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The Morse index theorem for Carnot-Carathéodory spaces

By

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1. Introduction

Let M be a Riemannian manifold and H be a distribution on M, which is *bracket-generating*, that is, for every point p of M local sections of H near p span together with all their commutators the tangent space T_pM of M at p. A curve c in M is called *horizontal* if c is tangent almost everywhere to H. By Chow's theorem, any two points of M can be joined by a horizontal curve. Thus we can define a Carnot-Carathéodory distance, $d_C(p,q) := \inf L(c)$, where c is a horizontal curve joining p and q.

Some results on geodesics, which means locally minimizing curves, in Carnot-Carathéodory spaces have been already known. Strichartz [9] showed that every curve which is the projection of a curve satisfying a certain Hamiltonian equation on the cotangent bundle T^*M on M, is a geodesic. Hamenstädt [4] obtained the geodesic equation from the first variational formula and defined the Jacobi field from the second variational formula. Her equation is satisfied by a class of geodesics which are called normal. There are another kind of geodesics which are called abnormal. (See [8] [1])

This distribution H is said to be *strongly bracket-generating* if the tangent bundle TM is spanned by H and $[X,H]$ for every non-zero local section X of H. If H is strongly bracket-generating, every gedesic is normal, that is, satisfies the geodesic equation. Hence it is a critical point of the energy functional $E(c) = \frac{1}{2}\int \|\dot{c}\|^2 dt$ on the horizontal path space and has Jacobi fields. Then we can define the conjugate point and the index of the geodesic in the same way as those of the geodesic for the usual Riemannian metric. We will prove the following Morse index theorem for the Carnot-Carathéodory spaces.

Theorem 1. *The index of the geodesic γ is finite and equal to the number of points which are conjugate to $\gamma(0)$, counted with its multiplicity.*

Zhong Ge [11] proved this theorem for the case that M is a total space of a principal G-bundle for a compact Lie grop G and H gives a fat connection, i.e.,

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the horizontal bundle H is strongly bracket-generating.

By Theorem 1, we can easily obtain the following corollaries.

Corollary 2. *There are at most finitely many points which are conjugate to $\gamma(0)$ along the gedesic γ.*

Corollary 3. *The geodesic is not length-minimizing beyond its conjugate point.*

2. Geodesics and its Jacobi fields

Fix a point p in M. Choose the orthonormal frame $\{X_1,\cdots,X_m\}$ around p such that $\{X_1,\cdots,X_k\}$ is an orthonormal frame of H. Let $\{\theta^1,\cdots,\theta^m\}$ be the dual coframe of $\{X_1,\cdots,X_m\}$ and $\theta=(\theta^1,\cdots,\theta^m)$ be a 1-form with values in $\mathbf{R}^m$.

Let $\alpha:(-\varepsilon,\varepsilon)\times[0,1]\to M$ be a horizontal variation of $\gamma=\alpha(0)$ with a variational vector field X, and define

$$\mathscr{D}_\gamma X=\frac{\partial}{\partial s}\theta\left(\frac{\partial\alpha}{\partial t}\right)\bigg|_{s=0}=\frac{d}{dt}\theta(X)-2d\theta(\dot\gamma,X).$$

Then we obtain the first variational formula:

$$\frac{d}{ds}E(c_s)\bigg|_{s=0}=\int_0^1\langle\mathscr{D}_\gamma X,\theta(\dot\gamma)\rangle dt,$$

where $c_s(t)=\alpha(s,t)$.

We denote a 1-form a with values in $m\times m$ matrices by the following equation:

$$2d\theta(u,v)=-a(u)\theta(v)$$

for u, $v\in T_qM$ and $q\in M$. According to Hamenstädt [4], the geodesic equation, the second variational formula and the Jacobi equation are given as follows:

2.1. The geodesic equation.

$$\begin{cases}\theta(\dot\gamma)=P\varphi\\ \frac{d}{dt}\varphi-a^*(\dot\gamma)\varphi=0,\end{cases}\tag{1}$$

where $a^*(\dot\gamma)$ is an adjoint matrix of $a(\dot\gamma)$, φ is a curve on $\boldsymbol{R}^m$ and P is a projection from $\boldsymbol{R}^m$ to $\boldsymbol{R}^k$ as the subspace of $\boldsymbol{R}^m$.

2.2. The second variational formula. For a variation $\alpha:(-\varepsilon,\varepsilon)\times(-\varepsilon,\varepsilon)\times[0,1]\to M$ of the geodesic γ with variational vector fields $X=\frac{\partial\alpha}{\partial s}|_{(s,u)=(0,0)}$ and $Y=\frac{\partial\alpha}{\partial u}|_{(s,u)=(0,0)}$,

$$\frac{\partial^2}{\partial s\partial u}E(c_{s,u})\bigg|_{(s,u)=(0,0)}=\int_0^1\{\langle\mathscr{D}_\gamma X,\mathscr{D}_\gamma Y-a^*(Y)\varphi\rangle-\langle\theta(X),2A^*(\dot\gamma,Y)\varphi\rangle\}dt,\tag{2}$$

where $c_{s,u}(t)=\alpha(s,u,t)$ and A^* is the 2-form with values in $m\times m$ matrices, whose exact form is not our concern in this paper.

A vector field X along γ is said to be *horizontal* if $(I-P)\mathscr{D}_\gamma X=0$. It can be easily checked that X is horizontal if X is a variational vector field associated to a horizontal variation α.

Let $\mathscr{T}_\gamma$ be the vector space of horizontal vector fields along γ vanishing at the endpoints of γ. The right hand side of the second variational formula (2) can be seen as the symmetric bilinear form on $\mathscr{T}_\gamma$. So it is denoted by $D^2E(\gamma)$.

Definitions 4. *The index of the geodesic γ is the dimension of the maximal subspace of $\mathscr{T}_\gamma$ on which $D^2E(\gamma)$ is negative definite.*

Lemma 5 (Hamenstädt [4]). *There exists a positive number δ depending upon $\varphi(0)$ such that the Hessian $D^2E(\gamma|_{[0,\delta]})$ is positive definite.*

Remark. In the case where H is strongly bracket-generating, the set $\mathscr{C}_{pq}$ consisting of horizontal curves jointing p and q, is a "manifold". One can prove this fact by using the results of Hsu [5] and Lemma 7 below. Here a "manifold" means that any horizontal vector field along γ vanishing at the endpoints is a variational vector field associated to a horizontal variation α. Then $\mathscr{T}_\gamma$ can be seen as a tangent space of $\mathscr{C}_{pq}$ at γ.

The argument of Hamenstädt [4] holds good in the case where $\mathscr{C}_{pq}$ is a "manifold". (See [5] for more detailed argument)

2.3. The Jacobi equation.

$$\begin{cases}\mathscr{D}_\gamma Y=P(a^*(Y)\varphi+\psi)\\ \frac{d}{dt}\psi-a^*(\dot\gamma)\psi=-2A^*(\dot\gamma,Y)\varphi,\end{cases}\tag{3}$$

where ψ is a curve on $\boldsymbol{R}^m$. Jacobi fields are defined to be a variational vector field associated to a variation $\alpha:(-\varepsilon,\varepsilon)\times[0,1]\to M$ such that each curve $t\mapsto\alpha(s,t)$ is a geodesic. And the null space of the bilinear form $D^2E(\gamma)$ consists of all Jacobi fields along γ vanishing at its endpoints.

3. Some properties of Jacobi fields

According to the Jacobi equation (3), a Jacobi field is determined by the values of $Y(0)$ and $\psi(0)$. In terms of the covariant derivative, we have

Lemma 6. *Let Y be a Jacobi field along γ and $\frac{DY}{dt}$, $\frac{D^2Y}{dt^2}$ be its first and second covariant derivatives. If Y, $\frac{DY}{dt}$ and $\frac{D^2Y}{dt^2}$ vanish simultaneously at some point $\gamma(t_0)$, then Y is identically equal to zero.*

Proof. By the linearity of (3), we must only prove $\psi(t_0)=0$. From the theory of the connection, there exists uniquely a 1-form ω with values in $m\times m$

skew-symmetric matrices such that $d\theta = -\omega \wedge \theta$. And for any two vector fields X and Y, $\theta(\nabla_X Y) = X\theta(Y) + \omega(X)\theta(Y)$. Hence for any vector field X along the geodesic γ,

$$\mathscr{D}_\gamma X = \theta\left(\frac{DX}{dt}\right) - \omega(X)\theta(\dot{\gamma}).$$

Therefore, from (3), we have $P\psi(t_0) = 0$. By differentiating the first equation of (3), $P\psi'(t_0) = 0$. And from the second equation of (3), $Pa^*(\dot{\gamma}(t_0))\psi(t_0) = 0$. Lemma 6 then follows from the following:

Lemma 7. *Let ξ be a vector in $\boldsymbol{R}^m$ and u be a non-zero horizontal tangent vector at a point q in M. If both $P\xi$ and $Pa^*(u)\xi$ vanish, then ξ is zero.*

Proof. For any vectors v_1 and v_2 of H_q,

$$\langle \xi, \theta(v_1)\rangle = \langle a^*(u)\xi, \theta(v_2)\rangle = 0.$$

From the definitin of a^*,

$$\langle \xi, 2d\theta(u, v_2)\rangle = -\langle a^*(u)\xi, \theta(v_2)\rangle = 0.$$

Thus we have

$$\langle \xi, \theta(v_1) - 2d\theta(u, v_2)\rangle = 0$$

From the strongly bracket-generating hypothesis, there exist a tagent vector v'_1 at q and local sections U and V of H around q such that $U_q = u$ and

$$\begin{aligned}\xi &= \theta(v'_1 + [U, V]_q)\\ &= \theta(v'_1) + U_q\theta(V) - V_q\theta(U) - 2d\theta(U_q, V_q)\end{aligned}$$

Take two vectors v_1 and v_2 of H_q such that

$$\theta(v_1) = \theta(v'_1) + U_q\theta(V) - V_q\theta(U)$$

and $v_2 = V_q$. Then $\xi = \theta(v_1) - 2d\theta(u, v_2)$. Hence we have $\xi = 0$.

Remark. If the Jacobi field Y is identically zero, so is $P\psi$. By the proof of Lemma 6, ψ is identically equal to zero. Thus the Jacobi field Y is *uniquely* determined by $Y(0)$ and $\psi(0)$. Therefore the dimension of the linear space consisting of all Jacobi fields along a non-constant geodesic γ is just equal to $2\dim M$ and that of the linear subspace consisting of al Jacobi fields vanishing at $\gamma(0)$ is equal to $\dim M$.

Definition 8. *The point q is conjugate to p along γ if there exists a non-zero Jacobi fields Y which vanishes at p and q. The multiplicity of q as a conjugate point is equal to the dimension of the vector space consisting of all such Jacobi fields.*

Lemma 9. *If q is not conjugate to p along γ, then every Jacobi field along γ is uniquely determined by its values at p and q.*

This lemma follows from the fact that there are "enough" Jacobi fields along γ, which is stated in the remark above. But if we do not assume that the distribution is strongly bracket-generating, Lemma 9 does not hold even for normal geodesic.

4. Proof of Theorem 1

Let $i(s)$ be the index of the restricted geodesic $\gamma|_{[0,s]}$ and $n(s)$ be the dimension of the null space of the bilinear form $D^2E(\gamma|_{[0,s]})$. Here we will give the proof of the finiteness of $i(1)$ and the following formula:

$$i(1) = \sum_{0<s<1} n(s). \tag{4}$$

4.1. Let Δ be the division of the unit interval $0=t_0<t_1<\cdots<t_N=1$ such that $D^2E(\gamma|_{[t_{j-1},t_j]})$ is positive definite for each j. (By Lemma 5, we can choose such a division Δ.) And let $\mathscr{T}^\Delta_\gamma$ be the subspace of $\mathscr{T}_\gamma$ consisting of all vector fields Y and γ such that $Y|_{[t_{j-1},t_j]}$ is a Jacobi field along $\gamma|_{[t_{j-1},t_j]}$. For any $X\in\mathscr{T}_\gamma$ and any $Y\in\mathscr{T}^\Delta_\gamma$,

$$\begin{aligned} D^2E(\gamma)(X,Y) &= \sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\left\{\langle\frac{d}{dt}\theta(X)+a(\dot\gamma)\theta(X),\psi\rangle+\langle\theta(X),\psi'-a^*(\dot\gamma)\psi\rangle\right\}dt \\ &= \sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\frac{d}{dt}\langle\theta(X),\psi\rangle dt \\ &= \sum_{j=1}^{N-1}\langle\theta(X(t_j)),\psi(t_j-0)-\psi(t_j+0)\rangle \end{aligned}$$

where $\psi|_{[t_{j-1},t_j]}$ is a curve on $\boldsymbol{R}^m$ associated to the Jacobi field $Y|_{[t_{j-1}t_j]}$. (Here we must remark that $\mathscr{D}_\gamma X$ and $\mathscr{D}_\gamma Y$ are curves in $\boldsymbol{R}^k=P(\boldsymbol{R}^m)$.)

Let Ω be the subspace of $\mathscr{T}_\gamma$ consisting of all the horizontal vector fields vanishing at $t=t_1,\cdots,t_{N-1}$. By Lemma 9, the vector space $\mathscr{T}_\gamma$ splits as the direct sum $\mathscr{T}^\Delta_\gamma\oplus\Omega$. Moreover, by the above computation, we can see these subspaces are orthogonal to each other with respect to the bilinear form $D^2E(\gamma)$. From the choice of the division Δ, $D^2E(\gamma)$ is positive definite on Ω. Thus the index of γ is equal to the index of the bilinear form $D^2E(\gamma)$ restricted on $\mathscr{T}^\Delta_\gamma$. The dimension of $\mathscr{T}^\Delta_\gamma$ is finite because $\mathscr{T}^\Delta_\gamma$ is isomorphic to $T_{\gamma(t_1)}M\oplus\cdots\oplus T_{\gamma(t_{N-1})}M$ from Lemma 9. Hence the index $i(1)$ is finite.

4.2. If $\tau<\tau'$, each element of $\mathscr{T}_{\gamma|_{[0,\tau]}}$ can be extended to a vector field which vanishes identically on the interval $[\tau,\tau']$ since it vanishes at $\gamma(\tau)$. So $\mathscr{T}_{\gamma|_{[0,\tau]}}$ can be seen as a subspace of $\mathscr{T}_{\gamma|_{[0,\tau']}}$. The bilinear form $D^2E(\gamma|_{[0,\tau']})$ is negative definite on the subspace of $\mathscr{T}_{\gamma|_{[0,\tau]}}$ on which $D^2E(\gamma|_{[0,\tau]})$ is negative definite. Hence $i(t)$ is non-decreasing.

4.3. We may assme that the division Δ is chosen so that $t_i<\tau<t_{i+1}$. Let Δ_τ be the division of the interval $[0,\tau]$, $0=t_0<t_1<\cdots<t_i<\tau$. According to Lemma 9, a broken Jacobi field is uniquely determined by its values at the break points $\gamma(t_j)$. Thus the vector space $\mathscr{T}^{\Delta_\tau}_{\gamma|_{[0,\tau]}}$ is isomorphic to the following vector space:

$$\Sigma_i(\Delta)=T_{\gamma(t_1)}M\oplus\cdots\oplus T_{\gamma(t_i)}M.$$

Let B_τ be the bilinear form on $\Sigma_i(\Delta)$ associated to $D^2E(\gamma|_{[0,\tau]})$. Then $i(\tau)$ is equal to the index of B_τ. Since the bilinear form B_t depends continuously on t, for any sufficiently small $\varepsilon>0$, $B_{\tau-\varepsilon}$ is negative definite on the subspace of $\Sigma_i(\Delta)$ on which B_τ is negative definite. Therefore $i(\tau-\varepsilon)\geq i(\tau)$. Since $i(t)$ is non-decreasing, $i(\tau-\varepsilon)=i(\tau)$. Namely $i(t)$ is left continuous.

4.4. For any sufficiently small $\varepsilon>0$ (we may assume that $t_i<\tau<\tau+\varepsilon<t_{i+1}$), $B_{\tau+\varepsilon}$ is positive definite on the subspace V_τ on which B_τ is positive definite. The dimension of V_τ is equal to $\dim\Sigma_i(\Delta)-i(\tau)-n(\tau)$. Hence

$$i(\tau+\varepsilon)\leq\dim\Sigma_i(\Delta)-\dim V_\tau=i(\tau)+n(\tau)$$

4.5. Let $\{X_1,\cdots,X_{i(\tau)}\}$ be a basis of the maximal subspace of $\mathscr{T}^{\Delta}_{\gamma|_{[0,\tau]}}$ on which $D^2E(\gamma|_{[0,\tau]})$ is negative definite, and $\{Y_1,\cdots,Y_{n(\tau)}\}$ be a basis of the nullspace of $D^2E(\gamma|_{[0,\tau]})$. Namely, $Y_1,\cdots,Y_{n(\tau)}$ are linearly independent Jacobi fields along $\gamma|_{[0,\tau]}$ vanishing at $\gamma(0)$ and $\gamma(\tau)$. Denote ψ_j by the curve on $\boldsymbol{R}^m$ such that the pair (Y_j,ψ_j) satisfies the equation (3). Since Y_j is *uniquely* determined by the value $(Y_j(\tau)=0,\ \psi_j(\tau))$, the vectors $\psi_1(\tau),\cdots,\psi_{n(\tau)}(\tau)$ are linearly independent. Hence, by Lemma 9, we can choose the elements $Z_1,\cdots,Z_{n(\tau)}$ of $\mathscr{T}^{\Delta}_{\gamma|_{[0,\tau+\varepsilon]}}$, so that $\langle\psi_i(\tau),\theta(Z_j(\tau))\rangle=\delta_{ij}$, where δ_{ij} is Kronecker's delta. Extend X_i's and Y_j's to vector fields vanishing identically between $\gamma(\tau)$ and $\gamma(\tau+\varepsilon)$ and define the vector fields W_j by $W_j=\frac{1}{c}Y_j-cZ_j$, for a small number c. Then it is easy to see that

$$D^2E((\gamma|_{[0,\tau+\varepsilon]})(X_i,W_j)=-cD^2E(\gamma|_{[0,\tau+\varepsilon]})(X_i,Z_j)$$

$$D^2E(\gamma|_{[0,\tau+\varepsilon]})(W_i,W_j)=-2\delta_{ij}+c^2D^2E(\gamma|_{[0,\tau+\varepsilon]})(Z_i,Z_j).$$

For a sufficiently small c, $\{X_1,\cdots,X_{i(\tau)},\ W_1,\cdots,W_{n(\tau)}\}$ spans a subspace of $\mathscr{T}^{\Delta}_{\gamma|_{[0,\tau+\varepsilon]}}$ on which $D^2E(\gamma|_{[0,\tau+\varepsilon]})$ is negative definite. Hence $i(\tau+\varepsilon)\geq i(\tau)+n(\tau)$. By the last subsection, this equality holds.

By Lemma 5, $i(t)=0$ if t is sufficiently small. Therefore the equation (4) holds.

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