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A topological characterization of *SU*

By

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1. Introduction

In the late 1950s, R. Bott in [4] showed that the stable homotopy of unitary group is periodic. This theorem has been extended, now it is well-known that the special unitary group is homotopy equivalent to its double loop space.

In [9], Toda gave a topological proof of periodicity of homotopy groups of the special unitary group. We will show that this result extends to homotopy equivalence and a characterization of the special unitary group.

Let $\boldsymbol{SU}$ be the special unitary group of infinite dimension. The cohomology ring of $\boldsymbol{SU}$ is given by the formula

$$H^*(\boldsymbol{SU};\boldsymbol{Z})=\bigwedge(x_3,x_5,\cdots,x_{2i+1},\cdots) \qquad \deg(x_{2i+1})=2i+1$$

and Bott periodicity theorems [4] of cmplex case:

$$\boldsymbol{BU}\simeq\Omega\boldsymbol{SU},\ \boldsymbol{U}\simeq\Omega\boldsymbol{BU}$$

are familiar to us.

Theorem 1.1. *Let X be a finite type CW complex which is a homotopy associative H-space. Suppose that X has following properties:*

(1) *X is simply cnnected;*
(2) *as an algebra $H^*(X;\boldsymbol{Z})=\bigwedge(x_3,x_5,\cdots,x_{2i+1},\cdots)$ where $|x_{2i+1}|=2i+1$ and each x_{2i+1} is a primitive element;*
(3) *there exist two maps*

$$j':\Sigma\boldsymbol{CP}^\infty\to X$$
$$\lambda':\Sigma^2X\to X$$

such that

(a) *$j'^*:H^*(X;\boldsymbol{Z})\to H^*(\Sigma\boldsymbol{CP}^\infty;\boldsymbol{Z})$ is epic and*
(b) *$(\lambda'\circ(\Sigma^2j'))^*:x_{2i+1}\mapsto\pm is_3t^{i-1}$,*

where s_3 is a generator of $H^*(S^3;\boldsymbol{Z})$ and t is a generator of $H^*(\boldsymbol{CP}^\infty;\boldsymbol{Z})$.

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Then the map

$$\widetilde{Ad^2\lambda'}: X \to \Omega^2(X\langle 3\rangle)$$

is a homotopy eqivalence.

Here, $X\langle 3\rangle$ denotes the 3-connected fiber space of X. Ad denotes a natural equivalence

$$[\Sigma A, B] \xrightarrow{\sim} [A, \Omega B].$$

The map $\widetilde{Ad^2\lambda'}: X \to \Omega^2(X\langle 3\rangle)$ is a lifting of $Ad^2\lambda': X \to \Omega^2 X$.

Theorem 1.2. *If X satisfies the above conditions, then X is homotoopy equivalent to $\boldsymbol{SU}$.*

Remark 1.3. *By using* [9] *Lemma* 3.1, *we can omit primitivity of x_{2i+1} in condition* (2).

In Section 2, we will prove Theorem 1.1. The two maps j' and λ' are naturally constructed in the case of $\boldsymbol{SU}$. This is discussed in Section 3. Miller [7] showed that $\boldsymbol{U}$ is stably homotopy equivalent to $\vee_{k\geq 1}\boldsymbol{BU}_k^{ad_k}$. Moreover when $k=1$, the inclusion map $\boldsymbol{BU}_1^{ad_1} \to \boldsymbol{U}$ is induced by an unstable map. Making use of these facts, we will build a homotopy equivalent map $\boldsymbol{SU} \to X$ in Section 4.

Through this paper, we denote a generator of a cohomology ring $H^*(S^n; \boldsymbol{Z})$ (resp. $H^*(\boldsymbol{CP}^\infty; \boldsymbol{Z})$) by s_n (resp. t). And let η (resp. η_∞) be the canonical line bundle over $\boldsymbol{CP}^1$ (resp. $\boldsymbol{CP}^\infty$) whose first Chern class is s_2 (resp. t). Let q be the natural inclusion map $\boldsymbol{SU} \to \boldsymbol{U}$ by which we identify x_{2i+1} in $H^*(\boldsymbol{SU}; \boldsymbol{Z})$ with that of $H^*(\boldsymbol{U}; \boldsymbol{Z})$.

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2. The Periodicity

In this section, we will prove Theorem 1.1.

First we describe elementary results from general theory of spectral sequences and Hopf-algebras. Let X be a finite CW complex. We assume that X is an associative H-space,

$$H^*(X; \boldsymbol{Z}) = \bigwedge(x_3, x_5, \cdots, x_{2i+1}, \cdots),$$

and that each x_{2i+1} is a primitive element of degree $2i+1$.

Lemma 2.1. *We get the following properties:*

(1) $H_*(X; \boldsymbol{Z}) = \bigwedge(\beta_3, \beta_5, \cdots, \beta_{2i+1}, \cdots)$

where β_{2i+1} is the dual of x_{2i+1} and a primitive element for each i.

(2) $H_*(\Omega X; \boldsymbol{Z}) = \boldsymbol{Z}[b_2, b_4, \cdots, b_{2i}, \cdots]$
where $b_{2i} = \tau(\beta_{2i+1})$ is the image of the homology transgression map of β_{2i+1}.

Moreover we assume that there exists a map

$$j' : \Sigma \boldsymbol{CP}^\infty \to X$$

such that $j'^* : H^*(X; \boldsymbol{Z}) \to H^*(\Sigma \boldsymbol{CP}^\infty; \boldsymbol{Z})$ is epic. Then we get

(3) *the coproduct of b_{2i} is*

$$\Delta_*(b_{2i}) = \sum_{j+k=i} b_{2j} \otimes b_{2k}.$$

See [8] for proofs of (1) and (2).

Proof of (3). We define $\tau_i \in H_{2i}(\boldsymbol{CP}^\infty; \boldsymbol{Z})$ to be the dual of $t^i \in H^{2i}(\boldsymbol{CP}^\infty; \boldsymbol{Z})$. From assumption we obtain

$$Adj'_*(\tau_i) = b_{2i}$$

by changing the sign of b_{2i} if necessary. By definition, the coproduct of τ_i is given by

$$\Delta_*(\tau_i) = \sum_{j+k=i} \tau_j \otimes \tau_k.$$

Adj'_* is a homomorphism of coalgebra. Hence we come to the conclusion.

Lemma 2.2. *On condition that Lemma* 2.1 (3) *holds, we obtain the following properties*:

(1) $H^*(\Omega X; \boldsymbol{Z}) = \boldsymbol{Z}[y_2, y_4, \cdots, y_{2i}, \cdots]$
where y_{2i} is the dual of ${b_2}^i$ and the coproduct of y_{2i} is

$$\mu(y_{2i}) = \sum_{j+k=i} y_{2j} \otimes y_{2k}.$$

(2) $H^*(\Omega^2 X; \boldsymbol{Z}) = \bigwedge(z_1, z_3, \cdots, z_{2i-1}, \cdots)$
where $z_{2i-1} = \sigma(y_{2i})$ is the image of the cohomology suspension map of y_{2i} and z_{2i-1} is a primitive element for each i.

Next we compute the image of the cohomology suspension of x_{2i+1}.

Lemma 2.3. *Let $t_1, t_2, \cdots, t_n$ be indeterminate elements and σ_i its i-th fundamental symmetric function. We define an integral polynomial S_n of n variables by*

$$t_1^n + t_2^n + \cdots + t_n^n = S_n(\sigma_1, \sigma_2, \cdots, \sigma_n).$$

Then the image of the cohomology suspension map

$$\sigma : H^{2i+1}(X; \boldsymbol{Z}) \to H^{2i}(\Omega X; \boldsymbol{Z})$$

of x_{2i+1} *is*

$$\sigma(x_{2i+1}) = S_i(y_2, y_4, \cdots, y_{2i}).$$

Proof. By [8] Chap. 4 Lemma 2.16, $\sigma(x_{2i+1})$ is a primitive element. By [8] Chap. 4 Theorem 2.18, the submodule of all primitive elements of $H^{2i}(\Omega X; \boldsymbol{Z})$ is generated by $S_i(y_2, y_4, \cdots, y_{2i})$. Then there exists an integer a such that

$$\sigma(x_{2i+1}) = aS_i(y_2, y_4, \cdots, y_{2i}).$$

We compute by Newton formula,

$$S_i(y_2, y_4, \cdots, y_{2i}) = y_2^i + \text{ other terms orthogonal to } b_{2i}.$$

Then

$$\begin{aligned} a &= a\langle S_i(y_2, y_4, \cdots, y_{2i}), b_{2i}\rangle \\ &= \langle \sigma(x_{2i+1}), b_{2i}\rangle \\ &= \langle x_{2i+1}, \beta_{2i+1}\rangle \\ &= 1. \end{aligned}$$

Proof of Theorem 1.1. First we denote

$$\begin{aligned} &k = \lambda' \circ \Sigma^2 j' : \Sigma^3 \boldsymbol{CP}^\infty \to \Sigma^2 X \to X \\ &k_1 = Adk = Ad\,\lambda' \circ \Sigma j' : \Sigma^2 \boldsymbol{CP}^\infty \to \Sigma X \to \Omega X \\ &k_2 = Adk_1 = Ad^2\,\lambda' \circ j' : \Sigma^1 \boldsymbol{CP}^\infty \to X \to \Omega^2 X. \end{aligned}$$

We will show that the image of k_2^* includes all generators of $H^*(\Sigma \boldsymbol{CP}^\infty; \boldsymbol{Z})$. Consider the following commutative diagram:

$$\begin{array}{ccc} H^{2i+1}(X; \boldsymbol{Z}) & \xrightarrow{k^*} & H^{2i+1}(\Sigma \boldsymbol{CP}^\infty; \boldsymbol{Z}) \\ \downarrow \sigma & & \downarrow \text{suspension iso.} \\ H^{2i}(\Omega X; \boldsymbol{Z}) & \xrightarrow{k_1^*} & H^{2i}(\Sigma^2 \boldsymbol{CP}^\infty; \boldsymbol{Z}). \end{array}$$

By the condition (3b) and Lemma 2.3, we obtain

$$k_1^*(S_i(y_2, y_4, \cdots, y_{2i})) = \pm i s_2 t^{i-1}.$$

Since in $\tilde{H}^*(\Sigma CP^\infty; Z)$ cup products vanish, k_1^* maps decomposable elements to 0. By Newton formula, we have

$$S_i(y_2, y_4, \cdots, y_{2i})) = \pm i y_{2i} + \quad \text{(decomposable elements)}.$$

Then

$$\pm k_1^*(i y_{2i}) = \pm i s_2 t^{i-1}$$
$$k_1^*(y_{2i}) = \pm s_2 t^{i-1}.$$

Consider the following commutative diagram:

$$\begin{array}{ccc} H^{2i}(\Omega X; Z) & \xrightarrow{k_1^*} & H^{2i}(\Sigma^2 CP^\infty; Z) \\ \downarrow \sigma & & \downarrow \text{suspension iso.} \\ H^{2i-1}(\Omega^2 X; Z) & \xrightarrow{k_2^*} & H^{2i-1}(\Sigma CP^\infty; Z). \end{array}$$

Then we obtain

$$k_2^*(z_{2i-1}) = \pm s_1 t^i.$$

Consequently k_2^* is an epimorphism. Condition (3a) implies that j'^* is a bijection between generators of algebras. Then Image $(\widetilde{Ad^2 \lambda'})^*$ includes all generators $\{x_{2i+1}\}$ of algebra $H^*(X; Z)$. So does $\widetilde{Ad^2 \lambda'})^*: H^*(\Omega^2(X\langle 3\rangle); Z) \to H^*(X; Z)$. By [6] Lemma 2.2, $\widetilde{Ad^2 \lambda'}^*$ is an isomorphism at each degree. Since both sides are free Z modules, $\widetilde{Ad^2 \lambda'}_*$ is an isomorphism also. Since X and $\Omega^2 X\langle 3\rangle$ are simply connected, $\widetilde{Ad^2 \lambda'}$ is a homotopy equivalence by J. H. C. Whitehead's theorem.

3. Periodicity theorem for *SU*

We will prove the periodicity theorem for SU by using Theorem 1.1. In other words, we will construct two maps j'_{SU} and λ'_{SU} which satisfy conditions of Theorem 1.1.

Let ξ_n be the universal vector bundle over BU_n. We define

$$\xi := \lim_{\rightarrow} (\xi_n - n),$$

where n is the trivial vector bundle of n dimension. Let λ be the classifying map of $(\eta - 1)\hat{\otimes}\xi \in \tilde{K}(\Sigma^2 BU)$. This map λ has a lift $\tilde{\lambda}: \Sigma^2 BU \to BSU$. And we denote the classifying map of the canonical line bundle η_∞ over CP^∞ by $j: CP^\infty \to BU$.

We now come to construct two maps j'_{SU} and λ'_{SU}. We define

$$j'_{SU} := Ad^2 \tilde{\lambda} \circ (\Sigma j) \qquad (1)$$

$$\lambda'_{SU} := Ad^2 \tilde{\lambda} \circ (\Sigma q) \circ (\Sigma \pi) \qquad (2)$$

where $q: BSU \to BU$ is the natural inclusion and $\pi: \Sigma\Omega BSU \to BSU$ represents $Ad^{-1}(1_{\Omega BSU})$.

Lemma 3.1. *The map*

$$j'_{SU}{}^*: H^*(SU; \boldsymbol{Z}) \to H^*(\Sigma \boldsymbol{CP}^\infty; \boldsymbol{Z})$$

is an epimorphism.

Proof. First we consider the following commuttive diagram:

$$\begin{array}{ccc} H^*(\boldsymbol{BSU}; \boldsymbol{Z}) & \xrightarrow{(Ad^{-1}j'_{SU})^*} & H^*(\Sigma \boldsymbol{CP}^\infty; \boldsymbol{Z}) \\ \downarrow \sigma & & \downarrow \text{suspension iso.} \\ H^{*-1}(\boldsymbol{SU}; \boldsymbol{Z}) & \xrightarrow{j'_{SU}{}^*} & H^{*-1}(\Sigma \boldsymbol{CP}^\infty; \boldsymbol{Z}). \end{array}$$

By (1), $Ad^{-1}j'_{SU} = \tilde{\lambda} \circ (\Sigma^2 j)$ and $q \circ \tilde{\lambda} = \lambda$, the proof completes if the map

$$H^*(\boldsymbol{BU}; \boldsymbol{Z}) \xrightarrow{\lambda^*} H^*(\Sigma^2 \boldsymbol{BU}; \boldsymbol{Z}) \xrightarrow{(\Sigma^2 j)^*} H^*(\Sigma^2 \boldsymbol{CP}^\infty; \boldsymbol{Z}) \tag{3}$$

is epic. We will calculate the image of total Chern class $c = 1 + c_1 + c_2 + \cdots \in H^*(\boldsymbol{BU}; \boldsymbol{Z})$. This is just the total Chern class of the virtual bundle $(\eta - 1)\hat{\otimes}(\eta_\infty - 1)$ classified by the map $\lambda \circ \Sigma^2 j$. Since

$$c((\eta - 1)\hat{\otimes}(\eta_\infty - 1)) = 1 + \sum_{i=1}^{\infty} (-1)^i s_2 t^i$$

(See [6]), we obtain

$$(\lambda \circ \Sigma^2 j)^*(c_{i+1}) = (-1)^i s_2 t^i.$$

Lemma 3.2. $(\lambda'_{SU} \circ \Sigma^2 j'_{SU})^*(x_{2i+1}) = \pm i s_3 t^{i-1}$

Proof. Since $\sigma(c_{i+1}) = x_{2i+1}$, it suffices to show that

$(Ad^{-1}\lambda'_{SU} \circ \Sigma^3 j'_{SU})^*(c_{i+1}) = \pm i s_4 t^{i-1}$.

By (1) and (2) we can easily get the following factorizations,

$$\Sigma^3 j'_{SU} = \Sigma^3 Ad^2 \tilde{\lambda} \circ \Sigma^4 j$$

$$Ad^{-1}\lambda'_{SU} = \tilde{\lambda} \circ \Sigma^2 q \circ \Sigma^2 \pi.$$

Then we get

$$\begin{aligned} Ad^{-1}\lambda'_{SU} \circ \Sigma^3 j'_{SU} &= \tilde{\lambda} \circ \Sigma^2 q \circ \Sigma^2 \pi \circ \Sigma^3 Ad \tilde{\lambda} \circ \Sigma^4 j \\ &= \tilde{\lambda} \circ \Sigma^2 q \circ \Sigma^2 \tilde{\lambda} \circ \Sigma^4 j \\ &= \tilde{\lambda} \circ \Sigma^2 \lambda \circ \Sigma^4 j. \end{aligned}$$

We will calculate the total Chern class of the virtual bundle $(\eta - 1)\hat{\otimes}(\eta - 1)\hat{\otimes}(\eta_\infty - 1)$ over $\Sigma^4 \boldsymbol{CP}^\infty \cong \Sigma^2 \Sigma^2 \boldsymbol{CP}^\infty$ classified by $\lambda \circ \Sigma^2 \lambda \circ \Sigma^4 j$. We write $s \in H^2(S^2; \boldsymbol{Z})$ to be the cohomology of the first component of $S^2 \wedge S^2$ and s' to be that of the second. Note that $s = c_1(\eta \hat{\otimes} 1)$, $s' = c_1(1 \hat{\otimes} \eta)$, and recall that $t = c_1(\eta_\infty)$.

$$
\begin{aligned}
&c((\eta-1)\hat{\otimes}(\eta-1)\hat{\otimes}(\eta_\infty-1))\\
&= c(\eta\hat{\otimes}\eta\hat{\otimes}\eta_\infty)c(1\hat{\otimes}\eta\hat{\otimes}\eta_\infty)^{-1}c(\eta\hat{\otimes}1\hat{\otimes}\eta_\infty)^{-1}c((\eta\hat{\otimes}\eta\hat{\otimes}1)^{-1}\\
&\quad c(\eta\hat{\otimes}1\hat{\otimes}1)c(1\hat{\otimes}\eta\hat{\otimes}1)c(1\hat{\otimes}1\hat{\otimes}\eta_\infty)\\
&= (1+s+s'+t)(1+s'+t)^{-1}(1+s+t)^{-1}(1+s+s')^{-1}\\
&\quad (1+s)(1+s')(1+t)\\
&= (1+s)(1+s(1+s'+t)^{-1})(1-s(1+s+t)^{-1})(1-s(1+s+s')^{-1})\\
&= 1+s+s(1+s'+t)^{-1}-s(1+s+t)^{-1}-s(1+s+s')^{-1}\\
&= 1+s\\
&\quad +s(1+\sum_{i=1}^{\infty}(-1)^i(s'+t)^i)\\
&\quad +s(1+\sum_{i=1}^{\infty}(-1)^i(s+t)^i)\\
&\quad -s(1+\sum_{i=1}^{\infty}(-1)^i(s+s')^i)\\
&= 1+s\\
&\quad +s(1+\sum_{i=1}^{\infty}(-1)^i(t^i+is't^{i-1})\\
&\quad -s(1+\sum_{i=1}^{\infty}(-1)^i(t^i+ist^{i-1}))\\
&\quad -s(1-s-s'+2ss')\\
&= 1+\sum_{i=2}^{\infty}(-1)^i iss't^{i-1},
\end{aligned}
$$

because $s^2 = 0$ and $s'^2 = 0$. Thus we obtain

$$(Ad^{-1}\lambda'_{SU}\circ\Sigma^3 j'_{SU})^*(c_{i+1}) = (-1)^i is_4 t^{i-1}$$

where $s_4 \in H^4(S^4; \boldsymbol{Z})$ is $ss' \in H^4(S^2 \wedge S^2; \boldsymbol{Z})$.

4. Characterization theorem

Theorem 1.1 implies that X is homotopy equivalent to its double loop space. Furthermore, this theorem gives a characterization of $\boldsymbol{SU}$. In this section we will show it.

In [7] and [5], there exists a stable homotopy equivalence

$$U \cong \bigvee_{k \geq 1} BU_k^{ad_k}.$$

When $k=1$, the inclusion map is induced by an unstable map

$$BU_1^{ad_1} \cong \Sigma CP^\infty \xrightarrow{j} U.$$

Remark that $BU_1 \cong CP^\infty$ and its adjoint representation of U_1 is trivial. Thus its Thom space is ΣCP^∞. There exists a stable map $p: U \to \Sigma CP^\infty$ such that $p \circ (\Sigma^\infty j) = id$. Note that j^* is epic in the cohomology rings. We will use the same symbol for the corresponding adjoint map of a stable map.

Lemma 4.1. *The map*

$$\Sigma CP^\infty \xrightarrow{j'_{SU}} SU \xrightarrow{q} U \xrightarrow{p} \Sigma CP^\infty$$

is a stable homotopy equivalence.

Proof. First we show

$$p^*(s_1 t^i) = \pm x_{2i+1} \pm (\text{decomposable elements}) \text{ for } i \geq 1.$$

this follows immediately from the fact that $j^*(x_{2i+1}) = \pm s_1 t^i$ and cup products vanish in $\tilde{H}^*(\Sigma CP^\infty; Z)$.

Since $j'_{SU}{}^*$ maps decomposable elements to zero,

$$\begin{aligned} & j'_{SU}{}^* \circ q^* \circ p^*(s_1 t^i) \\ &= j'_{SU}{}^* \circ q^*(\pm x_{2i+1} \pm (\text{decomposable elements})) \\ &= j'_{SU}(\pm x_{2i+1} \pm (\text{decomposable elements})) \\ &= \pm s_1 t^i. \end{aligned}$$

Thus this map induces an isomorphism between the cohomology rings. ΣCP^∞ is simply connected and its homology group is free. Then this map is stably homotopy equivalent by J. H. C. Whitehead's theorem.

Now we label this stable homotopy equivalent map by h. We assume that X satisfies the condition of Theorem 1.1 and we write

$$H^*(X; Z) = \bigwedge(x'_3, x'_5, \cdots, x'_{2i+1}, \cdots).$$

By Theorem 1.1, X is an infinite loop space. Thus we can define

$$\xi: Q(X) \to X$$

such that

$$\xi \circ \iota \simeq 1_X,$$

where $Q(X) := \varinjlim \Omega^n \Sigma^n X$ and $\iota := \varinjlim Ad^n(1_{\Sigma^n X})$. Now we prove Theorem 1.2. All we have to do is to prove the next lemma.

Lemma 4.2. *The map*

$$\phi : SU \xrightarrow{q} U \xrightarrow{p} Q(\Sigma CP^\infty) \xrightarrow{h^{-1}} Q(\Sigma CP^\infty) \xrightarrow{Q(j')} Q(X) \xrightarrow{\xi} X$$

is homotopy equivalent.

Proof. We consider the following commutative diagram:

$$\begin{array}{ccc}
\Sigma CP^\infty & \xrightarrow{j'_{SU}} & SU \\
 & & \downarrow q \\
\| & & U \\
 & & \downarrow p \\
\Sigma CP^\infty & \xrightarrow{h} & Q(\Sigma CP^\infty) \\
\| & & \downarrow h^{-1} \\
\Sigma CP^\infty & \xrightarrow{\iota} & Q(\Sigma CP^\infty) \\
\downarrow j' & & \downarrow Q(j') \\
X & \xrightarrow{\iota} & Q(X) \\
 & \diagdown\!\!\diagdown & \downarrow \xi \\
 & & X
\end{array}$$

where ι is the natural inclusion map. Commutativity is trivial by definition. We now see above diagram through cohomology functor. Both j' and j'_{SU} induce bijections between generators of cohomology rings. Then the map ϕ of right column induces a bijection between generators, too. Thus ϕ^* is an isomorphism of cohomology at each degree. Since both X and SU are simply connected, ϕ is a homotopy equivalence by J.H.C.Whitehaead's theorem.

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