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Isomorphisms of $\text{Alg}\mathscr{L}_{2n(k)}$

By

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1. Introduction

The study of non-self-adjoint operator algebras on Hilbert space was begun in 1974 by Arveson[1]. Recently, such algebras have been found to be of use in physics, in electrical engineering, and in general systems theory. Of particular interest to mathematicians are reflexive algebras with commutative lattices of invariant subspaces.

First we will introduce terminologies which are used in this paper. Let $\mathscr{H}$ be a complex Hilbert space and let $\mathscr{A}$ be a subset of $\mathscr{B}(\mathscr{H})$, the class of all bounded operators acting on $\mathscr{H}$. If $\mathscr{A}$ is a vector space over $\mathbf{C}$ and if $\mathscr{A}$ is closed under the composition of maps, then $\mathscr{A}$ is called an algebra. $\mathscr{A}$ is called a self-adjoint algebra provided A^* is in $\mathscr{A}$ for every A in $\mathscr{A}$. Otherwise, $\mathscr{A}$ is called a non-self-adjoint algebra. If $\mathscr{L}$ is a lattice of orthogonal projections acting on $\mathscr{H}$, then $\text{Alg}\mathscr{L}$ denotes the algebra of all bounded operators acting on $\mathscr{H}$ that leave invariant every orthogonal projection in $\mathscr{L}$. A subspace lattice $\mathscr{L}$ is a strongly closed lattice of orthogonal projections acting on $\mathscr{H}$, containing 0 and 1. Dually, if $\mathscr{A}$ is a subalgebra of $\mathscr{B}(\mathscr{H})$, then Lat $\mathscr{A}$ is the lattice of all orthogonal projections invariant for each operator in $\mathscr{A}$. An algebra $\mathscr{A}$ is reflexive if $\mathscr{A}=$ $\text{AlgLat}\mathscr{A}$ and a lattice $\mathscr{L}$ is reflexive if $\mathscr{L}=\text{LatAlg}\mathscr{L}$. A lattice $\mathscr{L}$ is a commutative subspace lattice, or CSL, if each pair of projections in $\mathscr{L}$ commutes; $\text{Alg}\mathscr{L}$ is then called a CSL-algebra. If $x_1, x_2, \ldots, x_n$ are vectors in some Hilbert space, then $[x_1, x_2, \ldots, x_n]$ denotes the closed subspace generated by the vectors $x_1, x_2, \ldots, x_n$.

Let $\mathscr{H}$ be a complex Hilbert space and let $\mathscr{L}_{2n}$ be the subspace lattice of orthogonal projections generated by $\{[e_1], [e_3], \ldots, [e_{2n-1}], [e_1, e_2, e_3], [e_3, e_4, e_5], \ldots, [e_{2n-3}, e_{2n-2}, e_{2n-1}], [e_{2n-1}, e_{2n}]\}$ with an orthonormal basis $\{e_1, e_2, \ldots, e_{2n}\}$. Then algebras $\text{Alg}\mathscr{L}_{2n}$ are important classes of non-self-adjoint operator algebras. These algebras possess many surprising properties related to isometries, isomorphisms, cohomology, and extreme points.

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Let $\mathscr{H}$ be a complex Hilbert space with an orthonormal basis $\{e_1, e_2, \dots, e_{2n}\}$. Let $\text{Alg}\mathscr{L}_{2n(k)}$ be a subalgebra of $\mathscr{B}(\mathscr{H})$ such that an operator A is in $\text{Alg}\mathscr{L}_{2n(k)}$ if and only if A is in $\text{Alg}\mathscr{L}_{2n}$ and one off-diagonal entry of A is zero. Such $\text{Alg}\mathscr{L}_{2n(k)}$ are also interesting algebras which have been found to be of use in physics, in electrical engineering and in other fields.

In this paper, we will investigate isomorphisms of these algebras. Let $\mathscr{L}_{2n(i)}$ be the subspace lattice of orthogonal projections generated by $\{[e_1], [e_3], \dots, [e_{2n-1}], [e_1, e_2, e_3], [e_3, e_4, e_5], \dots, [e_{2i-3}, e_{2i-2}, e_{2i-1}], [e_{2i-1}, e_{2i}], [e_{2i+1}, e_{2i+2}, e_{2i+3}], \dots, [e_{2n-1}, e_{2n}]\}$ $(i=1, 2, \dots, n-1)$ where $e_{-1}=0$ and $e_0=0$ whenever $i=1$. Let $\mathscr{B}_{2n(i)}$ be the algebra consisting of all bounded operators acting on a $2n$-dimensional complex Hilbert space $\mathscr{H}$, that are of the form

$$
\begin{array}{r}
\\ \\ \\ \\ \\ \\ 2i+1 \\ \\ \\ \\
\end{array}
\begin{pmatrix}
* & * & & \overset{2i}{\cdot} & & & & \\
& * & & \cdot & & & & \\
& * & & \cdot & & & & \\
& & \ddots & & & & & \\
& & & * & & & & \\
& & & * & & & & \\
\cdot & \cdot & \cdot & 0 & * & * & & \\
& & & & & & \ddots & \\
& & & & & & & * \\
& & & & & & & *
\end{pmatrix},
$$

where all non-starred entries are zero and with an orthonormal basis $\{e_1, e_2, \dots, e_{2n}\}$.

2. Isometries of $\text{Alg}\mathscr{L}_{2n(k)}$

Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be commutative subspace lattices. By an isometry $\varphi: \text{Alg}\mathscr{L}_1 \to \text{Alg}\mathscr{L}_2$ we mean a strictly algebraic isometry, that is, a bijective, linear, multiplicative map. An isometry $\varphi: \text{Alg}\mathscr{L}_1 \to \text{Alg}\mathscr{L}_2$ is said to be spatially implemented if there is a bounded invertible operator T such that $\varphi(A) = TAT^{-1}$ for all A in $\text{Alg}\mathscr{L}_1$.

Lemma 2.1. (1) $\text{Alg}\mathscr{L}_{2n(i)} = \mathscr{B}_{2n(i)}$,
(2) $\text{Lat}\mathscr{B}_{2n(i)} = \mathscr{L}_{2n(i)}$.

Let l and j be positive integers. Then E_{lj} is the matrix whose (l, j)-component is 1 and all other components are 0.

Theorem 2.2. *Let* $\varphi: \text{Alg}\mathscr{L}_{2n(i)} \to \text{Alg}\mathscr{L}_{2n(i)}$ *be an isometry such that* $\varphi(E_{ll}) = E_{ll}$ *for all* $l = 1, 2, \dots, 2n$. *Then there exist non-zero complex numbers* α_{lj} *such that* $\varphi(E_{lj}) = \alpha_{lj}E_{lj}$ *for all* E_{lj} *in* $\text{Alg}\mathscr{L}_{2n(i)}$.

Proof. Let $\varphi(E_{ll})=E_{ll}$ for all l. Then $\varphi(E_{lj})=\varphi(E_{ll}E_{lj}E_{jj})=E_{ll}\varphi(E_{lj})E_{jj}$ for E_{lj} in $\mathrm{Alg}\mathscr{L}_{2n(i)}$. So if we compare the (l, j)-component of $\varphi(E_{lj})$ with that of $E_{ll}\varphi(E_{lj})E_{jj}$, $\varphi(E_{lj})=\alpha_{lj}E_{lj}$ for some non-zero complex number α_{lj}.

Theorem 2.3. *Let* $\varphi: \mathrm{Alg}\mathscr{L}_{2n(i)}\to\mathrm{Alg}\mathscr{L}_{2n(i)}$ *be an isometry such that* $\varphi(E_{ll})=E_{ll}$ *for all* $l=1, 2, \dots, 2n$ *and let* $\varphi(E_{lj})=\alpha_{lj}E_{lj}$, $\alpha_{lj}\neq 0$ *for all* E_{lj} *in* $\mathrm{Alg}\mathscr{L}_{2n(i)}$. *Then there exists a diagonal operator* T *such that* $\varphi(A)=TAT^{-1}$ *for all* A *in* $\mathrm{Alg}\mathscr{L}_{2n(i)}$. *Here* T *is a* $2n\times 2n$ *diagonal operator whose*

(1) $(1, 1)$*-component is* 1 *and* $(2i+1, 2i+1)$*-component is* 1,

(2) $(2, 2)$*-component is* α_{12}^{-1},

(3) $(2j, 2j)$*-component is* $\left(\prod_{k=1}^{j}\alpha_{2k-1, 2k}\right)^{-1}\left(\prod_{k=1}^{j-1}\alpha_{2k+1, 2k}\right)$ $(j=1, 2, \dots, i)$ *and*

$(2j-1, 2j-1)$*-component is* $\left(\prod_{k=1}^{j-1}\alpha_{2k-1, 2k}\right)^{-1}\left(\prod_{k=1}^{j-1}\alpha_{2k+1, 2k}\right)$ $(j=2, \dots, i)$,

(4) $(2i+2j, 2i+2j)$*-component is*

$\left(\prod_{k=1}^{j}\alpha_{2i+2k-1, 2i+2k}\right)^{-1}\left(\prod_{k=1}^{j}\alpha_{2i+2k-1, 2i+2k-2}\right)$ $(j=1, 2, \dots, n-i)$

and $(2i+2j-1, 2i+2j-1)$*-component is*

$\left(\prod_{k=1}^{j-1}\alpha_{2i+2k-1, 2i+2k}\right)^{-1}\left(\prod_{k=1}^{j}\alpha_{2i+2k-1, 2i+2k-2}\right)$ $(j=2, \dots, n-i)$.

Proof. Let $A=(a_{lj})$ be in $\mathrm{Alg}\mathscr{L}_{2n(i)}$. Then $\varphi(A)=(\alpha_{lj}a_{lj})$. Let $T=(t_{ll})$ be a $2n\times 2n$-diagonal matrix such that $t_{ll}\neq 0$ for all $l=1, 2, \dots, 2n$. Then $TAT^{-1}=(t_{ll}a_{lj}t_{jj}^{-1})$. So if the linear system for unknown variables $t_{ll}(l=1, 2, \dots, 2n)$

$$
\begin{aligned}
&\alpha_{12}=t_{11}t_{22}^{-1},\\
&\alpha_{32}=t_{33}t_{22}^{-1},\ \alpha_{34}=t_{33}t_{44}^{-1},\\
&\quad\vdots\\
&\alpha_{2i-1, 2i-2}=t_{2i-1, 2i-1}t_{2i-2, 2i-2}^{-1},\ \alpha_{2i-1, 2i}=t_{2i-1, 2i-1}t_{2i, 2i}^{-1},\\
&\alpha_{2i+1, 2i+2}=t_{2i+1, 2i+1}t_{2i+2, 2i+2}^{-1},\\
&\alpha_{2i+3, 2i+2}=t_{2i+3, 2i+3}t_{2i+2, 2i+2}^{-1},\ \alpha_{2i+3, 2i+4}=t_{2i+3, 2i+3}t_{2i+4, 2i+4}^{-1},\\
&\quad\vdots\\
&\alpha_{2n-1, 2n-2}=t_{2n-1, 2n-1}t_{2n-2, 2n-2}^{-1} \text{ and } \alpha_{2n-1, 2n}=t_{2n-1, 2n-1}t_{2n, 2n}^{-1}
\end{aligned}
$$

has solutions, then $\varphi(A)=TAT^{-1}$ for all A in $\mathrm{Alg}\mathscr{L}_{2n(i)}$.

Put $t_{11}=1$ and $t_{2i+1, 2i+1}=1$. Then from the above relations

$$
\begin{aligned}
&t_{22}=\alpha_{12}^{-1},\ t_{33}=\alpha_{12}^{-1}\alpha_{32},\ t_{44}=\alpha_{12}^{-1}\alpha_{32}\alpha_{34}^{-1},\\
&\quad\vdots\\
&t_{2i-2, 2i-2}=\alpha_{12}^{-1}\alpha_{32}\alpha_{34}^{-1}\cdots\alpha_{2i-3, 2i-2}^{-1},\ t_{2i-1, 2i-1}=\alpha_{12}^{-1}\alpha_{32}\alpha_{34}^{-1}\cdots\alpha_{2i-1, 2i-2},\\
&t_{2i, 2i}=\alpha_{12}^{-1}\alpha_{32}\alpha_{34}^{-1}\cdots\alpha_{2i-1, 2i}^{-1},\ t_{2i+2, 2i+2}=\alpha_{2i+1, 2i+2}^{-1},\\
&t_{2i+3, 2i+3}=\alpha_{2i+1, 2i+2}^{-1}\alpha_{2i+3, 2i+2},\ t_{2i+4, 2i+4}=\alpha_{2i+1, 2i+2}^{-1}\alpha_{2i+3, 2i+2}\alpha_{2i+3, 2i+4}^{-1},
\end{aligned}
$$

$$\vdots$$

$$t_{2n-2,\,2n-2}=\alpha_{2i+1,\,2i+2}^{-1}\alpha_{2i+3,\,2i+2}\alpha_{2i+3,\,2i+4}^{-1}\cdots\alpha_{2n-3,\,2n-2}^{-1},$$

$$t_{2n-1,\,2n-1}=\alpha_{2i+1,\,2i+2}^{-1}\alpha_{2i+3,\,2i+2}\alpha_{2i+3,\,2i+4}^{-1}\cdots\alpha_{2n-3,\,2n-2}^{-1}\alpha_{2n-1,\,2n-2},$$

$$t_{2n,\,2n}=\alpha_{2i+1,\,2i+2}^{-1}\alpha_{2i+3,\,2i+2}\alpha_{2i+3,\,2i+4}^{-1}\cdots\alpha_{2i-3,\,2n-2}^{-1}\alpha_{2n-1,\,2n-2}\alpha_{2n-1,\,2n}^{-1}\ .$$

Hence $\varphi(A)=TAT^{-1}$ for all A in $\mathrm{Alg}\mathscr{L}_{2n(i)}$.

Lemma 2.4([10]). *Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be commutative subspace lattices on Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively, and suppose that $\varphi:\mathrm{Alg}\mathscr{L}_1\to\mathrm{Alg}\mathscr{L}_2$ is an algebraic isomorphism. Let $\mathscr{M}$ be a maximal abelian self-adjoint subalgebra (masa) contained in $\mathrm{Alg}\mathscr{L}_1$. Then there exist a bounded invertible operator $Y:\mathscr{H}_1\to\mathscr{H}_2$ and an automorphism $\rho:\mathrm{Alg}\mathscr{L}_1\to\mathrm{Alg}\mathscr{L}_1$ such that*

(i) *$\rho(M)=M$ for all M in $\mathscr{M}$ and*
(ii) *$\varphi(A)=Y\rho(A)Y^{-1}$ for all A in $\mathrm{Alg}\mathscr{L}_1$.*

Theorem 2.5. *Let $\varphi:\mathrm{Alg}\mathscr{L}_{2n(i)}\to\mathrm{Alg}\mathscr{L}_{2n(i)}$ be an isomorphism. Then there exists an invertible operator T such that $\varphi(A)=TAT^{-1}$ for all A in $\mathrm{Alg}\mathscr{L}_{2n(i)}$.*

Proof. Since $(\mathrm{Alg}\mathscr{L}_{2n(i)})\cap(\mathrm{Alg}\mathscr{L}_{2n(i)})^*$ is a masa of $\mathrm{Alg}\mathscr{L}_{2n(i)}$ and E_{ll} is in $(\mathrm{Alg}\mathscr{L}_{2n(i)})\cap(\mathrm{Alg}\mathscr{L}_{2n(i)})^*$ for all $l=1, 2, \dots, 2n$, by Lemma 2.4 there exist an invertible operator Y in $\mathscr{B}(\mathscr{H})$ and an automorphism ρ: $\mathrm{Alg}\mathscr{L}_{2n(i)}\to\mathrm{Alg}\mathscr{L}_{2n(i)}$ such that $\rho(E_{ll})=E_{ll}$ and $\varphi(A)=Y\rho(A)Y^{-1}$ for all A in $\mathrm{Alg}\mathscr{L}_{2n(i)}$ and $l(l=1, 2, \dots, 2n)$. By Theorem 2.3, $\rho(A)=SAS^{-1}$ for some invertible operator S. Hence $\varphi(A)=Y\rho(A)Y^{-1}=YSAS^{-1}Y^{-1}$. Let $T=YS$. Then $\varphi(A)=TAT^{-1}$ for all A in $\mathrm{Alg}\mathscr{L}_{2n(i)}$.

Theorem 2.6. *Let $\varphi:\mathrm{Alg}\mathscr{L}_{2n(i)}\to\mathrm{Alg}\mathscr{L}_{2n(i)}$ be an isomorphism. Then there exists an invertible operator T in $\mathrm{Alg}\mathscr{L}_{2n(i)}$, all of whose diagonal components are non-zero, such that $\varphi(A)=TAT^{-1}$ for all A in $\mathrm{Alg}\mathscr{L}_{2n(i)}$.*

Proof. Let $\varphi:\mathrm{Alg}\mathscr{L}_{2n(i)}\to\mathrm{Alg}\mathscr{L}_{2n(i)}$ be an isomorphism. Then by Theorem 2.5 there exists an invertible operator T such that $\varphi(A)=TAT^{-1}$ for all A in $\mathrm{Alg}\mathscr{L}_{2n(i)}$. Let $A=(a_{pj})$ and $\varphi(A)=(b_{pj})$ be in $\mathrm{Alg}\mathscr{L}_{2n(i)}$ and let $T=(t_{pj})$. Then $\varphi(A)T=TA\dots(*)$.

(1) $t_{2j,\,2m-1}=0$ for all j, m.

First, we will show that $t_{2j,\,1}=0$ for all j. Suppose that $t_{2j,\,1}\neq 0$ for some j. Comparing the $(2j,\ 1)$-component of $\varphi(A)T$ with that of TA, $t_{2j,\,1}b_{2j,\,2j}=t_{2j,\,1}a_{11}$. Since $t_{2j,\,1}\neq 0$, $a_{11}=b_{2j,\,2j}\dots(*_1)$.

Comparing the$(2j,\ 3)$-component of $\varphi(A)T$ with that of TA, $b_{2j,\,2j}t_{2j,\,3}=t_{2j,\,3}a_{33}$. So $t_{2j,\,3}(b_{2j,\,2j}-a_{33})=t_{2j,\,3}(a_{11}-a_{33})=0$ by $(*_1)$. Since the equation $(*)$ holds for all A in $\mathrm{Alg}\mathscr{L}_{2n(i)}$, $t_{2j,\,3}=0$.

Comparing the$(2j,\ 2)$-component of $\varphi(A)T$ with that of TA,

$$b_{2j,2j}t_{2j,2}=t_{2j,1}a_{12}+t_{2j,2}a_{22}+t_{2j,3}a_{32} \text{ if } i\neq 1 \text{ and}$$
$$b_{2j,2j}t_{2j,2}=t_{2j,1}a_{12}+t_{2j,2}a_{22} \text{ if } i=1.$$

Since $b_{2j,2j}=a_{11}$, $t_{2j,2}(a_{11}-a_{22})=t_{2j,1}a_{12}+t_{2j,3}a_{32}$ $(i\neq 1)$ and $t_{2j,2}(a_{11}-a_{22})=t_{2j,1}a_{12}$ $(i=1)$. Since the equation $(*)$ holds for all A in Alg$\mathscr{L}_{2n(i)}$, we have a contradiction if $a_{11}=a_{22}$ and $a_{12}\neq 0$. Thus $t_{2j,1}=0$. We want to show that if $t_{2j,2l-1}=0$, then $t_{2j,2l+1}=0$ for all l. Suppose that $t_{2j,2l-1}=0$ and $t_{2j,2l+1}\neq 0$ for some l. Comparing the $(2j, 2l+1)$-component of $\varphi(A)T$ with that of TA, $a_{2l+1,2l+1}=b_{2j,2j}$.

Comparing the $(2j, 2l)$-component of $\varphi(A)T$ with that of TA $(l\neq i)$,

$$b_{2j,2j}t_{2j,2l}=t_{2j,2l-1}a_{2l-1,2l}+t_{2j,2l}a_{2l,2l}+t_{2j,2l+1}a_{2l+1,2l}.$$

Since $t_{2j,2l-1}=0$ and $b_{2j,2j}=a_{2l+1,2l+1}$,

$$t_{2j,2l}(a_{2l+1,2l+1}-a_{2l,2l})=t_{2j,2l+1}a_{2l+1,2l} \ (l\neq i).$$

Since the equation $(*)$ holds for all A in Alg$\mathscr{L}_{2n(i)}$, we have a contradiction.

Comparing the $(2j, 2l+2)$-component of $\varphi(A)T$ with that of TA $(l=i)$,

$$b_{2j,2j}t_{2j,2l+2}=t_{2j,2l+1}a_{2l+1,2l+2}+t_{2j,2l+2}a_{2l+2,2l+2}+t_{2j,2l+3}a_{2l+3,2l+2}.$$

Since $b_{2j,2j}=a_{2l+1,2l+1}$,

$$t_{2j,2l+2}(a_{2l+1,2l+1}-a_{2l+2,2l+2})=t_{2j,2l+1}a_{2l+1,2l+2}+t_{2j,2l+3}a_{2l+3,2l+2}.$$

Since the equation $(*)$ holds for all A in Alg$\mathscr{L}_{2n(i)}$, we have a contradiction. Thus if $t_{2j,2l-1}=0$, then $t_{2j,2l+1}=0$ for all l. Therefore $t_{2j,2m-1}=0$ for all j, m.

If $t_{2j,2m}\neq 0$, then

(2) $a_{2m,2m}=b_{2j,2j}$,
(3) $t_{2j,l}=0$ for all $l\,(l\neq 2m)$ and
(4) $t_{2l,2m}=0$ for all $l\,(l\neq j)$.

For, comparing the $(2j, 2m)$-component of $\varphi(A)T$ with that of TA,

$$b_{2j,2j}t_{2j,2m}=t_{2j,2m-1}a_{2m-1,2m}+t_{2j,2m}a_{2m,2m}+t_{2j,2m+1}a_{2m+1,2m} \text{ if } m\neq i \text{ and}$$
$$b_{2j,2j}t_{2j,2m}=t_{2j,2m-1}a_{2m-1,2m}+t_{2j,2m}a_{2m,2m} \text{ if } m=i.$$

Since $t_{2j,2m}\neq 0$, $t_{2j,2m-1}=0$ and $t_{2j,2m+1}=0$, $b_{2j,2j}=a_{2m,2m}$. Similarly, we can prove the following.

(2) $t_{2j,l}=0$ for all $l\,(l\neq 2m)$ and
(3) $t_{2l,2m}=0$ for all $l\,(l\neq j)$.

Similarly, we can prove the following. If $t_{2j-1,2m-1}\neq 0$, then

(5) $b_{2j-1,2j-1}=a_{2m-1,2m-1}$,

(6) $t_{l,\,2m-1}=0$ for all $l(l\neq 2j-1)$ and
(7) $t_{2j-1,\,2l-1}=0$ for all $l(l\neq m)$.

If $t_{11}\neq 0$ and $t_{2i+1,\,2i+1}\neq 0$, then T is in Alg$\mathscr{L}_{2n(i)}$.

For, let $t_{11}\neq 0$. Comparing the (1, 2)-component of $\varphi(A)T$ with that of TA, $t_{22}\neq 0$.

Suppose that $t_{11}\neq 0$, $t_{22}\neq 0$, ..., $t_{2j,\,2j}\neq 0$. Comparing the $(2j+1,\,2j)$-component of $\varphi(A)T$ with that of $TA\,(j\neq i)$, $t_{2j+1,\,2j+1}\neq 0$.

Suppose that $t_{11}\neq 0$, $t_{22}\neq 0$, ..., $t_{2l+1,\,2l+1}\neq 0$. Comparing the $(2l+1,\,2l+2)$-component of $\varphi(A)T$ with that of TA, $t_{2l+2,\,2l+2}\neq 0$. If $t_{2l-1,\,2l-1}\neq 0$, $t_{2l,\,2l}\neq 0$ and $t_{2l+1,\,2l+1}\neq 0$, then we can get $t_{2j-1,\,2l}=0$ for all $j(j\neq l$ and $j\neq l+1)$ by comparing the $(2j-1,\,2l)$-component of $\varphi(A)T$ with that of TA.

Finally, suppose that $t_{2i+1,\,2i}\neq 0$. Comparing the $(2i+1,\,2i)$-component of $\varphi(A)T$ with that of TA,

$$b_{2i+1,\,2i}t_{2i,\,2i}+b_{2i+1,\,2i+1}t_{2i+1,\,2i}+b_{2i+1,\,2i+2}t_{2i+2,\,2i}$$
$$=t_{2i+1,\,2i-1}a_{2i-1,\,2i}+t_{2i+1,\,2i}a_{2i,\,2i}+t_{2i+1,\,2i+1}a_{2i+1,\,2i}.$$

Since $a_{2i+1,\,2i}=0$, $b_{2i+1,\,2i}=0$, $t_{2i+2,\,2i}=0$, $t_{2i+1,\,2i-1}=0$ and $b_{2i+1,\,2i+1}=a_{2i+1,\,2i+1}$, $t_{2i+1,\,2i}\cdot(a_{2i+1,\,2i+1}-a_{2i,\,2i})=0$.

Since the equation (*) holds for all A in Alg$\mathscr{L}_{2n(i)}$, we have a contradiction. Thus $t_{2i+1,\,2i}=0$. Hence T is in Alg$\mathscr{L}_{2n(i)}$.

We want to prove that $t_{11}\neq 0$ and $t_{2i+1,\,2i+1}\neq 0$.

It is easily verified that both $t_{2j-1,\,1}$ and $t_{2j-2,\,2}$ can not be non-zero, and both $t_{2j-1,\,1}$ and $t_{2j,\,2}$ can not be non-zero $(j\geq 2)$.

If $t_{11}=0$, then $t_{2k-1,\,1}\neq 0$ for some k. Suppose that $t_{2k-2,\,2}=0$ and $t_{2k,\,2}=0$. Comparing the $(2k-1,\,2)$-component of $\varphi(A)T$ with that of TA, we have $t_{2k-1,\,2}(a_{11}-a_{22})=t_{2k-1,\,1}a_{12}$ which is a contradiction. Thus $t_{2k-2,\,2}\neq 0$ or $t_{2k,\,2}\neq 0$... $(*_2)$. But this contradicts the just above fact. Hence $t_{11}\neq 0$. By a simple but tedious calculation, it is verified that both $t_{2l-1,\,2i+1}$ and $t_{2l-2,\,2i+2}$ cannot be non-zero, and both $t_{2l-1,\,2i+1}$ and $t_{2l,\,2i+2}$ cannot be non-zero $(l\neq i+1)$. Suppose that $t_{2i+1,\,2i+1}=0$. Then $t_{2l+1,\,2i+1}\neq 0$ for some $l(l=1,\,2,\,\ldots,\,n)$.

If $t_{2l+1,\,2i+1}\neq 0$ for some $l(l=1,\,2,\,\ldots,\,n)$, then with an argument similar to $(*_2)$ we have a contradiction. Hence $t_{2i+1,\,2i+1}\neq 0$.

Let $\mathscr{H}$ be a complex Hilbert space with an orthonormal basis $\{e_1,\,e_2,\,\ldots,\,e_{2n}\}$ and let $\mathscr{L}_{2n(i+1)}$ be the subspace lattice of orthogonal projections generated by $\{[e_1],\,[e_3],\,\ldots,\,[e_{2n-1}],\,[e_1,\,e_2,\,e_3],\,[e_3,\,e_4,\,e_5],\,\ldots,\,[e_{2i-1},\,e_{2i},\,e_{2i+1}],\,[e_{2i+2},\,e_{2i+3}],\,[e_{2i+3},\,e_{2i+4},\,e_{2i+5}],\,\ldots,\,[e_{2n-1},\,e_{2n}]\}$ $(i=0,\,1,\,2,\,\ldots,\,n-1)$ where $e_{-1}=0$ and $e_0=0$ whenever $i=0$.

Let $\mathscr{B}_{2n(i+1)}$ be the algebra consisting of all bounded operators, acting on a $2n$-dimensional complex Hilbert space $\mathscr{H}$, that are of the form

$$\begin{array}{c} \\ \\ \\ \\ 2i+1 \\ \\ \\ \\ \\ \\ \end{array}\begin{pmatrix} * & * & & & & \cdot & & \\ & * & & & & \cdot & & \\ & * & & & & \cdot & & \\ & & \ddots & & & & & \\ \cdot & \cdot & \cdot & * & * & 0 & & \\ & & & & & * & & \\ & & & & & * & & \\ & & & & & & \ddots & \\ & & & & & & & * \\ & & & & & & & * \end{pmatrix}^{\;2i+2},$$

where all non-starred entries are zero and with an orthonormal basis $\{e_1, e_2, \dots, e_{2n}\}$. By the arguments similar to those of Lemma 2. 1, Theorems 2. 2, 2. 3, 2. 5, we can obtain the following.

Lemma 2. 7. (1) $\mathrm{Alg}\mathscr{L}_{2n(i+1)}=\mathscr{B}_{2n(i+1)}$,
(2) $\mathrm{Lat}\mathscr{B}_{2n(i+1)}=\mathscr{L}_{2n(i+1)}$.

Theorem 2. 8. *Let* φ: $\mathrm{Alg}\mathscr{L}_{2n(i+1)}\to\mathrm{Alg}\mathscr{L}_{2n(i+1)}$ *be an isomorphism such that* $\varphi(E_{ll})=E_{ll}$ *for all* $l=1, 2, \dots, 2n$. *Then there exist non-zero complex numbers* α_{lj} *such that* $\varphi(E_{lj})=\alpha_{lj}E_{lj}$ *for all* E_{lj} *in* $\mathrm{Alg}\mathscr{L}_{2n(i+1)}$.

Theorem 2. 9. *Let* φ: $\mathrm{Alg}\mathscr{L}_{2n(i+1)}\to\mathrm{Alg}\mathscr{L}_{2n(i+1)}$ *be an isomorphism such that* $\varphi(E_{ll})=E_{ll}$ *for all* $l=1, 2, \dots, 2n$ *and let* $\varphi(E_{lj})=\alpha_{lj}E_{lj}$, $\alpha_{lj}\neq 0$ *for all* E_{lj} *in* $\mathrm{Alg}\mathscr{L}_{2n(i+1)}$. *Then there exists a diagonal operator* T *such that* $\varphi(A)=TAT^{-1}$ *for all* A *in* $\mathrm{Alg}\mathscr{L}_{2n(i+1)}$.

Theorem 2. 10. *Let* φ : $\mathrm{Alg}\mathscr{L}_{2n(i+1)}\to\mathrm{Alg}\mathscr{L}_{2n(i+1)}$ *be an isomorphism. Then there exists an invertible operator* T *such that* $\varphi(A)=TAT^{-1}$ *for all* A *in* $\mathrm{Alg}\mathscr{L}_{2n(i+1)}$.

Example 2. 11. Let

$$T=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Define φ: $\mathrm{Alg}\mathscr{L}_{2\times3(1+1)}\to\mathrm{Alg}\mathscr{L}_{2\times3(1+1)}$ by $\varphi(A)=TAT^{*}=TAT^{-1}$ for all A in $\mathrm{Alg}\mathscr{L}_{2\times3(1+1)}$. Then φ is spatially implemented but T is not in $\mathrm{Alg}\mathscr{L}_{2\times3(1+1)}$.

Example 2. 12. Let

$$T=\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Define φ: Alg$\mathscr{L}_{2\times 3(1+1)} \to$ Alg$\mathscr{L}_{2\times 3(1+1)}$ by $\varphi(A)=TAT^*=TAT^{-1}$ for all A in Alg$\mathscr{L}_{2\times 3(1+1)}$. Then φ is spatially implemented but T is not in Alg$\mathscr{L}_{2\times 3(1+1)}$.

Example 2.13. Let

$$T=\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Define φ: Alg$\mathscr{L}_{2\times 3(1+1)} \to$ Alg$\mathscr{L}_{2\times 3(1+1)}$ by $\varphi(A)=TAT^*=TAT^{-1}$ for all A in Alg$\mathscr{L}_{2\times 3(1+1)}$. Then φ is spatially implemented but T is not in Alg$\mathscr{L}_{2\times 3(1+1)}$.

Theorem 2.14. *Let* φ : Alg$\mathscr{L}_{2n(i+1)} \to$ Alg$\mathscr{L}_{2n(i+1)}$ *be an isomorphism. Then we can get the following*:

(1) $t_{2j,2m-1}=0$ *for all* j *and* m.

If $t_{2j,2m}\neq 0$, *then*

(2) $a_{2m,2m}=b_{2j,2j}$,
(3) $t_{2j,l}=0$ *for all* $l(l\neq 2m)$ *and*,
(4) $t_{2l,2m}=0$ *for all* $l(l\neq j)$.

If $t_{2j-1,2m-1}\neq 0$, *then*

(5) $a_{2m-1,2m-1}=b_{2j-1,2j-1}$,
(6) $t_{l,2m-1}=0$ *for all* $l(l\neq 2j-1)$ *and*
(7) $t_{2j-1,2l-1}=0$ *for all* $l(l\neq m)$.

Proof. (1) $t_{2j,2m-1}=0$ for all j and m.
First, we will show that $t_{2j,1}=0$ for all j. Suppose that $t_{2j,1}\neq 0$ for some j. Comparing the $(2j, 1)$-component of $\varphi(A)T$ with that of TA, $b_{2j,2j}t_{2j,1}=t_{2j,1}a_{11}$. Since $t_{2j,1}\neq 0$, $a_{11}=b_{2j,2j}$. Comparing the $(2j+1, 1)$-component of $\varphi(A)T$ with that of TA, $b_{2j+1,2j}t_{2j,1}+b_{2j+1,2j+1}t_{2j+1,1}+b_{2j+1,2j+2}t_{2j+2,1}=t_{2j+1,1}a_{11}$. Since $a_{11}=b_{2j,2j}$, we have $b_{2j+1,2j}t_{2j,1}=t_{2j+1,1}(b_{2j,2j}-b_{2j+1,2j+1})-b_{2j+1,2j+2}t_{2j+2,1}$ $(i\neq 1$ or $j\neq 1)$ and $b_{2j+1,2j}t_{2j,1}=t_{2j+1,1}(b_{2j,2j}-b_{2j+1,2j+1})$ $(i=1$ and $j=1)$. We have a contradiction. Hence $t_{2j,1}=0$ for all j.

Next we want to show that if $t_{2j,2l-1}=0$, then $t_{2j,2l+1}=0$ for all $l(l=2, \dots, n-1)$.

Suppose that $t_{2j,2l-1}=0$ and $t_{2j,2l+1}\neq 0$ for some $l(l=2, \dots, n-1)$. Comparing the $(2j, 2l+1)$-component of $\varphi(A)T$ with that of TA,

$$b_{2j,2j}t_{2j,2l+1}=t_{2j,2l+1}a_{2l+1,2l+1}.$$

Since $t_{2j,2l+1}\neq 0$, $b_{2j,2j}=a_{2l+1,2l+1}$.

Comparing the $(2j+1, 2l+1)$-component of $\varphi(A)T$ with that of TA,

$$b_{2j+1,2j}t_{2j,2l+1}+b_{2j+1,2j+1}t_{2j+1,2l+1}+b_{2j+1,2j+2}t_{2j+2,2l+1}=t_{2j+1,2l+1}a_{2l+1,2l+1}.$$

Since $a_{2l+1,2l+1}=b_{2j,2j}$,

$b_{2j+1,2j}t_{2j,2l+1}=t_{2j+1,2l+1}(b_{2j,2j}-b_{2j+1,2j+1})-b_{2j+1,2j+2}t_{2j+2,2l+1}$ $(i\neq 1$ or $j\neq 1)$ and
$b_{2j+1,2j}t_{2j,2l+1}=t_{2j+1,2l+1}(b_{2j,2j}-b_{2j+1,2j+1})$ $(i=1$ and $j=1)$.

We have a contradiction. Hence if $t_{2j,2l-1}=0$, then $t_{2j,2l+1}=0$ for all l.

By the arguments similar to those of Theorem 2.6, we can obtain the following.

If $t_{2j,2m}\neq 0$, then

(2) $a_{2m,2m}=b_{2j,2j}$,
(3) $t_{2j,l}=0$ for all $l(l\neq 2m)$ and
(4) $t_{2l,2m}=0$ for all $l(l\neq j)$.

If $t_{2j-1,2m-1}\neq 0$, then

(5) $b_{2j-1,2j-1}=a_{2m-1,2m-1}$,
(6) $t_{l,2m-1}=0$ for all $l(l\neq 2j-1)$ and
(7) $t_{2j-1,2l-1}=0$ for all $l(l\neq m)$.

As special cases, we will consider the cases that are $i=0$ and $i=n-1$.

Theorem 2.15. *Let* $\varphi:\mathrm{Alg}\mathscr{L}_{2n(0+1)}\to\mathrm{Alg}\mathscr{L}_{2n(0+1)}$ *be an isomorphism. Then there exists an invertible operator* S *in* $\mathrm{Alg}\mathscr{L}_{2n(0+1)}$ *such that* $\varphi(A)=SAS^{-1}$ *or* $\varphi(A)=(SU)A(SU)^{-1}$ *for all* A *in* $\mathrm{Alg}\mathscr{L}_{2n(0+1)}$, *where*

$$U=\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}$$

is a $2n\times 2n$ *matrix.*

Proof. Let $\varphi:\mathrm{Alg}\mathscr{L}_{2n(0+1)}\to\mathrm{Alg}\mathscr{L}_{2n(0+1)}$ be an isomorphism. Then by Theorem 2.10, there exists an invertible operator T such that $\varphi(A)=$

TAT^{-1} for all A in $\mathrm{Alg}\mathcal{L}_{2n(0+1)}$. Let $A=(a_{ij})$ and $\varphi(A)=(b_{ij})$ be in $\mathrm{Alg}\mathcal{L}_{2n(0+1)}$ and let $T=(t_{ij})$. Then $\varphi(A)T=TA$ for all A in $\mathrm{Alg}\mathcal{L}_{2n(0+1)}$.

We can get the following by Theorem 2.14:

(1) $t_{2j,2m-1}=0$ for all j and m.

If $t_{2j,2m}\neq 0$, then

(2) $a_{2m,2m}=b_{2j,2j}$,
(3) $t_{2j,l}=0$ for all $l(l\neq 2m)$ and
(4) $t_{2l,2m}=0$ for all $l(l\neq j)$.

If $t_{2j-1,2m-1}\neq 0$, then

(5) $a_{2m-1,2m-1}=b_{2j-1,2j-1}$,
(6) $t_{l,2m-1}=0$ for all $l(l\neq 2j-1)$ and
(7) $t_{2j-1,2l-1}=0$ for all $l(l\neq m)$.

If $t_{2l-1,2l-1}\neq 0$, $t_{2l,2l}\neq 0$ and $t_{2l+1,2l+1}\neq 0$, then

(8) $t_{2j-1,2l}=0$ for all $j(j\neq l$ and $j\neq l+1)$.

We want to prove that $t_{12}=0$ if $t_{11}\neq 0$. Suppose that $t_{11}\neq 0$ and $t_{12}\neq 0$. Comparing the $(1,2)$-component of $\varphi(A)T$ with that of TA, $b_{11}t_{12}=t_{12}a_{22}+t_{13}a_{32}$. Since $t_{11}\neq 0$, $b_{11}=a_{11}$. Since $t_{13}=0$, we have $t_{12}(a_{22}-a_{11})=0$ which is a contradiction. Hence $t_{12}=0$. If $t_{11}\neq 0$ and $t_{22}\neq 0$, then $t_{kk}\neq 0$ for all $k(k=1,2,\ldots,2n)$. Hence if $t_{11}\neq 0$ and $t_{22}\neq 0$, then T is $\mathrm{Alg}\mathcal{L}_{2n(0+1)}$ by (1), (2), ..., (8). So we can take $S=T$ in this case. If $t_{11}\neq 0$ and $t_{22}=0$, then let

$$U=\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}$$

be a $2n\times 2n$ matrix. Define $\varphi_1:\mathrm{Alg}\mathcal{L}_{2n(0+1)}\to\mathrm{Alg}\mathcal{L}_{2n(0+1)}$ by $\varphi_1(A)=UAU^{-1}$ for all A in $\mathrm{Alg}\mathcal{L}_{2n(0+1)}$. Then φ_1 is an isomorphism and $\varphi_1\varphi(A)=(UT)A(UT)^{-1}$ for all A in $\mathrm{Alg}\mathcal{L}_{2n(0+1)}$. If $t_{2n,2}\neq 0$, then UT belongs to $\mathrm{Alg}\mathcal{L}_{2n(0+1)}$ because $t_{2n,2}$ is the $(2,2)$-component of UT. In this case, we can take $S=TU$. Since $U^2=I$, $S=U(UT)U$ and so belongs to $\mathrm{Alg}\mathcal{L}_{2n(0+1)}$ and $T=SU$. Hence $\varphi(A)=TAT^{-1}=(SU)A(SU)^{-1}$ for all A in $\mathrm{Alg}\mathcal{L}_{2n(0+1)}$.

We want to prove that $t_{11}\neq 0$, and $t_{2n,2}\neq 0$ if $t_{22}=0$.

First, we will prove that $t_{2n,2}\neq 0$ if $t_{22}=0$.

Suppose that $t_{2,2j+2}\neq 0$, $t_{3,2j+1}=0$ and $t_{3,2j+3}=0$ for some $j(j=1,2,\ldots,n-2)$. Comparing the $(3,2j+2)$-component of $\varphi(A)T$ with that of TA,

$$b_{32}t_{2,2j+2}+b_{33}t_{3,2j+2}+b_{34}t_{4,2j+2}=t_{3,2j+1}a_{2j+1,2j+2}+t_{3,2j+2}a_{2j+2,2j+2}+t_{3,2j+3}a_{2j+3,2j+2}.$$

Since $t_{3\,2j+1}=0$, $t_{3\,2j+3}=0$ and $t_{2\,2j+2}\neq 0$, $t_{4\,2j+2}=0$ and hence we have $t_{3\,2j+2}(b_{22}-b_{33})=b_{32}t_{2\,2j+2}$ which is a contradiction. Hence if $t_{2\,2j+2}\neq 0$, then $t_{3\,2j+1}\neq 0$ or $t_{3\,2j+3}\neq 0$ for some $j(j=1, 2, \ldots, n-2)$. Suppose that $t_{2\,2j+2}\neq 0$, $t_{3\,2j+1}\neq 0$ and $t_{4\,2j}=0$. Comparing the $(3, 2j)$-component of $\varphi(A)T$ with that of TA,

$$b_{32}t_{2\,2j}+b_{33}t_{3\,2j}+b_{34}t_{4\,2j}=t_{3\,2j-1}a_{2j-1\,2j}+t_{3\,2j}a_{2j\,2j}+t_{3\,2j+1}a_{2j+1\,2j}.$$

Since $t_{2\,2j+2}\neq 0$, $t_{4\,2j}=0$ and $t_{3\,2j+1}\neq 0$, $t_{2\,2j}=0$, $t_{3\,2j-1}=0$ and $b_{33}=a_{2j+1\,2j+1}$. So we have $t_{3\,2j}(a_{2j+1\,2j+1}-a_{2j\,2j})=t_{3\,2j+1}a_{2j+1\,2j}$ which is a contradiction. If we continue this process, then we can get the following ; If $t_{2\,2j+2}\neq 0$ and $t_{3\,2j+1}\neq 0$, then $t_{2+l,\,2j+2-l}\neq 0$ for all $l(l=1, 2, \ldots, 2j)$. In particular, $t_{2j+2\,2}\neq 0$ and $t_{2j+1\,3}\neq 0$. Comparing the $(2j+3, 2)$-component of $\varphi(A)T$ with that of TA,

$$b_{2j+3\,2j+2}t_{2j+2\,2}+b_{2j+3\,2j+3}t_{2j+3\,2}+b_{2j+3\,2j+4}t_{2j+4\,2}=t_{2j+3\,1}a_{12}+t_{2j+3\,2}a_{22}+t_{2j+3\,3}a_{32}.$$

Since $t_{2j+1\,3}\neq 0$, $t_{2j+3\,3}=0$. So we have $t_{2j+3\,2}(b_{2j+2\,2j+2}-b_{2j+3\,2j+3})=b_{2j+3\,2j+2}t_{2j+2\,2}$ which is a contradiction. Suppose that $t_{2\,2j+2}\neq 0$, $t_{3\,2j+3}\neq 0$ and $t_{4\,2j+4}=0$. Comparing the $(3, 2j+4)$-component of $\varphi(A)T$ with that of TA,

$$b_{32}t_{2\,2j+4}+b_{33}t_{3\,2j+4}+b_{34}t_{4\,2j+4}=t_{3\,2j+3}a_{2j+3\,2j+4}+t_{3\,2j+4}a_{2j+4\,2j+4}+t_{3\,2j+5}a_{2j+5\,2j+4}.$$

Since $t_{2\,2j+2}\neq 0$, $t_{2\,2j+4}=0$. Since $t_{3\,2j+3}\neq 0$, $t_{3\,2j+5}=0$. Since $t_{3\,2j+3}\neq 0$, $b_{33}=a_{2j+3\,2j+3}$. So we have $t_{3\,2j+4}(a_{2j+3\,2j+3}-a_{2j+4\,2j+4})=t_{3\,2j+3}a_{2j+3\,2j+4}$ which is a contradiction.

If we continue this process, we can get the following ; If $t_{2\,2j+2}\neq 0$ and $t_{3\,2j+3}\neq 0$, then $t_{2+l,\,2j+2+l}\neq 0$ for all $l(l=1, 2, \ldots, 2n-2j-2)$. In particular, $t_{2n-2j,\,2n}\neq 0$ and $t_{2n-2j-1\,2n-1}\neq 0$. Comparing the $(2n-2j+1, 2n)$-component of $\varphi(A)T$ with that of TA,

$$\begin{aligned}b_{2n-2j+1\,2n-2j}t_{2n-2j\,2n}&+b_{2n-2j+1\,2n-2j+1}t_{2n-2j+1\,2n}+b_{2n-2j+1,\,2n-2j+2}t_{2n-2j+2\,2n}\\&=t_{2n-2j+1\,2n-1}a_{2n-1,\,2n}+t_{2n-2j+1\,2n}a_{2n,\,2n}.\end{aligned}$$

Since $t_{2n-2j,\,2n}\neq 0$ and $t_{2n-2j-1\,2n-1}\neq 0$, $t_{2n-2j+2\,2n}=0$ and $t_{2n-2j+1\,2n-1}=0$ and $a_{2n,\,2n}=b_{2n-2j,\,2n-2j}$. So we have $t_{2n-2j+1,\,2n}(b_{2n-2j,\,2n-2j}-b_{2n-2j+1\,2n-2j+1})=b_{2n-2j+1\,2n-2j}t_{2n-2j,\,2n}$ which is a contradiction. Therefore if $t_{22}=0$, then $t_{2\,2l}=0$ for all $l=2, \ldots, n-1$. Hence $t_{2\,2n}\neq 0$ if $t_{22}=0$.

Suppose that $t_{2\,2n}\neq 0$ and $t_{3\,2n-1}=0$. Comparing the $(3, 2n)$-component of $\varphi(A)T$ with that of TA,

$$b_{32}t_{2\,2n}+b_{33}t_{3\,2n}+b_{34}t_{4\,2n}=t_{3\,2n-1}a_{2n-1\,2n}+t_{3\,2n}a_{2n,\,2n}.$$

Since $t_{2\,2n}\neq 0$, $t_{4\,2n}=0$ and $a_{2n,\,2n}=b_{22}$. So we have $t_{3\,2n}(b_{22}-b_{33})=b_{32}t_{2\,2n}$ which is a contradiction. Hence if $t_{2\,2n}\neq 0$, then $t_{3\,2n-1}\neq 0$. If $t_{2\,2n}\neq 0$ and $t_{3\,2n-1}\neq 0$, then $t_{4\,2n-2}\neq 0$. Suppose that $t_{2\,2n}\neq 0$, $t_{3\,2n-1}\neq 0$ and $t_{4\,2n-2}=0$. Comparing the $(3, 2n-2)$-component of $\varphi(A)T$ with that of TA,

$$b_{32}t_{2\,2n-2}+b_{33}t_{3\,2n-2}+b_{34}t_{4\,2n-2}=t_{3\,2n-3}a_{2n-3\,2n-2}+t_{3\,2n-2}a_{2n-2\,2n-2}+t_{3\,2n-1}a_{2n-1\,2n-2}.$$

Since $t_{2\,2n}\neq 0$, $t_{2\,2n-2}=0$ and $t_{4\,2n-2}=0$. Since $t_{3\,2n-1}\neq 0$, $b_{33}=a_{2n-1,\,2n-1}$ and $t_{3\,2n-3}=$

0. So we have $t_{3,2n-2}(a_{2n-1,2n-1}-a_{2n-2,2n-2})=t_{3,2n-1}a_{2n-1,2n-2}$ which is a contradiction. If we continue this process, we can get the following; If $t_{2,2n}\neq 0$, then $t_{2+l,2n-l}\neq 0$ for all $l(l=1,2,\ldots,2n-2)\ldots(*)$. In particular, $t_{2n,2}$ is non-zero. In consequence, if t_{22} is zero, then $t_{2n,2}$ is non-zero.

Finally, we want to show that t_{11} is non-zero.

Suppose $t_{13}\neq 0$. Comparing the (1, 2)-component of $\varphi(A)T$ with that of TA, $b_{11}t_{12}+b_{12}t_{22}=t_{11}a_{12}+t_{12}a_{22}+t_{13}a_{32}$. Since $t_{13}\neq 0$, $a_{33}=b_{11}$. So $t_{12}(a_{33}-a_{22})=t_{13}a_{32}$ which is a contradiction. Hence $t_{13}=0$. Similarly we can prove that $t_{12}=0$ by comparing the (1, 2)-component of $\varphi(A)T$ with that of TA.

Suppose that $t_{1,2k+1}\neq 0$ for some $k(k=2,\ldots,n-1)$. Comparing the (1, $2k$)-component of $\varphi(A)T$ with that of TA,

$$b_{11}t_{1,2k}+b_{12}t_{2,2k}=t_{1,2k-1}a_{2k-1,2k}+t_{1,2k}a_{2k,2k}+t_{1,2k+1}a_{2k+1,2k}.$$

Since $t_{1,2k+1}\neq 0$, $b_{11}=a_{2k+1,2k+1}$ and $t_{1,2k-1}=0$. So we have $t_{1,2k}(a_{2k+1,2k+1}-a_{2k,2k})=t_{1,2k+1}a_{2k+1,2k}$ which is a contradiction. Hence $t_{1,2k+1}=0$ for all $k=1,2,\ldots,n-1$.

Suppose that $t_{1,2k}\neq 0$ for some $k(k=2,\ldots,n)$. Comparing the (1, $2k$)-component of $\varphi(A)T$ with that of TA,

$$b_{11}t_{1,2k}+b_{12}t_{2,2k}=t_{1,2k-1}a_{2k-1,2k}+t_{1,2k}a_{2k,2k}+t_{1,2k+1}a_{2k+1,2k}.$$

Since $b_{12}=0$, $t_{1,2k-1}=0$ and $t_{1,2k+1}=0$, $b_{11}t_{1,2k}=a_{2k,2k}t_{1,2k}\ldots(*_1)$. If $t_{22}=0$, then $t_{2+2n-2k,2k}\neq 0$ by $(*)$. So $a_{2k,2k}=b_{2+2n-2k,2+2n-2k}$. Hence $t_{1,2k}(b_{2+2n-2k,2+2n-2k}-b_{11})=0$ by $(*_1)$ which is a contradiction. Suppose that $t_{22}\neq 0$ and $t_{33}=0$. Comparing the (3, 2)-component of $\varphi(A)T$ with that of TA,

$$b_{32}t_{22}+b_{33}t_{32}+b_{34}t_{42}=t_{31}a_{12}+t_{32}a_{22}+t_{33}a_{32}.$$

Since $t_{22}\neq 0$, $t_{42}=0$ and $a_{22}=b_{22}$. So we have $t_{32}(b_{22}-b_{33})=b_{32}t_{22}$ which is a contradiction. Thus if $t_{22}\neq 0$, then $t_{33}\neq 0$. Suppose that $t_{22}\neq 0$, $t_{33}\neq 0$ and $t_{44}=0$. Comparing the (3, 4)-component of $\varphi(A)T$ with that of TA,

$$b_{32}t_{24}+b_{33}t_{34}+b_{34}t_{44}=t_{33}a_{34}+t_{34}a_{44}+t_{35}a_{54}.$$

Since $t_{33}\neq 0$, $b_{33}=a_{33}$ and $t_{35}=0$. Since $t_{22}\neq 0$, $t_{24}=0$. So we get $t_{34}(a_{33}-a_{44})=a_{34}t_{33}$ which is a contradiction. So if $t_{22}\neq 0$ and $t_{33}\neq 0$, then $t_{44}\neq 0$. Hence if we continue this process, then we can get the following; If $t_{22}\neq 0$, then $t_{ll}\neq 0$ for all $l(l=2,3,\ldots,2n)$. Since $t_{2k,2k}\neq 0$, $a_{2k,2k}=b_{2k,2k}$. So $t_{1,2k}(b_{2k,2k}-b_{11})=0$ by $(*_1)$ which is a contradiction. Hence $t_{11}\neq 0$.

Theorem 2.16. *Let* $\varphi:\mathrm{Alg}\mathscr{L}_{2n((n-1)+1)}\to\mathrm{Alg}\mathscr{L}_{2n((n-1)+1)}$ *be an isomorphism. Then there exists an invertible operator* S *in* $\mathrm{Alg}\mathscr{L}_{2n((n-1)+1)}$ *such that* $\varphi(A)=SAS^{-1}$ *or* $\varphi(A)=(SU)A(SU)^{-1}$ *for all* A *in* $\mathrm{Alg}\mathscr{L}_{2n((n-1)+1)}$, *where*

$$U=\begin{pmatrix} 0 & \cdot & \cdot & \cdot & \cdots & \cdot & 0 & 1 & 0 \\ 0 & \cdot & \cdot & \cdot & \cdots & 0 & 1 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & 1 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & \cdot & \cdot & \cdots & \cdot & 0 & 0 & 0 \\ 0 & \cdot & \cdot & \cdot & \cdots & \cdot & 0 & 0 & 1 \end{pmatrix}$$

is a $2n\times 2n$ *matrix.*

Proof. Let φ: $\mathrm{Alg}\mathcal{L}_{2n((n-1)+1)}\rightarrow\mathrm{Alg}\mathcal{L}_{2n((n-1)+1)}$ be an isomorphism. Then by Theorem 2.10, there exists an invertible operator T such that $\varphi(A)=TAT^{-1}$ for all A in $\mathrm{Alg}\mathcal{L}_{2n((n-1)+1)}$. Let $A=(a_{pj})$ and $\varphi(A)=(b_{pj})$ be in $\mathrm{Alg}\mathcal{L}_{2n((n-1)+1)}$ and let $T=(t_{pj})$. Then $\varphi(A)T=TA$ for all A in $\mathrm{Alg}\mathcal{L}_{2n((n-1)+1)}$. We can get the following by Theorem 2.14.

(1) $t_{2j,2m-1}=0$ for all j and m.

If $t_{2j,2m}\neq 0$, then

(2) $a_{2m,2m}=b_{2j,2j}$,
(3) $t_{2j,l}=0$ for all $l(l\neq 2m)$ and
(4) $t_{2l,2m}=0$ for all $l(l\neq j)$.

If $t_{2j-1,2m-1}\neq 0$, then

(5) $a_{2m-1,2m-1}=b_{2j-1,2j-1}$,
(6) $t_{l,2m-1}=0$ for all $l(l\neq 2j-1)$ and
(7) $t_{2j-1,2l-1}=0$ for all $l(l\neq m)$.

If $t_{2l-1,2l-1}\neq 0$, $t_{2l,2l}\neq 0$ and $t_{2l+1,2l+1}\neq 0$, then (8) $t_{2j-1,2l}=0$ for all $j(j\neq l$ and $j\neq l+1)$. If $t_{2n-1,2n-1}\neq 0$, then $t_{2n-2,2n-2}\neq 0$. Suppose that $t_{2n-1,2n-1}\neq 0$ and $t_{2n-2,2n-2}=0$. Comparing the $(2n-1,2n-2)$-component of $\varphi(A)T$ with that of TA,

$$\begin{aligned} &b_{2n-1,2n-2}t_{2n-2,2n-2}+b_{2n-1,2n-1}t_{2n-1,2n-2} \\ &\quad =t_{2n-1,2n-3}a_{2n-3,2n-2}+t_{2n-1,2n-2}a_{2n-2,2n-2}+t_{2n-1,2n-1}a_{2n-1,2n-2}. \end{aligned}$$

Since $t_{2n-1,2n-1}\neq 0$, $b_{2n-1,2n-1}=a_{2n-1,2n-1}$ and $t_{2n-1,2n-3}=0$. So we get $t_{2n-1,2n-2}\cdot(a_{2n-1,2n-1}-a_{2n-2,2n-2})=t_{2n-1,2n-1}a_{2n-1,2n-2}$ which is a contradiction.

Similarly, we can prove that if $t_{2n-1,2n-1}\neq 0$ and $t_{2n-2,2n-2}\neq 0$, then $t_{2n-3,2n-3}\neq 0$.

So if $t_{2n,2n}\neq 0$ and $t_{2n-1,2n-1}\neq 0$, then $t_{kk}\neq 0$ for all $k(k=1,2,\ldots,2n)$ and $t_{2n-1,2n}=0$ by comparing the $(2n-1,2n)$-component of $\varphi(A)T$ with that of TA. Hence T is in $\mathrm{Alg}\mathcal{L}_{2n((n-1)+1)}$ by (1), (2), ..., (8). In this case, we can take $S=T$. If $t_{2n,2n}\neq 0$ and $t_{2n-1,2n-1}=0$, then let

$$U=\begin{pmatrix} 0 & \cdot & \cdot & \cdot & \cdots & \cdot & 0 & 1 & 0 \\ 0 & \cdot & \cdot & \cdot & \cdots & 0 & 1 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & 1 & 0 & 0 & 0 \\ \cdots & & & & & & & & \cdots \\ 1 & 0 & \cdot & \cdot & \cdots & \cdot & 0 & 0 & 0 \\ 0 & \cdot & \cdot & \cdot & \cdots & \cdot & 0 & 0 & 1 \end{pmatrix}$$

be a $2n\times 2n$ matrix. Define φ_1: $\mathrm{Alg}\mathscr{L}_{2n((n-1)+1)}\to\mathrm{Alg}\mathscr{L}_{2n((n-1)+1)}$ by $\varphi_1(A)=UAU^{-1}$ for all A in $\mathrm{Alg}\mathscr{L}_{2n((n-1)+1)}$. Then φ_1 is an isomorphism and $\varphi_1\varphi(A)=(UT)A(UT)^{-1}$ for all A in $\mathrm{Alg}\mathscr{L}_{2n((n-1)+1)}$ and $t_{1,2n-1}$ is the $(2n-1, 2n-1)$-component of UT. If $t_{1,2n-1}\neq 0$, then UT belongs to $\mathrm{Alg}\mathscr{L}_{2n((n-1)+1)}$. In this case we can take $S=TU$. Since $U^2=I$, $S=U(UT)U$ and so belongs to $\mathrm{Alg}\mathscr{L}_{2n((n-1)+1)}$ and $T=SU$. Hence $\varphi(A)=TAT^{-1}=(SU)A(SU)^{-1}$ for all A in $\mathrm{Alg}\mathscr{L}_{2n((n-1)+1)}$. With the same proof as some part of Theorem 2.15, we can prove that if $t_{2n-1,2n-1}=0$, then $t_{2n,2n}\neq 0$ and $t_{1,2n-1}\neq 0$.

Let I_m be the $m\times m$ identity matrix and let J_m be the $m\times m$ backward identity matrix.

Theorem 2.17. *Let φ : $\mathrm{Alg}\mathscr{L}_{2n(i+1)}\to\mathrm{Alg}\mathscr{L}_{2n(i+1)}$ be an isomorphism $(1\leq i\leq n-2)$. Then there exists an invertible operator S in $\mathrm{Alg}\mathscr{L}_{2n(i+1)}$ such that $\varphi(A)=SAS^{-1}$ or $\varphi(A)=(SU)A(SU)^{-1}$ or $\varphi(A)=(SV)A(SV)^{-1}$ or $\varphi(A)=(SW)A(SW)^{-1}$, where*

$$U=\begin{pmatrix} I_{2i+1} & 0 \\ 0 & J_{2n-2i-1}\end{pmatrix},\ V=\begin{pmatrix} J_{2i+1} & 0 \\ 0 & I_{2n-2i-1}\end{pmatrix} \text{ and } W=\begin{pmatrix} J_{2i+1} & 0 \\ 0 & J_{2n-2i-1}\end{pmatrix}.$$

Proof. Let φ: $\mathrm{Alg}\mathscr{L}_{2n(i+1)}\to\mathrm{Alg}\mathscr{L}_{2n(i+1)}$ be an isomorphism. Then by Theorem 2.10, there exists an invertible operator T such that $\varphi(A)=TAT^{-1}$ for all A in $\mathrm{Alg}\mathscr{L}_{2n(i+1)}$. Let $A=(a_{pj})$ and $\varphi(A)=(b_{pj})$ be in $\mathrm{Alg}\mathscr{L}_{2n(i+1)}$ and let $T=(t_{pj})$. Then $\varphi(A)T=TA$ for all A in $\mathrm{Alg}\mathscr{L}_{2n(i+1)}$. By Theorem 2.14, we can get the following:

(1) $t_{2j,2m-1}=0$ for all j and m.

If $t_{2j,2m}\neq 0$, then

(2) $a_{2m,2m}=b_{2j,2j}$,
(3) $t_{2j,l}=0$ for all $l(l\neq 2m)$ and
(4) $t_{2l,2m}=0$ for all $l(l\neq j)$.

If $t_{2j-1,2m-1}\neq 0$, then

(5) $a_{2m-1,2m-1}=b_{2j-1,2j-1}$,
(6) $t_{l,2m-1}=0$ for all $l(l\neq 2j-1)$ and
(7) $t_{2j-1,2l-1}=0$ for all $l(l\neq m)$.

If $t_{2l-1,2l-1}\neq 0$, $t_{2l,2l}\neq 0$ and $t_{2l+1,2l+1}\neq 0$, then

(8) $t_{2j-1,2l}=0$ for all $j(j\neq l$ and $j\neq l+1)$.

If $t_{11}\neq 0$ and $t_{2i+2,2i+2}\neq 0$, then T is in $\mathrm{Alg}\mathscr{L}_{2n(i+1)}$ as in the proof of Theorem 2.15. In this case, we can take $S=T$. If $t_{11}\neq 0$ and $t_{2i+2,2i+2}=0$, then let $U=\begin{pmatrix} I_{2i+1} & 0 \\ 0 & J_{2n-2i-1}\end{pmatrix}$. Define $\varphi_1:\mathrm{Alg}\mathscr{L}_{2n(i+1)}\rightarrow\mathrm{Alg}\mathscr{L}_{2n(i+1)}$ by $\varphi_1(A)=UAU^{-1}$ for all A in $\mathrm{Alg}\mathscr{L}_{2n(i+1)}$. Then φ_1 is an isomorphism, $\varphi_1\varphi(A)=(UT)A(UT)^{-1}$ for all A in $\mathrm{Alg}\mathscr{L}_{2n(i+1)}$ and $t_{2n,2i+2}$ is the $(2i+2, 2i+2)$-component of UT. If $t_{2n,2i+2}\neq 0$, then UT belongs to $\mathrm{Alg}\mathscr{L}_{2n(i+1)}$. In this case, we can take $S=TU$. Since $U^2=I$, $S=U(UT)U$ and so belongs to $\mathrm{Alg}\mathscr{L}_{2n(i+1)}$ and $T=SU$. Hence $\varphi(A)=TAT^{-1}=(SU)A(SU)^{-1}$ for all A in $\mathrm{Alg}\mathscr{L}_{2n(i+1)}$. If $t_{11}=0$ and $t_{2i+2,2i+2}\neq 0$, then let $V=\begin{pmatrix} J_{2i+1} & 0 \\ 0 & I_{2n-2i-1}\end{pmatrix}$. Define $\varphi_2:\mathrm{Alg}\mathscr{L}_{2n(i+1)}\rightarrow\mathrm{Alg}\mathscr{L}_{2n(i+1)}$ by $\varphi_2(A)=VAV^{-1}$ for all A in $\mathrm{Alg}\mathscr{L}_{2n(i+1)}$. Then φ_2 is an isomorphism, $\varphi_2\varphi(A)=(VT)A(VT)^{-1}$ for all A in $\mathrm{Alg}\mathscr{L}_{2n(i+1)}$ and $t_{2i+1,1}$ is the $(1, 1)$-component of VT. If $t_{2i+1,1}\neq 0$, then VT belongs to $\mathrm{Alg}\mathscr{L}_{2n(i+1)}$. In this case, we can take $S=TV$. Since $V^2=I$, $S=V(VT)V$ and so belongs to $\mathrm{Alg}\mathscr{L}_{2n(i+1)}$ and $T=SV$. Hence $\varphi(A)=TAT^{-1}=(SV)A(SV)^{-1}$ for all A in $\mathrm{Alg}\mathscr{L}_{2n(i+1)}$.

If $t_{11}=0$ and $t_{2i+2,2i+2}=0$, then let $W=\begin{pmatrix} J_{2i+1} & 0 \\ 0 & J_{2n-2i-1}\end{pmatrix}$. Define $\varphi_3:\mathrm{Alg}\mathscr{L}_{2n(i+1)}\rightarrow\mathrm{Alg}\mathscr{L}_{2n(i+1)}$ by $\varphi_3(A)=WAW^{-1}$ for all A in $\mathrm{Alg}\mathscr{L}_{2n(i+1)}$. Then φ_3 is an isomorphism, $\varphi_3\varphi(A)=(WT)A(WT)^{-1}$ for all A in $\mathrm{Alg}\mathscr{L}_{2n(i+1)}$, $t_{2i+1,1}$ is the $(1, 1)$-component of WT and $t_{2n,2i+2}$ is the $(2i+2, 2i+2)$-component of WT. If $t_{2i+1,1}\neq 0$ and $t_{2n,2i+2}\neq 0$, then WT belongs to $\mathrm{Alg}\mathscr{L}_{2n(i+1)}$. In this case, we can take $S=TW$. Since $W^2=I$, $S=W(WT)W$ and so belongs to $\mathrm{Alg}\mathscr{L}_{2n(i+1)}$ and $T=SW$. Hence $\varphi(A)=TAT^{-1}=(SW)A(SW)^{-1}$ for all A in $\mathrm{Alg}\mathscr{L}_{2n(i+1)}$.

We want to prove that $t_{2i+1,1}\neq 0$ if $t_{11}=0$ and $t_{2n,2i+2}\neq 0$ if $t_{2i+2,2i+2}=0$.

First we want to prove that if $t_{11}=0$, then $t_{2i+1,1}\neq 0$.

Suppose that $t_{11}=0$. Then $t_{2k+1,1}\neq 0$ for some $k(k=1, 2, \ldots, n-1)$. If $t_{2k+1,1}\neq 0$, then we can prove that $t_{2k,2}\neq 0$ or $t_{2k+2,2}\neq 0(k\neq i, k=1, 2, \ldots, n-1)$ by comparing the $(2k+1, 2)$-component of $\varphi(A)T$ with that of TA. If $t_{2k+1,1}\neq 0$ and $t_{2k,2}\neq 0$, then $t_{2k-1,3}\neq 0$ by comparing the $(2k-1, 2)$-component of $\varphi(A)T$ with that of TA. If $t_{2k,2}\neq 0$ and $t_{2k-1,3}\neq 0$, then $t_{2k-2,4}\neq 0$ by comparing the $(2k-1, 4)$-component of $\varphi(A)T$ with that of TA. If we continue this process, then we can prove that if $t_{2k+1,1}\neq 0$ and $t_{2k,2}\neq 0$, then $t_{2k+1-l,1+l}\neq 0$ for all $l(l=1, 2, \ldots, 2k)$...(**i**).

If $t_{2k+1,1}\neq 0$ and $t_{2k+2,2}\neq 0$, then $t_{2k+3,3}\neq 0$ by comparing the $(2k+3, 2)$-component of $\varphi(A)T$ with that of TA. If $t_{2k+2,2}\neq 0$ and $t_{2k+3,3}\neq 0$, then $t_{2k+4,4}\neq 0$ by comparing the $(2k+3, 4)$-component of $\varphi(A)T$ with that of TA. So if we continue this process, then we can prove that if $t_{2k+1,1}\neq 0$ and $t_{2k+2,2}\neq 0$, then $t_{2k+l,l}\neq 0$ for all $l(l=1, 2, \ldots, 2n-2k)$...(**ii**). In the case (**i**), if $k<i$, then

$t_{1,\,2k+1} \neq 0$ and $t_{2,\,2k} \neq 0$. Comparing the $(1,\,2k+2)$-component of $\varphi(A)T$ with that of TA,

$$b_{11}t_{1,\,2k+2}+b_{12}t_{2,\,2k+2}=t_{1,\,2k+1}a_{2k+1,\,2k+2}+t_{1,\,2k+2}a_{2k+2,\,2k+2}+t_{1,\,2k+3}a_{2k+3,\,2k+2}.$$

Since $b_{11}=a_{2k+1,\,2k+1}$, we have $t_{1,\,2k+2}(a_{2k+1,\,2k+1}-a_{2k+2,\,2k+2})=t_{1,\,2k+1}a_{2k+1,\,2k+2}$ which is a contradiction.

If $i<k$, then $t_{2i+3,\,2k-2i-1} \neq 0$, $t_{2i+2,\,2k-2i} \neq 0$ and $t_{2i+1,\,2k-2i+1} \neq 0$. By (7) $t_{2i+3,\,2k-2i+1}=0$. We want to prove that $t_{2j+1,\,2k-2i+1}=0$ for all $j=0,\,1,\,2,\,\dots,\,n-1$.

Suppose that $t_{2j+1,\,2k-2i+1} \neq 0$ for some j, Comparing the $(2j+1,\,2k-2i)$-component of $\varphi(A)T$ with that of TA,

$$\begin{aligned}&b_{2j+1,\,2j}t_{2j,\,2k-2i}+b_{2j+1,\,2j+1}t_{2j+1,\,2k-2i}+b_{2j+1,\,2j+2}t_{2j+2,\,2k-2i}\\&\quad=t_{2j+1,\,2k-2i-1}a_{2k-2i-1,\,2k-2i}+t_{2j+1,\,2k-2i}a_{2k-2i,\,2k-2i}+t_{2j+1,\,2k-2i+1}a_{2k-2i+1,\,2k-2i}.\end{aligned}$$

Since $t_{2i+2,\,2k-2i} \neq 0$, $t_{2j,\,2k-2i}=0$ and $t_{2j+2,\,2k-2i}=0\,(i \neq j)$. Since $t_{2i+3,\,2k-2i-1} \neq 0$, $t_{2j+1,\,2k-2i-1}=0$. Since $t_{2j+1,\,2k-2i+1} \neq 0$, $b_{2j+1,\,2j+1}=a_{2k-2i+1,\,2k-2i+1}$. If $i=j$, then $b_{2j+1,\,2j+2}=0$. Hence we have $t_{2j+1,\,2k-2i}(a_{2k-2i+1,\,2k-2i+1}-a_{2k-2i,\,2k-2i})=t_{2j+1,\,2k-2i+1}a_{2k-2i+1,\,2k-2i}$ which is a contradiction. Thus $t_{2j+1,\,2k-2i+1}=0$ for all $j=0,\,1,\,2,\,\dots,\,n-1$. It is a contradiction.

In the case (**ii**), if $k<i$, then $t_{2i,\,2i-2k} \neq 0$ and $t_{2i+1,\,2i-2k+1} \neq 0$.

Comparing the $(2i+1,\,2i-2k+2)$-component of $\varphi(A)T$ with that of TA,

$$\begin{aligned}&b_{2i+1,\,2i}t_{2i,\,2i-2k+2}+b_{2i+1,\,2i+1}t_{2i+1,\,2i-2k+2}+b_{2i+1,\,2i+2}t_{2i+2,\,2i-2k+2}\\&\quad=t_{2i+1,\,2i-2k+1}a_{2i-2k+1,\,2i-2k+2}+t_{2i+1,\,2i-2k+2}a_{2i-2k+2,\,2i-2k+2}\\&\qquad+t_{2i+1,\,2i-2k+3}a_{2i-2k+3,\,2i-2k+2}.\end{aligned}$$

Since $t_{2i,\,2i-2k} \neq 0$, $t_{2i,\,2i-2k+2}=0$. Since $t_{2i+1,\,2i-2k+1} \neq 0$, $t_{2i+1,\,2i-2k+3}=0$ and $b_{2i+1,\,2i+1}=a_{2i-2k+1,\,2i-2k+1}$. Since $b_{2i+1,\,2i+2}=0$, we have

$$t_{2i+1,\,2i-2k+2}(a_{2i-2k+1,\,2i-2k+1}-a_{2i-2k+2,\,2i-2k+2})=t_{2i+1,\,2i-2k+1}a_{2i-2k+1,\,2i-2k+2}.$$

It is a contradiction.

If $i<k$, then $t_{2k+2i,\,2i} \neq 0$, $t_{2k+2i+1,\,2i+1} \neq 0$ and $t_{2k+2i+2,\,2i+2} \neq 0$.

Comparing the $(2k+2i+1,\,2i+2)$-component of $\varphi(A)T$ with that of TA,

$$\begin{aligned}&b_{2k+2i+1,\,2k+2i}t_{2k+2i,\,2i+2}+b_{2k+2i+1,\,2k+2i+1}t_{2k+2i+1,\,2i+2}+b_{2k+2i+1,\,2k+2i+2}t_{2k+2i+2,\,2i+2}\\&\quad=t_{2k+2i+1,\,2i+1}a_{2i+1,\,2i+2}+t_{2k+2i+1,\,2i+2}a_{2i+2,\,2i+2}+t_{2k+2i+1,\,2i+3}a_{2i+3,\,2i+2}.\end{aligned}$$

Since $t_{2k+2i,\,2i+2}=0$, $t_{2k+2i+1,\,2i+3}=0$ and $a_{2i+2,\,2i+2}=b_{2k+2i+2,\,2k+2i+2}$,

$$t_{2k+2i+1,\,2i+2}(b_{2k+2i+2,\,2k+2i+2}-b_{2k+2i+1,\,2k+2i+1})=b_{2k+2i+1,\,2k+2i+2}t_{2k+2i+2,\,2i+2}.$$

It is a contradiction. Thus if $t_{11}=0$, $t_{2i+1,\,1} \neq 0 \dots (*_0)$.

Next we want to prove that if $t_{2i+2,\,2i+2}=0$, then $t_{2n,\,2i+2} \neq 0$.

Suppose that $t_{2i+2,\,2i+2}=0$.

If $t_{11} \neq 0$, then $t_{jj} \neq 0$ for all $j\,(j=1,\,2,\,\dots,\,2i+1)$. So $t_{2i+2,\,2}=0$, $t_{2i+2,\,4}=0$, $\dots$,

$t_{2i+2\,2i}=0$ and $t_{2i+2\,2i+2}=0 \ldots (*_1)$.

If $t_{2i+2\,2i+2j}\neq 0$ for some $j(j=2, \ldots, n-i-1)$, then $t_{2i+3\,2i+2j-1}\neq 0$ or $t_{2i+3\,2i+2j+1}\neq 0$ by comparing the $(2i+3, 2i+2j)$-component of $\varphi(A)T$ with that of TA.

If $t_{2i+2\,2i+2j}\neq 0$ and $t_{2i+3\,2i+2j-1}\neq 0$, then $t_{2i+4\,2i+2j-2}\neq 0$ by comparing the $(2i+3, 2i+2j-2)$-component of $\varphi(A)T$ with that of TA. If $t_{2i+3\,2i+2j-1}\neq 0$ and $t_{2i+4\,2i+2j-2}\neq 0$, then $t_{2i+5\,2i+2j-3}\neq 0$ by comparing the $(2i+5, 2i+2j-2)$-component of $\varphi(A)T$ with that of TA.

If we continue this process, then we can get the following ; If $t_{2i+2\,2i+2j}\neq 0$ and $t_{2i+3\,2i+2j-1}\neq 0$, then $t_{2i+2+l,\,2i+2j-l}\neq 0$ for all $l(l=1, 2, \ldots, 2j-2)$. In particular $t_{2i+2j+1\,2i+1}\neq 0$ and $t_{2i+2j,\,2i+2}\neq 0$.

Comparing the $(2i+2j+1, 2i+2)$-component of $\varphi(A)T$ with that of TA,

$$b_{2i+2j+1,\,2i+2j}t_{2i+2j,\,2i+2}+b_{2i+2j+1,\,2i+2j+1}t_{2i+2j+1,\,2i+2}+b_{2i+2j+1\,2i+2j+2}t_{2i+2j+2\,2i+2}$$
$$=t_{2i+2j+1\,2i+1}a_{2i+1,\,2i+2}+t_{2i+2j+1\,2i+2}a_{2i+2\,2i+2}+t_{2i+2j+1,\,2i+3}a_{2i+3\,2i+2}.$$

Since $t_{2i+2j,\,2i+2}\neq 0$, $t_{2i+2j+2\,2i+2}=0$ and $a_{2i+2\,2i+2}=b_{2i+2j,\,2i+2j}$.

Since $t_{2i+2j+1\,2i+1}\neq 0$, $t_{2i+2j+1\,2i+3}=0$. Since $a_{2i+1\,2i+2}=0$, we have $t_{2i+2j+1\,2i+2}(b_{2i+2j,\,2i+2j}-b_{2i+2j+1\,2i+2j+1})=b_{2i+2j+1\,2i+2j}t_{2i+2j,\,2i+2}$ which is a contradiction.

If $t_{2i+2\,2i+2j}\neq 0$ and $t_{2i+3\,2i+2j+1}\neq 0$, then $t_{2i+4\,2i+2j+2}\neq 0$ by comparing the $(2i+3, 2i+2j+2)$-component of $\varphi(A)T$ with that of TA. If $t_{2i+3\,2i+2j+1}\neq 0$ and $t_{2i+4\,2i+2j+2}\neq 0$, then $t_{2i+5\,2i+2j+3}\neq 0$ by comparing the $(2i+5, 2i+2j+2)$-component of $\varphi(A)T$ with that of TA. So if we continue this process, then we have the following. If $t_{2i+2\,2i+2j}\neq 0$ and $t_{2i+3\,2i+2j+1}\neq 0$, then $t_{2i+2+l,\,2i+2j+l}\neq 0$ for all $l(l=1, 2, \ldots, 2n-2i-2j)$. In particular, $t_{2n-2j+2\,2n}\neq 0$ and $t_{2n-2j+1\,2n-1}\neq 0$. Comparing the $(2n-2j+3, 2n)$-component of $\varphi(A)T$ with that of TA,

$$b_{2n-2j+3\,2n-2j+2}t_{2n-2j+2\,2n}+b_{2n-2j+3\,2n-2j+3}t_{2n-2j+3\,2n}+b_{2n-2j+3\,2n-2j+4}t_{2n-2j+4\,2n}$$
$$=t_{2n-2j+3\,2n-1}a_{2n-1,\,2n}+t_{2n-2j+3\,2n}a_{2n,\,2n}.$$

Since $t_{2n-2j+2\,2n}\neq 0$, $t_{2n-2j+4\,2n}=0$ and $a_{2n,\,2n}=b_{2n-2j+2\,2n-2j+2}$. Since $t_{2n-2j+1\,2n-1}\neq 0$, $t_{2n-2j+3\,2n-1}=0$. So we have $t_{2n-2j+3\,2n}(b_{2n-2j+2\,2n-2j+2}-b_{2n-2j+3\,2n-2j+3})=b_{2n-2j+3\,2n-2j+2}t_{2n-2j+2\,2n}$ which is a contradiction. Thus $t_{2i+2\,2i+2j}=0$ for all $j(j=1, 2, \ldots, n-i-1) \ldots (*_2)$. Hence $t_{2i+2\,2j}=0$ for all $j(j=1, 2, \ldots, n-1)$ by $(*_1)$ and $(*_2)$ and hence $t_{2i+2\,2n}\neq 0$.

If $t_{2i+2\,2n}\neq 0$, then $t_{2i+3\,2n-1}\neq 0$ by comparing the $(2i+3, 2n)$-component of $\varphi(A)T$ with that of TA.

If $t_{2i+2\,2n}\neq 0$ and $t_{2i+3\,2n-1}\neq 0$, then $t_{2i+4\,2n-2}\neq 0$ by comparing the $(2i+3, 2n-2)$-component of $\varphi(A)T$ with that of TA. So if we continue this process, then we have the following. If $t_{2i+2\,2n}\neq 0$, then $t_{2i+2+l,\,2n-l}\neq 0$ for all $l(l=1, 2, \ldots, 2n-2i-2)$. In particular, $t_{2n,\,2i+2}\neq 0 \ldots (*_3)$. If $t_{11}=0$, then $t_{2i+1\,1}\neq 0$ by $(*_0)$. Suppose that $t_{2i+1,\,1}\neq 0$ and $t_{2i,\,2}=0$. Comparing the $(2i+1, 2)$-component of $\varphi(A)T$ with that of TA,

$$b_{2i+1,\,2i}t_{2i,\,2}+b_{2i+1,\,2i+1}t_{2i+1,\,2}+b_{2i+1,\,2i+2}t_{2i+2,\,2}=t_{2i+1,\,1}a_{12}+t_{2i+1,\,2}a_{22}+t_{2i+1,\,3}a_{32}.$$

Since $t_{2i+1,\,1}\neq 0$, $t_{2i+1,\,3}=0$ and $b_{2i+1,\,2i+1}=a_{11}$. Since $t_{2i,\,2}=0$ and $b_{2i+1,\,2i+2}=0$, we have $t_{2i+1,\,2}(a_{11}-a_{22})=t_{2i+1,\,1}a_{12}$ which is a contradiction.

If $t_{2i+1,\,1}\neq 0$ and $t_{2i,\,2}\neq 0$, then $t_{2i-1,\,3}\neq 0$ by comparing the $(2i-1,\,2)$-component of $\varphi(A)T$ with that of TA. If we continue this process, then we have the following. If $t_{2i+1,\,1}\neq 0$ and $t_{2i,\,2}\neq 0$, then $t_{2i+1-l,\,1+l}\neq 0$ for all $l(l=0,\,1,\,2,\,\ldots,\,2i)$. By $(*_2)$ $t_{2i+2,\,2i+2j}=0$ for all $j(j=1,\,2,\,\ldots,\,n-i-1)$. So $t_{2i+2,\,2j}=0$ for all $j(j=1,\,2,\,\ldots,\,n-1)$ and hence $t_{2i+2,\,2n}\neq 0$. By $(*_3)$ $t_{2n,\,2i+2}\neq 0$.

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