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An Appell-Humbert theorem for hyperelliptic surfaces

By

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0. Introduction

Let $S \to B$ be a hyperelliptic surface over a smooth elliptic curve B defined over the field of complex numbers. The aim of this paper is to give a description of the Picard group of S in terms of hermitian forms and multiplicators, similar to Appell-Humbert for complex tori. The main tool used here is the cohomology of the groups and the ideas are similar to those used in [3], [9].

In the first section we recall some fundamental facts on hyperelliptic surfaces, such as the classification theorem and their fundamental groups.

In section 2, we get a description of the group of line bundles whose first Chern classes are torsion elements in the Néron-Severi group, which is usually denoted by $\mathrm{Pic}^{\tau}(S)$ and in the third section, which plays an important role for our purpose, we obtain a description of $\mathrm{Num}(S)$ in terms of hermitian forms.

The fourth section is devoted to the Appell-Humbert theorem and the final section presents some direct applications of it such as computing Tors $H^2(S, \mathbf{Z})$, finding a basis in $\mathrm{Num}(S)$ (see, also [10]) and computing the space of global sections for the line bundles over S numerically equivalent to a multiple of the fiber of $S \to B$.

1. Preliminaries and notations

There are many approaches concerning the theory of hyperelliptic surfaces ([1], [2], [6], [10], [12], [15]). Firstly, we recall the definition used by Suwa (cf. [12]).

Definition 1.1. A *hyperelliptic surface* is an elliptic bundle S over an elliptic curve B with $b_1(S) = 2$.

Theorem 1.2 (cf. [12]). *Any hyperelliptic surface can be expressed as a quotient of an abelian variety A by the group generated by an automorphism g_5 of A. The period matrix of A and the automorphism g_5 are given as follows:*

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(a1) $\begin{pmatrix} 1 & 0 & \alpha & 0 \\ 0 & 1 & 0 & \beta \end{pmatrix}$ (a2) $\begin{pmatrix} 1 & 0 & \alpha & 0 \\ 0 & 1 & \frac{1}{2} & \beta \end{pmatrix}$

$g_5(u, z) = \left(u + \frac{1}{2}, -z\right)$

(b1) $\begin{pmatrix} 1 & 0 & \alpha & 0 \\ 0 & 1 & 0 & \rho \end{pmatrix}$ (b2) $\begin{pmatrix} 1 & 0 & \alpha & 0 \\ 0 & 1 & \frac{1-\rho}{3} & \rho \end{pmatrix}$

$g_5(u, z) = \left(u + \frac{1}{3}, \rho z\right)$, *where* $\rho = e^{\frac{2\pi i}{3}}$

(c1) $\begin{pmatrix} 1 & 0 & \alpha & 0 \\ 0 & 1 & 0 & i \end{pmatrix}$ (c2) $\begin{pmatrix} 1 & 0 & \alpha & 0 \\ 0 & 1 & \frac{1+i}{2} & i \end{pmatrix}$

$g_5(u, z) = \left(u + \frac{1}{4}, iz\right)$

(d1) $\begin{pmatrix} 1 & 0 & \alpha & 0 \\ 0 & 1 & 0 & \rho \end{pmatrix}$

$g_5(u, z) = \left(u + \frac{1}{6}, -\rho z\right)$.

We say that S is *of the first type* if S is of type (a1), (b1), (c1) or (d1) and S is *of the second type* otherwise.

For the sake of simplicity, we shall use the following notations:

$$\beta = \begin{cases} \text{arbitrary} & (a1),\ (a2) \\ \rho & (b1),\ (b2),\ (d1) \\ i & (c1),\ (c2) \end{cases} \qquad d = \begin{cases} 1/2 & (a2) \\ (1-\rho)/3 & (b2) \\ (1+i)/2 & (c2) \\ 0 & \text{for the other cases} \end{cases}$$

$$\xi = \begin{cases} -1 & (a1),\ (a2) \\ \rho & (b1),\ (b2) \\ i & (c1),\ (c2) \\ -\rho & (d1) \end{cases} \qquad c = \begin{cases} 1/2 & (a1),\ (a2) \\ 1/3 & (b1),\ (b2) \\ 1/4 & (c1),\ (c2) \\ 1/6 & (d1) \end{cases}$$

$l = 1/c$.

So, S is the quotient of $\mathbf{C}^2$ by a group G of holomorphic automorphisms of $\mathbf{C}^2$ generated by g_i, $i = \overline{1, 5}$, where $g_1(u, z) = (u+1, z)$, $g_2(u, z) = (u, z+1)$, $g_3(u, z) = (u+\alpha, z+d)$, $g_4(u, z) = (u, z+\beta)$ and $g_5(u, z) = (u+c, \xi z)$.

For the next elementary result, see [14]:

Lemma 1.3. *The relations between generators are:*
g_1, g_2, g_3 and g_4 *commute to each other*, $g_5{}^l = g_1$ *and*

$$(a1)\quad \begin{aligned} g_2g_5 &= g_5g_2^{-1} \\ g_3g_5 &= g_5g_3 \\ g_4g_5 &= g_5g_4^{-1} \end{aligned} \qquad (a2)\quad \begin{aligned} g_2g_5 &= g_5g_2^{-1} \\ g_3g_5 &= g_5g_3g_2^{-1} \\ g_4g_5 &= g_5g_4^{-1} \end{aligned}$$

$$(b1)\quad \begin{aligned} g_2g_5 &= g_5g_2^{-1}g_4^{-1} \\ g_3g_5 &= g_5g_3 \\ g_4g_5 &= g_5g_2 \end{aligned} \qquad (b2)\quad \begin{aligned} g_2g_5 &= g_5g_2^{-1}g_4^{-1} \\ g_3g_5 &= g_5g_3g_2^{-1} \\ g_4g_5 &= g_5g_2 \end{aligned}$$

$$(c1)\quad \begin{aligned} g_2g_5 &= g_5g_4^{-1} \\ g_3g_5 &= g_5g_3 \\ g_4g_5 &= g_5g_2 \end{aligned} \qquad (c2)\quad \begin{aligned} g_2g_5 &= g_5g_4^{-1} \\ g_3g_5 &= g_5g_3g_4^{-1} \\ g_4g_5 &= g_5g_2 \end{aligned}$$

$$(d1)\quad \begin{aligned} g_2g_5 &= g_5g_2g_4 \\ g_3g_5 &= g_5g_3 \\ g_4g_5 &= g_5g_2^{-1} \end{aligned}$$

From the lemma above, one may see that any element $g \in G$ has a unique expression as a product $g = g_2^{l_2}g_4^{l_4}g_3^{l_3}g_5^{l_5}$. The action of a such g on $\mathbf{C}^2$ is given by

$$g(u, z) = (u + l_3\alpha + l_5c,\ \xi^{l_5}z + l_2 + l_4\beta + l_3d).$$

Another way of representing the hyperelliptic surface S is as follows. Let $\Gamma = \mathbf{Z} + \mathbf{Z}\beta$, $\Lambda = \mathbf{Z}\alpha + \mathbf{Z}c$, $\Lambda_1 = \mathbf{Z}\alpha + \mathbf{Z}$ and

$$\Lambda_2 = \begin{cases} 2\mathbf{Z}\alpha + \mathbf{Z} & (a2),\ (c2) \\ 3\mathbf{Z}\alpha + \mathbf{Z} & (b2) \\ \mathbf{Z}\alpha + \mathbf{Z} = \Lambda_1 & \text{otherwise} \end{cases}$$

Let $\Delta = \mathbf{C}/\Lambda_2$ and $E = \mathbf{C}/\Gamma$. Then S can be expressed as $S = (\Delta \times E)/\mathcal{G}$ where $\mathcal{G}$ is a finite translations group of Δ, acting on E not by translations only, given by the Bagnera-deFranchis table (see for example [1], [2], [10]).

Moreover, $\Delta/\mathcal{G} \cong B$, $E/\mathcal{G} \cong \mathbf{P}^1$ and S has two fibrations: first of them is $S \to B$ from the definition 1.1, with fiber E, and the other one is $S \to \mathbf{P}^1$ with generic fiber Δ. Since Λ is the lattice of B, the short exact sequence of homotopy groups of the first fibration leads us to the following extension:

$$0 \longrightarrow \Gamma \xrightarrow{\ j\ } G \xrightarrow{\ \pi\ } \Lambda \longrightarrow 0$$

where $j(\gamma) = g_2^{l_2}g_4^{l_4}$ and $\pi(g) = l_3\alpha + l_5c$.

Choosing as a cross-section of π the map s: $\Lambda \to G$, $s(\lambda) = g_3^{l_3}g_5^{l_5}$ for $\lambda = \alpha l_3 + cl_5 \in \Lambda$, we see that if S is of the first type, then s is a morphism of groups.

Next, we identify an element $\gamma \in \Gamma$ with $j(\gamma) \in G$ and $\lambda \in \Lambda$ with $s(\lambda) \in G$. In other words, we make no distinctions between $\gamma = l_2 + l_4\beta$ and $g_2^{l_2}g_4^{l_4}$ or between $\lambda = l_3\alpha + l_5 c$ and $g_3^{l_3}g_5^{l_5}$. So $\lambda\lambda'$ is the same as $s(\lambda)s(\lambda')$ and by $\lambda + \lambda'$ we mean $s(\lambda + \lambda')$. This convention simplifies our formulae and produces no ambiguity.

The natural action of an element $\lambda \in \Lambda$ on Γ is given by $\lambda\gamma\lambda^{-1} = \xi^{l_5}\gamma$. If we write $\lambda\lambda' = h(\lambda, \lambda')(\lambda + \lambda')$, then $h(\lambda, \lambda') = (\xi^{l_5} - 1)l_3'd$.

Next, let us point out the following useful lemma

Lemma 1.4. *Let* $v \in \mathrm{Hom}(G, \mathbf{C}^*)$. *Then*

(a1) $v(g_2) = \pm 1$, $v(g_4) = \pm 1$; (a2) $v(g_2) = 1$, $v(g_4) = \pm 1$;

(b1) $v(g_2) = v(g_4)$, $v(g_2)^3 = 1$; (b2) $v(g_2) = 1$, $v(g_4) = 1$;

(c1) $v(g_2) = v(g_4)$, $v(g_2) = \pm 1$; (c2) $v(g_2) = 1$, $v(g_4) = 1$;

(d1) $v(g_2) = 1$, $v(g_4) = 1$

2. The group $\mathrm{Pic}^\tau(S)$

The vanishing of the cohomology groups $H^i(\mathbf{C}^2, \mathbf{Z})$, $H^i(\mathbf{C}^2, \mathbf{C})$, $H^i(\mathbf{C}^2, \mathcal{O}_{\mathbf{C}^2})$, $H^i(\mathbf{C}^2, \mathcal{O}^*_{\mathbf{C}^2})$, $H^i(\mathbf{C}^2, \mathbf{C}^*)$ for all $i \geq 1$ yields to the natural isomorphisms (see [9]):
$H^i(S, \mathbf{Z}) \cong H^i(G, \mathbf{Z})$, $H^i(S, \mathbf{C}) \cong H^i(G, \mathbf{C})$, $H^i(S, \mathbf{C}^*) \cong H^i(G, \mathbf{C}^*)$, $H^i(S, \mathcal{O}_S) \cong H^i(G, H)$, $H^i(S, \mathcal{O}^*_S) \cong H^i(G, H^*)$, where $H^* = H^0(\mathbf{C}^2, \mathcal{O}^*_{\mathbf{C}^2})$.

The exponential sequence

$$0 \longrightarrow \mathbf{Z} \longrightarrow \mathcal{O}_S \xrightarrow{exp} \mathcal{O}^*_S \longrightarrow 0$$

gives rise to the cohomology sequence

$$\ldots \longrightarrow H^1(S, \mathcal{O}_S) \longrightarrow \mathrm{Pic}(S) \xrightarrow{c_1} H^2(S, \mathbf{Z}) \longrightarrow 0.$$

Recall that the universal coefficients theorem leads us to

Lemma 2.1. Tors $H^2(S, \mathbf{Z}) \cong \mathrm{Ker}(i\colon H^2(S, \mathbf{Z}) \to H^2(S, \mathbf{C}))$.

For any $L \in \mathrm{Pic}(S)$, $c_1(L)$ denotes the Chern class of L and $\mathrm{Pic}^0(S) = \mathrm{Ker}(c_1)$. The subgroup $\mathrm{Pic}^\tau(S) \subset \mathrm{Pic}(S)$ (see [3]) is defined as $\mathrm{Ker}(ic_1)$ (where $i\colon H^2(S, \mathbf{Z}) \to H^2(S, \mathbf{C})$ is the canonical homomorphism) and this is the group of the elements $L \in \mathrm{Pic}(S)$ such that $c_1(L)$ is a torsion element in $H^2(S, \mathbf{Z})$ (as we saw in Lemma 2.1.). Then $\mathrm{Pic}^\tau(S) = \zeta(H^1(S, \mathbf{C}^*))$ where ζ is the natural morphism $H^1(S, \mathbf{C}^*) \to H^i(S, \mathcal{O}^*_S)$ (see [3]).

Let us compute next $\mathrm{Ker}(\zeta)$, by using the isomorphisms from the beginning of this section. So, $v \in \mathrm{Ker}(\zeta)$ if and only if there is $h \in H^*$ such that

$$(1) \qquad h(g(u, z)) = v(g)h(u, z), \text{ for all } g \in G, \ (u, z) \in \mathbf{C}^2.$$

By taking the logarithmic derivatives $\omega_1 = h'_u/h$ and $\omega_2 = h'_z/h$ (in order to eliminate v from (1)), these functions verify the following relations:

$$\begin{aligned}
(2) \qquad & \omega_i(u, z) = \omega_i(u+1, z), \\
& \omega_i(u, z) = \omega_i(u, z+1), \\
& \omega_i(u, z) = \omega_i(u, z+\beta), \\
& \omega_i(u, z) = \omega_i(u+\alpha, z+d), \ i = 1, 2 \\
(3) \qquad & \omega_1(u, z) = \omega_1(u+c, \xi z) \\
(4) \qquad & \omega_2(u, z) = \xi\omega_2(u+c, \xi z)
\end{aligned}$$

for all $(u, z) \in \mathbf{C}^2$.

From (2), if we take $K \subset \mathbf{C}^2$ a compact set with $K + (\Gamma \times \Lambda) = \mathbf{C}^2$ and apply the maximum principle, we deduce that ω_i are contants.

From (4) it follows that $\omega_2 = 0$, so h does not depend on z. This means that there is a holomorphic function $\tilde{h}$ on $\mathbf{C}$ such that $h(u, z) = \tilde{h}(u)$, for all u, $z \in \mathbf{C}$. Moreover, since $\tilde{h}'/\tilde{h}$ is constant, we get $h(u, z) = e^{2\pi i(au+b)}$ with $(a, b) \in \mathbf{C}^2$. Then, by denoting $v_i = v(g_i)$, we have $v_2 = 1$, $v_4 = 1$, $v_3 = e^{2\pi i a\alpha}$, $v_5 = e^{2\pi iac}$, where $a \in \mathbf{C}$.

Then we proved the following:

Lemma 2.2. $\operatorname{Ker}(\zeta) = \{v \in \operatorname{Hom}(G, \mathbf{C}^*) : v(g) = e^{2\pi i a\lambda}, g = \gamma\lambda \in G, a \in \mathbf{C}\}$.

Next, we try to describe $\operatorname{Pic}^\tau(S) \cong \operatorname{Hom}(G, \mathbf{C}^*)/\operatorname{Ker}(\zeta)$.

Let $v \in \operatorname{Hom}(G, \mathbf{C}^*)$. If S is of the first type, s is a morphism, so $v(\lambda\lambda') = v(\lambda + \lambda')$.

Otherwise, we know that $\lambda\lambda' = h(\lambda, \lambda')(\lambda+\lambda')$ where $h(\lambda, \lambda') = (\xi^{l_5} - 1) l'_3 d \in \Gamma$. But, if S is of type $(a2)$, then $h(\lambda, \lambda')$ depends only on g_2 and, by taking into account Lemma 1.4., it follows that $v(h(\lambda, \lambda')) = 1$. If S is of type $(b2)$ or $(c2)$, then again from Lemma 1.4. we have $v(h(\lambda, \lambda')) = 1$.

In any case we obtained $v(\lambda\lambda') = v(\lambda + \lambda')$.

Now, we write $v(\lambda) = e^{2\pi i r(\lambda)}$. Since $r(\lambda) + r(\lambda') - r(\lambda + \lambda') \in \mathbf{Z}$, for all $\lambda, \lambda' \in \Lambda$, $\varphi := \operatorname{Im} r$ must be $\mathbf{Z}$-linear. Then φ has a unique $\mathbf{R}$-linear extension $\tilde{\varphi}: \mathbf{C} \to \mathbf{R}$. We define $k: \mathbf{C} \to \mathbf{C}$, $k(u) = \tilde{\varphi}(iz) + i\tilde{\varphi}(z)$ which is $\mathbf{C}$-linear and $\tilde{r} := i\tilde{\varphi} - k$ is real-valued.

The function k being $\mathbf{C}$-linear, there exists $a \in \mathbf{C}$ such that $k(u) = au$, for all $u \in \mathbf{C}$ and we take $v_0 \in \operatorname{Ker}(\zeta)$, $v_0(g) = e^{2\pi i a\lambda}$. Then $\alpha_G := v/v_0$ has the property that $\alpha_G(\lambda) \in U(1)$, for any $\lambda \in \Lambda$ and it is uniquely determined by this property in the class of v in $\operatorname{Hom}(G, \mathbf{C}^*)/\operatorname{Ker}(\zeta)$.

Then we have

$$\operatorname{Pic}^\tau(S) \cong \{\alpha_G \in \operatorname{Hom}(G, \mathbf{C}^*), \ \alpha_G(\lambda) \in U(1), \text{ for all } \lambda \in \Lambda\}.$$

Moreover, $\alpha_G(\gamma) \in U(1)$, for all $\alpha_G \in \operatorname{Hom}(G, \mathbf{C}^*)$, so we got

Proposition 2.3. *There is a canonical isomorphism*

$$\Psi^{\tau}\colon \operatorname{Hom}(G, U(1)) \xrightarrow{\sim} \operatorname{Pic}^{\tau}(S).$$

3. The group $\operatorname{Num}(S)$

In this section we shall give a description of $\operatorname{Num}(S)$ in terms of hermitian forms related to Λ_1 and Γ. It is well-known (see, for example [10]) that $\operatorname{Num}(S) \cong H^2(S, \mathbf{Z})/\operatorname{Tors} H^2(S, \mathbf{Z})$ and, as we saw in section 2, the cohomology of S is computed by cohomology of groups.

The inclusion $j\colon \Gamma \to G$ induces a morphism of restriction $\operatorname{res}_\Gamma\colon H^2(G, \mathbf{Z}) \to H^2(\Gamma, \mathbf{Z})$.

The map $s|_{\Lambda_1}\colon \Lambda_1 \to G$ is a group homomorphism, so it induces another morphism of restriction $\operatorname{res}_{\Lambda_1}\colon H^2(G, \mathbf{Z}) \to H^2(\Lambda_1, \mathbf{Z})$.

According to [9], Chapter I, Appendix, we have classical isomorphisms

(5) $$H^2(\Gamma, \mathbf{Z}) \cong \{H_\Gamma\colon \mathbf{C}^2 \to \mathbf{C} \text{ hermitian, } \operatorname{Im} H_\Gamma(\Gamma \times \Gamma) \subset \mathbf{Z}\},$$
(6) $$H^2(\Lambda, \mathbf{Z}) \cong \{H_\Lambda\colon \mathbf{C}^2 \to \mathbf{C} \text{ hermitian, } \operatorname{Im} H_\Lambda(\Lambda_1 \times \Lambda_1) \subset \mathbf{Z}\}.$$

Let us explain the morphisms $\operatorname{res}_\Gamma$ and $\operatorname{res}_{\Lambda_1}$ (cf. [9], Chapter I) passing through the above isomorphisms.

Starting with $F \in H^2(G, \mathbf{Z})$, we construct $A_\Gamma F\colon \Gamma \times \Gamma \to \mathbf{C}$, $A_\Gamma F(\gamma, \gamma') = F(\gamma', \gamma) - F(\gamma, \gamma')$, bilinear and antisymmetric which can be extended to $E_\Gamma\colon \mathbf{C}^2 \to \mathbf{C}$, $\mathbf{R}$-bilinear and antisymmetric verifying $E_\Gamma(ix, iy) = E(x, y)$ for any $x, y \in \mathbf{C}$. Then $H_\Gamma\colon \mathbf{C}^2 \to \mathbf{C}$ defined by $H_\Gamma(x, y) := E_\Gamma(ix, y) + iE_\Gamma(x, y)$ is a hermitian form on $\mathbf{C}^2$ with $\operatorname{Im} H_\Gamma = E_\Gamma$ and H_Γ will be $\operatorname{res}_\Gamma F$ modulo canonical isomorphism (5).

By applying the same argument for Λ_1, $\operatorname{res}_\Gamma$ and $\operatorname{res}_{\Lambda_1}$ will induce a morphism

$$\chi\colon H^2(G, \mathbf{Z}) \to \mathcal{N}_1$$

where

$$\mathcal{N}_1 := \{(H_\Gamma, H_\Lambda),\ H_\Gamma, H_\Lambda \text{ hermitian forms on } \mathbf{C}^2 \text{ with } \operatorname{Im} H_\Gamma(\Gamma \times \Gamma) \subset \mathbf{Z},\ \operatorname{Im} H_\Lambda(\Lambda_1 \times \Lambda_1) \subset \mathbf{Z}\}.$$

We denote by

$$\mathcal{N}S := \begin{cases} \{(H_\Gamma, H_\Lambda) \in \mathcal{N}_1,\ H_\Gamma(1,1)\operatorname{Im}\beta \in 2\mathbf{Z}\}, & \text{type } (a2) \\ \{(H_\Gamma, H_\Lambda) \in \mathcal{N}_1,\ H_\Gamma(1,1)\operatorname{Im}\rho \in 3\mathbf{Z}\}, & \text{type } (b2) \\ \{(H_\Gamma, H_\Lambda) \in \mathcal{N}_1,\ H_\Gamma(1,1) \in 2\mathbf{Z},\ 2\operatorname{Im} H_\Lambda(\Lambda \times \Lambda) \subset \mathbf{Z}\}, & \text{type } (c2) \\ \{(H_\Gamma, H_\Lambda) \in \mathcal{N}_1,\ \operatorname{Im} H_\Lambda(\Lambda \times \Lambda) \subset \mathbf{Z}\}, & \text{otherwise.} \end{cases}$$

Now, we can state the main theorem of this section:

Theorem 3.1. *χ induces an isomorphism* $\tilde{\chi}\colon \operatorname{Num}(S) \xrightarrow{\sim} \mathcal{N}S$.

Proof. Because $\mathcal{N}_1$ has no torsion it follows that Tors $H^2(G, \mathbf{Z}) \subset \mathrm{Ker}(\chi)$. So it remains to prove that $\mathrm{Ker}(\chi) \subset$ Tors $H^2(G, \mathbf{Z})$ and $\chi(H^2(G, \mathbf{Z})) = \mathcal{NS}$.

Let F be a normalized cocycle in $H^2(G, \mathbf{Z})$. Then F is the Chern class of a line bundle. If we represent this line bundle as a cocycle $\{e_g\}_g \in H^1(G, H^*)$ then, by standard diagram chasing, we get

(7)
$$F(g, g') = f_g(g'(u, z)) - f_{gg'}(u, z) + f_{g'}(u, z) \in \mathbf{Z}, \text{ for all } u, z \in \mathbf{C}, g, g' \in G,$$

where f_g: $\mathbf{C}^2 \to \mathbf{C}$ is a holomorphic function with $e^{2\pi i f_g} = e_g$ for any $g \in G$ (see, for example [3], [9]).

Now, we divide the proof into two cases corresponding to the two different kinds of hyperelliptic surfaces.

Case 1. S is of the first type.

Let us notice that, in this case, s is a morphism and, by denoting res_Λ the corresponding map from $H^2(G, \mathbf{Z})$ to $H^2(\Lambda, \mathbf{Z})$ we have the following commutative diagram, coming from the inclusion $\Lambda_1 \subset \Lambda$.

$$\begin{array}{ccc} H^2(\Lambda, \mathbf{Z}) & \hookrightarrow & H(\Lambda_1, \mathbf{Z}) \\ \mathrm{res}_\Lambda \nwarrow & & \nearrow \mathrm{res}_{\Lambda_1} \\ & H^2(G, \mathbf{Z}) & \end{array}$$

Then it is obvious that $\chi(H^2(G, \mathbf{Z})) \subset \mathcal{NS}$.

Step 1. Our next goal is to find f_g and thus to get a nice form of (7).

Since the restricition of F to Γ and Λ are 2-cocycles, it follows (see [9], Chapter I) that

(8)
$$f_\gamma(u, z) = \frac{1}{2i} H_\Gamma(z, \gamma) + \beta_\Gamma(u, \gamma), \text{ for all } \gamma \in \Gamma,$$

(9)
$$f_\lambda(u, z) = \frac{1}{2i} H_\Lambda(u, \lambda) + \beta_\Lambda(z, \lambda), \text{ for all } \lambda \in \Lambda,$$

where $\beta_\Gamma(\,.\,, \gamma)$, $\beta_\Lambda(\,.\,, \lambda)$ are holomorphic functions on $\mathbf{C}$.

Next, we write $\equiv$ for congruence modulo $\mathbf{Z}$. From (7) it follows that, for any $g = \gamma\lambda \in G$, we have

(10)
$$f_\gamma(\lambda(u, z)) - f_g(u, z) + f_\lambda(u, z) \equiv 0,$$

so

(11)
$$f_g(u, z) \equiv \frac{1}{2i} H_\Gamma(\xi^{l_5} z, \gamma) + \frac{1}{2i} H_\Lambda(u, \lambda) + \beta_\Gamma(u + \lambda, \gamma) + \beta_\Lambda(z, \lambda),$$

The relation (7) can be read as

$$f_{gg'}(u, z) \equiv f_g(g'(u, z)) + f_{g'}(u, z), \; g, g' \in G.$$

By replacing f_g from (7) in the above formula, we have

(12)
$$\beta_\Gamma(u + \lambda + \lambda', \gamma + \xi^{l_5}\gamma') + \beta_\Lambda(z, \lambda + \lambda') \equiv$$

$$\frac{1}{2i}H_\Gamma(\xi^{l_5}\gamma',\gamma)+\frac{1}{2i}H_\Lambda(\lambda',\lambda)+\beta_\Gamma(u+\lambda+\lambda',\gamma)$$
$$+\beta_\Gamma(u+\lambda',\gamma')+\beta_\Lambda(\xi^{l_5}z+\gamma',\lambda)+\beta_\Lambda(z,\lambda').$$

Let us denote by $\varepsilon_\Gamma(.,\gamma)$ and $\varepsilon_\Lambda(.,\gamma)$ the derivatives of $\beta_\Gamma(.,\gamma)$ and $\beta_\Lambda(.,\lambda)$ respectively. Then, from (12) we obtain:

(13) $$\varepsilon_\Gamma(u+\lambda+\lambda',\gamma+\xi^{l_5}\gamma')=\varepsilon_\Gamma(u+\lambda+\lambda',\gamma)+\varepsilon_\Gamma(u+\lambda',\gamma')$$

(14) $$\varepsilon_\Lambda(z,\lambda+\lambda')=\xi^{l_5}\varepsilon_\Lambda(\xi^{l_5}z+\gamma',\lambda)+\varepsilon_\Lambda(z,\lambda')$$

and from these relations we can describe β_Γ and β_Λ.

Firstly, we determine β_Γ.

In (13), we choose $\lambda=\lambda'=0$ and we get

(15) $$\varepsilon_\Gamma(u,\gamma+\gamma')=\varepsilon_\Gamma(u,\gamma)+\varepsilon_\Gamma(u,\gamma') \text{ for all } \gamma,\gamma'\in\Gamma,$$

which means that $\varepsilon_\Gamma(u,.)$: $\Gamma\to\mathbf{C}$ is a morphism of groups.

In (13) we choose $\lambda'=0$ and it follows

(16) $$\varepsilon_\Gamma(u+\lambda,\gamma+\xi^{l_5}\gamma')=\varepsilon_\Gamma(u+\lambda,\gamma)+\varepsilon_\Gamma(u,\gamma').$$

From (15) and (16) we deduce that

(17) $$\varepsilon_\Gamma(u+\lambda,\xi^{l_5}\gamma')=\varepsilon_\Gamma(u,\gamma').$$

We choose $\lambda\in\Lambda_1$ in (17), so $\varepsilon_\Gamma(u+\lambda,\gamma')=\varepsilon_\Gamma(u,\gamma')$, for all $\lambda\in\Lambda_1$, $\gamma'\in\Gamma$, $u\in$ C. By standard arguments, we can prove that $\varepsilon_\Gamma(.,\gamma')$ is a constant function, so we write $\varepsilon_\Gamma(\gamma)$ instead of $\varepsilon_\Gamma(u,\gamma)$. On the other side, if we apply (15) and (17) again, ε_Γ must be identically equal to zero and β_Γ does not depend on u. Then we write $\beta_\Gamma(\gamma)$ instead of $\beta_\Gamma(u,\gamma)$.

Next, we determine β_Λ. We choose $\lambda=\lambda'=0$ and $\gamma'=0$ in (14) so $\varepsilon_\Lambda(z,0)=0$, for all $z\in\mathbf{C}$. We apply these relations to (14) for $\lambda'=0$ and we obtain

$$\varepsilon_\Lambda(z,\lambda)=\varepsilon_\Lambda(z+\gamma',\lambda), \text{ for all } \lambda\in\Lambda,\ \gamma'\in\Gamma.$$

For the same reason as above, ε_Λ does not depend on u and hence we write $\varepsilon_\Lambda(\lambda)$ instead of $\varepsilon_\Lambda(z,\lambda)$. With this notation, we turn back to (14) which becomes

(18) $$\varepsilon_\Lambda(\lambda+\lambda')=\xi^{l_5}\varepsilon_\Lambda(\lambda)+\varepsilon_\Lambda(\lambda').$$

An easy computation in (18) will show that $\varepsilon_\Lambda(\lambda)=\frac{1-\xi^{l_5}}{1-\xi}\varepsilon_\Lambda(c)$ and $\beta_\Lambda(z,\lambda)=\frac{1-\xi^{l_5}}{1-\xi}\varepsilon_\Lambda(c)z+\beta_\Lambda(\lambda)$.

Then we get

$$(19)\qquad f_g(u,z)=\frac{1}{2i}H_\Gamma(\xi^{l_5}z,\gamma)+\frac{1}{2i}H_\Lambda(u,\lambda)+\beta_\Gamma(\gamma)$$
$$+\frac{1-\xi^{l_5}}{1-\xi}\varepsilon_\Lambda(c)z+\beta_\Lambda(\gamma)+\text{const}(g),\ \text{for all } g\in G,$$

where const$(g)\in\mathbf{Z}$, for all $g\in G$, and (7) becomes

$$(20)\quad F(g,g')=\frac{1}{2i}H_\Lambda(\lambda',\lambda)+\frac{\xi^{l_5}}{2i}H_\Gamma(\gamma',\gamma)+\beta_\Lambda(\lambda)+\beta_\Lambda(\lambda')-\beta_\Lambda(\lambda+\lambda')$$
$$+\beta_\Gamma(\gamma)+\beta_\Gamma(\gamma')-\beta_\Gamma(\gamma+\xi^{l_5}\gamma')+\frac{1-\xi^{l_5}}{1-\xi}\varepsilon_\Lambda(c)\gamma'$$
$$+\text{const}(g)+\text{const}(g')-\text{const}(gg'),\ \text{for all } g,g'\in G$$

Since const (g) + const (g') − const (gg') is a coboundary in $C^2(G,\mathbf{Z})$, we can ignore this term, without changing the cohomology class of F in $H^2(G,\mathbf{Z})$.

Let $r(g):=\beta_\Lambda(\lambda)+\beta_\Gamma(\gamma)+\frac{1}{1-\xi}\varepsilon_\Lambda(c)\gamma$, $r_\Gamma(\gamma):=r(\gamma)=\beta_\Gamma(\gamma)+\frac{1}{1-\xi}\varepsilon_\Lambda(c)$ γ and $r_\Lambda(\lambda):=r(\lambda)=\beta_\Lambda(\lambda)$.

With this notations, (20) gives rise to the final formula for F

$$(21)\qquad F(g,g')=\frac{1}{2i}H_\Lambda(\lambda',\lambda)+\frac{\xi^{l_5}}{2i}H_\Gamma(\gamma',\gamma)+r(g)+r(g')-r(gg')\in\mathbf{Z}.$$

and thus, if we replace β_Γ by r_Γ, we may always suppose that $\varepsilon_\Lambda(c)=0$.

From (21), one may see that if $H_\Gamma=0$ and $H_\Gamma=0$, then $F(g,g')=r(g)+r(g')-r(gg')$, which means that the cohomology class of F in $H^2(G,\mathbf{C})$ equals to zero. Then, by means of Lemma 2.1., F represents a torsion class in $H^2(G,\mathbf{Z})$. Thus we proved that Ker$(\chi)\subset$Tors $H^2(G,\mathbf{Z})$.

Step 2. It remains to prove that $\mathcal{NS}\subset\chi(H^2(G,\mathbf{Z}))$.

We check that for given $(H_\Gamma,H_\Lambda)\in\mathcal{NS}$, there exist $r_\Gamma:\Gamma\to\mathbf{C}$ and $r_\Lambda:\Lambda\to\mathbf{C}$ such that, by defining $r(g)=r_\Gamma(\gamma)+r_\Lambda(\lambda)$, for any $g=\gamma\lambda$ then

$$(22)\qquad \frac{1}{2i}H(\lambda',\lambda)+\frac{\xi^{l_5}}{2i}H_\Gamma(\gamma',\gamma)+r(g)+r(g')-r(gg')\in\mathbf{Z}.$$

Let us set

$$b_\Gamma(\gamma)=ir_\Gamma(\gamma)-\frac{1}{4}H_\Gamma(\gamma,\gamma),\ \text{for all } \gamma\in\Gamma,$$
$$b_\Lambda(\lambda)=ir_\Lambda(\lambda)-\frac{1}{4}H_\Lambda(\lambda,\lambda),\ \text{for all } \lambda\in\Lambda.$$

One may see that (22) is equivalent to the following three relations:

$$(23)\qquad b_\Gamma(\xi\gamma)-b_\Gamma(\gamma)\in i\mathbf{Z},$$

$$(24)\qquad b_\Gamma(\gamma)+b_\Gamma(\gamma')-b_\Gamma(\gamma+\gamma')+\frac{1}{2}iE_\Gamma(\gamma,\gamma')\in i\mathbf{Z},\ \text{for all } \gamma,\gamma'\in\Gamma$$

$$(25)\qquad b_\Lambda(\lambda)+b_\Lambda(\lambda')-b_\Lambda(\lambda+\lambda')+\frac{1}{2}iE_\Lambda(\lambda,\lambda')\in i\mathbf{Z},\ \text{for all } \lambda,\lambda'\in\Lambda.$$

Then, the problem of finding r_Γ and r_Λ such that (22) is true reduces to searching for b_Γ and b_Λ which satisfy (23), (24) and (25).

By using (24), a straighforward computation shows that (23) is equivalent to

$$(26)\qquad \begin{array}{ll} S \text{ of type } (a1) & 2b_\Gamma(1),\ 2b_\Gamma(\beta) \in i\mathbf{Z}, \\ S \text{ of type } (b1) & b_\Gamma(1) - b_\Gamma(\rho) \in i\mathbf{Z},\ 3b_\Gamma(1) - \frac{i\sqrt{3}}{4}H_\Gamma(1, 1) \in i\mathbf{Z}, \\ S \text{ of type } (c1) & 2b_\Gamma(1) \in i\mathbf{Z},\ b_\Gamma(1) - b_\Gamma(i) \in i\mathbf{Z}, \\ S \text{ of type } (d1) & b_\Gamma(1) + b_\Gamma(\rho) \in i\mathbf{Z},\ b_\Gamma(1) + \frac{i\sqrt{3}}{4}H_\Gamma(1, 1) \in i\mathbf{Z}. \end{array}$$

If we fix $b_\Lambda(c)$, $b_\Lambda(\alpha)$, $b_\Gamma(1)$ and $b_\Gamma(\beta) \in \mathbf{C}$ such that (26) is verified and we set:

$$b_\Gamma(\gamma) := l_2 b_\Gamma(1) + l_4 b_\Gamma(\beta) + \frac{1}{2} i l_2 l_4 E_\Gamma(1, \beta), \text{ for all } \gamma = l_2 + l_4\beta,$$

$$b_\Lambda(\lambda) := l_3 b_\Gamma(\alpha) + l_5 b_\Lambda(c) + \frac{1}{2} i l_3 l_5 E_\Lambda(c, \alpha), \text{ for all } \lambda = l_3\alpha + l_5 c,$$

then it is obvious that b_Γ and b_Λ are the functions we were looking for.

Case 2. S is of the second type.

The proof is similar to the proof of *Case 1.*, but it needs more computations.

As in the previous case, we try to find a decent form of f_g.

Since the restriction of F to Γ and Λ_1 are cocycles, then we must have, as in the first case

$$(27)\qquad f_\gamma(u, z) = \frac{1}{2i} H_\Gamma(z, \gamma) + \beta_\Gamma(u, \gamma), \text{ for all } \gamma \in \Gamma,$$

$$(28)\qquad f_{\lambda_1}(u, z) = \frac{1}{2i} H_\Lambda(u, \lambda_1) + \beta_\Lambda(z, \lambda_1), \text{ for all } \lambda_1 \in \Lambda_1,$$

where $\beta_\Gamma(\,.\,, \gamma)$, $\beta_\Lambda(\,.\,, \lambda_1)$ are holomorphic functions on $\mathbf{C}$. Let us denote by $\varepsilon_\Gamma(\,.\,, \gamma)$, $\varepsilon_\Lambda(\,.\,, \lambda_1)$ the derivatives of $\beta_\Gamma(\,.\,, \gamma)$ and $\beta_\Lambda(\,.\,, \lambda_1)$ respectively.

Step 1. We show that $\varepsilon_\Gamma(\,.\,, .\,)$ and $\varepsilon_\Lambda(\,.\,, .\,)$ are constants in their first variable and group homomorphism to $\mathbf{C}$ in their second variable.

For $g = \gamma\lambda \in G$ with $\lambda \in \Lambda_1$, then g is also equal to $\lambda\gamma$ and we apply (7) two times

$$f_g(u, z) \equiv f_\gamma(\lambda(u, z)) + f_\lambda(u, z) \equiv f_\lambda(\gamma(u, z)) + f_\gamma(u, z)$$

to get the following:

$$(29)\qquad \frac{1}{2i} H_\Gamma(l_3 d, \gamma) + \beta_\Gamma(u + \lambda, \gamma) + \beta_\Lambda(z, \lambda) \equiv \beta_\Gamma(u, \gamma) + \beta_\Lambda(z + \gamma, \lambda),\ \lambda \in \Lambda_1.$$

By taking the derivatives with respect to u and z respectively in (29) it follows that $\varepsilon_\Gamma(u+\lambda, \gamma)=\varepsilon_\Gamma(u, \gamma)$ and $\varepsilon_\Lambda(z+\gamma, \gamma)=\varepsilon_\Lambda(z, \lambda)$, for all $\gamma\in\Gamma$, $\lambda\in\Lambda_1$, $u, z\in\mathbf{C}$ and thus ε_Γ and ε_Λ are constant in their first variable.

Then we write $\varepsilon_\Gamma(\gamma)$ instead of $\varepsilon_\Gamma(u, \gamma)$ and $\varepsilon_\Lambda(\lambda)$ instead of $\varepsilon_\Lambda(z, \lambda)$ and by denoting $\beta_\Gamma(\gamma)=\beta_\Gamma(0, \gamma)$ and $\beta_\Lambda(\lambda)=\beta_\Lambda(0, \lambda)$, we deduce that

$$\beta_\Gamma(u, \gamma)=\varepsilon_\Gamma(\gamma)u+\beta_\Gamma(\gamma) \tag{30}$$

$$\beta_\Lambda(z, \lambda)=\varepsilon_\Lambda(\lambda)z+\beta_\Lambda(\lambda). \tag{31}$$

Next, we turn back to (7) and we choose $g, g'\in G$, $g=\gamma\lambda$, $g'=\gamma'\lambda'$ with $\lambda, \lambda'\in\Lambda_1$. Then we obtain

$$\begin{aligned}&\frac{1}{2i}H_\Gamma(l_3d, \gamma')+\varepsilon_\Gamma(\gamma+\gamma')(u+\lambda+\lambda')+\varepsilon_\Lambda(\lambda+\lambda')z\\&+\beta_\Gamma(\gamma+\gamma')+\beta_\Lambda(\lambda+\lambda')\equiv\frac{1}{2i}H_\Gamma(\gamma', \gamma)+\frac{1}{2i}H_\Lambda(\lambda', \lambda)\\&+\varepsilon_\Gamma(\gamma)(u+\lambda+\lambda')+\varepsilon_\Gamma(\gamma')(u+\lambda')+\varepsilon_\Lambda(\lambda)(z+\gamma'+l'_3d)\\&+\varepsilon_\Lambda(\lambda')z+\beta_\Gamma(\gamma)+\beta_\Gamma(\gamma')+\beta_\Lambda(\lambda)+\beta_\Lambda(\lambda')\end{aligned} \tag{32}$$

Now, we take the derivatives with respect to u and z respectively in (32) and it follows that $\varepsilon_\Gamma\in\mathrm{Hom}(\Gamma, \mathbf{C})$ and $\varepsilon_\Lambda\in\mathrm{Hom}(\Lambda_1, \mathbf{C})$.

If we apply (30) and (31) in (29) we obtain the following relation:

$$\frac{1}{2i}H_\Gamma(l_3d, \gamma)-\varepsilon_\Lambda(\lambda)\gamma+\varepsilon_\Gamma(\gamma)\lambda\equiv 0, \text{ for all } \lambda\in\Lambda_1,\ \gamma\in\Gamma. \tag{33}$$

Step 2. We prove that β_Λ can be extended to β_Λ: $\mathbf{C}\times\Lambda\to\mathbf{C}$, also holomorphic in the first variable such that

$$f_\lambda(u, z)=\frac{1}{2i}H_\Lambda(u, \lambda)+\beta_\Lambda(z, \lambda), \text{ for all } \lambda\in\Lambda.$$

In fact, by taking into account (7) and (28), it is sufficient to prove this only for $\lambda=c$.

Let $\eta_\lambda=\dfrac{\partial f_\lambda}{\partial u}$, $\mu_\lambda=\dfrac{\partial^2 f_\lambda}{\partial u^2}$ and $\nu_\lambda=\dfrac{\partial^2 f_\lambda}{\partial u\partial z}$, for all $\lambda\in\Lambda$.

By using induction on m, one may apply (7) several times to prove that

$$f_{mc}\equiv\sum_{k=0}^{m-1}f_c(u+kc, \xi^k z), \text{ for all } m\in\mathbf{N}, \tag{34}$$

which implies

$$\eta_{mc}=\sum_{k=0}^{m-1}\eta_c(u+kc, \xi^k z), \tag{35}$$

$$\mu_{mc}=\sum_{k=0}^{m-1}\mu_c(u+kc, \xi^k z), \text{ for all } m\in\mathbf{N}. \tag{36}$$

In particular, for $mc=n\in\mathbf{N}$, we get

$$\sum_{k=0}^{m-1}\eta_c(u+kc,\ \xi^k z)=\frac{1}{2i}H_\Lambda(1,\ n), \tag{37}$$

$$\sum_{k=0}^{m-1}\mu_c(u+kc,\ \xi^k z)=0. \tag{38}$$

Our next goal is to prove that η_c is a constant and then, from (37), we deduce that this constant must be equal to $\frac{1}{2i}H_\Gamma(1,\ c)$ and this step will be finished.

We apply (7) for $l_3\alpha$, l_5c and then, for $\lambda=l_3\alpha+l_5c$, we have

$$\begin{aligned} f_\lambda(u,\ z) &\equiv f_{l_3\alpha}(u+l_5c,\ \xi^{l_5}z)+f_{l_5c}(u,\ z)\\ &\equiv f_{l_5c}(u+l_3\alpha,\ z+l_3d)+f_{l_3\alpha}(u,\ z). \end{aligned} \tag{39}$$

But $l_3\alpha\in\Lambda_1$ and, by meaning of (28) and (39) the following two formulae hold:

$$\eta_{l_5c}(u,\ z)=\eta_{l_5c}(u+l_3\alpha,\ z+l_3d), \tag{40}$$
$$\mu_{l_5c}(u,\ z)=\mu_{l_5c}(u+l_3\alpha,\ z+l_3d),\ \text{for all } l_3,\ l_5\in\mathbf{Z}. \tag{41}$$

We apply again (7) for l_5c and mc, where we choose m such that $mc=n\in\mathbf{Z}\subset\Lambda_1$. A similar argument as in (39) leads us to

$$\eta_{l_5c}(u,\ z)=\eta_{l_5c}(u+n,\ z), \tag{42}$$
$$\mu_{l_5c}(u,\ z)=\mu_{l_5c}(u+n,\ z),\ \text{for all } l_5,\ n\in\mathbf{Z}. \tag{43}$$

Applying (7) for γ, λ and $g=\gamma\lambda$, we obtain

$$f_g(u,\ z)\equiv\frac{1}{2i}H_\Gamma(\xi^{l_5}z+l_3d,\ \gamma)+\varepsilon_\Gamma(\gamma)\ (u+\lambda)+\beta_\Gamma(\gamma)+f_\lambda(u,\ z). \tag{44}$$

Again in (7), we take $g=\gamma\lambda$, $g'=\gamma'\lambda'$ with $l'_3=0$ (and this implies that $h(\lambda,\ \lambda')=0$) and $(l_5+l'_5)c\in\mathbf{Z}\subset\Lambda_1$ and use (44) and (28):

$$\begin{aligned} &\frac{1}{2i}H_\Gamma(z+l_3d,\ \gamma+\xi^{l_5}\gamma')+\frac{1}{2i}H_\Lambda(u,\ \lambda+\lambda')+\varepsilon_\Gamma(\gamma+\xi^{l_5}\gamma')\ (u+\lambda+\lambda')\\ &+\beta_\Gamma(\gamma+\xi^{l_5}\gamma')+\beta_\Lambda(z,\ \lambda+\lambda')\equiv\frac{1}{2i}H_\Gamma(z+\xi^{l_5},\ \gamma',\ \gamma)+\frac{1}{2i}H_\Gamma(\xi^{l'_5}z,\ \gamma')\\ &+\varepsilon_\Gamma(\gamma)\ (u+\lambda+\lambda')+\varepsilon_\Gamma(\gamma')\ (u+\lambda')+\beta_\Gamma(\gamma)+\beta_\Gamma(\gamma')+f_{\lambda'}(u,\ z)\\ &+f_\lambda(u+\lambda',\ \xi^{l'_5}z+\gamma'). \end{aligned} \tag{45}$$

Then,

$$\begin{aligned} &\varepsilon_\Gamma(\gamma+\xi^{l_5}\gamma')+\frac{1}{2i}H_\Lambda(1,\ \lambda+\lambda')=\varepsilon_\Gamma(\gamma)\\ &+\varepsilon_\Gamma(\gamma')+\eta_\lambda(u+\lambda',\ \xi^{l'_5}z+\gamma')+\eta_{\lambda'}(u,\ z) \end{aligned} \tag{46}$$

and

(47) $$\mu_\lambda(u+\lambda', \xi^{l_5'}z+\gamma') = -\mu_{\lambda'}(u, z).$$

In particular, for all $u, z \in \mathbf{C}$, $\gamma' \in \Gamma$, $l_5, l_5' \in \mathbf{Z}$ such that $(l_5+l_5')c \in \mathbf{Z}$ we have

(48) $$\mu_{l_5'c}(u, z) = -\mu_{l_5c}(u+l_5'c, \xi^{l_5'}z+\gamma').$$

From this relation, one may immediatelly obtain that

(49) $$\mu_{l_5'c}(u, z) = \mu_{l_5'c}(u+n, z+\gamma), \text{ for all } \gamma \in \Gamma, n \in \mathbf{Z}.$$

We apply (43) and (49) for $l_5'=1$ to deduce that $\mu_c(u, z)$ does not depend on z and we write $\mu_c(u) = \mu_c(u, z)$. Now, we take into account (41) and (43) which show us that $\mu_c(u+\lambda) = \mu_c(u)$, for any $\lambda \in \Lambda_1$. But this means nothing else than μ_c is a constant. From (38), this constant must be zero, so η_c depends only on z, say $\eta_c(z) = \eta_c(u, z)$. In fact, it is easy to see that η_λ depends only on z, for any $\lambda \in \Lambda$.

Then ν_λ will depend only on z for any $\lambda \in \Lambda$ and, from (46), we have

(50) $$\nu_\lambda(\xi^{l_5'}z+\gamma') = -\nu_{\lambda'}(z), \text{ for all } z \in \mathbf{C}, \gamma' \in \Gamma,$$

as soon as $l_3'=0$ and $(l_5+l_5')c \in \mathbf{Z}$.

In particular, for all $z \in \mathbf{C}$, $\gamma' \in \Gamma$, $l_5, l_5' \in \mathbf{Z}$ such that $(l_5+l_5')c \in \mathbf{Z}$ we have

$$\nu_{l_5'c}(z) = -\nu_{l_5c}(\xi^{l_5'}z+\gamma').$$

As we have already seen for μ_c, we see that ν_c must be a constant and, by means of (40), η_c must be a constant too.

Step 3. Next, we try to find β_Λ and thus to get the finest form of F.

If we apply (46) for $l_5 = -l_5' = 1$ and $l_3 = 0$, then we get $\varepsilon_\Gamma(\gamma+\xi\gamma') = \varepsilon_\Gamma(\gamma) + \varepsilon_\Gamma(\gamma')$, for all $\gamma, \gamma' \in \Gamma$. Since ε_Γ is a morphism, it must be identically zero.

So, we find the following relation for f_g:

(51) $$f_g(u, z) \equiv \frac{1}{2i}H_\Gamma(\xi^{l_5}z+l_3d, \gamma) + \frac{1}{2i}H_\Lambda(u, \lambda) + \beta_\Gamma(\gamma) + \beta_\Lambda(z, \lambda).$$

Let $\varepsilon_\Lambda(z, \lambda) = \frac{\partial\beta_\Lambda}{\partial z}(z, \lambda)$. We turn again to (7) to replace f_g obtained in (51) and then, by taking the derivatives with respect to z, we get

(52) $$\frac{\xi^{l_5+l_5'}}{2i}H_\Gamma(1, h(\lambda, \lambda')) + \varepsilon_\Lambda(z, \lambda+\lambda') = \xi^{l_5'}\varepsilon_\Lambda(\xi^{l_5'}z+\gamma'+l_3'd, \lambda) + \varepsilon_\Lambda(z, \lambda').$$

By using the same computations as before, one may see that ε_Λ does not depend on z, so we write $\varepsilon_\Lambda(\lambda) = \varepsilon_\Lambda(z, \lambda)$ and

(53) $$\varepsilon_\Lambda(\lambda) = \frac{1}{2i}H_\Gamma(1, l_3d) + \frac{1-\xi^{l_5}}{1-\xi}\varepsilon_\Lambda(c),$$

(54) $$\beta_\Lambda(z,\lambda)=\frac{\xi^{l_5}}{2i}H_\Gamma(z,l_3d)+\frac{1-\xi^{l_5}}{1-\xi}\varepsilon_\Lambda(c)z+\beta_\Lambda(\lambda),$$

where $\beta_\Lambda(\lambda):=\beta_\Lambda(0,\lambda)$.

In particular, for $\lambda\in\Lambda_1$, we have $\varepsilon_\Lambda(\lambda)=\frac{1}{2i}H_\Gamma(1,l_3d)$ and, by applying (33), we get the following extra-condition for H_Γ:

(55) $$\frac{1}{2i}H_\Gamma(l_3d,\gamma)-\frac{1}{2i}H_\Gamma(\gamma,l_3d)\in\mathbf{Z},\text{ for all }\gamma\in\Gamma,\ l_3\in\mathbf{Z},$$

which is equivalent to

(56) $$\begin{aligned}&(a2)\quad H_\Gamma(1,1)\operatorname{Im}\beta\in2\mathbf{Z},\\&(b2)\quad H_\Gamma(1,1)\operatorname{Im}\rho\in3\mathbf{Z},\\&(c2)\quad H_\Gamma(1,1)\in2\mathbf{Z}.\end{aligned}$$

Next, we turn back to (7).

Firstly, let us notice that (51) is read here

(57) $$\begin{aligned}f_g(u,z)=&\frac{1}{2i}H_\Gamma(\xi^{l_5}z+l_3d,\gamma)+\beta_\Gamma(\gamma)+\frac{1}{2i}H_\Lambda(u,\lambda)+\frac{\xi^{l_5}}{2i}H_\Gamma(z,l_3d)\\&+\frac{1-\xi^{l_5}}{1-\xi}\varepsilon_\Lambda(c)z+\beta_\lambda(\lambda)+\operatorname{const}(g),\end{aligned}$$

where $\operatorname{const}(g)\in\mathbf{Z}$. As in the proof of *Case 1*, we may suppose that $\operatorname{const}(g)=0$, without changing the cohomology class of F in $H^2(G,\mathbf{Z})$.

Let us set $r(g):=\beta_\Lambda(\gamma)+\beta_\Gamma(\gamma)+\frac{1}{1-\xi}\varepsilon_\Lambda(c)(\gamma+l_3d)$ and $r_\Lambda(\lambda):=r(\lambda)=\beta_\Lambda(\lambda)+\frac{1}{1-\xi}\varepsilon_\Lambda(c)l_3d$, $r_\Gamma(\gamma):=r(\gamma)=\beta_\Gamma(\gamma)+\frac{1}{1-\xi}\varepsilon_\Lambda(c)\gamma$. Then, we may suppose that $\varepsilon_\Lambda(c)=0$ and we find the following final formula for F:

(58) $$\begin{aligned}F(g,g')=&\frac{1}{2i}H_\Lambda(\lambda',\lambda)+\frac{\xi^{l_5}}{2i}H_\Gamma(\gamma'+l_3'd,\gamma)+\frac{1}{2i}H_\Gamma(l_3d,\gamma)\\&+\frac{1}{2i}H_\Gamma(l_3'd,\gamma')-\frac{1}{2i}H_\Gamma((l_3+l_3')d,\gamma+\xi^{l_5}\gamma'+h(\lambda,\lambda'))\\&+\frac{\xi^{l_5}}{2i}H_\Gamma(\gamma'+l_3'd,l_3d)+r(g)+r(g')-r(gg')\in\mathbf{Z}.\end{aligned}$$

From (58), one may see that if $H_\Lambda=0$ and $H_\Gamma=0$, then F has the cohomology class in $H^2(G,\mathbf{C})$ equal to zero, so the cohomology class of F in $H^2(G,\mathbf{Z})$ is a torsion element. This fact shows that $\operatorname{Ker}(\chi)\subset\operatorname{Tors}H^2(G,\mathbf{Z})$.

Step 4. We show next $\mathcal{N}S=\chi(H^2(G,\mathbf{Z}))$.

"$\supset$". Let $(H_\Gamma,H_\Lambda)=\chi(F)$ where $F\in H^2(G,\mathbf{Z})$. We have already seen in *Step 3* that (56) must be true. It remains to prove that $2\operatorname{Im}H_\Lambda(\Lambda\times\Lambda)\subset\mathbf{Z}\times\mathbf{Z}$ if S is of type $(c2)$. In fact, we have some more relations which lead us to the conclusion and which are also useful for the Appell-Humbert Theorem.

Let $b_\Gamma(\gamma)=ir_\Gamma(\gamma)-\frac{1}{4}H_\Gamma(\gamma,\gamma)$ and $b_\Lambda(\lambda)=ir_\Lambda(\lambda)-\frac{1}{4}H_\Lambda(\lambda,\lambda)$. As in the case when S is of the first type, we have the following relations:

$$\begin{aligned}(59)\qquad &S \text{ of type } (a2) \quad 2b_\Gamma(1),\ 2b_\Gamma(\beta)\in i\mathbf{Z},\\ &S \text{ of type } (b2) \quad b_\Gamma(1)-b_\Gamma(\rho)\in i\mathbf{Z},\ 3b_\Gamma(1)-\frac{i\sqrt{3}}{4}H_\Gamma(1,1)\in i\mathbf{Z},\\ &S \text{ of type } (c2) \quad 2b_\Gamma(1)\in i\mathbf{Z},\ b_\Gamma(1)-b_\Gamma(i)\in i\mathbf{Z}.\end{aligned}$$

We start from the relation $F(\lambda',\lambda)-F(\lambda,\lambda')\in\mathbf{Z}$, for all $\lambda,\lambda'\in\Lambda$, we replace F from the formula (58) for $\gamma=\gamma'=0$, $l'_5=l_3=0$ and we use (55) to get

$$(60)\qquad iE_\Lambda(l_5c,l'_3\alpha)+b_\Gamma(h(l_5c,l'_3\alpha))+\frac{1}{4}H_\Gamma(1,1)l'^2_3|d|^2(\overline{\xi^{l_5}}-\xi^{l_5})\in i\mathbf{Z},$$

for all $l_5, l'_3\in\mathbf{Z}$.

This condition is equivalent to

$$\begin{aligned}(61)\qquad &S \text{ of type } (a2) \quad b_\Gamma(1)+iE_\Lambda(c,\alpha)\in i\mathbf{Z},\\ &S \text{ of type } (b2) \quad b_\Gamma(1)+iE_\Lambda(c,\alpha)-\frac{i\sqrt{3}}{12}H_\Gamma(1,1)\in i\mathbf{Z},\\ &S \text{ of type } (c2) \quad -b_\Gamma(1)+iE_\Lambda(c,\alpha)-\frac{i}{4}H_\Gamma(1,1)\in i\mathbf{Z}\end{aligned}$$

and, because of (56) and (59), if S is of type $(c2)$ then $2E_\Lambda(c,\alpha)\in\mathbf{Z}$.

Moreover, from (55), (58) and (60), we have the following relation for b_Λ:

$$\begin{aligned}(62)\qquad &b_\Lambda(\lambda)+b_\Lambda(\lambda')-b_\Lambda(\lambda+\lambda')+\frac{1}{2}iE_\Lambda(l'_5c,l_3\alpha)+iE_\Lambda(l_5c,l'_3\alpha)\\ &+\frac{1}{2}H_\Gamma(l_3d,l'_3d)\in i\mathbf{Z},\ \text{for all } \lambda,\lambda'\in\Lambda.\end{aligned}$$

"$\subset$". To prove this inclusion, we have to prove that if $(H_\Gamma, H_\Lambda)\in\mathcal{N}S$, then there exist r_Γ and r_Λ such that

$$\begin{aligned}(63)\qquad &\frac{1}{2i}H_\Lambda(\lambda',\lambda)+\frac{\xi^{l_5}}{2i}H_\Gamma(\gamma'+l'_3d,\gamma)+\frac{1}{2i}H_\Gamma(l_3d,\gamma)\\ &+\frac{1}{2i}H_\Gamma(l'_3d,\gamma')-\frac{1}{2i}H_\Gamma((l_3+l'_3)d,\gamma+\xi^{l_5}\gamma'+h(\lambda,\lambda'))\\ &+\frac{\xi^{l_5}}{2i}H_\Gamma(\gamma'+l'_3d,l_3d)+r_\Lambda(\lambda)+r_\Lambda(\lambda')-r_\Lambda(\lambda+\lambda')\\ &+r_\Gamma(\gamma)+r_\Gamma(\gamma')-r_\Gamma(\gamma+\xi^{l_5}\gamma'+h(\lambda,\lambda'))\in\mathbf{Z}.\end{aligned}$$

We start with $b_\Gamma(1)$ and $b_\Gamma(\beta)$ such that (59) and (61) are satisfied. We set, as in the first case,

$$(64)\qquad b_\Gamma(\gamma)=l_2b_\Gamma(1)+l_4b_\Gamma(\beta)+\frac{1}{2}il_2l_4E_\Gamma(1,\beta)$$

and this b_Γ will satisfy the following relation:

$$b_\Gamma(\gamma)+b_\Gamma(\gamma')-b_\Gamma(\gamma+\gamma')+\frac{1}{2}iE_\Gamma(\gamma,\gamma')\in i\mathbf{Z}, \tag{65}$$

$$b_\Gamma(\xi\gamma)-b_\Gamma(\gamma)\in i\mathbf{Z}. \tag{66}$$

We define

$$r_\Gamma(\gamma)=-ib_\Gamma(\gamma)-\frac{i}{4}H_\Gamma(\gamma,\gamma). \tag{67}$$

Next, we start with $r_\Lambda(\alpha)$ and $r_\Lambda(c)$ in $\mathbf{C}$ and we take

$$r_\Lambda(\lambda)=\frac{(l_3-1)l_3}{4i}H_\Lambda(\alpha,\alpha)+\frac{(l_5-1)l_5}{4i}H_\Lambda(c,c)+\frac{(l_3-1)l_3}{4i}H_\Gamma(d,d) \\ +\frac{1}{2i}H_\Lambda(l_5c,l_3\alpha)+l_3r_\Lambda(\alpha)+l_5r_\Lambda(c) \tag{68}$$

A straightforward computation, by using the relations (55), (60), (64), (65), (66), (67) and (68) leads us to the conclusion.

We denote by Ψ'': $\mathcal{N}S\tilde{\to}\mathrm{Num}(S)$ the isomorphism obtained in Theorem 3.1.

4. Appell-Humbert theorem

Keeping the notations in the previous sections, we define $\alpha_\Gamma(\gamma):=e^{2\pi b_\Gamma(\gamma)}$ and $\alpha_\Lambda(\lambda):=e^{2\pi b_\Lambda(\lambda)}$. Recall that, since $b_\Gamma(\xi\gamma)-b_\Gamma(\gamma)\in i\mathbf{Z}$, b_Γ must be purely imaginary.

If S is of the first type, then α_Γ and α_Λ satisfy the following relations:

$$\alpha_\Lambda(\lambda+\lambda')=\alpha_\Lambda(\lambda)\alpha_\Lambda(\lambda')e^{\pi iE_\Lambda(\lambda,\lambda')} \tag{69}$$

$$\alpha_\Gamma(\gamma+\gamma')=\alpha_\Gamma(\gamma)\alpha_\Gamma(\gamma')e^{\pi iE_\Gamma(\gamma,\gamma')} \tag{70}$$

$$\alpha_\Gamma(\xi\gamma)=\alpha_\Gamma(\gamma), \tag{71}$$

where $(H_\Gamma,H_\Lambda)\in\mathcal{N}S$.

If S is of the second type, then α_Γ and α_Λ satisfy the following relations:

$$\alpha_\Lambda(\lambda+\lambda')=\alpha_\Lambda(\lambda)\alpha_\Lambda(\lambda')e^{\pi iE_\Lambda(l'_5c,l_3\alpha)+\pi iE_\Lambda(l_5c,l'_3\alpha)+\pi H_\Gamma(l_3d,l'_3d)} \tag{72}$$

$$\alpha_\Gamma(\gamma+\gamma')=\alpha_\Gamma(\gamma)\alpha_\Gamma(\gamma')e^{\pi iE_\Gamma(\gamma,\gamma')} \tag{73}$$

$$\alpha_\Gamma(\xi\gamma)=\alpha_\Gamma(\gamma) \tag{74}$$

and

$$\alpha_\Gamma(1)=\begin{cases} e^{-2\pi iE_\Lambda(c,\alpha)} & S \text{ of type } (a2) \\ e^{-2\pi iE_\Lambda(c,\alpha)+\pi\frac{i\sqrt{3}}{6}H_\Gamma(1,1)} & S \text{ of type } (b2) \\ e^{-2\pi iE_\Lambda(c,\alpha)-\pi\frac{i}{2}H_\Gamma(1,1)} & S \text{ of type } (c2) \end{cases} \tag{75}$$

where $(H_\Gamma,H_\Lambda)\in\mathcal{N}S$.

Let $\mathcal{P}_1 = \{$Group of data $(H_\Gamma, H_\Lambda, \alpha_\Gamma, \alpha_\Lambda)\}$ with natural group operation and $\mathcal{P} = \mathcal{P}_1/\sim$ where $(H_\Gamma, H_\Lambda, \alpha_\Gamma, \alpha_\Lambda) \sim (H'_\Gamma, H'_\Lambda, \alpha'_\Gamma, \alpha'_\Lambda)$ if and only if $H_\Gamma = H'_\Gamma$, $H_\Lambda = H'_\Lambda$, $\alpha_\Gamma = \alpha'_\Gamma$ and there exists $a \in \mathbf{C}$ such that $\alpha_\Lambda(\lambda) = \alpha'_\Lambda(\lambda) e^{2\pi i a \lambda}$, for any $\lambda \in \Lambda$. For simplicity, we shall denote by $(H_\Gamma, H_\Lambda, \alpha_\Gamma, \widehat{\alpha_\Lambda})$ instead of $(H_\Gamma, \widehat{H_\Lambda, \alpha_\Gamma}, \alpha_\Lambda)$ and $\alpha_\Lambda \sim \alpha'_\Lambda$ for the equivalence.

Remark 4.1. By using a classical argument that have been already used in section 2 (cf. [9], Chàpter I), one may see that if S is of the second type and $H_\Gamma = 0$ or if S is of the first type, then exists a unique α'_Λ such that $\alpha_\Lambda \sim \alpha'_\Lambda$ and $\alpha'_\Lambda(\lambda) \in U(1)$, for all $\lambda \in \Lambda$.

This argument allows us many times to suppose that the multiplicators appearing in theorems of Appell-Humbert kind are $U(1)$-valued (see [9] for tori and [3] for primary Kodaira surfaces).

Lemma 4.2. *We have an exact short sequence*

$$0 \longrightarrow \mathrm{Hom}(G, U(1)) \xrightarrow{\mu} \mathcal{P} \xrightarrow{\eta} \mathcal{NS} \longrightarrow 0$$

where η is the canonical projection and $\mu(\alpha_G) = (0, 0, \alpha_G|_\Gamma, \alpha_G|_\Lambda)$.

Proof. The morphism η is surjective from the proof of the Theorem 3.1. By the above remark, μ is injective. Since $\eta\mu = 0$ it remains to check that $\mathrm{Ker}(\eta) \subset \mu(\mathrm{Hom}(G, U(1)))$.

Indeed, let $(0, 0, \alpha_\Gamma, \widehat{\alpha_\Lambda}) \in P$. Since the corresponding hermitian forms are equal to zero, it follows that $\alpha_\Gamma \in \mathrm{Hom}(\Gamma, U(1))$ and $\alpha_\Lambda \in \mathrm{Hom}(\Lambda, \mathbf{C}^*)$. From Remark 4.1., $\widehat{\alpha_\Lambda}$ has a representative that is $U(1)$-valued, say α'_Λ.

Then we define $\alpha_G(g) := \alpha_\Gamma(\lambda)\alpha'_\Lambda(\lambda) \in U(1)$, for any $g = \gamma\lambda \in G$, which is an element of $\mathrm{Hom}(G, U(1))$ and satisfies $\mu(\alpha_G) = (0, 0, \alpha_\Gamma, \widehat{\alpha_\Lambda})$.

Theorem 4.3. *There is the following isomorphism of exact sequences:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}(G, U(1)) & \longrightarrow & \mathcal{P} & \longrightarrow & \mathcal{NS} & \longrightarrow & 0 \\ & & \wr\downarrow \Psi' & & \wr\downarrow \Psi & & \wr\downarrow \Psi'' & & \\ 0 & \longrightarrow & \mathrm{Pic}^\tau(S) & \longrightarrow & \mathrm{Pic}(S) & \longrightarrow & \mathrm{Num}(S) & \longrightarrow & 0 \end{array}$$

where Ψ' is the isomorphism from section 2, Ψ'' is the isomorphism from section 3 and Ψ maps an element $(H_\Gamma, H_\Lambda, \alpha_\Gamma, \widehat{\alpha_\Lambda}) \in \mathcal{P}$ to the cocycle $\{e_g\}_g \in H^1(G, H^*)$ given by

$$e_g(u, z) = \alpha_\Gamma(\gamma)\alpha_\Lambda(\lambda) e^{\pi H_\Lambda(u,\lambda) + \pi H_\Gamma(\xi'^s z + \gamma, \gamma + l_3 d) - \frac{\pi}{2} H_\Gamma(\gamma,\gamma) + \frac{\pi}{2} H_\Lambda(\lambda,\lambda)}$$

Proof. All we have to check is that Ψ is well-defined, so let us suppose that $(H_\Gamma, H_\Lambda, \alpha_\Gamma, \widehat{\alpha_\Lambda})$ maps by Ψ to $\{e_g\}_g \in H^1(G, H^*)$ and we change the representative of α_Λ by α'_Λ. If $e''_g = \dfrac{\alpha_\Lambda(\lambda)}{\alpha'_\Lambda(\lambda)} \overset{not}{=} \alpha''_\Lambda(\lambda)$, then is is easy to see that $\{e''_g\}_g$ is a coboundary in $C^1(G, H^*)$.

Indeed, there exists $a\in\mathbf{C}$ such that $\alpha''_\Lambda(\lambda)=e^{2\pi ia\lambda}$ and we chose $h(u,z)=e^{2\pi iau}$. Then, $e''_g=h(g(u,z))h^{-1}(u,z)$, for $u, z\in\mathbf{C}$, $g\in G$.

Definition 4.4. For any $(H_\Gamma, H_\Lambda, \alpha_\Gamma, \widehat{\alpha_\Lambda})\in\mathcal{P}$, the line bundle over S associated to the cocycle $\{e_g\}_g=\Psi(H_\Gamma, H_\Lambda, \alpha_\Gamma, \widehat{\alpha_\Lambda})\in H^1(G, H^*)$ will be denoted by $L(H_\Gamma, H_\Lambda, \alpha_\Gamma, \widehat{\alpha_\Lambda})$.

Remark 4.5. $L(H_\Gamma, H_\Lambda, \alpha_\Gamma, \widehat{\alpha_\Lambda})$ is the quotient of $\mathbf{C}^2\times\mathbf{C}$ given by the equivalence relation $((u,z),w)\sim(g(u,z), e_g(u,z)w)$, for any $g\in G$.

5. Applications

The first application of Appell-Humbert theorem is a description of Tors $H^2(G,\mathbf{Z})$ and its generators in terms of the groups cohomology (see, also [10], [12] for a precised characterisation).

By taking into account that torsion cocycles F are given by the vanishing of their corresponding hermitian forms H_Γ and H_Λ, one may obtain very easy the following table (see, also [5] for a similar result on primary Kodaira surfaces):

Type	Tors $H^2(G,\mathbf{Z})$	Action of generators of Tors $H^2(G,\mathbf{Z})$ on (g,g')
$(a1)$	$\mathbf{Z}_2\times\mathbf{Z}_2$	$(1-(-1)^{l_5})l'_2/2$ and $(1-(-1)^{l_5})l'_4/2$
$(a2)$	$\mathbf{Z}_2$	$(1-(-1)^{l_5})l'_4/2$
$(b1)$	$\mathbf{Z}_3$	$(\mathrm{Re}((1-\rho^{l_5})\gamma')+\sqrt{3}\mathrm{Im}((1-\rho^{l_5})\gamma'))/3$
$(b2)$	0	0
$(c1)$	$\mathbf{Z}_2$	$(\mathrm{Re}((1-i^{l_5})\gamma')+\mathrm{Im}((1-i^{l_5})\gamma'))/2$
$(c2)$	0	0
$(d1)$	0	0

Next, we may apply Appell-Humbert theorem to compute a basis in Num (S) (see, also [10], Therrem 1.4.).

Let us denote by q the cardinal of $\mathcal{G}$.

If we fix isomorphisms $H^2(\Gamma,\mathbf{Z})\cong H^2(E,Z)\overset{deg}{\cong}\mathbf{Z}$ and $H^2(\Lambda_2,\mathbf{Z})\cong H^2(\Delta,\mathbf{Z})\overset{deg}{\cong}\mathbf{Z}$, then the inclusions $\mathcal{NS}\subset\mathcal{N}_1\subset\mathcal{N}_2=\mathbf{Z}\oplus\mathbf{Z}$ become:

Type	$\mathcal{N}_1$	$\mathcal{NS}$	q	basis in $\mathcal{NS}$	
				e_1	e_2
$(a1)$	$\mathbf{Z}\oplus\mathbf{Z}$	$\mathbf{Z}\oplus 2\mathbf{Z}$	2	(1, 0)	(0, 2)
$(a2)$	$\mathbf{Z}\oplus 2\mathbf{Z}$	$2\mathbf{Z}\oplus 2\mathbf{Z}$	4	(2, 0)	(0, 2)
$(b1)$	$\mathbf{Z}\oplus\mathbf{Z}$	$\mathbf{Z}\oplus 3\mathbf{Z}$	3	(1, 0)	(0, 3)
$(b2)$	$\mathbf{Z}\oplus 3\mathbf{Z}$	$3\mathbf{Z}\oplus 3\mathbf{Z}$	9	(3, 0)	(0, 3)
$(c1)$	$\mathbf{Z}\oplus\mathbf{Z}$	$\mathbf{Z}\oplus 4\mathbf{Z}$	4	(1, 0)	(0, 4)
$(c2)$	$\mathbf{Z}\oplus 2\mathbf{Z}$	$2\mathbf{Z}\oplus 4\mathbf{Z}$	8	(2, 0)	(0, 4)
$(d1)$	$\mathbf{Z}\oplus\mathbf{Z}$	$\mathbf{Z}\oplus 6\mathbf{Z}$	6	(1, 0)	(0, 6)

It is easy to determine the numerical classes of $\mathcal{O}_S(E)$ and $\mathcal{O}_S(\Delta)$ in $\mathcal{N}S$. Indeed, according to [10], since the intersection number $E.\Delta$ is equal to q, then via isomorphism $N_2 \cong \mathbf{Z} \oplus \mathbf{Z}$, we have $c_1(E) = (0, q)$ and $c_1(\Delta) = (q, 0)$.

Then, by using the previous table, we get the following (compare also with [10], Theorem 1.4):

Type	Basis of Num (S)	
$(a1)$	$1/2\Delta$	E
$(a2)$	$1/2\Delta$	$1/2E$
$(b1)$	$1/3\Delta$	E
$(b2)$	$1/3\Delta$	$1/3E$
$(c1)$	$1/4\Delta$	E
$(c2)$	$1/4\Delta$	$1/2E$
$(d1)$	$1/6\Delta$	E

The next application of Appell-Humbert theorem is computing the space of global sections of some line bundles over S.

As we saw, any element $L \in \mathrm{Pic}(S)$ can be written as $L = L(H_\Gamma, H_\Lambda, \alpha_\Gamma, \widehat{\alpha_\Lambda})$, where $(H_\Gamma, H_\Lambda, \alpha_\Gamma, \widehat{\alpha_\Lambda}) \in \mathcal{P}$.

From [10], Theorem 1.4., the numerical type of L is of form $c_1(L) = a\Delta + bE$, where $a, b \in \mathbf{Q}$, or $c_1(L) = a_1 e_1 + b_1 e_2$ with $a_1, b_1 \in \mathbf{Z}$. According to [10], Lemma 1.3., if $H^0(L) \neq 0$, then $a, b \geq 0$, which is equivalent to the inequalities $H_\Gamma(1, 1) \geq 0$, $H_\Lambda(1, 1) \geq 0$. If $a, b > 0$, then L is ample (cf. [10], Lemma 1.3) and $h^0(L) = abq = a_1 b_1 > 0$, so it remains to study the cases $a = 0$, $b > 0$ and $a > 0$, $b = 0$.

Here we shall compute $H^0(L)$ for $a = 0$, $b > 0$. Before stating our result, let us introduce the following notion:

Definition 5.1. Let $(H_\Gamma, H_\Lambda, \alpha_\Gamma, \widehat{\alpha_\Lambda}) \in \mathcal{P}$. Any holomorphic function θ: $\mathbf{C}^2 \to \mathbf{C}$ such that

$$\theta(g(u, z)) = e_g(u, z)\,\theta(u, z), \text{ for all } g \in G,\ u, z \in \mathbf{C} \tag{76}$$

is called a *θ-function* for the data $(H_\Gamma, H_\Lambda, \alpha_\Gamma, \widehat{\alpha_\Lambda})$.

It is easy to see that there is a natural one-to one correspondence between θ-functions for $(H_\Gamma, H_\Lambda, \alpha_\Gamma, \widehat{\alpha_\Lambda})$ and sections of $L(H_\Gamma, H_\Lambda, \alpha_\Gamma, \widehat{\alpha_\Lambda})$.

Proposition 5.2. *If $c_1(L) = bE$, $b > 0$ then $h^0(L) \neq 0$ if and only if α_Γ is identically equal to* 1.

In this case, $b \in \mathbf{Z}$ and there is a natural isomorphism $H^0(L) \cong H^0(L(H_\Gamma, \alpha_\Lambda))$, where $L(H_\Lambda, \alpha_\Lambda)$ is the bundle over $\mathbf{C}/\Lambda$ associated to the hermitian form H_Λ and the multiplicator α_Λ.

Proof. The equality $a = 0$ is equivalent to $H_\Gamma = 0$ and then α_Γ: $\Gamma \to U(1)$ is a morphism of groups with $\alpha_\Gamma(\xi\gamma) = \alpha_\Gamma(\gamma)$, for any $\gamma \in \Gamma$. On the other hand, from Remark 4.1., we may suppose that α_Λ is $U(1)$-valued. Moreover, since $H_\Gamma=0$ then

$$e_g(u, z) = \alpha_\Gamma(\gamma)\alpha_\Lambda(\lambda)e^{\pi H_\Lambda(u,\lambda)+\frac{\pi}{2}H_\Lambda(\lambda,\lambda)}$$

for both types of hyperelliptic surfaces.

Claim 1. If α_Γ is identically equal to 1 then $E_\Lambda(\Lambda\times\Lambda)\subset\mathbf{Z}$ and

$$\alpha_\Lambda(\lambda+\lambda') = \alpha_\Lambda(\lambda)\alpha_\Lambda(\lambda')e^{\pi i E_\Lambda(\lambda,\lambda')}.$$

Proof of Claim 1. For the case when S is of the first type, this is nothing else than the definition. If S is of the second type, then $H_\Gamma=0$ implies that $1=\alpha_\Gamma(1)=e^{-2\pi i E_\Lambda(c,\alpha)}$ so $E_\Lambda(c, \alpha)\in\mathbf{Z}$ i.e. $E_\Lambda(\Lambda\times\Lambda)\in\mathbf{Z}$. Because $E_\Lambda(c, \alpha)\in\mathbf{Z}$, we apply (72) to get $\alpha_\Lambda(\lambda+\lambda')=\alpha_\Lambda(\lambda)\alpha_\Lambda(\lambda')e^{\pi i E_\Lambda(\lambda,\lambda')}$.

Claim 2. The condition $b\in\mathbf{Z}$ is equivalent $E_\Lambda(\Lambda\times\Lambda)\subset\mathbf{Z}$.

Now, we turn back to the proof of Proposition 5.2.
"$\Rightarrow$". If $h^0(L)>0$, then there exists a θ-function for $(0, H_\Lambda, \alpha_\Gamma, \widehat{\alpha_\Lambda})$, say θ, non-identically zero. Then, for all $u, z\in\mathbf{C}$, $\gamma\in\Gamma$, $\lambda\in\Lambda$, θ must satisfy

(77) $$\theta(u+\lambda, \xi^{l_5}z+\gamma+l_3d) = \alpha_\Gamma(\gamma)\alpha_\Lambda(\gamma)e^{\pi H_\Lambda(u,\lambda)+\frac{\pi}{2}H_\Lambda(\lambda,\lambda)}\theta(u, z).$$

If we take $\lambda=0$ in (77), it follows that

(78) $$\theta(u, z+\gamma) = \alpha_\Gamma(\gamma)\theta(u, z), \text{ for all } u, z\in\mathbf{C}, \gamma\in\Gamma.$$

Since α_Γ is $U(1)$-valued, then we can apply maximum principle in (78) to conclude that θ does not depend on z i.e. $\theta(u, z)=\theta(u)$, $z\in\mathbf{C}$. The condition (78) implies also that α_Γ must be identically equal to 1. Moreover, (77) becomes

(79) $$\theta(u+\lambda) = \alpha_\Lambda(\lambda)e^{\pi H_\Lambda(u,\lambda)+\frac{\pi}{2}H_\Lambda(\lambda,\lambda)}\theta(u).$$

From (79) and *Claim 1.* we deduce that θ is in fact a θ-function for the data $(H_\Lambda, \alpha_\Lambda)$ with respect to the lattice Λ.
"$\Leftarrow$". We apply again *Claim 1.* and then we can choose $\theta\in H^0(H_\Lambda, \alpha_\Lambda)$. It is easy to see that if we define $\theta(u, z)=\theta(u)$, then θ is also a θ-function for the data $(0, H_\Lambda, 1, \alpha_\Lambda)$.

For the final part of proposition, we apply *Claim 2.* and [9], Chapter I.

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