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The structure of dualizing complex for a ring which is (S_2)

By

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Dualizing complexes were introduced by Grothendieck and Hartshorne in [7] for use in algebraic geometry and were studied afterwards by R. Y. Sharp and a number of authors, in a series of papers in commutative algebra.

The aim of this paper is to discuss more thoroughly about dualizing complexes. After introductory section 1, in section 2, we shall find out that dualizing complex of a ring is isomorphic to a Cousin complex of a certain module in certain cases. Cousin complexes were introduced in [7] and it has a commutative algebra analogous given by R. Y. Sharp in [14]. Incidentally any Cousin complex of a finitely generated module is a complex of modules of generalized fractions; so that it makes each term and each morphism of dualizing complex more clarified (see 3.2). As a result, we generalize a result of H. Zakeri [24, 3.6] which shows that each indecomposable injective module over a Gorenstein ring is expressible in terms of a module of generalized fractions. More precisely, we prove that each indecomposable injective module over a ring which is (S_2) and possesses a dualizing complex is expressible as a module of generalized fractions. It worth noting that [24] generalizes [22]. Note that finding a precise description of indecomposable injective module has been the main objects of [10], [4] and [8].

Finally, we prove that if a local ring A is (S_2) and has the canonical module K, then a necessary and sufficient condition for A to possess a dualizing complex is that the Cousin complex of K with respect to a certain filtration has finitely generated homology modules. In particular, if K is a generalized Cohen-Macaulay module, we show that A is a generalized Cohen-Macaulay ring and it possesses dualizing complex (see 3.4 and 3.5).

Throughout this paper, A denotes a commutative Noetherian ring with non-zero identity and M denotes an A-module.

1. Reminder and Preliminaries

In this section we recall some definitions and facts about Cousin complexes and dualizing complexes.

1.1. Definition. A filtration of $\mathrm{Spec}(A)$ [19, 1.1] is a descending sequence $\mathcal{F} = (F_i)_{i\geq 0}$ of subsets of $\mathrm{Spec}(A)$, so that

$$F_0 \supseteq F_1 \supseteq \cdots \supseteq F_i \supseteq F_{i+1} \supseteq \cdots,$$

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with the property that, for each $i \geq 0$, each member of $\partial F_i = F_i - F_{i+1}$ is a minimal member of F_i with respect to inclusion. We say that $\mathscr{F}$ admits M if $\operatorname{Supp}_A(M) \subseteq F_0$. Suppose $\mathscr{F}$ is a filtration of $\operatorname{Spec}(A)$ that admits M. The Cousin complex $C(\mathscr{F}, M)$ for M with respect to $\mathscr{F}$ has the form

$$0 \xrightarrow{d^{-2}} M \xrightarrow{d^{-1}} M^0 \xrightarrow{d^0} M^1 \longrightarrow \cdots \longrightarrow M^n \xrightarrow{d^n} M^{n+1} \longrightarrow \cdots$$

with $M^n = \bigoplus_{\mathfrak{p} \in \partial F_n} (\operatorname{Coker} d^{n-2})_{\mathfrak{p}}$ for all $n \geq 0$. The homomorphisms in this complex have the following properties: for $m \in M$ and $\mathfrak{p} \in \partial F_0$, the component of $d^{-1}(m)$ in $M_{\mathfrak{p}}$ is $m/1$; for $n > 0$, $x \in M^{n-1}$ and $\mathfrak{q} \in \partial F_n$, the component of $d^{n-1}(x)$ in $(\operatorname{Coker} d^{n-2})_{\mathfrak{q}}$ is $\bar{x}/1$, where $^- : M^{n-1} \to \operatorname{Coker} d^{n-2}$ is the canonical epimorphism. The fact that such a complex can be constructed is explained in [19, 1.3].

1.2. Definition [15, (2.4)]. A *dualizing complex* $I^\cdot$ for A is a complex of A-modules and A-homomorphisms such that

(i) each I^i is an injective A-module;

(ii) $I^\cdot$ is a bounded complex;

(iii) for each i, $H^i(I^\cdot)$, the i-th cohomology module of $I^\cdot$ is finitely generated A-module;

(iv) whenever $X^\cdot$ is a bounded complex of A-modules and A-homomorphisms with the property that all its cohomology modules are finitely generated, the morphism of complexes

$$\theta(X^\cdot; I^\cdot)^\cdot : X^\cdot \to \operatorname{Hom}_A([\operatorname{Hom}_A(X^\cdot, I^\cdot)], I^\cdot),$$

described in [15, (2.3) (ii)] is a quasi-isomorphism.

It is known that a Noetherian ring A possesses a dualizing complex if and only if it possesses a fundamental dualizing complex $I^\cdot$, say, satisfying (i), (ii) and (iii) above and the following condition (iv)′:

$$\text{(iv)}' \qquad \bigoplus_{i \in \mathbf{Z}} I^i \cong \bigoplus_{\mathfrak{p} \in \operatorname{Spec}(A)} E(A/\mathfrak{p}),$$

where $E(A/\mathfrak{p})$ is the injective envelope of $A/\mathfrak{p}$ as A-module, i.e. each prime ideal of A occurs in exactly one term of $I^\cdot$, and there it occurs exactly once (see [6, (3.6)] and [18, (1.2)]). Fundamental dualizing complex for a local ring A (if exists) is unique up to isomorphism of complexes and shifting (see [15, (4.5)] and [6, 4.2]). A fundamental dualizing complex $I^\cdot$ for a ring A can be normalized, so that $\sup\{i : I^i \neq 0\} = \dim A$ (see [6, 4.3]). It therefore follows from [16, (3.3)] that, for a local ring $(A, \mathfrak{m})$ with $\dim A = d$, $I^\cdot$ is a normalized fundamental dualizing complex (abbr. NFDC) for A if and only if (i), (ii), (iii) of (1.2) and the following condition (iv)″ are satisfied:

$$\text{(iv)}'' \qquad I^i \cong \bigoplus_{\substack{\mathfrak{p} \in \operatorname{Spec}(A) \\ \dim(A/\mathfrak{p}) = d-i}} E(A/\mathfrak{p}), \quad i = 0, 1, \ldots, d.$$

In this case, $H^0(I^\cdot)$, the initial non-zero cohomology module of $I^\cdot$, is the canonical module of A.

Recall that, for a given integer $n > 0$, an A-module M is said to be (S_n) if $\operatorname{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) \geq \min\{n, \dim M_\mathfrak{p}\}$ for all $\mathfrak{p} \in \operatorname{Supp}_A(M)$. Denote, by $\operatorname{Min}(A)$, the set of all minimal elements in $\operatorname{Spec}(A)$ and, in case where M is of finite dimension, we put $\operatorname{Assh}(M) = \{\mathfrak{p} \in \operatorname{Ass}_A(M) : \dim A/\mathfrak{p} = \dim M\}$.

The following remarks are needed for our process.

1.3. Remarks. Let $(A, \mathfrak{m})$ be a local ring of dimension d and K be its canonical module.

(i) (See [1, (1.7) and (1.9)]) The following statements are equivalent:

(a) $\operatorname{Min}(A) = \operatorname{Assh}(A)$;

(b) $\operatorname{Supp}_A(K) = \operatorname{Spec}(A)$;

(c) For any $\mathfrak{p} \in \operatorname{Spec}(A)$, $\dim A_\mathfrak{p} + \dim(A/\mathfrak{p}) = d$.

(ii) (See [1, (1.7)]) $\operatorname{Assh}(A) = \operatorname{Ass}_A(K)$.

(iii) (See [2, 1.1]) If A is (S_2), then $\operatorname{Ass}(A) = \operatorname{Assh}(A)$.

(iv) (See, [2, 1.4]) The following statements are equivalent:

(a) A is (S_2);

(b) $H^d_\mathfrak{m}(K) \cong E_A(A/\mathfrak{m})$, where $H^d_\mathfrak{m}(K)$ is the d-th local cohomology module of K with respect to $\mathfrak{m}$.

2. Connection between dualizing and Cousin complexes

In this section we establish, in certain situation, a connection between dualizing complex of a ring A and the Cousin complex of the canonical module of A with respect to a certain filtration. But, first we need the following preparatory result.

2.1. Proposition. *Let $\mathcal{F} = (F_i)_{i \geq 0}$ be a filtration of $\operatorname{Spec}(A)$ that admits M. Let*

$$X^\bullet : 0 \xrightarrow{e^{-2}} X^{-1} \xrightarrow{e^{-1}} X \xrightarrow{e^0} X^1 \xrightarrow{e^1} \cdots \longrightarrow X^i \xrightarrow{e^i} X^{i+1} \longrightarrow \cdots$$

be a complex of A-modules and A-homomorphisms such that $X^{-1} = M$, and, for each $i \geq 0$, the following two conditions hold:

(a) $\operatorname{Supp}_A(X^i) \subseteq F_i$;

(b) *The natural A-homomorphism $\xi(X^i) : X^i \to \bigoplus_{\mathfrak{p} \in \partial F_i} (X^i)_\mathfrak{p}$ such that, for $x \in X^i$ and $\mathfrak{p} \in \partial F_i$, the component of $\xi(X^i)(x)$ in the summand $(X^i)_\mathfrak{p}$ is $x/1$ (it follows from condition (a) and [14, (2.2) and (2.3)] that there is such an A-homomorphism), is an isomorphism.*

Then there is a (unique) morphism of complexes

$$\Psi = (\psi^i)_{i \geq -2} : C(\mathcal{F}, M) \to X^\bullet$$

from the Cousin complex of M with respect to $\mathcal{F}$ to $X^\bullet$ such that $\psi^{-1} : M \to M$ is the identity mapping on M (Ψ is called a morphism of complexes over Id_M). Moreover

(i) *Ψ is an epimorphism of complexes if and only if $\operatorname{Supp}_A(\operatorname{Coker} e^{i-1}) \subseteq F_{i+1}$, for all $i \geq 0$.*

(ii) *Ψ is an isomorphism of complexes if and only if* $\operatorname{Supp}_A(\operatorname{Coker} e^{i-1}) \subseteq F_{i+1}$ *and* $\operatorname{Supp}_A(H^{i-1}(X^\cdot)) \subseteq F_{i+1}$, *for all* $i \geq 0$.

Proof. The proof of existence and uniqueness of Ψ is a straightforward adoptation of arguments given in [12, (3.3)] and is left to the reader.

(i) Use the notation

$$0 \xrightarrow{d^{-2}} M \xrightarrow{d^{-1}} M^0 \xrightarrow{d^0} M^1 \longrightarrow \cdots \longrightarrow M^i \xrightarrow{d^i} M^{i+1} \longrightarrow \cdots$$

for the Cousin complex $C(\mathscr{F}, M)$. By using [12, (1.2) (ii)], it is easy to see that $\psi^i = (\xi(X^i))^{-1} \circ (\bigoplus_{\mathfrak{p} \in \partial F_i} (\psi^i)_\mathfrak{p}) \circ \xi(M^i)$ for each $i \geq 0$, where $\xi(M^i) : M^i \to \bigoplus_{\mathfrak{p} \in \partial F_i} (M^i)_\mathfrak{p}$ and $\xi(X^i) : X^i \to \bigoplus_{\mathfrak{p} \in \partial F_i} (X^i)_\mathfrak{p}$ are the natural A-isomorphisms and $(\psi^i)_\mathfrak{p} : (M^i)_\mathfrak{p} \to (X^i)_\mathfrak{p}$ is the induced A-homomorphism. Hence ψ^i is an epimorphism if and only if $(\psi^i)_\mathfrak{p}$ is an epimorphism, for all $\mathfrak{p} \in \partial F_i$.

For each $i \geq 0$, since $\operatorname{Supp}_A(\operatorname{Coker} d^{i-1}) \subseteq F_{i+1}$ (see [12, (1.1)]), we have, for each $\mathfrak{p} \in \partial F_i$, the commutative diagram

$$\begin{array}{ccccc} (M^{i-1})_\mathfrak{p} & \xrightarrow{(d^{i-1})_\mathfrak{p}} & (M^i)_\mathfrak{p} & \longrightarrow & 0 \\ \downarrow{\scriptstyle (\psi^{i-1})_\mathfrak{p}} & & \downarrow{\scriptstyle (\psi^i)_\mathfrak{p}} & & \\ (X^{i-1})_\mathfrak{p} & \xrightarrow{(e^{i-1})_\mathfrak{p}} & (X^i)_\mathfrak{p} & & \end{array}$$

with top exact row. Now, from $\operatorname{Supp}_A(\operatorname{Coker} e^{i-1}) \subseteq F_{i+1}$ for all $i \geq 0$, it follows, by induction on n, that $(\psi^n)_\mathfrak{p}$ is an epimorphism for all $\mathfrak{p} \in \partial F_n$. Conversely, if Ψ is an epimorphism, then, for each $i \geq 0$ and each $\mathfrak{p} \in \partial F_i$, $(e^{i-1})_\mathfrak{p}$ is epimorphism. Since $\operatorname{Supp}_A(\operatorname{Coker} e^{i-1}) \subseteq F_i$, then $\operatorname{Supp}_A(\operatorname{Coker} e^{i-1}) \subseteq F_{i+1}$.

(ii) It follows from [12, (1.1), (3.1) and (3.3)].

2.2. Lemma. *Suppose $U' \subseteq U$ are subsets of* $\operatorname{Spec}(A)$ *such that each element of $U - U'$ is minimal (with respect to inclusion) in U. Assume $E = \bigoplus_{\mathfrak{p} \in U - U'} E(A/\mathfrak{p})$ and $\operatorname{Supp}_A(E) \subseteq U$. Then the natural A-homomorphism $\xi(E) : E \to \bigoplus_{\mathfrak{p} \in U - U'} (E)_\mathfrak{p}$ such that for $x \in E$ and $\mathfrak{p} \in U - U'$, the component of $\xi(E)(x)$ in the summand $(E)_\mathfrak{p}$ is $x/1$ (it follows from assumption and* [14, (2.2) *and* (2.3)] *that there is such an A-homomorphism), is an isomorphism.*

Proof. It follows from [14, (2.5)] that $\operatorname{Supp}_A(\operatorname{Ker} \xi(E)) \subseteq U'$ and $\operatorname{Supp}_A(\operatorname{Coker} \xi(E)) \subseteq U'$. On the other hand, we have $\operatorname{Ass}_A(\operatorname{Ker} \xi(E)) \subseteq \operatorname{Ass}_A(E) = U - U'$. Therefore $\operatorname{Ker} \xi(E) = 0$. Hence the sequence

$$0 \longrightarrow E \xrightarrow{\xi(E)} \bigoplus_{\mathfrak{p} \in U - U'} (E)_\mathfrak{p} \xrightarrow{\pi} \operatorname{Coker} \xi(E) \longrightarrow 0$$

is split exact sequence, where π is the natural epimorphism. Therefore, since $(E)_\mathfrak{p} \cong E(A/\mathfrak{p})$, for all $\mathfrak{p} \in U - U'$, we have

$$\operatorname{Ass}_A(\operatorname{Coker} \xi(E)) \subseteq \bigcup_{\mathfrak{p} \in U - U'} \operatorname{Ass}_A((E)_\mathfrak{p}) = U - U'.$$

Therefore $\operatorname{Coker} \xi(E) = 0$.

Let A be a ring possessing a dualizing complex, so that A possesses a NFDC. By shifting this complex, we may assume that

$$I^{\cdot} : 0 \longrightarrow I^0 \xrightarrow{\delta^0} I^1 \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^{l-1}} I^l \longrightarrow 0$$

is a fundamental dualizing complex, $I^0 \neq 0$, $I^l \neq 0$, and $I^i = 0$ for all $i < 0$ or $i > l$. For each $\mathfrak{p} \in \operatorname{Spec}(A)$, let $t(\mathfrak{p}; I^{\cdot})$ be the unique integer t for which $E(A/\mathfrak{p})$ is a summand of I^t (see [16, page 208]).

The following result is essential in the rest of the paper and we quote it for the convinience of the reader.

2.3. Lemma [16, (3.3)]. *With the above notation, suppose that* $\mathfrak{p}$ *and* $\mathfrak{q}$ *are prime ideals of* A *such that* $\mathfrak{p} \subset \mathfrak{q}$ *and there is no prime strictly between* $\mathfrak{p}$ *and* $\mathfrak{q}$. *Then*

$$t(\mathfrak{q}; I^{\cdot}) = t(\mathfrak{p}; I^{\cdot}) + 1.$$

For each $i \geq 0$, set $T_i := \{\mathfrak{p} \in \operatorname{Spec}(A) : t(\mathfrak{p}; I^{\cdot}) \geq i\}$. Then, in view of 2.3, it is easy to see that $\mathscr{T} = (T_i)_{i \geq 0}$ is a filtration of $\operatorname{Spec}(A)$. We refer to $\mathscr{T}$ as *the dualizing filtration of* $\operatorname{Spec}(A)$ *with respect to* $I^{\cdot}$.

2.4. Theorem. *Let* A *be a (not necessarily local) ring, possessing a fundamental dualizing complex*

$$I^{\cdot} : 0 \longrightarrow I^0 \xrightarrow{\delta^0} I^1 \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^{l-1}} I^l \longrightarrow 0$$

with $I^0 \neq 0$, $I^l \neq 0$, *and* $I^i = 0$, *for all* i, $i < 0$ *or* $l < i$. *Set* $K := \operatorname{Ker} \delta^0$. *Then the following statements are true*:

(i) *There exists a (unique) homomorphism of complexes (over* Id_K*)*

$$\Psi = (\psi^i)_{i \geq -2} : C(\mathscr{T}, K) \to I^*$$

from the Cousin complex of K *with respect to* $\mathscr{T}$, *the dualizing filtration of* $\operatorname{Spec}(A)$ *with respect to* $I^{\cdot}$, *to the extended complex*

$$I^* : 0 \xrightarrow{\delta^{-2}} K \xrightarrow{\delta^{-1}} I^0 \xrightarrow{\delta^0} I^1 \longrightarrow \cdots \longrightarrow I^l \longrightarrow 0,$$

of $I^{\cdot}$, *where* δ^{-1} *is inclusion map.*

(ii) $\operatorname{Min}(A) = \operatorname{Ass}_A(K)$ *if and only if* $\mathscr{T} = \mathscr{H}$, *where* $\mathscr{H} = (H_i)_{i \geq 0}$ *is the height filtration of* $\operatorname{Spec}(A)$, *i.e.* $H_i = \{\mathfrak{p} \in \operatorname{Spec}(A) : \operatorname{ht}(\mathfrak{p}) \geq i\}$, *for each* $i \geq 0$.

(iii) *If* $\operatorname{Min}(A) = \operatorname{Ass}_A(K)$, *then* A *is* (S_1) *if and only if* Ψ *is an epimorphism.*

(iv) *If* $\operatorname{Min}(A) = \operatorname{Ass}_A(K)$, *then* A *is* (S_2) *if and only if* Ψ *is an isomorphism.*

Proof. (i) It is clear that $I^i = \bigoplus_{\mathfrak{p} \in \partial T_i} E(A/\mathfrak{p})$, for all $i \geq 0$. The claim follows from 2.1 and 2.2.

(ii) For each $\mathfrak{p} \in \operatorname{Spec}(A)$ such that $t(\mathfrak{p}; I^{\cdot}) = 0$, we have $K_{\mathfrak{p}} \cong (I^0)_{\mathfrak{p}} \cong E(A/\mathfrak{p})$; so that $\operatorname{Ass}_A(I^0) = \operatorname{Ass}_A(K)$. Also, by 2.3, $\operatorname{Ass}_A(I^0) \subseteq \operatorname{Min}(A)$. Hence $\operatorname{Min}(A) = \operatorname{Ass}_A(K)$ if and only if $t(\mathfrak{p}; I^{\cdot}) = \operatorname{ht}(\mathfrak{p}) = 0$, for all $\mathfrak{p} \in \operatorname{Min}(A)$. Now, using 2.3, the result follows.

(iii) Assume that A is (S_1). By 2.1(i), it is enough to show that $\operatorname{Supp}_A(\operatorname{Coker}\delta^{i-1}) \subseteq H_{i+1}$ for each $i \geq 0$. It is clear that $\operatorname{Supp}_A(\operatorname{Coker}\delta^{-1}) \subseteq H_1$. Suppose that $i > 0$ and $\mathfrak{p} \in \partial H_i$. Then $\operatorname{ht}(\mathfrak{p}) = i$; so that $\operatorname{depth}(A_\mathfrak{p}) > 0$. Since $(I^\cdot)_\mathfrak{p}$ is a dualizing complex for $A_\mathfrak{p}$ (see [15, (4.2)]) and $(I^i)_\mathfrak{p} \neq 0$, by [3, (2.5)], we have $D_\mathfrak{p}(H^i((I^\cdot)_\mathfrak{p})) \cong H^0_{\mathfrak{p}A_\mathfrak{p}}(A_\mathfrak{p}) = 0$, where $D_\mathfrak{p}(-)$ is the Matlis duality functor in $A_\mathfrak{p}$ and $H^0_{\mathfrak{p}A_\mathfrak{p}}(A_\mathfrak{p})$ is the 0-th local cohomology module of $A_\mathfrak{p}$ with respect to $\mathfrak{p}A_\mathfrak{p}$. Hence $H^i((I^\cdot)_\mathfrak{p}) = 0$. This implies that $\mathfrak{p} \notin \operatorname{Supp}_A(\operatorname{Coker}\delta^{i-1})$. Now, from $\operatorname{Supp}_A(\operatorname{Coker}\delta^{i-1}) \subseteq H_i$ we get $\operatorname{Supp}_A(\operatorname{Coker}\delta^{i-1}) \subseteq H_{i+1}$.

Conversely, assume that Ψ is an epimorphism of complexes. Let $\mathfrak{p} \in \operatorname{Spec}(A)$. We may assume that $\dim A_\mathfrak{p} > 0$. Set $i = \operatorname{ht}(\mathfrak{p})$. Then $\mathfrak{p} \in \partial H_i$; so that, by 2.1(i), $\mathfrak{p} \notin \operatorname{Supp}_A(\operatorname{Coker}\delta^{i-1})$. Thus the induced $A_\mathfrak{p}$-homomorphism $\delta^{i-1}_\mathfrak{p} : (I^{i-1})_\mathfrak{p} \to (I^i)_\mathfrak{p}$ is an epimorphism. This shows that $H^0_{\mathfrak{p}A_\mathfrak{p}}(A_\mathfrak{p}) \cong D_\mathfrak{p}(H^i((I^\cdot)_\mathfrak{p})) = 0$ [3, (2.5)]. Thus $\operatorname{depth}(A_\mathfrak{p}) \geq 1$ and A is (S_1).

(iv) Assume A is (S_2). We have, by (iii) and 2.1(i),

$$\operatorname{Supp}_A(H^{i-1}(I^*)) \subseteq \operatorname{Supp}_A(\operatorname{Coker}\delta^{i-2}) \subseteq H_i.$$

Consequently, in view of 2.1(ii), it is enough to show that $\operatorname{Supp}_A(H^{i-1}(I^*)) \subseteq H_{i+1}$ for all $i \geq 0$. We have $H^{-1}(I^*) = 0 = H^0(I^*)$. Let us assume $i \geq 2$ and $\mathfrak{p} \in \partial H_i$. From the NFDC

$$(I^\cdot)_\mathfrak{p} : 0 \longrightarrow (I^0)_\mathfrak{p} \longrightarrow \cdots \longrightarrow (I^{i-2})_\mathfrak{p} \xrightarrow{\delta^{i-2}_\mathfrak{p}} (I^{i-1})_\mathfrak{p} \xrightarrow{\delta^{i-1}_\mathfrak{p}} (I^i)_\mathfrak{p} \longrightarrow 0$$

for $A_\mathfrak{p}$ and the fact that $\operatorname{depth} A_\mathfrak{p} \geq \min\{2, \dim A_\mathfrak{p}\} \geq 2$, again by [3, (2.5)], we get $H^{i-1}(I)^\cdot_\mathfrak{p} = 0$. Therefore $\mathfrak{p} \notin \operatorname{Supp}_A(H^{i-1}(I^*))$. This contradiction shows that $\operatorname{Supp}_A(H^{i-1}(I^*)) \subseteq H_{i+1}$.

Conversely, assume that Ψ is an isomorphism of complexes. Let $\mathfrak{p} \in \operatorname{Spec}(A)$. By (iii), we may assume that $\dim(A_\mathfrak{p}) > 1$. Set $i = \operatorname{ht}(\mathfrak{p})$. Then $\mathfrak{p} \in \partial H_i$; so that, by 2.1(ii), $\mathfrak{p} \notin \operatorname{Supp}_A(H^{i-1}(I^*))$ and $\mathfrak{p} \notin \operatorname{Supp}_A(\operatorname{Coker}\delta^{i-1})$. Hence, from the complex $(I^\cdot)_p$, we get $H^0_{\mathfrak{p}A_\mathfrak{p}}(A_\mathfrak{p}) \cong D_\mathfrak{p}(H^i((I^\cdot)_\mathfrak{p})) = 0$, and $H^1_{\mathfrak{p}A_\mathfrak{p}}(A_\mathfrak{p}) \cong D_\mathfrak{p}(H^{i-1}(I)^\cdot_\mathfrak{p}) = 0$. Thus $\operatorname{depth} A_\mathfrak{p} \geq 2$ and A is (S_2).

Now, by using 2.3 and 1.3, it is straightforward to prove the following corollary.

2.5. Corollary. *Let $(A, \mathfrak{m})$ be a local ring with $\dim A = d$. Suppose that A possesses a dualizing complex and that*

$$I^\cdot : 0 \longrightarrow I^0 \xrightarrow{\delta^0} I^1 \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^{d-1}} I^d \longrightarrow 0$$

is NFDC for A. Let $K = \operatorname{Ker}\delta^0$, and

$$I^* : 0 \longrightarrow K \xrightarrow{\delta^{-1}} I^0 \xrightarrow{\delta^0} I^1 \longrightarrow \cdots \xrightarrow{\delta^{d-1}} I^d \longrightarrow 0$$

be the extension of $I^\cdot$, where δ^{-1} is inclusion map. (It is known that K is the canonical module of A.) Set $\mathcal{D} = (D_i)_{i \geq 0}$ be the dimension filtration of $\operatorname{Spec}(A)$, i.e. $D_i = \{\mathfrak{p} \in \operatorname{Spec}(A) : \dim(A/\mathfrak{p}) \leq d - i\}$. Then the following statements are true.

(i) *There exists a (unique) homomorphism of complexes*

$$\Psi = (\psi^i)_{i \geq -2} : C(\mathscr{D}, K) \to I^*$$

from the Cousin complex of K with respect to $\mathscr{D}$ to I^* (over Id_K).

(ii) $\mathrm{Min}(A) = \mathrm{Ass}_A(K)$ *if and only if* $\mathscr{D} = \mathscr{H}$, *where $\mathscr{H}$ is the height filtration of* $\mathrm{Spec}(A)$.

(iii) *If* $\mathrm{Min}(A) = \mathrm{Ass}_A(K)$, *then A is (S_1) if and only if Ψ is an epimorphism.*

(iv) *A is (S_2) if and only if Ψ is an isomorphism.*

3. Applications

In this section we provide some applications of the presentation of dualizing complex by Cousin complex.

Note that if M is an A-module with $\mathrm{Ass}_A(M)$ contains only finitely many minimal members, then the Cousin complex of M with respect to a filtration admitting M is isomorphic to a complex of modules of generalized fractions [12, (3.4)]. Thus we may find a description, in terms of a complex of modules of generalized fractions, of a dualizing complex for a ring which is (S_2). This will help us to understand each term and each morphism of dualizing complex more explicitly; so that it makes easier to work with.

The concept of a chain of triangular subsets on A is explained in [11, page 420]. Such a chain $\mathscr{U} = (U_i)_{i \geq 0}$ determines a complex of modules of generalized fractions

$$C(\mathscr{U}, M) : 0 \longrightarrow M \xrightarrow{e^0} U_1^{-1}M \xrightarrow{e^1} \cdots \longrightarrow U_i^{-i}M \xrightarrow{e^i} U_{i+1}^{-i-1}M \longrightarrow \cdots$$

in which $e^0(m) = m/(1)$ for all $m \in M$ and

$$e^i\left(\frac{m}{(u_1, \ldots, u_i)}\right) = \frac{m}{(u_1, \ldots, u_i, 1)}$$

for all $i \geq 1$, $m \in M$, and $(u_1, \ldots, u_i) \in U_i$. When working with a complex of modules of generalized fractions $C(\mathscr{U}, M)$, as above, we are regarding the term $U_1^{-1}M$ as being in the 0-th position, so that $H^i(C(\mathscr{U}, M)) = \mathrm{Ker}\, e^{i+1}/\mathrm{Im}\, e^i$ for $i \geq 0$, and $H^{-1}(C(\mathscr{U}, M)) = \mathrm{Ker}\, e^0$.

In the rest of this paper, M denotes a non-zero finitely generated A-module with $\dim M = n$. We need the following lemma.

3.1. Lemma. *Let A be a local ring with maximal ideal* $\mathfrak{m}$.

(i) *Let* $\mathscr{F} = (F_i)_{i \geq 0}$, *with* $F_i = \{\mathfrak{p} \in \mathrm{Supp}_A(M) : \dim(M/\mathfrak{p}M) \leq n - i\}$; *so that $\mathscr{F}$ is a filtration of* $\mathrm{Spec}(A)$ *which admits M. Assume, for each* $i \geq 1$, $W_i = \left\{(w_1, \ldots, w_i) \in A^i : \dim\left(\frac{M}{(w_1, \ldots, w_j)M}\right) \leq n - j, \text{ for all } j \text{ with } 1 \leq j \leq i\right\}$ *(We adopt the convention whereby dimension of the zero module is $-\infty$). Then $\mathscr{W} = (W_i)_{i \geq 1}$ is a chain of triangular subsets on A, and there exists a (unique)*

isomorphsim of complexes (over Id_M*)*

$$\Theta = (\theta^i)_{i\geq -2} : C(\mathscr{W}, M) \to C(\mathscr{F}, M)$$

from the complex of generalized fractions $C(\mathscr{W}, M)$ *to the Cousin complex* $C(\mathscr{F}, M)$.

(ii) *Let, for each* $i \geq 1$,

$$U_i = \{(x_1, \ldots, x_i) \in A^i : \text{there exists } j \text{ with } 0 \leq j \leq i \text{ such that } x_1, \ldots, x_j \text{ is an s.s.o.p. for } M \text{ and } x_{j+1} = \cdots = x_i = 1\}$$

where 's.s.o.p' stands for "subset of a system of parameters". Then $\mathscr{U} = (U_i)_{i\geq 1}$ *is a chain of triangular subsets on* A, *and the two complexes of modules of generalized fractions* $C(\mathscr{U}, M)$ *and* $C(\mathscr{W}, M)$ *are isomorphic (over the* Id_M).

Proof. Let $b_1, \ldots, b_j \in \mathfrak{m}$, then, by [9, 13.4],

(a) $n - j \leq \dim(M/(b_1, \ldots, b_j)M)$;

(b) $n - j = \dim(M/(b_1, \ldots, b_j)M)$ if and only if $b_1, \ldots, b_j$ is an s.s.o.p. for M.

(i) It is straightforward to see that, for each $i \geq 1$,

$$W_i = \{(v_1, \ldots, v_i) \in A^i : \text{for every } j \text{ with } 1 \leq j \leq i,\ Av_1 + \cdots + Av_j \nsubseteq \mathfrak{p}, \text{ for all } \mathfrak{p} \in \partial F_{j-1} \cap \mathrm{Supp}_A(M)\}.$$

Hence the claim follows from [12, (3.4)].

(ii) By [21, 1.2 and 1.4], $\mathscr{U}$ is a chain of triangular subsets on A. For each $i \geq 1$, since $U_i \subseteq W_i$, there is an A-homomorphism

$$\varphi^i : U_i^{-i}M \to W_i^{-i}M$$

which is such that $\varphi^i\left(\dfrac{m}{(u_1, \ldots, u_i)}\right) = \dfrac{m}{(u_1, \ldots, u_i)}$, for all $m \in M$ and all $(u_1, \ldots, u_i) \in U_i$. Now, it is clear that

$$\Phi = (\varphi^i)_{i\geq 1} : C(\mathscr{U}, M) \to C(\mathscr{W}, M)$$

is a morphism of complexes. We show that each φ^i is an isomorphism.

Let $\dfrac{m}{(w_1, \ldots, w_i)} \in W_i^{-i}M$ be a non-zero element. Then w_j is not a unit, for all j with $1 \leq j < i$. Hence, either $w_1, \ldots, w_i \in \mathfrak{m}$ which shows that $(w_1, \ldots, w_i) \in U_i$, or $w_1, \ldots, w_{i-1} \in \mathfrak{m}$ and w_i is a unit. Therefore $w_1, \ldots, w_{i-1}$ is an s.s.o.p. for M, and

$$\varphi^i\left(\frac{w_i^{-1}m}{(w_1, \ldots, w_{i-1}, 1)}\right) = \frac{w_i^{-1}m}{(w_1, \ldots, w_{i-1}, 1)} = \frac{m}{(w_1, \ldots, w_i)}, \quad (\text{in } W_i^{-i}M).$$

This shows that φ^i is an epimorphism.

Suppose, for $\dfrac{m}{(u_1,\dots,u_i)} \in U_i^{-i}M$, we have $\varphi^i\left(\dfrac{m}{(u_1,\dots,u_i)}\right) = 0$, i.e. $\dfrac{m}{(u_1,\dots,u_i)} = 0$ (in $W_i^{-i}M$). Then there exists $(w_1,\dots,w_i) \in W_i$, and an $i \times i$ lower triangular matrix H with entries in A such that $H[u_1,\dots,u_i]^T = [w_1,\dots,w_i]^T$ and $|H| \cdot m \in \sum_{j=1}^{i-1} w_j M$. Since $(u_1,\dots,u_i) \in U_i$, there exists j with $0 \le j \le i$ such that $u_1,\dots,u_j$ is an s.s.o.p. for M and $u_{j+1} = \cdots = u_i = 1$. If $j < i-1$, then $\dfrac{m}{(u_1,\dots,u_i)} = 0$ (in $U_i^{-i}M$). If $j = i$, then we have $w_1,\dots,w_i \in \mathfrak{m}$ and therefore $w_1,\dots,w_i$ is an s.s.o.p. for M, so that $(w_1,\dots,w_i) \in U_i$. Thus $\dfrac{m}{(u_1,\dots,u_i)} = 0$, in $U_i^{-i}M$. Finally, if $j = i-1$, then $w_1,\dots,w_{i-1} \in \mathfrak{m}$. We may assume that w_i is a unit. Taking the $i \times i$ diagonal matrix $K = \operatorname{diag}[1,1,\dots,1,w_i^{-1}]$, we have $KH[u_1,\dots,u_i]^T = [w_1,\dots,w_{i-1},1]^T$ and $|K|\,|H|m \in \sum_{j=1}^{i-1}(w_i^{-1}w_j)M = \sum_{j=1}^{i-1} w_j M$. This shows that $\dfrac{m}{(u_1,\dots,u_i)} = 0$ in $U_i^{-i}M$.

The following proposition gives the first application of the connection between dualizing and Cousin complexes.

3.2. Proposition. (i) *Let the situation and conventions be as in* 2.4. *Let* A *be* (S_2) *and* $\operatorname{Min}(A) = \operatorname{Ass}_A(K)$ *(this condition would be satisfied if* A *is local and* (S_2), *by* 1.3*). Assume, for each* $i \ge 1$, *that*

$$V_i = \{(v_1,\dots,v_i) \in A^i : \operatorname{ht}_A((v_1,\dots,v_j)) \ge j, \text{ for all } j \text{ with } 1 \le j \le i\},$$

(we adopt the convention whereby $\operatorname{ht}_A A = \infty$*). Then* $\mathscr{V} = (V_i)_{i\in\mathbf{N}}$ *is a chain of triangular subsets on* A, *and there is a unique isomorphism of complexes from* $C(\mathscr{V}, K)$, *the complex of modules of generalized fractions, to* I^*, *the extended complex of* $I^\cdot$ *(over the* Id_K*).*

(ii) *Let the situation and conventions be as in 2.5. Let, for each* $i \ge 1$,

$$Z_i = \{(z_1,\dots,z_i) \in A^i : \text{there exists } j \text{ with } 0 \le j \le i \text{ such that } z_1,\dots,z_j \text{ is an s.s.o.p. for } A \text{ and } z_{j+1} = \cdots = z_i = 1\}$$

Then $\mathscr{Z} = (Z_i)_{i\ge 1}$ *is a chain of triangular subsets on* A *and there exists a unique morphism of complexes*

$$\Lambda = (\lambda^i)_{i\ge -1} : C(\mathscr{Z}, K) \to I^*,$$

from the complex of modules of generalized fractions $C(\mathscr{Z}, K)$ *to* I^* *(over the* Id_K*). Moreover if* A *is* (S_2), *then* Λ *is an isomorphism.*

Proof. (i) Since $\operatorname{Supp}_A(K) = \operatorname{Spec}(A)$, it is easy to see that, for each $i \ge 1$,

$$V_i = \{(v_1,\dots,v_i) \in A^i : \text{for each } j \text{ with } 1 \le j \le i, Av_1 + \cdots + Av_j \not\subseteq \mathfrak{p} \text{ for all } \mathfrak{p} \in \partial H_{j-1} \cap \operatorname{Supp}_A(K)\},$$

where $\mathscr{H} = (H_i)_{i\geq 0}$ is the height filtration of $\operatorname{Spec}(A)$. Hence it follows from [12, (3.4)] that the Cousin complex $C(\mathscr{H}, K)$ is isomorphic to $C(\mathscr{V}, K)$. The claim follows from 2.4.

(ii) It follows from 3.1 that the Cousin complex $C(\mathscr{D}, A)$ is isomorphic (over Id_A) to $C(\mathscr{Z}, A)$, where $\mathscr{D} = (D_i)_{i\geq 0}$ is the dimension filtration of $\operatorname{Spec}(A)$. Hence, by [19, 1.7],

$$C(\mathscr{Z}, K) \cong C(\mathscr{Z}, A) \otimes_A K \cong C(\mathscr{D}, A) \otimes_A K \cong C(\mathscr{D}, K).$$

Therefore, the claim follows from 2.5.

We are now able to give a description in terms of modules of generalized fractions of indecomposable injective modules over certain rings. This generalizes [24, (3.6)].

3.3. Corollary. *Let A be a ring which is (S_2) and possessing a fundamental dualizing complex*

$$I^{\cdot} : 0 \to I^0 \to I^1 \to \cdots \to I^l \to 0.$$

Set $K = H^0(I^{\cdot})$. Let $\mathfrak{p} \in \operatorname{Spec}(A)$ with $r = \operatorname{ht}\mathfrak{p}$. Then the injective envelope $E(A/\mathfrak{p})$ may be viewed in terms of a module of generalized fractions. More precisely

$$E(A/\mathfrak{p}) \cong (U_r \times \{1\})^{-r-1}(H^{t-r}(I^{\cdot}))_{\mathfrak{p}},$$

where $t = t(\mathfrak{p}; I^{\cdot})$ and, for $i \geq 1$,

$$U_i = \left\{ \left(\frac{x_1}{1}, \ldots, \frac{x_i}{1}\right) \in (A_{\mathfrak{p}})^i : \operatorname{ht}_{A_{\mathfrak{p}}}\left(\left(\frac{x_1}{1}, \ldots, \frac{x_j}{1}\right)\right) \geq j, \text{ for all } j \text{ with } 1 \leq j \leq i \right\}.$$

Moreover

(i) *If $\operatorname{Min}(A) = \operatorname{Ass}_A(K)$ (this condition would be satisfied if A is local), then*

$$E(A/\mathfrak{p}) \cong (V_r \times (A - \mathfrak{p}))^{-r-1}K,$$

where, for $i \geq 1$,

$$V_i = \{(v_1, \ldots, v_i) \in A^i : \operatorname{ht}_A((v_1, \ldots, v_j)) \geq j, \text{for all } j \text{ with } 1 \leq j \leq i\}.$$

(ii) *If A is local, then*

$$E(A/\mathfrak{p}) \cong (Z_r \times (A - \mathfrak{p}))^{-r-1}K,$$

where, for $i \geq 1$,

$$Z_i = \{(z_1, \ldots, z_i) \in A^i : \text{there exists } j \text{ with } 0 \leq j \leq i, \text{ such that } z_1, \ldots, z_j \text{ is an s.s.o.p. for } A \text{ and } z_{j+1} = \cdots = z_i = 1\}.$$

Proof. We first prove (i). By 3.2 (i) and [24, (2.1) and (2.2)],

$$E(A/\mathfrak{p}) \cong E_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}) \cong (V_{r+1}^{-r-1}K)_{\mathfrak{p}} \cong (V_{r+1}[A - \mathfrak{p}])^{-r-1}K$$

(as $A_{\mathfrak{p}}$-modules), where

$$V_{r+1}[A-\mathfrak{p}] = \{(v_1,\dots,v_{r+1}s) \in A^{r+1} : (v_1,\dots,v_{r+1}) \in V_{r+1} \text{ and } s \in A-\mathfrak{p}\}.$$

Define the natural A-homomorphism

$$\eta : (V_r \times (A-\mathfrak{p}))^{-r-1}K \to (V_{r+1}[A-\mathfrak{p}])^{-r-1}K,$$

by $\eta(x/(v_1,\dots,v_r,s)) = x/(v_1,\dots,v_r,s)$, for all $x \in K$, $(v_1,\dots,v_r) \in V_r$ and $s \in A-\mathfrak{p}$. We show that η is an isomorphism.

Let $\alpha = x/(v_1,\dots,v_r,v_{r+1}s) \in (V_{r+1}[A-\mathfrak{p}])^{-r-1}K$ be a non-zero element. If $v_i \notin \mathfrak{p}$, for some i with $1 \le i \le r$, then $\alpha = v_i x/(v_1,\dots,v_r,v_{r+1}(sv_i)) = 0$ in $(V_{r+1}[A-\mathfrak{p}])^{-r-1}K$. Hence $Av_1 + \cdots + Av_r \subseteq \mathfrak{p}$, and $v_{r+1} \notin \mathfrak{p}$. Therefore $(v_1,\dots,v_r,v_{r+1}s) \in V_r \times (A-\mathfrak{p})$. This shows that η is an epimorphism.

Suppose, for $x/(v_1,\dots,v_r,s) \in (V_r \times (A-\mathfrak{p}))^{-r-1}K$, we have $\beta := x/(v_1,\dots,v_r,s) = 0$ (in $(V_{r+1}[A-\mathfrak{p}])^{-r-1}K$). Then there exists $(w_1,\dots,w_r,w_{r+1}t) \in V_{r+1}[A-\mathfrak{p}]$ and an $(r+1)\times(r+1)$ lower triangular matrix H with entries in A such that $H[v_1,\dots,v_r,s]^T = [w_1,\dots,w_{r+1}t]^T$ and $|H|\cdot x \in \sum_{j=1}^{r} w_j K$. If $v_i \notin \mathfrak{p}$, for some i with $1 \le i \le r$, then $\beta = 0$ (in $(V_r \times (A-\mathfrak{p}))^{-r-1}K$). Hence $Av_1 + \cdots + Av_r \subseteq \mathfrak{p}$. Therefore, $Aw_1 + \cdots + Aw_r \subseteq \mathfrak{p}$, and $w_{r+1} \notin \mathfrak{p}$. Hence $(w_1,\dots,w_r,w_{r+1}t) \in V_r \times (A-\mathfrak{p})$. This shows that $\beta = 0$ (in $(V_r \times (A-\mathfrak{p}))^{-r-1}K$).

Now, we prove (ii). Let $\mathscr{Z} = (Z_i)_{i\in\mathbf{N}}$ and $\mathscr{V} = (V_i)_{i\in\mathbf{N}}$ be as in assumption. Then, by [5, 3.3] and 3.2,

$$(Z_r \times \{1\})^{-r-1}K \cong (V_r \times \{1\})^{-r-1}K.$$

Consequently, in view of [24, (2.1) and (2.2)], we have

$$\begin{aligned}(Z_r \times (A-\mathfrak{p}))^{-r-1}K &\cong ((Z_r \times \{1\})^{-r-1}K)_{\mathfrak{p}}\\ &\cong ((V_r \times \{1\})^{-r-1}K)_{\mathfrak{p}}\\ &\cong (V_r \times (A-\mathfrak{p}))^{-r-1}K \cong E(A/\mathfrak{p}),\end{aligned}$$

as $A_{\mathfrak{p}}$-modules.

Finally, we consider the general case. After localizing $I^{\cdot}$ at $\mathfrak{p}$, we get the dualizing complex

$$I^{\cdot}_{\mathfrak{p}} : 0 \to I^{t-r}_{\mathfrak{p}} \to I^{t-r+1}_{\mathfrak{p}} \to \cdots \to I^{t}_{\mathfrak{p}} \to 0.$$

for $A_{\mathfrak{p}}$, where $t = t(\mathfrak{p}; I^{\cdot})$. Note that if $i < t-r$, then $I^i_{\mathfrak{p}} = 0$. For otherwise $(E(A/\mathfrak{q}))_{\mathfrak{p}} \neq 0$ for some $\mathfrak{q} \in \operatorname{Spec}(A)$ with $t(\mathfrak{q}; I^{\cdot}) = i$. Then $\mathfrak{q} \subseteq \mathfrak{p}$. Let $\mathfrak{q} = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_s = \mathfrak{p}$ be a saturated chain of elements of $\operatorname{Spec}(A)$. Then $i = t(\mathfrak{q}; I^{\cdot}) = t(\mathfrak{p}; I^{\cdot}) - s < t-r$. Hence $s > \operatorname{ht}\mathfrak{p}$ which is a contradiction. It is clear that $I^i_{\mathfrak{p}} = 0$, for all $i > t$.

In view of (i), we have

$$E(A/\mathfrak{p}) \cong (U'_r \times (A_{\mathfrak{p}} - \mathfrak{p}A_{\mathfrak{p}}))^{-r-1}(H^{t-r}(I^{\cdot}))_{\mathfrak{p}},$$

where

$$U_r' = \left\{ \left(\frac{x_1}{s_1}, \ldots, \frac{x_r}{s_r} \right) \in (A_\mathfrak{p})^r : \mathrm{ht}_{A_\mathfrak{p}} \left(\frac{x_1}{s_1}, \ldots, \frac{x_j}{s_j} \right) \geq j, \text{ for all } j \text{ with } 1 \leq j \leq r \right\}.$$

It is easy to see that

$$(U_r' \times (A_\mathfrak{p} - \mathfrak{p}A_\mathfrak{p}))^{-r-1}(H^{t-r}(I^\cdot))_\mathfrak{p} \cong (U_r \times \{1\})^{-r-1}(H^{t-r}(I^\cdot))_\mathfrak{p},$$

where U_r is as in the assumption.

The following result gives an equivalent condition for a Cousin complex to be a dualizing complex, in certain rings.

3.4. Corollary. *Let $(A, \mathfrak{m})$ be a local ring of $\dim A = d$. Assume that A has canonical module K and A is (S_2). Denote by $\mathscr{D} = (D_i)_{i \geq 0}$ the dimension filtration of* Spec(A). *Write the Cousin complex $C(\mathscr{D}, K)$ as*

$$0 \longrightarrow K \xrightarrow{d^{-1}} K^0 \xrightarrow{d^0} K^1 \longrightarrow \cdots \longrightarrow K^i \xrightarrow{d^i} K^{i+1} \longrightarrow \cdots$$

and denote by $K^\cdot$ the induced complex

$$K^\cdot : 0 \longrightarrow K^0 \xrightarrow{d^0} K^1 \longrightarrow \cdots \longrightarrow K^i \xrightarrow{d^i} K^{i+1} \longrightarrow \cdots.$$

Then the following statements are equivalent.

(i) *A possesses a dualizing complex;*

(ii) *$K^\cdot$ is a dualizing complex for A;*

(iii) *$H^i(C(\mathscr{D}, K))$ is finitely generated A-module for all $i \geq 1$.*

Proof. If A possesses a dualizing complex, then, by 2.5, $K^\cdot$ is also a dualizing complex for A. So it is enough to show that (iii) implies (ii). From 1.3, we have $D_i = \{\mathfrak{p} \in \mathrm{Supp}_A(K) : \mathrm{ht}_K \mathfrak{p} \geq i\}$ for all $i \geq 0$. Also, by [1, (1.10)], K is (S_2). It therefore follows from [20, 4.4] that

$$H^{-1}(C(\mathscr{D}, K)) = 0 = H^0(C(\mathscr{D}, K)).$$

Thus all cohomology modules of the complex $K^\cdot$ are finitely generated A-modules.

Let $i \geq 0$ and $\mathfrak{p} \in \partial D_i$. By [17, page 21], $(\mathrm{Coker}\, d^{i-2})_\mathfrak{p} \cong H^i_{\mathfrak{p}A_\mathfrak{p}}(K_\mathfrak{p})$. Since $K_\mathfrak{p}$ is the canonical module of $A_\mathfrak{p}$ [1, 4.3] and $A_\mathfrak{p}$ is (S_2), by 1.3 (iv), $H^i_{\mathfrak{p}A_\mathfrak{p}}(K_\mathfrak{p}) \cong E_{A_\mathfrak{p}}(A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p})$. This shows that

$$K^i \cong \bigoplus_{\mathfrak{p} \in \partial D_i} (\mathrm{Coker}\, d^{i-2})_\mathfrak{p} \cong \bigoplus_{\mathfrak{p} \in \partial D_i} E(A/\mathfrak{p}).$$

In what follows, we assume that $(A, \mathfrak{m})$ is local. For the final application of our approach of dualizing complex, we need some preparatory notions from [21, (1.8)]. (i) A finite dimensional A-module M is called a *generalized Cohen-Macaulay* (abbr. g.CM) if there exists $r \geq 1$ such that, for each system of parameters $x_1, \ldots, x_n$ for M and for all $i = 1, \ldots, n$,

$$\mathfrak{m}^r[((Ax_1 + \cdots + Ax_{i-1})M : x_i)/(Ax_1 + \cdots + Ax_{i-1})M] = 0,$$

where $n = \dim M$. Note that, by [13, (3.2) and (3.3)], M is g.CM module if and only if $H^i_\mathfrak{m}(M)$ is of finite length for all $i = 0, 1, \ldots, n-1$. (ii) [13, (2.1)] A

sequence $x_1, \ldots, x_s$ of elements of $\mathfrak{m}$ is said to be a *filter-M-regular sequence* if

$$\operatorname{Supp}_A(((Ax_1 + \cdots + Ax_{i-1})M : x_i)/(Ax_1 + \cdots + Ax_{i-1})M) \subseteq \{\mathfrak{m}\},$$

for all $i = 1, \ldots, s$ (here and above, $Ax_1 + \cdots + Ax_{i-1}$ is to be interperted as 0 when $i = 1$). (iii) [13, (2.3)] M is said to be an *f-module* if every system of parameters for M constitutes a filter-M-regular sequence.

Now, we present a partial converse of [23, (3.5)] which is also a generalization of [2, (2.3)].

3.5. Theorem. *Assume $(A, \mathfrak{m})$ is local ring, $d = \dim A > 0$, A is (S_2), and A has canonical module K. Let $C(\mathscr{D}, K)$ be the Cousin complex of K with respect to the dimension filtration $\mathscr{D}$ of* $\operatorname{Spec}(A)$ *and $K^\bullet$ as in 3.4. If K is g.CM A-module, then $K^\bullet$ is a fundamental dualizing complex for A and A is a g.CM ring.*

Proof. In view of the preceding paragraph of 3.5, K is an f-module. By 1.3, $\operatorname{Supp}_A(K/\mathfrak{a}K) = \operatorname{Supp}_A(A/\mathfrak{a}A)$, for all ideals $\mathfrak{a}$ of A; so that $\dim K = d$ and for each $b_1, \ldots, b_j \in \mathfrak{m}$, $b_1, \ldots, b_j$ is an s.s.o.p. for A if and only if it is an s.s.o.p. for K. Take $\mathscr{Z} = (Z_i)_{i \geq 1}$ as in 3.2. Then, by [21, (2.3) and (2.4)], $H^{i-1}(C(\mathscr{Z}, K))$, the i-th cohomology module of $C(\mathscr{Z}, K)$, is isomorphic to $H^i_{\mathfrak{m}}(K)$ for all $i = 0, 1, \ldots, d-1$. As we have seen in the proof of 3.2 (ii), $C(\mathscr{D}, K) \cong C(\mathscr{Z}, K)$. Hence

$$H^{i-1}(C(\mathscr{D}, K)) \cong H^i_{\mathfrak{m}}(K),$$

for all $i = 0, 1, \ldots, d-1$. Also, we have $\operatorname{Supp}_A(H^{d-1}(C(\mathscr{D}, K)) = \varnothing$ and $\operatorname{Supp}_A(H^d(C(\mathscr{D}, K)) = \varnothing$. Therefore all cohomology modules of the Cousin complex $C(\mathscr{D}, K)$ are finitely generated. Now the first claim follows by 3.4.

By [3, (2.5)], we have

$$\operatorname{Hom}_A(H^{i-1}(C(\mathscr{D}, K)), E(A/\mathfrak{m})) \cong H^{d-i+1}_{\mathfrak{m}}(A),$$

for all $i = 2, \ldots, d-1$. It therefore follows from [9, 18.6(ii)] that $H^j_{\mathfrak{m}}(A)$ is of finite length, for all $j = 2, \ldots, d-1$. If $d \leq 2$, then A is CM ring and there is nothing to prove. If $d \geq 3$, then depth $A \geq 2$, so that $H^0_{\mathfrak{m}}(A) = 0 = H^1_{\mathfrak{m}}(A)$. Thus A is g.CM ring.

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