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Periodic commuting squares of finite von Neumann algebras

Dedicated to Professor Takeshi Hirai on his sixtieth birthday

By

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0. Introduction

In his paper [4], Jones introduced an index for a pair of type II_1 factors and showed that the index value less than 4 is equal to $4\cos^2(\pi/n)$ for some integer $n\geq 3$. Since then the interests of study in the theory of operator algebras have been gradually extended from a single factor to a pair of factors. Pimsner-Popa [7] showed for a pair of factors $N\subset M$ with finite index, the existence of a special orthonormal basis, called Pimsner-Popa basis, of M as an N-module. Kosaki [5] extended index theory to arbitary factors and gave the definition of an index depending on a conditional expectation. In the case of C^*-algebras, Watatani defined an index by using a quasi-basis.

However it is not easy to calculate explicitly the index even for a pair of II_1 factors from the definition itself or from such a basis. So many index formulas were given by Pimsner-Popa [7], Wenzl [13], Ocneanu [6] and the present author [10] respectively. In the preceding paper [10], we treat a pair of factors $N\subset M$ generated by the increasing sequences $\{M_n\}_{n\in N}$ and $\{N_n\}_{n\in N}$ of finite direct sums of II_1 factors such that the diagram

$$\text{(A)}\qquad \begin{array}{ccc} M_n & \subset & M_{n+1} \\ \cup & & \cup \\ N_n & \subset & N_{n+1} \end{array}$$

is a commuting square for any $n\in \boldsymbol{N}$, and obtained the following

Theorem. *Let $\{M_n\}_{n\in N}$ and $\{N_n\}_{n\in N}$ be increasing sequences of finite direct sums of II_1 factors such that the diagram (A) is a commuting square for any $n\in\mathbf{N}$. Set $M=(\cup M_n)''$ and $N=(\cup N_n)''$. If a certain periodicity condition (Condition I in 1.4 below) holds, then there exists $n_0\in\mathbf{N}$ such that*

$$[M: N]=[M_n: N_n]\qquad \textit{for } n\geq n_0.$$

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In this paper we study commuting squares which generate increasing sequences satisfying the above periodicity condition.

Let us explain more exactly, let a diagram

$$\text{(C)}\qquad \begin{array}{ccc} A_0 & \subset & B_0 \\ \cap & & \cap \\ A_1 & \subset & B_1 \end{array}$$

be a commuting square of finite direct sums of factors. By iterating the basic construction, we get projections $e_n = e_{B_{n-1}}$ and finite von Neumann algebras $B_{n+1} = \langle B_n, e_n \rangle$ and put $A_{n+1} = (A_n \cup \{e_n\})''$ for $n \in \mathbf{N}$.

Definition 2.1. A commuting square (C) is periodic if, for any $n \in \mathbf{N}$,
(i) trace matrices $T_{A_n}^{A_{n+1}}$ and $T_{B_n}^{B_{n+1}}$ are periodic modulo 2, and
(ii) $T_{A_n}^{A_{n+2}}$ and $T_{B_n}^{B_{n+2}}$ are primitive.

We give a neccesary and sufficient condition for a commuting square to be periodic.

Theorem 2.1. *A commuting square (C) is periodic if and only if there exists a positive constant λ such that $F_{A_0}^{A_1} = \lambda I_n$ and $F_{B_0}^{B_1} = \lambda I_m$, where $n = \dim_{\mathbf{C}} Z(A_0)$, $m = \dim_{\mathbf{C}} Z(B_0)$ and I_n is the identity matrix in $M_n(\mathbf{C})$.*

Moreover increasing sequences constructed from a periodic commuting square satisty the periodicity condition.

Futhermore we consider a periodic commuting square, in which only one von Neumann algebra among the four is not a factor, and show properties of such squares.

Theorem 3.2. *Let $N \subset M \subset L$ be II_1 factors such that $[L: M] = [M: N] = 2$, and K be a nonfactor intermediate von Neumann algebra for $N \subset L$. Suppose that the diagram*

$$\begin{array}{ccc} N & \subset & M \\ \cap & & \cap \\ K & \subset & L \end{array}$$

is a periodic commuting square. Then there exists an outer action α of $\mathbf{Z}_2$ on N such that

$$\begin{array}{ccc} N & \subset & M \\ \cap & & \cap \\ K & \subset & L \end{array} \cong \begin{array}{ccc} N & \subset & N \rtimes_\alpha \mathbf{Z}_2 \\ \cap & & \cap \\ (N \cup \{\mu\})'' & \subset & N \rtimes_\alpha \mathbf{Z}_2 \rtimes_{\hat{\alpha}} \widehat{\mathbf{Z}_2} \end{array},$$

where μ is the implementing unitary for $\hat{\alpha}$.

This paper consists of three sections. In §1, we recall the notations (trace matrix, index matrix, the basic construction and Markov trace etc.) concerning inclusions of finite direct sums of finite factors, and list up some of their properties. Section 2 contains the definition of a periodic commuting square. We give a neccesary and sufficient condition for a commuting square to be periodic and show the symmetry of such squares. In the last section, we give some examples and consider a periodic commuting square consisting of three II_1 factors and one nonfactor von Neumann algebra. In particular we study the periodic commuting square, in which all inclusions have positive real numbers less than 4 as indices, and give its characterizations.

1. Preliminaries

1.1. Inclusions of von Neumann algebras. Let $M=\bigoplus_{j=1}^{m}M_j$ be a finite direct sums of finite factors and $\{q_j; j=1,\cdots,m\}$ the corresponding minimal central projections. Since the normalized trace on a finite factor is unique, a trace tr on M is specified by a column vector $\vec{s}=(\mathrm{tr}(q_1)\cdots\mathrm{tr}(q_m))^t$, called the trace vector.

Let $N=\bigoplus_{i=1}^{n}N_i\subset M$ be another finite direct sum of finite factors having the same identity and $\{p_i; i=1,\cdots,n\}$ the corresponding minimal central projections. We assume that the trace on N is the restriction of the trace tr and denote by $\vec{t}$ the trace vector for N.

The inclusion $N\subset M$ is represented by two matrices, one is the index matrix and the other is the trace matrix. The index matrix $\Lambda_N^M=(\lambda_{ij})$ is defined by

$$\lambda_{ij}=\begin{cases}[M_{p_iq_j}: N_{p_iq_j}]^{1/2} & \text{if } p_iq_j\neq 0,\\ 0 & \text{if } p_iq_j=0,\end{cases}$$

and the trace matrix $T_N^M=(t_{ij})$ is defined by $t_{ij}=\mathrm{tr}_{M_j}(p_iq_j)$, where tr_{M_j} is the normalized trace on M_j. The following properties are easy consequences of the definitions.

(1.1) $\lambda_{ij}\in\{0\}\cup\{2\cos(\pi/n);\, n\geq 3\}\cup[2,\infty]$.

(1.2) The trace matrix T_N^M is column-stochastic, i.e., $t_{ij}\geq 0$ and $\sum_{i=1}^{n}t_{ij}=1$ for all j.

(1.3) The equality $\vec{t}=T_N^M\vec{s}$ holds.

(1.4) If $N\subset M\subset L$ are finite direct sums of finite factors, then $T_N^L=T_N^M T_M^L$.

1.2. Basic construction. Now we suppose that N is of finite index in M in the sense of [3] , i.e., there is a faithful representation π of M on a Hilbert space such that $\pi(N)'$ is finite. Then the algebra $\langle M, e_N\rangle$ obtained by

the basic construction for $N\subset M$ is a finite direct sum of finite factors and the corresponding minimal central projections are $J_Mp_1J_M, \cdots, J_Mp_nJ_M$, where J_M is the canonical conjugation on $L^2(M, \mathrm{tr})$. The following properties comes from the definitions:
(1.4) $e_Nxe_N=E_N(x)e_N$ for $x\in M$,
(1.5) $e_NJ_Mp_iJ_M=e_Np_i$ for all i.
We now list up some of properties concerning the index matrix and the trace matrix for $M\subset\langle M, e_N\rangle$:

(1.6) $$\Lambda_M^{\langle M, e_N\rangle}=(\Lambda_N^M)^t,$$

(1.7) $$T_M^{\langle M, e_N\rangle}=\tilde{T}_N^M F_N^M,$$

where $(\tilde{T}_N^M)_{ji}=\begin{cases} t_{ij}^{-1}\lambda_{ij}^2 & p_iq_j\neq 0, \\ 0 & p_iq_j=0, \end{cases}$ $F_N^M=\mathrm{diag}(\varphi_1, \cdots, \varphi_n)$, $\varphi_i=(\sum_j(\tilde{T}_N^M)_{ji})^{-1}$,

(1.8) for any trace Tr on $\langle M, e_N\rangle$, $\mathrm{Tr}(e_NJ_Mp_iJ_M)=\varphi_i\mathrm{Tr}(J_Mp_iJ_M)$.
The index $[M: N]$ is defined as follows:
(1.9) $[M: N]=r(\tilde{T}_N^M T_N^M)$, where $r(T)$ is the spectral radius of T.

1.3. Markov traces. A trace tr is called a Markov trace of modulus β for the pair $N\subset M$, if there exists a trace Tr on $\langle M, e_N\rangle$ such that tr is the restriction of Tr and $\beta\mathrm{Tr}(xe_N)=\mathrm{tr}(x)$ for $x\in M$. The following are important properties of Markov traces.
(1.10) The trace tr is a Markov trace of modulus β if and only if
$\tilde{T}_N^M T_N^M\vec{s}=\beta\vec{s}$.
(1.11) If inclusion $N\subset M$ is connected, i.e., $Z(N)\cap Z(M)=\mathbf{C}$, there exists a uninue normalized Markov trace for $N\subset M$. Moreover it is faithful and has modulus $[M: N]$.

1.4. Index formula. We consider two increasing sequences $\{M_n\}_{n\in N}$ and $\{N_n\}_{n\in N}$ of finite direct sums of finite factors. Assume that the traces on M_n and N_{n+1} are restrictions of the one on M_{n+1} and that the diagram

$$\begin{array}{ccc} M_n & \subset & M_{n+1} \\ \cup & & \cup \\ N_n & \subset & N_{n+1} \end{array}$$

is a communing square, i.e., $E_{N_n}^{M_n}E_{M_n}^{M_{n+1}}=E_{N_n}^{N_{n+1}}E_{N_{n+1}}^{M_{n+1}}$, where conditional expectations $E_{N_n}^{M_n}$, $E_{M_n}^{M_{n+1}}$, $E_{N_n}^{N_{n+1}}$ and $E_{N_{n+1}}^{M_{n+1}}$ are trace invariant.

We deal with following condition.

Condition I (Periodicity): There exist $n_0\geq 1$ and $p\geq 1$. such that for any $n\geq n_0$,

(1) $T_{N_n}^{N_{n+1}}$, $T_{M_n}^{M_{n+1}}$ and $F_{N_n}^{M_n}$ are periodic modulo p, and

(2) $T_{N_n}^{N_{n+p}}$ and $T_{M_n}^{M_{n+p}}$ are primitive.

Now we put $M=(\cup M_n)''$ and $N=(\cup N_n)''$. If Condition I holds, then

(1.12) M and N are II_1 factors,

and for all $n\geq n_0$.

(1.13) $$[M:N]=[M_n:N_n],$$

(1.14) $$(M_n\cup\{e_N\})''\cong\langle M_n, e_{N_n}\rangle.$$

2. Periodic commuting squares

Let a diagram

(C) $$\begin{array}{ccc} A_0 & \subset & B_0 \\ \cap & & \cap \\ A_1 & \subset & B_1 \end{array}$$

be of finite direct sums of finite factors, and suppose that all indices of inclusions are finite and that the diagram is a commuting square with respect to a Markov trace tr on B_1 for $B_0\subset B_1$.

By iterating the basic construction, we get projections $e_n=e_{B_{n-1}}$ and finite von Neumann algebras $B_{n+1}=\langle B_n, e_n\rangle$ and then put $A_{n+1}=(A_n\cup\{e_n\})''$ for $n\in\mathbf{N}$.

Definition 2.1. A commuting square (C) is periodic if for any $n\in\mathbf{N}$

(i) trace matrices $T_{A_n}^{A_{n+1}}$ and $T_{B_n}^{B_{n+1}}$ are periodic modulo 2, and

(ii) $T_{A_n}^{A_{n+2}}$ and $T_{B_n}^{B_{n+2}}$ are primitive.

Remark 2.1. If a commuting (C) is periodic, then for any $n\in\mathbf{N}$ a commuting square

$$\begin{array}{ccc} A_n & \subset & B_n \\ \cap & & \cap \\ A_{n+1} & \subset & B_{n+1} \end{array}$$

is periodic. Moreover by Theorem 2.3 of [8] we see that a commuting square $\begin{array}{ccc} A_0 & \subset & B_0 \\ \cap & & \cap \\ A_n & \subset & B_n \end{array}$ is periodic for any $n\in\mathbf{N}$.

Remark 2.2. If a commuting square (C) is periodic, then it holds that $\dim_{\mathbf{C}} Z(A_0)=\dim_{\mathbf{C}} Z(A_2)$. By [10], this is equivalent to $A_2\cong\langle A_1, e_{A_0}\rangle$, and the map θ: $\langle A_1, e_{A_0}\rangle\rightarrow A_2$, defined by $\theta(\sum_{i=1}^{n}x_ie_{A_0}y_i)=\sum_{i=1}^{n}x_ie_{B_0}y_i$ for $x_i, y_i\in A_1$, is a $*$-isomorphism. So it follows that the central support of e_{B_0} in A_2 is equal to 1, and hence the commuting square $\begin{array}{ccc} A_0 & \subset & B_0 \\ \cap & & \cap \\ A_1 & \subset & B_1 \end{array}$ is nondegenerate, i.e., $\overline{\mathrm{sp}A_1B_0}=B_1$, where sp$A$ denotes the linear span of A.

Example 2.1. Let $N\subset M$ be II_1 factors with finite index and $L=(N\cup\{e_N\})''$. If $[M:N]\geq 2$, then L has a cannonical decomposition as a direct sum of two II_1 factors. The diagram

$$\begin{array}{ccc} N & \subset & M \\ \cap & & \cap \\ L & \subset & \langle M, e_N\rangle \end{array}$$

is a commuting square, and it is periodic if and only if $[M:N]=2$.

Lemma 2.1. *Assume that trace matrices $T_{A_n}^{A_{n+1}}$ and $T_{B_n}^{B_{n+1}}$ are periodic modulo 2 for any $n\in\mathbf{N}$. Then the following are equivalent:*

(i) *$T_{A_n}^{A_{n+2}}$ and $T_{B_n}^{B_{n+2}}$ are primitive for any $n\in\mathbf{N}$;*

(ii) *$Z(A_0)\cap Z(A_1)=Z(B_0)\cap Z(B_1)=\mathbf{C}$, i.e., inclusions $A_0\subset A_1$ and $B_0\subset B_1$ are connected.*

proof. By assumption, (i) is equivalent to primitivity of $T_{A_0}^{A_2}$ and $T_{B_0}^{B_2}$. Since $T_{A_0}^{A_2}=T_{A_0}^{A_1}T_{A_1}^{\langle A_1,e_{A_0}\rangle}$ and $T_{B_0}^{B_2}=T_{B_0}^{B_1}T_{B_1}^{\langle B_1,e_{B_0}\rangle}$, $T_{A_0}^{A_2}$ and $T_{B_0}^{B_2}$ are primitive if and only if $T_{A_0}^{A_1}$ and $T_{B_0}^{B_1}$ are indecomposable. It is easy to see that indecomposability of $T_{A_0}^{A_1}$ and $T_{B_0}^{B_1}$ is equivalent to (ii).

In the following of this section, we assume that all inclusions are connected.

Lemma 2.2. *Let* tr *be a normalized Markov trace on B_1 for $B_0\subset B_1$ and $\{p_i;\ i=1,\cdots,n\}$ minimal central projections of A_0, and set $\varphi_i=(F_{A_0}^{A_1})_{ii}$ for $i=1,\cdots,n$. Then the following are equivalent:*

(i) $A_2\cong\langle A_1, e_{A_0}\rangle$;

(ii) $[B_1:B_0]=\sum_{i=1}^{n}\varphi_i^{-1}\mathrm{tr}(p_i)$.

Proof. Let Tr be the Markov extension of tr to $\langle B_1, e_{B_0}\rangle$. By the proof of Lemma 2.1 in [10], there exists a central projection z of A_2 such that $A_2z\cong\langle A_1, e_{A_0}\rangle$, and $\mathrm{Tr}(z)=[B_1:B_0]^{-1}\sum_{i=1}^{n}\varphi_i^{-1}\mathrm{tr}(p_i)$. So, (ii) is equivalent to $\mathrm{Tr}(z)=1$, i.e., $z=1$, and hence the statement follows.

Proposition 2.1. *Let $A_2=(A_1\cup e_{B_0})''$ and $B_2=\langle B_1, e_{B_0}\rangle$, and suppose that A_2 is $*$-isomorphic to $\langle A_1, e_{A_0}\rangle$. Then we have*
(i) $[A_1: A_0]=[B_1: B_0]$,
(ii) $T_{A_2}^{B_2}=(F_{A_0}^{A_1})^{-1}T_{A_0}^{B_0}F_{B_0}^{B_1}$,
(iii) $\Lambda_{A_2}^{B_2}=\Lambda_{A_0}^{B_0}$.

Proof. (i) Let tr be a normalized Markov trace, of modulus $[B_1: B_0]$, on B_1 for $B_0\subset B_1$ and denote its extension to B_2 by Tr. Since $A_2\cong\langle A_1, e_{A_0}\rangle$, we have a $*$-isomorpism θ: $\langle A_1, e_{A_0}\rangle\rightarrow A_2$ such that $\theta(\sum_{i=1}^{n}x_ie_{A_0}y_i)=\sum_{i=1}^{n}x_ie_{B_0}y_i$ for $x_i, y_i\in A_1$. Then the trace Tr' on $\langle A_1, e_{A_0}\rangle$ defined by $\mathrm{Tr}'=\mathrm{Tr}\circ\theta$ is a normalized Markov trace for $A_1\subset\langle A_1, e_{A_0}\rangle$ and has modulus equal to $[B_1: B_0]$. By uniqueness of modulus of normalized Markov trace, we obtain that $[A_1: A_0]=[B_1: B_0]$.

(ii) Let $\{p_i;\ i=1,\cdots,n\}$ and $\{q_j;\ j=1,\cdots,m\}$ be minimal central projections of A_0 and B_0 respectively, then $\{\tilde{p}_i=\theta(J_{A_1}p_iJ_{A_1});\ i=1,\cdots,n\}$ and $\{\tilde{q}_j=J_{B_1}q_jJ_{B_1};\ j=1,\cdots,m\}$ are those of A_2 and B_2 respectively, where J_{B_1} (resp. J_{A_1}) is the cannonical conjugation on $L^2(B_1, \mathrm{tr})$ (resp. $L^2(A_1, \mathrm{tr})$). Hence it holds that $(T_{A_2}^{B_2})_{ij}=\mathrm{tr}_{B_2\tilde{q}_j}(\tilde{p}_i\tilde{q}_j)=\mathrm{Tr}(\tilde{q}_j)^{-1}\mathrm{Tr}(\tilde{p}_i\tilde{q}_j)$. Define a trace Tr_{ij} on a factor $A_2\tilde{p}_i$ by $\mathrm{Tr}_{ij}(x)=\mathrm{Tr}(\tilde{p}_ix\tilde{q}_j)$ for $x\in A_2\tilde{p}_i$, then we see that $\mathrm{Tr}_{ij}(e_{B_0}\tilde{p}_i)=\mathrm{Tr}(\tilde{p}_i\tilde{q}_j)\mathrm{tr}_{A_2\tilde{p}_i}(e_{B_0}\tilde{p}_i)=\mathrm{Tr}(\tilde{p}_i\tilde{q}_j)\mathrm{Tr}(\tilde{p}_i)^{-1}\mathrm{Tr}(e_{B_0}\tilde{p}_i)$. Since $e_{B_0}\tilde{p}_i=e_{B_0}p_i$ and $e_{B_0}\tilde{p}_i\tilde{q}_j=e_{B_0}p_iq_j$, it follows that $\mathrm{Tr}(\tilde{p}_i\tilde{q}_j)=\mathrm{Tr}(e_{B_0}p_iq_j)\mathrm{Tr}(\tilde{p}_i)\mathrm{Tr}(e_{B_0}p_i)^{-1}$.

By using (1.8), we have that

$$
\begin{aligned}
(T_{A_2}^{B_2})_{ij}&=\mathrm{Tr}(\tilde{q}_j)^{-1}\mathrm{Tr}(e_{B_0}p_iq_j)\mathrm{Tr}(x\tilde{p}_i)\mathrm{Tr}(e_{B_0}p_i)^{-1}\\
&=\phi_j\mathrm{Tr}(e_{B_0}q_j)^{-1}\mathrm{Tr}(e_{B_0}p_iq_j)\varphi_i^{-1}\\
&=\varphi_i^{-1}\phi_j\mathrm{Tr}(q_j)^{-1}\mathrm{Tr}(p_iq_j)\\
&=\varphi_i^{-1}(T_{A_0}^{B_0})_{ij}\phi_j,
\end{aligned}
$$

where $\varphi_i=(F_{A_0}^{A_1})_{ii}$ and $\phi_j=(F_{B_0}^{B_1})_{jj}$.

(iii) Since $e_{B_0}\in B_0'\cap A_2$, we see that $(\Lambda_{A_2}^{B_2})_{ij}^2=[(B_{2\tilde{p}_i\tilde{q}_j})_{e_{B_0}\tilde{p}_i\tilde{q}_j}: (A_{2\tilde{p}_i\tilde{q}_j})_{e_{B_0}\tilde{p}_i\tilde{q}_j}]$ and $(\Lambda_{A_0}^{B_0})_{ij}^2=[(B_{0p_iq_j})_{e_{B_0}p_iq_j}: (A_{0p_iq_j})_{e_{B_0}p_iq_j}]$. By using (1.4) and (1.5), it holds that $(B_{2\tilde{p}_i\tilde{q}_j})_{e_{B_0}\tilde{p}_i\tilde{q}_j}=(B_{0p_iq_j})_{e_{B_0}p_iq_j}$ and $(A_{2\tilde{p}_i\tilde{q}_j})_{e_{B_0}\tilde{p}_i\tilde{q}_j}=(A_{0p_iq_j})_{e_{B_0}p_iq_j}$, so we have $\Lambda_{A_2}^{B_2}=\Lambda_{A_0}^{B_0}$.

Now we obtain a neccesary and sufficient condition for a commuting square to be periodic

Theorem 2.1. *A commuting square (C) is periodic if and only if there exists a positive constant λ such that $F_{A_0}^{A_1}=\lambda I_n$ and $F_{B_0}^{B_1}=\lambda I_m$, where $n=\dim_{\mathbf{C}}(A_0)$, $m=\dim_{\mathbf{C}}Z(B_0)$ and I_n is the identity matrix in $M_n(\mathbf{C})$. Moreover, in this case, the constant λ is equal to $[B_1: B_0]^{-1}$.*

Proof. Suppose that the commuting square (C) is periodic, then it holds that

$A_2 \cong \langle A_1, e_{A_0}\rangle$, $F_{A_{2k}}^{A_{2k+1}} = F_{A_0}^{A_1}$ and $F_{B_{2k}}^{B_{2k+1}} = F_{B_0}^{B_1}$ for any $k \in \mathbf{N}$. By using Proposition 2.1, we have that $T_{A_{2k}}^{B_{2k}} = (F_{A_0}^{A_1})^{-k} T_{A_0}^{B_1} (F_{B_0}^{B_1})^k$, that is, $(T_{A_{2k}}^{B_{2k}})_{ij} = \varphi_i^{-k} (T_{A_0}^{B_0})_{ij} \psi_j^k$ with $\varphi_i = (F_{A_0}^{A_1})_{ii}$ and $\psi_j = (F_{B_0}^{B_1})_{jj}$. Since a trace matrix is column-stochastic, we see that $\sum_{i=1}^{n} \varphi_i^{-k} (T_{A_0}^{B_0})_{ij} \psi_j^k = 1$ for any $k \in \mathbf{N}$, so that $\sum_{i=1}^{n} \lim_{k\to\infty} (\varphi_i^{-1}\psi_j)^k (T_{A_0}^{B_0})_{ij} = 1$. And by $\sum_{i=1}^{n} (T_{A_0}^{B_0})_{ij} = 1$, we obtain that $\varphi_i = \psi_j$ if $(T_{A_0}^{B_0})_{ij} \neq 0$. Because the inclusion $A_0 \subset B_0$ is connected, we conclude that $\varphi_i = \psi_j$ for all i and j.

Conversely, we assume that there exists a positive constant λ such that $F_{A_0}^{A_1} = \lambda I_n$ and $F_{B_0}^{B_1} = \lambda I_m$. By virtue of Lemma 2.2, it follows that $\lambda = [B_1 : B_0]^{-1}$ and $A_2 \cong \langle A_1, e_{A_0}\rangle$, so that $T_{A_1}^{A_2} = T_{A_1}^{\langle A_1, e_{A_0}\rangle}$. And by a simple calculation, we get $T_{A_k}^{A_{k+1}} = T_{A_{k+2}}^{A_{k+3}}$ and $T_{B_k}^{B_{k+1}} = T_{B_{k+2}}^{B_{k+3}}$ for any $k \in \mathbf{N}$. Since inclusions $A_0 \subset A_1$ and $B_0 \subset B_1$ are connected, it holds that $T_{A_k}^{A_{k+2}}$ and $T_{B_k}^{B_{k+2}}$ are primitive for $k \in \mathbf{N}$. Therefore the commuting square (C) is periodic.

Corollary 2.1. *Let a diagram*

$$\begin{array}{ccccc} A_0 & \subset & B_0 & \subset & C_0 \\ \cap & & \cap & & \cap \\ A_1 & \subset & B_1 & \subset & C_1 \end{array}$$

consist of commuting squares. If the two small commuting squares are periodic, then the big commuting square is periodic.

Proof. By using Theorem 2.1, it follows that $F_{A_0}^{A_1} = \lambda I_n$, $F_{B_0}^{B_1} = \lambda I_m$ and $F_{C_0}^{C_1} = \lambda I_l$ for some $\lambda > 0$, and hence the big commuting square is periodic.

The following theorem is one of main results of this section.

Theorem 2.2. *Let $\{e_n = e_{B_{n-1}}; n \in \mathbf{N}\}$ be projections and $\{B_{n+1} = \langle B_n, e_n\rangle ; n \in \mathbf{N}\}$ finite von Neumann algebras obtained by iterating the basic construction, and put $A_{n+1} = (A_n \cup \{e_n\})''$ for $n \in \mathbf{N}$. If the commuting square (C) is periodic, then two increasing sequences $\{A_n\}_{n=0,1,2,\cdots}$ and $\{B_n\}_{n=0,1,2,\cdots}$ satisfy Condition I.*

Proof. It is sufficient to prove that $F_{A_n}^{B_n}$ is periodic modulo 2. By using Proposition 2.1 and Theorem 2.1, we have $T_{A_2}^{B_2} = T_{A_0}^{B_0}$ and $\Lambda_{A_2}^{B_2} = \Lambda_{A_0}^{B_0}$. Hence it holds that $F_{A_2}^{B_2} = F_{A_0}^{B_0}$. Similarly we can easily see that $F_{A_n}^{B_n} = F_{A_{n+2}}^{B_{n+2}}$ for any $n \in \mathbf{N}$.

Corollary 2.2. *If a commuting square (C) is periodic, then*

$$[B_1 : A_1] = [B_0 : A_0].$$

Proof. Put $A = (\cup_n A_n)''$ and $B = (\cup_n B_n)''$. By the preceding theorem and (1.4), we obtain that $[B : A] = [B_n : A_n]$ for any n, so that $[B_1 : A_1] = [B_0 : A_0]$.

Proposition 2.2. *Set* $C_1 = \langle B_1, e_{A_1}\rangle$ *and* $C_0 = (B_0 \cup \{e_{A_1}\})''$. *If the commuting square* (C) *is periodic, then* $C_0 \cong \langle B_0, e_{A_0}\rangle$.

Proof. Let $\langle B, e_A\rangle$ be a II_1 factor obtained by the basic construction for $A \subset B$, tr a Markov trace on B, and $\{p_i;\ i=1, \cdots, n\}$ minimal central projections of A_0. From (1.14) and Theorem 2.2, it follows that $(B_0 \cup \{e_A\})'' \cong \langle B_0, e_{A_0}\rangle$. Hence, by Lemma 2.2, we have that $\sum_{i=1}^{n} \varphi_i^{-1} \mathrm{tr}\,(p_i) = [B\colon A] = [B_1\colon A_1]$, where $\varphi_i = (F_{A_0}^{B_0})_{ii}$. And since $\mathrm{tr}|_{B_1}$ is a Markov trace on B_1 for $A_1 \subset B_1$, we obtain that $(B_1 \cup \{e_{A_1}\})'' \cong \langle B_0, e_{A_0}\rangle$.

The periodic commuting squares have the symmetry as below.

Theorem 2.3. *Let*

$$\text{(C)} \qquad \begin{array}{ccc} A_0 & \subset & B_0 \\ \cap & & \cap \\ A_1 & \subset & B_1 \end{array}$$

be a diagram of finite direct sums of finite factors such that any inclusions are connected and indices are finite. Assume that this diagram is a periodic commuting square with respect to a Markov trace on B_1 *for* $B_0 \subset B_1$, *then the commuting square*

$$\text{(C}'\text{)} \qquad \begin{array}{ccc} A_0 & \subset & A_1 \\ \cap & & \cap \\ B_0 & \subset & B_1 \end{array}$$

is periodic.

Proof. Since the trace matrix $T_{A_k}^{A_{k+2}}$ is primitive and periodic modulo 2, there exists an integer k such that all entries of $T_{A_0}^{A_{2k}}$ are strictly positive. Because the commuting square $\begin{array}{ccc} A_0 & \subset & B_0 \\ \cap & & \cap \\ A_{2k} & \subset & B_{2k} \end{array}$ is periodic, we have $(B_0 \cup \{e_{A_{2k}}\})'' \cong \langle B_0, e_{A_0}\rangle$, by the preceding proposition. And by using Proposition 2.1, it follows that $T_{C_0}^{C_{2k}} = (F_{A_0}^{B_0})^{-1}\, T_{A_0}^{A_{2k}}\, F_{A_{2k}}^{B_{2k}}$, where $C_0 = (B_0 \cup \{e_{A_{2k}}\})''$ and $C_{2k} = \langle B_{2k}, e_{A_{2k}}\rangle$. Now denote by l a suffix such that φ_l is maximum in $\{\varphi_1, \cdots, \varphi_n\}$, and $F_{A_0}^{B_0}\,(=F_{A_{2k}}^{B_{2k}}) = \mathrm{diag}\,(\varphi_1, \cdots, \varphi_n)$. Then by (1.2) it holds that $\sum_{i=1}^{n} (T_{A_0}^{A_{2k}})_{il} = 1 = \sum_{i=1}^{n} (T_{C_0}^{C_{2k}})_{il} = \sum_{i=1}^{n} \varphi_i^{-1} (T_{A_0}^{A_{2k}})_{il} \varphi_l$. Since φ_l is maximum, we obtain that $\varphi_i = \varphi_l$ for $i=1, \cdots, n$, so that $F_{A_0}^{B_0} = \lambda I_n$ with $\lambda = \varphi_l$. In the same way, we get that $F_{A_1}^{B_1} = \lambda' I_m$ for some $\lambda' > 0$. Moreover, by using the above method to the commuting square (C), it follows that $\lambda = \lambda'$ and hence the commuting square (C′) is periodic.

3. Examples

In this section, we give some examples of periodic commuting squares and the classification of particular ones.

Proposition 3.1. *Let N be a II_1 factor, G a finite abelian group of outer automorphism of N and $N \rtimes G$, $N \rtimes G \rtimes \widehat{G}$ be crossed products. Further set $K = (N \cup \{\mu_\gamma;\ \gamma \in \widehat{G}\})''$, where μ_γ is the implementing unitary for $\gamma \in \widehat{G}$. Then the diagram*

$$\begin{array}{ccc} N & \subset & N \rtimes G \\ \cap & & \cap \\ K & \subset & N \rtimes G \rtimes \widehat{G} \end{array}$$

is a periodic commuting square.

Proof. Put $M = N \rtimes G$, $L = N \rtimes G \rtimes \widehat{G}$ and $n = |G|$. Then the cannonical conditional expectation E^L_M: $L \to M$, is defined by $E^L_M(\sum_{\gamma \in \widehat{G}} x_\gamma \mu_\gamma) = x_e$, where $x_\gamma \in M$ and e is the unit of $\widehat{G}$. Since $\mu_\gamma \in N'$, any element y of K is uniquely written in the form $y = \sum_{\gamma \in \widehat{G}} y_\gamma \mu_\gamma$, where $y_\gamma \in N$. Hence it follows that $E^L_M(K) = N$ so that the diagram is a commuting square. Now, for $g \in G$, define the projection p_g by $p_g = \frac{1}{|G|} \sum_{\gamma \in \widehat{G}} \langle g, \gamma \rangle \mu_\gamma$, then the projections $\{p_g;\ g \in G\}$ are minimal central projections of K and K is the direct sum of Np_g $(g \in G)$. So we have $T^K_N = \Lambda^K_N = (1 \cdots 1)$ and hence $F^K_N = (|G|^{-1})$. On the other hand, since $T^L_M = (1)$ and $\Lambda^L_M = (|G|^{1/2})$, it holds that $F^L_M = (|G|^{-1})$. Therefore the commuting square is periodic by Theorem 2.1.

The next example was suggested to me by H. Kosaki and it is generalization of the above one.

Proposition 3.2. *Let N be a II_1 factor, $G \supset H$ finite groups, α an outer action of G to N and denote by λ the action of G to $l^\infty(G/H)$ defined by the left multiplication.*
Then the diagram

$$\begin{array}{ccc} N \rtimes_\alpha H & \subset & (N \otimes l^\infty(G/H)) \rtimes_{\alpha \otimes \lambda} H \\ \cap & & \cap \\ N \rtimes_\alpha G & \subset & (N \otimes l^\infty(G/H)) \rtimes_{\alpha \otimes \lambda} G \end{array}$$

is a periodic commuting square.

Proof. Put $L=(N\otimes l^{\infty}(G/H))\rtimes_{\alpha\otimes\lambda}G$, $M=(N\otimes l^{\infty}(G/H))\rtimes_{\alpha\otimes\lambda}H$ and let μ_g be the implementing unitary for $\alpha_g\otimes\lambda_g$. We can easily see that the algebras $N\rtimes_{\alpha}H$, $N\rtimes_{\alpha}G$ and L are all II_1 factors and that $[L: N\rtimes_{\alpha}G]=[N\rtimes_{\alpha}G: N\rtimes_{\alpha}H]=|G/H|$. The canonical conditional expectation $E: L\rightarrow M$ is defined by $E(\sum_{g\in G}x_g\mu_g)=\sum_{g\in H}x_g\mu_g$, where $x_g\in N\otimes l^{\infty}(G/H)$. So it follows that $E(N\rtimes_{\alpha}G)\subset N\rtimes_{\alpha}H$, hence the diagram is a commuting square. Now put $m=[G: H]$ and $n=|H\backslash G/H|$, then there exist $g_1,\cdots,g_m\in G$ such that $G/H=\{g_1H,\cdots,g_mH\}$ and $H\backslash G/H=\{Hg_1H,\cdots,Hg_nH\}$. We define projections $p_1,\cdots,p_n$ of M by $p_i=\sum_{g_k\in Hg_iH} id\otimes\chi_{g_kH}$ and set $(G/H)_i=\{g_kH; g_k\in Hg_iH\}$, $N(i)=|(G/H)_i|$ for $i=1,\cdots,n$. By simple calculation we see that the projections $\{p_i; i=1,\cdots,n\}$ are minimal central projections of M. Since $M_{p_i}=(N\otimes l^{\infty}((G/H)_i))\rtimes_{\alpha\otimes\lambda}H$ and $(N\rtimes_{\alpha}H)_{p_i}=(N\otimes \mathbf{C}id_{l^2((G/H)_i)})\rtimes_{\alpha\otimes\lambda}H$, it follows that $[M_{p_i}: (N\rtimes_{\alpha}H)_{p_i}]=|(G/H)_i|=N(i)$, that is, $\Lambda^{M}_{N\rtimes_{\alpha}H}=(N(1)\cdots N(n))$. And by $T^{M}_{N\rtimes_{\alpha}H}=(1\cdots 1)$, we have that $(F^{M}_{N\rtimes_{\alpha}H})^{-1}=(\sum_{i=1}^{n}N(i))=(|G/H|)=(F^{L}_{N\rtimes_{\alpha}G})^{-1}$. Therefore, by Theorem 2.1, the diagram is a periodic commuting square.

Let $N\subset M\subset L$ be II_1 factors with finite indices and K a nonfactor intermediate von Neumann algebra for $N\subset L$. Now suppose that the diagram

$$\text{(D)}\qquad \begin{array}{ccc} N & \subset & M \\ \cap & & \cap \\ K & \subset & L \end{array}$$

is a commuting square. Then a necessary and sufficient condition for the above diagram to be periodic is given by the next proposition.

Proposition 3.3. *Let $\{p_i; i=1,\cdots,n\}$ be minimal central projections of K and* tr *a normalized trace on L. Then the commuting square (D) is periodic if and only if for any i*

$$[K_{p_i}: N_{p_i}]=[L: M]\,\mathrm{tr}(p_i) \text{ and } [L_{p_i}: K_{p_i}]=[M: N]\,\mathrm{tr}(p_i).$$

Proof. By a simple calculation it follows that $(F^{K}_{N})^{-1}=(\sum_{j=1}^{n}[K_{p_j}: N_{p_j}])$ and $(F^{L}_{M})^{-1}=\mathrm{diag}(\mathrm{tr}(p_1)^{-1}[L_{p_1}: M_{p_1}],\cdots,\mathrm{tr}(p_n)^{-1}[L_{p_n}: M_{p_n}])$.

Suppose that the commuting square is periodic. By using Theorem 2.1 we obtain that $\sum_{j=1}^{n}[K_{p_j}: N_{p_j}] = [L: M]$ and $[L_{p_i}: M_{p_i}] = [M: N]\,\mathrm{tr}(p_i)$ for any i. Now we set $Q=\langle L, e_K\rangle$, $P=(M\cup\{e_K\})''$ and $q_i=J_Lp_iJ_L$ for $i=1,\cdots,n$, where J_L is the canonical conjugation on $L^2(L, \mathrm{tr})$. Since $\Lambda_P^Q=\Lambda_K^N$ and $\Lambda_L^Q=(\Lambda_K^L)^t$, we have that $[Q_{q_i}: P_{q_i}]=[K_{p_i}: N_{p_i}]$ and $[Q_{q_i}: L_{q_i}]=[L_{p_i}: K_{p_i}]$. From the equation $[Q_{q_i}: P_{q_i}][P_{q_i}: N_{q_i}]=[Q_{q_i}: L_{q_i}][L_{q_i}: N_{q_i}]$, it follows that $[K_{p_i}: N_{p_i}]=[L: M]\,\mathrm{tr}(p_i)$.

Conversely, suppose that $[K_{p_i}: N_{p_i}]=[L: M]\,\mathrm{tr}(p_i)$ and $[L_{p_i}: K_{p_i}]=[M: N]\,\mathrm{tr}(p_i)$ for any i, then
$(F_K^L)^{-1}=\mathrm{diag}(\mathrm{tr}(p_1)^{-1}[L_{p_1}: M_{p_1}],\cdots,\mathrm{tr}(p_n)^{-1}[L_{p_n}: M_{p_n}])=[M: N]I_n$.
On the other hand, we have that $F_N^M=([M: N]^{-1})$. Hence the diagram is periodic by Theorem 2.1.

We see from the preceding theorem that trace matrices and index matrices for inclusions in a periodic commuting square as (D) are expressed by means of indices $[L: M]$, $[M: N]$ and the vector $\vec{t}=(\mathrm{tr}(p_1),\cdots,\mathrm{tr}(p_n))$. In the following we assume that $\mathrm{tr}(p_1)\leq\cdots\leq\mathrm{tr}(p_n)$.

Theorem 3.1. *Let $N\subset M\subset L$ be II_1 factors such that indices $[L: M]$ and $[M: N]$ are less than 4, and K a nonfactor intermediate von Neumann algebra for $N\subset L$. Suppose that a diagram*

$$\begin{array}{ccc} N & \subset & M \\ \cap & & \cap \\ K & \subset & L \end{array}$$

is a periodic commuting square. Then
(i) $[M: N]=[L: M]$,
(ii) *the pair* $([M: N];\vec{t})$ *is one of the following:*

$$\left(2;\left(\tfrac{1}{2},\tfrac{1}{2}\right)\right),\ \left(3;\left(\tfrac{1}{3},\tfrac{2}{3}\right)\right),\ \left(3;\left(\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3}\right)\right),\ \left(4\cos^2\tfrac{\pi}{10};\left(\frac{1}{4\cos^2\frac{\pi}{10}},\frac{\cos^2\frac{\pi}{5}}{\cos^2\frac{\pi}{10}}\right)\right).$$

Proof. Let $\{p_i; i=1,\cdots,n\}$ be minimal central projections of K and $\lambda_i=[K_{p_i}: N_{p_i}]$, where $n=\dim_{\mathbf{C}}Z(K)$. By assumption it holds that $\sum_{i=1}^{n}[K_{p_i}: N_{p_i}]=[L: M]<4$ so that $\lambda_i<4$, and $n=2$ or 3. Since $\lambda_i\in\{4\cos^2(\pi/m); m=3,4,\cdots\}$, we obtain that $([L: M];\lambda_1,\cdots,\lambda_n)=(2; 1,1)$, $(3; 1,2)$, $(3; 1,1,1)$ or $(4\cos^2(\pi/10); 1, 4\cos^2(\pi/5))$. Hence (ii) follows by $\lambda_i=[L: M]\,\mathrm{tr}(p_i)$. Since $[M: N]\,\mathrm{tr}(p_i)\in\{4\cos^2(\pi/m); m=3,4,\cdots\}$ for all i, we can easily see $[L: M]=[M: N]$.

Remark 3.1. The periodic commuting square in Proposition 3.1 corresponds to $([M:N];\vec{t})=\left(|G|;\left(\frac{1}{|G|},\cdots,\frac{1}{|G|}\right)\right)$, and the one in Proposition 3.2 with $G=S_3$ and $H=S_2$ corresponds to $\left(3;\left(\frac{1}{3},\frac{2}{3}\right)\right)$.

In the rest of this section we consider the classification of periodic commuting squares

$$\text{(E)}\qquad \begin{array}{ccc} N & \subset & M \\ \cap & & \cap \\ K & \subset & L \end{array}$$

corresponding to $([M:N];\vec{t})=\left(2;\left(\frac{1}{2},\frac{1}{2}\right)\right)$.

Since $N'\cap L\supset Z(K)\cong\mathbf{C}\oplus\mathbf{C}$and $[L:N]=4$, there exist a II_1 factor P and an automorphism α of P such that $(N\subset L)\cong(P_\alpha\subset P\otimes M_2(\mathbf{C}))$, where $P_\alpha=\left\{\begin{pmatrix} x & 0 \\ 0 & \alpha(x)\end{pmatrix}; x\in P\right\}$. By Theorem 5.4 of [11], we may assume that α is outer and $\alpha^2=id$. Moreover it follows that $(N\subset M\subset L)\cong(P_\alpha\subset Q\subset P\otimes M_2(\mathbf{C}))$, where $Q=\left\{\begin{pmatrix} x & y \\ \alpha(y) & \alpha(x)\end{pmatrix}; x, y\in P\right\}\cong P\rtimes\mathbf{Z}_2$. On the other hand, by Remark 5.5 of [11] we have that

$$\begin{array}{ccc} N & \subset & M \\ \cap & & \cap \\ K & \subset & L \end{array} \cong \begin{array}{ccc} P_\alpha & \subset & Q \\ \cap & & \cap \\ S & \subset & P\otimes M_2(\mathbf{C}) \end{array}$$

$$\cong \begin{array}{ccc} P & \subset & P\rtimes_\alpha\mathbf{Z}_2 \\ \cap & & \cap \\ (P\cup\{\mu\})'' & \subset & P\rtimes_\alpha\mathbf{Z}_2\rtimes_{\hat{\alpha}}\widehat{\mathbf{Z}_2} \end{array}$$

where $S=\left\{\begin{pmatrix} x & 0 \\ 0 & y\end{pmatrix}; x, y\in P\right\}$ and μ is the implementing unitary for $\hat{\alpha}$. Therefore the next theorem follows, which asserts that the periodic commuting square (E) is written in the form of the one in Proposition 3.1.

Theorem 3.2. *Let $N\subset M\subset L$ be II_1 factors such that $[L:M]=[M:N]=2$, and K a nonfactor intermediate von Neumann algebra for $N\subset L$. Suppose that the diagram (E) is a periodic commuting square. Then there exists an outer action of $\mathbf{Z}_2$ on N such that*

$$\begin{array}{ccccccc} N & \subset & M & & N & \subset & N \rtimes_{\alpha} \mathbf{Z}_2 \\ \cap & & \cap & \cong & \cap & & \cap \\ K & \subset & L & & (N \cup \{\mu\})'' & \subset & N \rtimes_{\alpha} \mathbf{Z}_2 \rtimes_{\hat{\alpha}} \widehat{\mathbf{Z}_2} \end{array},$$

where μ *is the implementing unitary for* $\hat{\alpha}$.

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