\}\mathbf{Z}_3^\wedge$	0

where the notation has been chosen in such a way that the mod 3 reduction sends a 3-adic class $\bar{x}$ to its mod 3 reduction x.

Proof. Because X is a simply connected, p-complete space such that H^jX is finite for all j, we can apply Proposition 5.7 in [1], and we get that $H^j(X;\mathbf{Z}_3^\wedge)$ is a finitely generated $\mathbf{Z}_3^\wedge$-module for all j. Hence we can apply the Bss.

The first differential is the primary Bockstein. It vanishes for all the algebra generators of $H^*X \cong H^*BSU(3,3)$ but (see 1.2)

$$\beta y_2 = y_3, \qquad \beta y_7 = w_8, \qquad \text{and} \qquad \beta y_8 = y_9.$$

Therefore at the B_2-stage of the spectral sequence all the elements are in even degree, hence all of them are permanent cycles and we get the desired result.

By the lemma above, we know that $H^3(X;\mathbf{Z}_3^\wedge) = \mathbf{Z}/3$. Let $X \to K(\mathbf{Z}_3^\wedge, 3)$ be the natural map induced by $\bar{y}_3$, the generator of $H^3(X;\mathbf{Z}_3^\wedge)$, and let Y be the homotopy fibre of that map. This map classifies a principal fibration,

$$(BS^1)_3^\wedge \xrightarrow{h} Y \xrightarrow{j} X. \tag{7}$$

We also know that $H^2X = \mathbf{Z}/3$. Let $X \to K(\mathbf{Z}/3,2)$ be the natural map induced by y_2, the generator of H^2X, and let F be the homotopy fibre of that map. This map classifies a principal fibration,

$$B\mathbf{Z}/3 \xrightarrow{f} F \xrightarrow{g} X. \tag{8}$$

As there exists a map $K(\mathbf{Z}/3,2) \xrightarrow{\xi} K(\mathbf{Z}_3^\wedge, 3)$ such that the following diagram commutes

$$\begin{array}{ccc} X & = & X \\ {\scriptstyle y_2}\downarrow & & \downarrow{\scriptstyle \bar{y}_3} \\ K(\mathbf{Z}/3,2) & \xrightarrow{\xi} & K(\mathbf{Z}_3^\wedge, 3), \end{array}$$

both fibrations (7) and (8) fit in the following commutative diagram,

$$\begin{array}{ccccc}
(S^1)^\wedge_3 & \longrightarrow & B\mathbf{Z}/3 & \longrightarrow & (BS^1)^\wedge_3 \\
\| & & \downarrow f & & \downarrow h \\
(S^1)^\wedge_3 & \longrightarrow & F & \longrightarrow & Y \\
& & \downarrow g & & \downarrow j \\
& & X & = & X
\end{array} \tag{9}$$

where both rows are also fibrations.

In this section we compute the cohomology of Y, H^*Y, additively.

Proposition 8.2. *The mod* 3 *Sss associated to the fibration* (7) *collapses at the* E_6*-stage and:*

$$E_\infty^{*,*} = \mathbf{F}_3[x_2, x_4, \tilde{x}_4, x_6, \tilde{x}_6, x_8, x_{12}, \tilde{x}_{18}, \tilde{x}_{20}]/H$$

where H *is the ideal generated by* $\tilde{x}_4 x_8^2 - \tilde{x}_{18}x_2, \tilde{x}_4 x_4 - \tilde{x}_6 x_2, \tilde{x}_4^3 - x_6 x_2^3, \tilde{x}_4\tilde{x}_{18} - \tilde{x}_{20}x_2$ *and* $x_2^3 x_{18} - x_8^3 + x_2 x_4 x_8^2 - x_4^3 x_{12}$.

Proof. Let us denote $H^*BS^1 = \mathbf{F}_3[v]$ where $|v| = 2$ and let $(E_*^{*,*}, d_*)$ be the mod 3 Sss associated to the fibration (7). The spectral sequence starts as:

$$E_2^{*,*} = \mathscr{H}^*(X; H^*BS^1) \cong H^*X \otimes H^*BS^1;$$

hence, in view of Corollary 4.10, we can write it as

$$E_2^{*,*} = C \oplus \{1, y_2, y_3, y_4, y_7, y_8, y_9, y_3 y_8\}\mathbf{F}_3[w_8, y_{12}, y_{18}, v],$$

where y_n has bidegree $(n, 0), w_8$ has bidegree $(8, 0), v$ has bidegree $(0, 2)$, and

$$\begin{aligned} C &= (A/A \cap B) \otimes \mathbf{F}_3[v] \\ &= (y_2^2 \mathbf{F}_3[y_2, y_4, y_8, y_{12}, y_{18}, v] \oplus \{y_2 y_4, y_4^2\}\mathbf{F}_3[y_4, y_8, y_{12}, y_{18}, v] \\ &\quad \oplus \{y_2 y_8, y_4 y_8, y_8^2\}\mathbf{F}_3[y_8, y_{12}, y_{18}, v])/r_{24} \end{aligned}$$

being $r_{24} = y_2^3 y_{18} - y_8^3 + y_2^2 y_4 y_8^2 - y_4^3 y_{12}$.

The first non trivial differential is the transgression of v, which is by construction y_3, that is, $d_3(v) = y_3$. This determines d_3 and:

$$\begin{aligned} E_4^{*,*} &= C \oplus \{y_2 v^2, y_3 v^2, y_4 v^2, y_7, y_8, y_9, y_3 y_8 v^2\}\mathbf{F}_3[w_8, y_{12}, y_{18}, v^3] \\ &\quad \oplus \{y_2, y_2 v, y_4, y_4 v\}\mathbf{F}_3[y_{12}, y_{18}, v^3], \end{aligned}$$

where

$$\{y_2, y_2 v, y_2 v^2, y_3 v^2, y_4, y_4 v, y_4 v^2, y_7, y_8, y_8^2 v, y_8^2 v^2, w_8, y_9, y_{12}, y_{18}, v^3\}$$

can be chosen as a set of algebra generators. It follows that $d_4 \equiv 0$ and $E_5^{*,*} = E_4^{*,*}$.

Lemma 8.3. *v^3 is a permanent cycle, and d_5 is determined by $d_5(y_3v^2) = w_8, d_5(y_2v^2) = \pm y_7$ and $d_5(y_4v^2) = \pm y_9$.*

Proof. Since v transgresses to y_3, v^3 must be transgressive too, in particular $d_5(v^3) = 0$, and by the Kudo transgression theorem, $d_5(y_2v^2) = \beta\mathscr{P}^1 y_3 = w_8$.

We look now at the Sss with coefficients in $\mathbf{Z}_3^\wedge$:

$$\hat{E}_2^{*,*} = \mathscr{H}^*(X; H^*(BS^1; \mathbf{Z}_3^\wedge)) \quad \Rightarrow \quad H^*(Y; \mathbf{Z}_3^\wedge).$$

It is well known that $H^*(BS^1; \mathbf{Z}_3^\wedge) = \mathbf{Z}_3^\wedge[\bar{v}]$ and, by Lemma 8.1, we also know $H^*(X; \mathbf{Z}_3^\wedge)$ for $* \leq 7$. The first non trivial differential is $\hat{d}_3\bar{v} = \bar{y}_3$ and we find that $\bar{v}^3$ should be a permanent cycle because $\hat{d}_3\bar{v}^3 = 3\bar{y}_3\bar{v}^2 = 0$ and there is no other possibilities for $\hat{d}_r\bar{v}^3$.

In addition, we know that the mod 3 reduction is a natural transformation of cohomology theories and therefore induces a map between the corresponding spectral sequences. Comparing both spectral sequences, we find that v^3 should be a permanent cycle in the mod 3 Sss as so is $\bar{v}^3$ in the 3-adic Sss.

Since $y_7 = \mathscr{P}^1 y_3$ and y_3 does not survive, y_7 has to "die" too. The unique possibilities are $d_5(y_2v^2) = \pm y_7$, by degree reasons.

At this point, $d_5(y_4v^2)$ could be $\pm y_9$ or trivial. Assume $d_5(y_4v^2) = 0$, in this case

$$y_8(y_3v^2) = y_7(y_4v^2)$$
$$d_5(y_8(y_3v^2)) = d_5(y_7(y_4v^2))$$
$$y_8w_8 = 0$$

which is impossible (recall that y_3v^2 and y_4v^2 are indecomposables at the E_5-stage). Hence $d_5(y_4v^2) = \pm y_9$.

By dimensional reasons we get that d_5 vanishes for the other algebra generators of $E_5^{*,*}$.

Now, we know d_5 over all generators so we can calculate $E_6^{*,*}$:

$$E_6^{*,*} = C \oplus \{y_2, y_2v, y_4, y_4v, y_8\}\mathbf{F}_3[y_{12}, y_{18}, v^3],$$

where

$$\{y_2, y_2v, y_4, y_4v, y_8, y_8^2v, y_8^2v^2, y_{12}, y_{18}, v^3\}$$

can be chosen as a set of algebra generators.

Now, dimensional arguments show that all the following differentials vanish, so $E_6^{*,*} = E_\infty^{*,*}$, that is,

$$E_\infty^{*,*} = \mathbf{F}_3[x_2, x_4, \tilde{x}_4, x_6, \tilde{x}_6, x_8, x_{12}, \tilde{x}_{18}, \tilde{x}_{20}]/H$$

where x_i is represented by y_i for $i \neq 6, x_6$ is represented by $v^3, \tilde{x}_4$ is represented by $y_2v, \tilde{x}_6$ is represented by $y_4v, \tilde{x}_{18}$ is represented by y_8^2v and $\tilde{x}_{20}$ is represented by $y_8^2v^2$, and H is the ideal generated by $\tilde{x}_4x_8^2 - \tilde{x}_{18}x_2, \tilde{x}_4x_4 - \tilde{x}_6x_2, \tilde{x}_4^3 - x_6x_2^3$, $\tilde{x}_4\tilde{x}_{18} - \tilde{x}_{20}x_2$ and $x_2^3x_{18} - x_8^3 + x_2x_4x_8^2 - x_4^3x_{12}$.

Remark 8.4. From the edge homomorphism of this Sss we know that $x_i = j^*(y_i)$ for $i \neq 6$ and $h^*(x_6) = v^3$.

Now we can calculate the Poincaré series of H^*Y.

Corollary 8.5. *The Poincaré series of H^*Y is*

$$P(H^*(Y;\mathbf{F}_3),t) = \frac{1}{(1-t^4)^2(1-t^6)^2(1-t^2)}.$$

Proof. Note that the calculation of mod 3 Sss in the proposition above does only depend on the $\mathscr{A}_3$-algebra structure of H^*X, hence the same result is obtained if we replace X by $BSU(3,3)$. In this last case, Y would be the classifying space of

$$G = \{(A,B) \in U(3) \times U(3) \mid \det(A) = \det(B)\},$$

and therefore the Poincaré series of H^*Y and of H^*BG should agree.

Finally, notice that G fits in a short exact sequence

$$SU(3) \to G \to U(3)$$

that induces a fibration

$$BSU(3) \to BG \to BU(3)$$

whose mod 3 Sss collapses at the E_2-stage and gives an algebra isomorphism $H^*BG \cong H^*BSU(3) \otimes H^*BU(3)$ from which one obtains

$$P(H^*Y,t) = P(H^*BG,t) = \frac{1}{(1-t^4)^2(1-t^6)^2(1-t^2)}.$$

9. Maximal tori

Let us consider V the toral elementary abelian 3-subgroup of $SU(3,3)$ of maximal rank and denote by i_V the inclusion. We can easily obtain that the centralizer of V in $SU(3,3)$ is T^4, the maximal torus.

Now, if X is a 3-complete space with $H^*X \cong_{\mathscr{A}_3} H^*BSU(3,3)$, Lannes' theory, [9], provides a map $\alpha : BV \to X$, that induces

$$\alpha^* \equiv Bi_V^* : H^*X \cong H^*BSU(3,3) \to H^*BV$$

in cohomology. Then, Lannes' T functor shows that $\mathrm{map}(BV,X)_\alpha \simeq (BT^4)^\wedge_3$.

Now, the mod 3 reduction of the natural representation of $\Sigma_3 \oplus \Sigma_3 \cong W_{SU(3,3)}$ on $\mathrm{Aut}(T^4)$ induces an action on BV and $\mathrm{map}(BV,X)_\alpha \simeq (BT^4)^\wedge_3$. This action looks in mod 3 cohomology like the honest one on $BT^4 = BC_{SU(3,3)}(V)$. According to Notbohm [18], the possible lattices associated such an action of $\Sigma_3 \oplus \Sigma_3$ are equivalent to those of $SU(3)^2, SU(3,3)$ or $SU(3) \times PU(3)$, whose mod 3 reductions are not equivalents. Therefore the action of $\Sigma_3 \oplus \Sigma_3$ on $\mathrm{map}(BV,X)_\alpha \simeq (BT^4)^\wedge_3$ should be equivalent to that on $BT^4 = BC_{SU(3,3)}(V)$.

Moreover, the evaluation map is $\Sigma_3 \oplus \Sigma_3$-equivariant and the action is lifted to homotopy pullbacks, so from fibration (7) we obtain the diagram

$$\begin{array}{ccc} (BS^1)^\wedge_3 & = & (BS^1)^\wedge_3 \\ \downarrow & & \downarrow \\ \Sigma_3 \oplus \Sigma_3 \circlearrowleft E & \xrightarrow{\widetilde{ev}} & Y \\ {\scriptstyle \tilde{j}}\downarrow & & {\scriptstyle j}\downarrow \\ \Sigma_3 \oplus \Sigma_3 \circlearrowleft \operatorname{map}(BV, X)_\alpha & \xrightarrow{ev} & X \end{array} \tag{10}$$

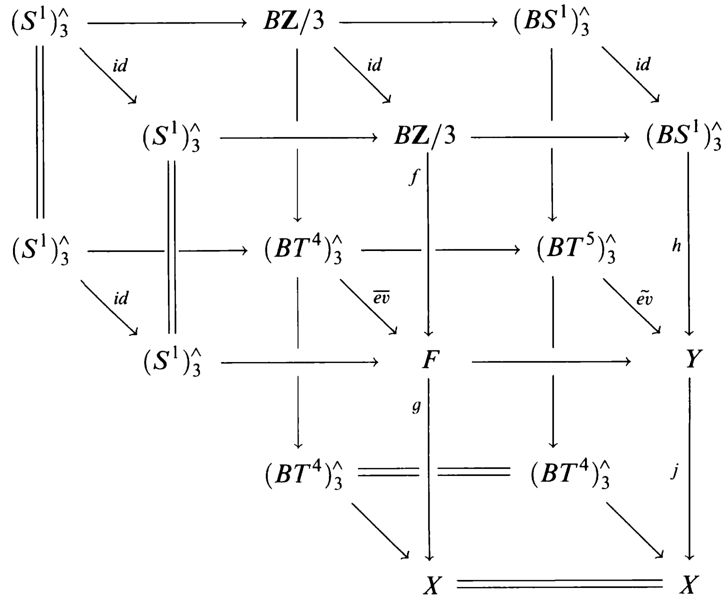
where all maps are equivariant. The pullback fibration is therefore classified by the composition map $(BV, X)_\alpha \xrightarrow{ev} X \xrightarrow{\bar{y}_3} K(\mathbf{Z}_3^\wedge, 3)$ which is clearly trivial, hence $E \simeq (BT^5)^\wedge_3$.

In the same way, from fibration (8) we obtain the diagram

$$\begin{array}{ccc} B\mathbf{Z}/3 & = & B\mathbf{Z}/3 \\ \downarrow & & \downarrow \\ \Sigma_3 \oplus \Sigma_3 \circlearrowleft \bar{E} & \xrightarrow{\overline{ev}} & F \\ {\scriptstyle \bar{g}}\downarrow & & {\scriptstyle g}\downarrow \\ \Sigma_3 \oplus \Sigma_3 \circlearrowleft \operatorname{map}(BV, X)_\alpha & \xrightarrow{ev} & X \end{array} \tag{11}$$

where all maps are equivariant. The pullback fibration is therefore classified by the composition $\operatorname{map}(BV, X)_\alpha \xrightarrow{ev} X \xrightarrow{y_2} K(\mathbf{Z}/3, 2)$. As this map represents a non splitting extension, we see that $\bar{E} \simeq (BT^4)^\wedge_3$.

Now, we can complete the diagram (9) to



(12)

We study the maps $\overline{ev}$ and $\widetilde{ev}$ to obtain information about the cohomology of F, H^*F.

Lemma 9.1. *The induced map in cohomology*

$$\widetilde{ev}^* : H^*Y \to H^*E \cong H^*BT^5$$

is injective.

Proof. Note that by construction $ev : \operatorname{map}(BV, X)_\alpha \simeq (BT^4)^\wedge_3 \to X$ looks in cohomology like the inclusion of the maximal torus in $SU(3,3)$, hence (see 6) we have that

$$ev^* : H^*X \to (H^*(BT^4)^\wedge_3)^{\Sigma_3 \oplus \Sigma_3}$$
$$\cong \mathbf{F}_3[z_2, z_4, z_8, z_{12}, z_{18}]/(z_2^3 z_{18} - z_8^3 + z_2^2 z_4 z_8^2 - z_4^3 z_{12})$$

is given by $ev^*(y_3) = ev^*(y_7) = ev^*(w_8) = ev^*(y_9) = 0$ and $ev^*(y_i) = z_i$ for $i = 2, 4, 8, 12, 18$. This determines the map of Sss's induced by the diagram (10). At the E_2-stage is

$$ev^* \otimes 1 : H^*X \otimes H^*BS^1 \to H^*BT^4 \otimes H^*BS^1$$

and then we find that the generators of the E_∞-stage of the first spectral sequence (see Section 8) are all detected in the second one which clearly collapses at the E_2-stage. But this last is a free commutative algebra, hence the map of spectral sequences becomes an injection at the E_∞-stage. The lemma follows by induction on the filtration degree of the spectral sequences.

Lemma 9.2. (1) *The induced map in cohomology*

$$\overline{ev}^* : H^*F \to H^*\bar{E} \cong H^*BT^4$$

is injective.

(2) *The Poincaré series of H^*F is*

$$P(H^*F, t) = \frac{1}{(1-t^4)^2(1-t^6)^2}$$

Proof. (2) Consider the mod 3 Sss associated to the fibration

$$(S^1)^\wedge_3 \to F \to Y$$

that appears in diagram (12). Let us denote $H^*S^1 = \Lambda_{\mathbf{F}_3}(u)$. The spectral sequence starts as

$$E_2^{*,*} = \mathscr{H}^*(Y; H^*S^1) \cong H^*Y \otimes H^*S^1.$$

The first non trivial differential is the transgression of u, which is by construction y_2, the generator of H^2Y. This determines d_2 and, because every element in H^*Y is regular (by Lemma 9.1 H^*Y is a subalgebra of a polynomial algebra), $E_3^{*,*} = H^*Y/J$ where J is the ideal generated by y_2. Clearly the following differentials vanish, so $E_\infty = E_3$ and because y_2 is regular,

$$P(H^*F, t) = P(H^*Y, t) \cdot (1 - t^2)$$

which gives the desired result.

(1) Consider the natural map of spectral sequences between the mod 3 Sss associated to both central rows in diagram (12). At the E_2-stage is

$$\widetilde{ev}^* \otimes 1 : H^*Y \otimes H^*S^1 \to H^*BT^5 \otimes H^*S^1$$

and, by Lemma 9.1, it is injective. Clearly, both spectral sequences collapse at the E_3-stage which gives us an injection at the E_∞-stage. Those E_∞-terms are concentrated in the zero horizontal line so that they actually coincide with the cohomology of the total spaces, thus proving the desired result.

Since $\overline{ev}$ is $\Sigma_3 \oplus \Sigma_3$-equivariant, we actually obtain that

$$\overline{ev}^* : H^*F \hookrightarrow (H^*\bar{E})^{\Sigma_3 \oplus \Sigma_3}.$$

It is therefore crucial to understand the action of $\Sigma_3 \oplus \Sigma_3$ on $H^*\bar{E} \cong H^*BT^4$. Let $L_{\bar{E}}$ denote the 3-adic lattice associated to the action of $\Sigma_3 \oplus \Sigma_3$ on $H^*(\bar{E}; \mathbf{Z}_3^\wedge)$. Because the 3-adic lattice associated to the action of $\Sigma_3 \oplus \Sigma_3$ on $H^*(\mathrm{map}\,(BV, X)_\alpha; \mathbf{Z}_3^\wedge)$ is equivalent to that of $SU(3,3)$, the left hand fibration of the diagram (11) provides an exact sequence of lattices:

$$0 \to L_{SU(3,3)} \to L_{\bar{E}} \to \mathbf{Z}/3 \to 0.$$

According to Notbohm [18], we get that $L_{\bar{E}}$ have to be the centerfree lattice associated to $L_{SU(3,3)}$ and therefore $L_{\bar{E}} \cong L_{SU(3)\times SU(3)}$ which gives us the precise action of $\Sigma_3 \oplus \Sigma_3$ on $H^*\bar{E} \cong H^*BT^4$. Now we can prove,

Proposition 9.3. *$H^*F \cong H^*B(SU(3) \times SU(3))$ as $\mathscr{A}_3$-algebras.*

Proof. By Lemma 9.2 (1), we know that $\overline{ev}^*$: $H^*F \hookrightarrow (H^*\bar{E})^{\Sigma_3 \oplus \Sigma_3}$. Because $(L_E)_3^\wedge \cong (L_{SU(3)\times SU(3)})_3^\wedge$, then

$$\begin{aligned}(H^*\bar{E})^{\Sigma_3 \oplus \Sigma_3} &\cong (H^*BT^4)^{W_{SU(3)\times SU(3)}} \\ &\cong \mathbf{F}_3[c_4, \tilde{c}_4, c_6, \tilde{c}_6] \cong H^*B(SU(3) \times SU(3)).\end{aligned}$$

Hence we have that

$$\overline{ev}^* : H^*F \hookrightarrow H^*B(SU(3) \times SU(3)),$$

and by Lemma 9.2 (2), the Poincaré series of both algebras agree, so $\overline{ev}^*$ induces the desired isomorphism.

10. The proof of Theorems 1.3 and 1.4

Proof of Theorem 1.3. Let X be a 3-complete space with $H^*X \cong H^*BSU(3,3)$ as $\mathscr{A}_3$-algebras. For such a space we have obtained a principal fibration (8):

$$BZ/3 \xrightarrow{f} F \xrightarrow{g} X$$

and proved (Proposition 9.3) that $H^*F \cong_{\mathscr{A}_3} H^*BSU(3)^2$. According to Notbohm ([17]) this implies that $F \simeq (BSU(3)^2)^\wedge_3$.

Following Dwyer and Zabrodsky's work [5] we have that

$$\operatorname{map}(B\mathbf{Z}/3, (BSU(3)^2)^\wedge_3)_c \simeq B(SU(3)^2)^\wedge_3$$

where c stands for the constant map. Therefore we can apply Zabrodsky's Lemma (see [11], [22]) and obtain that the composition of the homotopy equivalence $F \simeq (BSU(3)^2)^\wedge_3$ with the 3-completion of the standard $(BSU(3)^2) \to BSU(3,3)$ factors up to homotopy through X:

$$\begin{array}{ccccc} B\mathbf{Z}/3 & \longrightarrow & F & \xrightarrow{g} & X \\ \| & & \simeq\downarrow & & \downarrow \\ B\mathbf{Z}/3 & \longrightarrow & (BSU(3)^2)^\wedge_3 & \longrightarrow & BSU(3,3)^\wedge_3, \end{array}$$

from which $X \simeq BSU(3,3)^\wedge_3$.

Proof of Theorem 1.4. On the one hand,

$$H^*_{\mathbf{Q}_3}(BX) \cong H^*_{\mathbf{Q}_3}(BT_X)^{W_{SU(3,3)}} \cong \mathbf{Q}^\wedge_3[y_4, \tilde{y}_4, y_6, \tilde{y}_6]$$

and we get that $H^*_{\mathbf{Q}_3}X \cong \Lambda_{\mathbf{Q}^\wedge_3}(x_3, \tilde{x}_3, x_5, \tilde{x}_5)$. Hence the mod 3 Bss of H^*X should converge to $B_\infty \cong \Lambda_{\mathbf{F}_3}(x_3, \tilde{x}_3, x_5, \tilde{x}_5)$.

On the other hand, because the map $\pi_1 T \to \pi_1 X$ is surjective [14], and it factors through the covariants $(\pi_1 T)_{W_X} \cong (\pi_1 T)_{W_{SU(3,3)}} = \mathbf{Z}/3$, we have that either $\pi_1 X = 0$ or $\pi_1 X = \mathbf{Z}/3$.

If $\pi_1 X = 0$, according to Lin (see [10]) H^*X has only elementary 3-torsion and the first possible indecomposable in H^*X which is not in B_∞ appears in dimension 7. Hence $x_3, \tilde{x}_3, x_5$, and $\tilde{x}_5$ are in fact indecomposable elements of H^*X and $H^*X \cong \Lambda_{\mathbf{F}_3}(x_3, \tilde{x}_3, x_5, \tilde{x}_5)$. Using EMss we get then that $H^*BX \cong \mathbf{F}_3[y_4, \tilde{y}_4, y_6, \tilde{y}_6]$ and therefore by [17] $BX \simeq (BSU(3)^2)^\wedge_3$. But $SU(3,3)$ and $SU(3)^2$ have not the same Weyl group type at the prime 3. Hence $\pi_1 X = \mathbf{Z}/3$.

Let $\tilde{X}$ be the universal cover of X which is a simply connected 3-compact group such that the mod 3 Bss of $H^*\tilde{X}$ should also converge to $B_\infty \cong \Lambda_{\mathbf{F}_3}(x_3, \tilde{x}_3, x_5, \tilde{x}_5)$. By the argument above, $B\tilde{X} \simeq (BSU(3)^2)^\wedge_3$.

To finish the proof we follow exactly the same arguments we used in the proof of 1.3. Following Dwyer and Zabrodsky's work [5] we have that

$$\operatorname{map}(B\mathbf{Z}/3, (BSU(3)^2)^\wedge_3)_c \simeq (BSU(3)^2)^\wedge_3$$

where c stands for the constant map. Therefore we can apply Zabrodsky's Lemma (see [11] and [22]) and obtain that the composition of the homotopy equivalence $B\tilde{X} \simeq (BSU(3)^2)^\wedge_3$ with the 3-completion of the standard $BSU(3)^2 \to BSU(3,3)$ factors up to homotopy through BX:

$$\begin{array}{ccccc} B\mathbb{Z}/3 & \longrightarrow & B\tilde{X} & \xrightarrow{g} & BX \\ \| & & \simeq\downarrow & & \downarrow \\ B\mathbb{Z}/3 & \longrightarrow & BSU(3)^2)^\wedge_3 & \longrightarrow & BSU(3,3)^\wedge_3, \end{array}$$

from which $BX \simeq BSU(3,3)^\wedge_3$.

We finish this section proving Corollary 1.5.

Proof of Corollary 1.5. By [16], we have to prove that the classifying space of L is in the same genus as $BSU(3,3)$. The rational case and the p-completed cases, for $p \neq 3$, follow from [15] and the case $p = 3$ from Theorem 1.4. This completes the proof.

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