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On the Berry Conjecture

By

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1. Introduction

One of the most important problems proposed in the 20-th century seems to be concerned with the Montgomery Conjecture [22] (cf. the subsection 3-2 below). The main point might be to get a characterization of the distribution of the zeros of the Riemann zeta function $\zeta(s)$. Dyson (cf. the subsection 3-2 below) has noticed that the law in the Montgomery Conjecture is exactly the law under which the quantities which appear in the world of physics are distributed. On the other hand, from the side of physics, a striking conjecture has been proposed to the theory of $\zeta(s)$. It is a conjecture of Berry's [1], with which we are concerned in this article. The main purpose of the present article is to give a reaction to the Berry Conjecture from the side of mathematics. In fact, we shall give a proof of his conjecture at the level of the first approximation. We shall also realize that a part of the Berry Conjecture is deeply connected with the Montgomery Conjecture.

We start with describing the Berry Conjecture, namely, (19) of p.402 of Berry [1]. For this purpose we shall first introduce some notations. Let $N(T)$ denote the number of the zeros $\beta+i\gamma$ of $\zeta(s)$ in $0<\gamma<T$, $0<\beta<1$, when $T\neq\gamma$ for any γ. When $T=\gamma$, then we put

$$N(T)=\frac{1}{2}(N(T+0)+N(T-0)).$$

Let

$$S(T)=\frac{1}{\pi}\arg\zeta\left(\frac{1}{2}+iT\right) \quad \text{for} \quad T\neq\gamma,$$

where the argument is obtained by continuous variation along the straight lines joining 2, $2+iT$, and $\frac{1}{2}+iT$, starting with the value zero. When $T=\gamma$, then we put

$$S(T)=\frac{1}{2}(S(T+0)+S(T-0)).$$

Then the well known Riemann-von Mangoldt formula (cf. p.212 of Titchmarsh [26]) states that

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$$N(T)=\frac{1}{\pi}\vartheta(T)+1+S(T),$$

where $\vartheta(T)$ is the continuos function defined by

$$\vartheta(T)=\Im(\log\Gamma(\frac{1}{4}+\frac{iT}{2}))-\frac{1}{2}T\log\pi$$

with

$$\vartheta(0)=0,$$

$\Gamma(s)$ being the Γ-function. It is well known that

$$\vartheta(T)=\frac{T}{2}\log\frac{T}{2\pi}-\frac{T}{2}-\frac{\pi}{8}+\frac{1}{48T}+\frac{7}{5760T^3}+\cdots$$

and that

$$S(T)\ll\log T.$$

Now the Berry Conjecture may be stated, with a slight change of notations, as follows.

Berry Conjecture. For $T>T_0$ and for $0<\alpha\ll T\log T$, we have

$$\begin{aligned}V(\alpha,T)&\equiv\frac{1}{T}\int_0^T(N(B(x+\frac{\alpha}{2}))-N(B(x-\frac{\alpha}{2}))-\alpha)^2dx\\&=\frac{1}{\pi^2}\{\log(2\pi\alpha)-\mathrm{Ci}(2\pi\alpha)-2\pi\alpha\cdot\mathrm{Si}(2\pi\alpha)+\pi^2\alpha-\cos(2\pi\alpha)+1+C_0\}\\&\quad+\frac{1}{\pi^2}\{2\sum_{p^r<(\frac{T}{2\pi})^{\tau^*},r\geq1}\frac{\sin^2(\pi\alpha r\frac{\log p}{\log\frac{T}{2\pi}})}{r^2p^r}+\mathrm{Ci}(2\pi\alpha\tau^*)-\log(2\pi\alpha\tau^*)-C_0\}\\&=\widetilde{V}(\alpha,T,\tau^*),\quad\text{say,}\end{aligned}$$

where p runs over the prime numbers, r runs over the integers, $B(t)$ being the inverse function for $t>t_0$ of the function

$$A(t)=\frac{t}{2\pi}(\log\frac{t}{2\pi}-1)+\frac{7}{8},$$

τ^* (cf. (26) of Berry [1]) satisfies

$$\frac{\log2}{\log\frac{T}{2\pi}}\ll\tau^*\ll1,$$

we put

$$\mathrm{Ci}(x)=-\int_x^\infty\frac{\cos t}{t}dt$$

and

$$\mathrm{Si}(x)=\int_0^x \frac{\sin t}{t}dt$$

and C_o is the Euler constant.

We suppose below that τ^* satisfies

$$\tau^* \geq \frac{\log 2}{\log \frac{T}{2\pi}},$$

although it is not stated explicitly in Berry [1].

It may be stressed that the Berry Conjecture is concerned with the number variance $V(\alpha,T)$ for a whole range of α.

We call

$$\frac{1}{\pi^2}\{\log(2\pi\alpha)-\mathrm{Ci}(2\pi\alpha)-2\pi\alpha\cdot\mathrm{Si}(2\pi\alpha)+\pi^2\alpha-\cos(2\pi\alpha)+1+C_o\}$$

the GUE part and

$$\frac{1}{\pi^2}\{2\sum_{p^r<(\frac{T}{2\pi})^{\tau^*}, r\geq 1}\frac{\sin^2(\pi\alpha r\frac{\log p}{\log\frac{T}{2\pi}})}{r^2p^r}+\mathrm{Ci}(2\pi\alpha\tau^*)-\log(2\pi\alpha\tau^*)-C_o\}$$

the arithmetic part. The GUE part has no term containing T explicitly, although α may depend on T.

We shall first study $\tilde{V}(\alpha,T,\tau^*)$. We shall prove below (cf. Theorem 1 with Remark 1 in the section 2) that when α is small enough, namely, when $\alpha=o(\log T)$, then the GUE part dominates the arithmetic part. And that when α is large enough, namely, when $\log T \ll \alpha \ll T\log T$, then the GUE part and the arithmetic part are mixed and they produce the beautiful term

$$\frac{1}{\pi^2}\{\log\log T-\log|\zeta(1+i\frac{2\pi\alpha}{\log\frac{T}{2\pi}})|\}.$$

We shall next study a simplified number variance, namely, the first approximation of $V(\alpha,T)$. We shall prove below (cf. Theorems 2, 2′, 3, 3′, 3″ and 4 in the subsection 3-3) that it is controled by the same law as $\tilde{V}(\alpha,T,\tau^*)$. In this sense we may claim that we prove the Berry Conjecture at the level of the first approximation.

We should notice here that the simplified number variance for a shorter α is connected with the Montgomery Conjecture in two ways, as will be recalled in the subsection 3-2. There we shall realize that the Montgomery

Conjecture is waiting us at the critical point, indeed.

We shall also turn our attentions to a discrete version of the Berry Conjecture. It is inevitable to consider our problem also in this situation if one wants to try to understand the hidden part of the unknown story about the distribution of the zeros of $\zeta(s)$. Moreover, many numerical computations (cf. the subsection 4-2 below, for example,) await a theoretical explanation in this context.

Furthermore, by applying the techniques used in the analysis for the discrete number variance, we shall show (cf. Theorem 6′ in the section 5) that we can obtain the first main term for $V(\alpha,T)$ in the range $1 \ll \alpha \ll T^{1-\eta}$ with a positive constant $\frac{1}{2} \leq \eta < 1$, which coincides certainly with the first main term of the Berry Conjecture. The number variance $V(\alpha,T)$ for a shorter α is connected with the modified version of the Montgomery Conjecture (cf. the section 5 below.)

As have been already noticed above, numerical computations related with these problems have been presented by many mathematicians, for example, Odlyzko [23] and van de Lune, H. J. J. te Riele and D. T. Winter [19]. The latter will be recalled in the subsection 4-2. Numerical computations of the number variance by Odlyzko [23] will be reproduced in the theorems (cf. Theorems 2′, 3, 3′, 3″ and 4 in the subsection 3-3) and , as a result, in the theoretical graphs (cf. the section 6 below.)

All of our results in this article and the numerical computations mentioned above might permit us to say that the distribution of the zeros of $\zeta(s)$ may be controled by the eigen-values of the Gaussian Unitary Ensembles (GUE) as far as the problem is local within the magnitude of $0 < \alpha = o(\log T)$. And that as a trial of the unification of the local aspects and the global aspects, the Berry Conjecture has been proposed.

Finally, we notice that in some situations (cf. some parts of the subsection 3-2 and the section 4), it is simpler to use $N_+(T)$ in stead of $N(T)$, where we put

$$N_+(T) = \sum_{0<\gamma\leq T} \cdot 1.$$

The Riemann-von Mangoldt formula for $N_+(T)$ becomes

$$N_+(T) = \frac{1}{\pi}\vartheta(T) + 1 + S_+(T),$$

where $S_+(T) = \frac{1}{\pi}\arg\zeta(\frac{1}{2}+iT)$ as above when $T \neq \gamma$ and $S_+(T) = S_+(T+0)$ when $T = \gamma$.

2. The evaluation of the right hand side $\widetilde{V}(\alpha,T,\tau^*)$ of the Berry's formula

We shall first try to check how $\widetilde{V}(\alpha,T,\tau^*)$ in the Berry's formula looks like. We shall give the details of the proof as much as possible, for completeness. First of all

$$2\sum_{p^r<(\frac{T}{2\pi})^{\tau^*},r\geq 1}\frac{\sin^2(\pi\alpha r\frac{\log p}{\log\frac{T}{2\pi}})}{r^2p^r}=2\sum_{p<(\frac{T}{2\pi})^{\tau^*}}\frac{\sin^2(\pi\alpha\frac{\log p}{\log\frac{T}{2\pi}})}{p}+2\sum_{p^r<(\frac{T}{2\pi})^{\tau^*},r\geq 2}\frac{\sin^2(\pi\alpha r\frac{\log p}{\log\frac{T}{2\pi}})}{r^2p^r}.$$

When $0<\alpha\ll\log T$, then the first term is, by the Stieltjes integral,

$$\begin{aligned}
&\sum_{p\leq Y}\frac{1-\cos(h\log p)}{p}\\
&=\frac{1-\cos(h\log Y)}{Y\log Y}\sum_{p\leq Y}\log p-\int_2^Y(\sum_{p\leq t}\log p)(\frac{1-\cos(h\log t)}{t\log t})'dt\\
&=\frac{1-\cos(h\log Y)}{Y\log Y}(Y+R(Y))-\int_2^Y t(\frac{1-\cos(h\log t)}{t\log t})'dt\\
&\quad-\int_2^Y R(t)(\frac{1-\cos(h\log t)}{t\log t})'dt\\
&=\int_2^Y\frac{1-\cos(h\log t)}{t\log t}dt+R(Y)\frac{1-\cos(h\log Y)}{Y\log Y}+\frac{1-\cos(h\log 2)}{\log 2}\\
&\quad-h\int_2^Y\frac{R(t)}{t^2\log t}\sin(h\log t)dt+\int_2^Y\frac{R(t)}{t^2\log^2 t}(1-\cos(h\log t))(\log t+1)dt\\
&=\mathrm{Cin}(2\pi\alpha\tau^*)-\mathrm{Cin}(\frac{2\pi\alpha}{\log\frac{T}{2\pi}}\log 2)+R(Y)\frac{1-\cos(h\log Y)}{Y\log Y}+\frac{1-\cos(h\log 2)}{\log 2}\\
&\quad-h\int_2^Y\frac{R(t)}{t^2\log t}\sin(h\log t)dt+\int_2^Y\frac{R(t)}{t^2\log^2 t}(1-\cos(h\log t))(\log t+1)dt\\
&=\log(2\pi\alpha\tau^*)+C_0-\mathrm{Ci}(2\pi\alpha\tau^*)-\mathrm{Cin}(\frac{2\pi\alpha}{\log\frac{T}{2\pi}}\log 2)\\
&\quad+R(Y)\frac{1-\cos(h\log Y)}{Y\log Y}+\frac{1-\cos(h\log 2)}{\log 2}\\
&\quad-h\int_2^Y\frac{R(t)}{t^2\log t}\sin(h\log t)dt+\int_2^Y\frac{R(t)}{t^2\log^2 t}(1-\cos(h\log t))(\log t+1)dt,
\end{aligned}$$

where we put $\mathrm{Cin}(x)=\int_0^x\frac{1-\cos t}{t}dt$, $h=\frac{2\pi\alpha}{\log\frac{T}{2\pi}}$, $Y=(\frac{T}{2\pi})^{\tau^*}$ and

$$R(t)=\sum_{p\leq t}\log p-t.$$

Thus we get

$$\widetilde{V}(\alpha,T,\tau^*)=\frac{1}{\pi^2}\{\log(2\pi\alpha)-\mathrm{Ci}(2\pi\alpha)-2\pi\alpha\cdot\mathrm{Si}(2\pi\alpha)+\pi^2\alpha-\cos(2\pi\alpha)+1+C_o\}$$

$$+\frac{1}{\pi^2}\{2\sum_{p^r<(\frac{T}{2\pi})^{\tau^*},r\geq 2}\frac{\sin^2(\pi\alpha r\frac{\log p}{\log\frac{T}{2\pi}})}{r^2p^r}-\mathrm{Cin}(\frac{2\pi\alpha}{\log\frac{T}{2\pi}}\log 2)$$

$$+R(Y)\frac{1-\cos(h\log Y)}{Y\log Y}+\frac{1-\cos(h\log 2)}{\log 2}-h\int_2^Y\frac{R(t)}{t^2\log t}\sin(h\log t)\,dt$$

$$+\int_2^Y\frac{R(t)}{t^2\log^2 t}(1-\cos(h\log t))(\log t+1)\,dt\}.$$

When α satisfies further $\frac{\alpha}{\log T}\to 0$ as $T\to\infty$, then we see immediately, using the prime number theorem

$$\sum_{p\leq t}\log p=t+O(t\exp(-A\sqrt{\log t})),$$

that

$$\widetilde{V}(\alpha,T,\tau^*)=\frac{1}{\pi^2}\{\log(2\pi\alpha)-\mathrm{Ci}(2\pi\alpha)-2\pi\alpha\cdot\mathrm{Si}(2\pi\alpha)$$
$$+\pi^2\alpha-\cos(2\pi\alpha)+1+C_o+o(1)\},$$

because

$$\frac{1}{\pi^2}\{2\sum_{p^r<(\frac{T}{2\pi})^{\tau^*},r\geq 2}\frac{\sin^2(\pi\alpha r\frac{\log p}{\log\frac{T}{2\pi}})}{r^2p^r}-\mathrm{Cin}(\frac{2\pi\alpha}{\log\frac{T}{2\pi}}\log 2)$$

$$+R(Y)\frac{1-\cos(h\log Y)}{Y\log Y}+\frac{1-\cos(h\log 2)}{\log 2}-h\int_2^Y\frac{R(t)}{t^2\log t}\sin(h\log t)\,dt$$

$$+\int_2^Y\frac{R(t)}{t^2\log^2 t}(1-\cos(h\log t))(\log t+1)\,dt\}=o(1),$$

as $T\to\infty$. Here we notice that we always denote in this article some positive constants by A.

When

$$\log T\ll\alpha\ll\log T,$$

then the expression of $\widetilde{V}(\alpha,T,\tau^*)$ given above implies that

$$\widetilde{V}(\alpha,T,\tau^*)=\frac{1}{\pi^2}\log(2\pi\alpha)+O(1)=\frac{1}{\pi^2}\log\log T+O(1),$$

because

$$\pi^2\alpha-2\pi\alpha\cdot\mathrm{Si}(2\pi\alpha)=\cos(2\pi\alpha)+O(\frac{1}{\alpha})$$

and

$$\mathrm{Ci}(2\pi\alpha)=O(\frac{1}{\alpha}).$$

We suppose next that

$$\frac{\alpha}{\log T}\gg 1$$

and moreover, $\frac{\alpha}{\log T}$ is sufficiently large. Then

$$2\sum_{p^r<(\frac{T}{2\pi})^{r^*}, r\geq 1}\frac{\sin^2(\pi\alpha r\frac{\log p}{\log\frac{T}{2\pi}})}{r^2p^r}=\sum_{p\leq Y}\frac{1-\cos(h\log p)}{p}+O(1)$$

$$=\sum_{p\leq Y}\frac{1}{p}-\Re(\sum_{p\leq Y}\frac{1}{p^{1+ih}})+O(1).$$

We see as in p.220 of Gallagher and Mueller [12] that

$$\sum_{p\leq Y}\frac{1}{p^{1+ih}}=\sum_{p}\frac{1}{p^{1+\delta+ih}}-\sum_{p>Y}\frac{1}{p^{1+\delta+ih}}+(\sum_{p\leq Y}\frac{1}{p^{1+ih}}-\sum_{p\leq Y}\frac{1}{p^{1+\delta+ih}})$$

$$=\sum_{p}\frac{1}{p^{1+\delta+ih}}+O(1)=\log\zeta(1+\delta+ih)+O(1),$$

where we put

$$\delta=\frac{1}{\log Y}.$$

Since

$$\sum_{p\leq Y}\frac{1}{p}=\log\log Y+O(1),$$

we get

$$\sum_{p\leq Y}\frac{1-\cos(h\log p)}{p}=\log\log Y-\log|\zeta(1+\frac{1}{\log Y}+ih)|+O(1).$$

Hence, we get

$$2\sum_{p^r<(\frac{T}{2\pi})^{\tau^*},r\geq 1}\frac{\sin^2(\pi\alpha r\frac{\log p}{\log\frac{T}{2\pi}})}{r^2p^r}=\log\log(\frac{T}{2\pi})^{\tau^*}-\log|\zeta(1+\frac{1}{\tau^*\log\frac{T}{2\pi}}+ih)|+O(1)$$

$$=\log\tau^*+\log\log\frac{T}{2\pi}-\log|\zeta(1+\frac{1}{\tau^*\log\frac{T}{2\pi}}+ih)|+O(1).$$

Thus we get

$$\tilde{V}(\alpha,T,\tau^*)=\frac{1}{\pi^2}\{\log(2\pi\alpha)-\mathrm{Ci}(2\pi\alpha)-2\pi\alpha\cdot\mathrm{Si}(2\pi\alpha)$$
$$+\pi^2\alpha-\cos(2\pi\alpha)+1+C_o\}$$
$$+\frac{1}{\pi^2}\{\log\tau^*+\log\log\frac{T}{2\pi}-\log|\zeta(1+\frac{1}{\tau^*\log\frac{T}{2\pi}}+ih)|$$
$$+\mathrm{Ci}(2\pi\alpha\tau^*)-\log(2\pi\alpha\tau^*)-C_o+O(1)\}.$$

This implies that

$$\tilde{V}(\alpha,T,\tau^*)=\frac{1}{\pi^2}\{\log\log\frac{T}{2\pi}-\log|\zeta(1+\frac{1}{\tau^*\log\frac{T}{2\pi}}+ih)|\}+O(\frac{1}{\alpha\tau^*})+O(1)$$
$$=\frac{1}{\pi^2}\{\log\log T-\log|\zeta(1+\frac{1}{\tau^*\log\frac{T}{2\pi}}+ih)|+O(1)\}.$$

Since when $\log T\ll\alpha\ll\log T$, we have

$$\log|\zeta(1+\frac{1}{\tau^*\log\frac{T}{2\pi}}+ih)|=O(1),$$

we get the following as a conclusion.

Theorem 1. *Suppose that* $T>T_0$ *and* $\frac{\log 2}{\log\frac{T}{2\pi}}\leq\tau^*\ll 1$. *Then we have*

$$\tilde{V}(\alpha,T,\tau^*)=$$
$$\begin{cases}\frac{1}{\pi^2}\{\log(2\pi\alpha)-\mathrm{Ci}(2\pi\alpha)-2\pi\alpha\cdot\mathrm{Si}(2\pi\alpha)+\pi^2\alpha-\cos(2\pi\alpha)+1+C_o+o(1)\} \\ \qquad \text{if } 0<\alpha=o(\log T) \\ \frac{1}{\pi^2}\{\log\log T-\log|\zeta(1+\frac{1}{\tau^*\log\frac{T}{2\pi}}+i\frac{2\pi\alpha}{\log\frac{T}{2\pi}})|+O(1)\} \\ \qquad \text{if } \log T\ll\alpha\ll T\log T.\end{cases}$$

Remark 1. Theorem 1 can be simplified a little bit. We suppose that $1 \ll h \ll T$. By p.135 of Titchmarsh[26], we get

$$\log\zeta(1+\frac{1}{\log Y}+ih)-\log\zeta(1+ih)$$
$$=\int_1^{1+\frac{1}{\log Y}}\frac{\zeta'}{\zeta}(\sigma+ih)\,d\sigma \ll \frac{(\log h)^{\frac{2}{3}}\cdot(\log\log h)^{\frac{1}{3}}}{\log Y} \ll \frac{(\log T)^{\frac{2}{3}}\cdot(\log\log T)^{\frac{1}{3}}}{\log Y}.$$

Hence we get

$$\log|\zeta(1+\frac{1}{\log Y}+ih)|=\log|\zeta(1+ih)|+O(1)$$

if

$$(\frac{\log\log T}{\log T})^{\frac{1}{3}} \ll \tau^*.$$

Thus under the condition

$$(\frac{\log\log T}{\log T})^{\frac{1}{3}} \ll \tau^* \ll 1,$$

we get

$$\widetilde{V}(\alpha,T,\tau^*)$$
$$=\begin{cases} \frac{1}{\pi^2}\{\log(2\pi\alpha)-\mathrm{Ci}(2\pi\alpha)-2\pi\alpha\cdot\mathrm{Si}(2\pi\alpha)+\pi^2\alpha-\cos(2\pi\alpha)+1+C_0+o(1)\} \\ \qquad \text{if} \quad 0<\alpha=o(\log T) \\ \frac{1}{\pi^2}\{\log\log T-\log|\zeta(1+i\frac{2\pi\alpha}{\log\frac{T}{2\pi}})|+O(1)\} \\ \qquad \text{if} \quad \log T \ll \alpha \ll T\log T. \end{cases}$$

We understand from the above argument that when $0<\alpha \ll \log T$, then we need only such condition as $\frac{\log 2}{\log\frac{T}{2\pi}} \leq \tau^* \ll 1$.

Remark 2. By p.135 of Titchmarsh[26], we know that for $t>t_0$

$$\frac{1}{\zeta(1+it)} \ll (\log t)^{\frac{2}{3}}\cdot(\log\log t)^{\frac{1}{3}}$$

and

$$\zeta(1+it) \ll (\log t)^{\frac{2}{3}}.$$

Hence for $t>t_0$, we get

$$\log|\zeta(1+it)| \leq \frac{2}{3}\log\log t+\frac{1}{3}\log\log\log t+\log A<\log\log t-A.$$

If we assume the Riemann hypothesis, then it is well-known (cf. Theorem 14.9

of Titchmarsh [26]) that for $t>t_0$

$$\log|\zeta(1+it)|\ll\log\log\log t.$$

3. The evaluation of a simplified number variance

3-1. A simplified number variance. In the previous section, we have seen what the Berry Conjecture claims. In the present section and the section 5, we shall see what we can prove with respect to the Berry Conjecture.

We start with noticing that

$$V(\alpha,T)=\frac{1}{T}\int_0^T(S(B(x+\frac{\alpha}{2}))-S(B(x-\frac{\alpha}{2})))^2dx+O(\frac{\log^2 T}{T}),$$

since

$$N(T)=\frac{T}{2\pi}\log\frac{T}{2\pi}-\frac{T}{2\pi}+\frac{7}{8}+O(\frac{1}{T})+S(T)$$

as is mentioned in the introduction. We have not touched the last integral directly. However, we have given much study on the mean value

$$\int_0^T(S(t+\frac{2\pi\alpha}{\log\frac{T}{2\pi}})-S(t))^2dt,$$

which can be considered as the first approximation of the last integral and is called a simplified number variance in this article. We notice only that when we put $t=B(x)$,

$$B(x+\alpha)\sim t+\frac{2\pi\alpha}{\log\frac{t}{2\pi}}.$$

Moreover, we have mainly given much study for a shorter

$$\frac{2\pi\alpha}{\log\frac{T}{2\pi}}.$$

In this section, we shall evaluate the mean value

$$\int_0^T(S(t+\frac{2\pi\alpha}{\log\frac{T}{2\pi}})-S(t))^2dt$$

for the whole range of α and prove that it coincides, essentially, with what the Berry Conjecture claims.

In the section 5, we shall give our study on the original number variance.

$$\frac{1}{T}\int_0^T (S(B(x+\frac{\alpha}{2}))-S(B(x-\frac{\alpha}{2})))^2 dx.$$

3-2. A simplified number variance for a shorter α and the Montgomery Conjecture. In this subsection, we shall recall that the simplified number variance for a shorter α is connected with the Montgomery Conjecture.

We recall first that under the Riemann Hypothesis (R.H.), the author [6] [7] has shown, by applying Goldston [13], that

$$\int_0^T (S(t+\frac{2\pi\alpha}{\log\frac{T}{2\pi}})-S(t))^2 dt$$

$$=\frac{T}{\pi^2}\{\int_0^{2\pi\alpha}\frac{1-\cos a}{a}da+\int_1^\infty \frac{F(a)}{a^2}(1-\cos(2\pi\alpha a))da+o(1)\}.$$

for $0<\alpha=o(\log T)$, where $F(a)$ is the Montgomery's sum [22] defined by

$$F(a)\equiv F(a,T)\equiv\frac{1}{\frac{T}{2\pi}\log T}\sum_{0<\gamma,\gamma'\leq T}(\frac{T}{2\pi})^{ia(\gamma-\gamma')}w(\gamma-\gamma'),$$

γ and γ' running over the imaginary parts of the zeros of $\zeta(s)$ and

$$w(t)=\frac{4}{4+t^2}.$$

With respect to $F(a)$, Montgomery [22] and Goldston-Montgomery [14] have shown, under R. H., that for $0\leq a\leq 1$,

$$F(a)=a+O(\sqrt{\frac{\log\log T}{\log T}})+(1+O(\sqrt{\frac{\log\log T}{\log T}}))(\frac{T}{2\pi})^{-2a}\log\frac{T}{2\pi}.$$

For $a\geq 1$, Goldston [13] has shown, under R. H., that

$$\int_1^\infty \frac{F(a)}{a^2}da \quad \text{is bounded.}$$

For an individual value of $F(a)$ for $a\geq 1$, Montgomery [22] has conjectured the following.

Montgomery's conjecture.

$$F(a)=1+o(1) \quad \text{for} \quad a\geq 1$$

uniformly in bounded intervals.

Thus if we assume R. H. and the Montgomery's conjecture on $F(a)$, then we get for $0<\alpha=o(\log T)$,

$$\int_0^T (S(t+\frac{2\pi\alpha}{\log\frac{T}{2\pi}})-S(t))^2dt$$

$$=\frac{T}{\pi^2}\{\log(2\pi\alpha)+C_0-\mathrm{Ci}(2\pi\alpha)+1-\cos(2\pi\alpha)+\pi^2\alpha-2\pi\alpha\cdot\mathrm{Si}(2\pi\alpha)+o(1)\}.$$

The right hand side is nothing but the GUE part of the Berry's formula in the Berry Conjecture mentioned in the introduction. Moreover, the present range of α, namely, $0<\alpha=o(\log T)$, coincides exactly with the range of α of the appearance of the GUE part in Theorem 1 in the previous section.

It is highly probable that the higher moments (cf. Theorems 2, 2′ and 3 in the subsection 3-3 below) of

$$S(t+\frac{2\pi\alpha}{\log\frac{T}{2\pi}})-S(t)$$

might be the same as those comming from GUE.

We should recall here another approach. It has been observed by Gallagher-Mueller [12] that

$$\int_0^T (N_+(t+\frac{2\pi\alpha}{\log\frac{T}{2\pi}})-N_+(t))^2dt$$

$$=\frac{2\pi\alpha}{\log\frac{T}{2\pi}}\cdot\sum_{0<\gamma\le T}{}' m^2(\gamma)+2\int_0^{\frac{2\pi\alpha}{\log\frac{T}{2\pi}}}\sum_{\substack{0<\gamma,\gamma'\le T\\ 0<\gamma-\gamma'\le u}}\cdot 1du+O(\log^2T)$$

for $0<\frac{2\pi\alpha}{\log\frac{T}{2\pi}}\le 1$, where $T>T_0$, the dash indicates that we sum over the different γ's and $m(\gamma)$ denotes the multiplicity of γ. Concerning the integrand in the right hand side of the above equality, we have the following conjecture due to Montgomery [22].

Montgomery's pair correlation conjecture. For any $\alpha>0$,

$$\sum_{\substack{0<\gamma,\gamma'\le T\\ 0<\gamma-\gamma'\le\frac{2\pi\alpha}{\log\frac{T}{2\pi}}}}\cdot 1=\frac{T}{2\pi}\log T\{\int_0^\alpha(1-(\frac{\sin\pi t}{\pi t})^2)dt+o(1)\}.$$

As is noticed by Dyson, the density function

$$1-(\frac{\sin\pi t}{\pi t})^2$$

is exactly the density function of the pair correlation of the eigenvalues of Gaussian Unitary Ensembles.

By the Riemann-von Mangoldt formula for $N(T)$, we see that for any positive α

$$\sum_{\substack{0<\gamma,\gamma'\leq T\\ 0<\gamma-\gamma'\leq\frac{2\pi\alpha}{\log\frac{T}{2\pi}}}}\cdot 1=\sum_{0<\gamma\leq T}\ \sum_{\gamma-\frac{2\pi\alpha}{\log\frac{T}{2\pi}}\leq\gamma'<\gamma}\cdot 1=\frac{T}{2\pi}\log\frac{T}{2\pi}\int_0^{\alpha}dt$$

$$+\sum_{0<\gamma\leq T}(S(\gamma)-S(\gamma-\frac{2\pi\alpha}{\log\frac{T}{2\pi}}))+O(T).$$

Since, we [7] have shown, under R. H., that for $a\ll T^A$

$$\sum_{0<\gamma\leq T,\gamma+a>0}S(\gamma+a)\ll T\log T,$$

we get

$$\sum_{\substack{0<\gamma,\gamma'\leq T\\ 0<\gamma-\gamma'\leq\frac{2\pi\alpha}{\log\frac{T}{2\pi}}}}\cdot 1=\frac{T}{2\pi}\log\frac{T}{2\pi}\{\int_0^{\alpha}dt+O(1)\}.$$

Moreover, we (cf. pp.242-243 of [8]) have seen that the following is equivalent to the Montgomery's pair correlation conjecture: for any $\alpha>0$,

$$\sum_{0<\gamma\leq T}S(\gamma-\frac{2\pi\alpha}{\log\frac{T}{2\pi}})=\frac{T}{2\pi}\log\frac{T}{2\pi}\{\int_0^{\alpha}(\frac{\sin\pi t}{\pi t})^2dt+o(1)\}.$$

Now Gallagher-Mueller's Theorem 1 in [12] shows that if the Montgomery's pair correlation conjecture holds uniformly in each interval $0<\alpha_0\leq\alpha\leq\alpha_1<\infty$, then we have

$$\sum_{0<\gamma\leq T}{}' m^2(\gamma)=\frac{T}{2\pi}\log\frac{T}{2\pi}(1+o(1)).$$

Moreover, combining this with their observation mentioned above, they show in the same Theorem 1 under the Montgomery's pair correlation conjecture that for any bounded α,

$$\int_0^T(S(t+\frac{2\pi\alpha}{\log\frac{T}{2\pi}})-S(t))^2dt=\int_0^T(S_+(t+\frac{2\pi\alpha}{\log\frac{T}{2\pi}})-S_+(t))^2dt$$

$$=T\int_{-\infty}^{\infty}\min(|a|,\alpha)(\frac{\sin\pi a}{\pi a})^2da+o(T).$$

It is simple to see that

$$\int_{-\infty}^{\infty}\min(|a|,\alpha)(\frac{\sin\pi a}{\pi a})^2da$$

$$=\frac{1}{\pi^2}\{\log(2\pi\alpha)-\mathrm{Ci}(2\pi\alpha)-2\pi\alpha\cdot\mathrm{Si}(2\pi\alpha)+\pi^2\alpha-\cos(2\pi\alpha)+1+C_0\}.$$

To close this subsection, we may stress that Gallagher-Mueller's range of α is $C_1<\alpha<C_2$ with some positive constants C_1 and C_2, while our range of α is $0<\alpha=o(\log T)$. This difference is important when one sees the Berry Conjecture, in particular, our Theorem 1 proved in the section 2.

3-3. A simplified number variance for a whole range of α and a comparison with the Berry Conjecture. Here we shall evaluate

$$\int_0^T (S(t+\frac{2\pi\alpha}{\log\frac{T}{2\pi}})-S(t))^2 dt$$

for the whole range of α with or without assuming any unproved hypothesis. We denote $\frac{2\pi\alpha}{\log\frac{T}{2\pi}}$, sometimes, by h.

More generally and more recently, the author [10] has shown by refining the previous results that for $T>T_0$, for h in $0<h\ll T$ and for any integer k in $1\leq k\ll\frac{\log T}{\log\log T}$, we have

$$\int_0^T (S(t+h)-S(t))^{2k}dt=\frac{2k!}{(2\pi)^{2k}k!}2^kT(\mathrm{Cin}(h\log\frac{T}{2\pi})-\mathrm{Cin}(h\log 2))^k$$
$$+O(T(Ak)^k\{(\mathrm{Cin}(h\log\frac{T}{2\pi})-\mathrm{Cin}(h\log 2))^{k-\frac{1}{2}}$$
$$+(\mathrm{Cin}(h\log\frac{T}{2\pi})-\mathrm{Cin}(h\log 2))^{k-1}\log\log(h+3)+k^k$$
$$+(\log\log(h+3))^k\}).$$

To compare our results with Berry's $\tilde{V}(\alpha,T,\tau^*)$, it might be better to start with the following result which is written down in pp.182-183 of Fujii [10].

$$\int_0^T (S(t+h)-S(t))^{2k}dt=\frac{2k!}{(2\pi)^{2k}k!}2^kT\Xi^k+O(T(Ak)^k(k^k+\Xi^{k-\frac{1}{2}})),$$

where we put

$$\Xi\equiv\Xi(Z)\equiv\sum_{p\leq Z}\frac{1-\cos(h\log p)}{p},$$

$Z=(\frac{T}{2\pi})^{\frac{b}{k}}$ with some positive constant b and the dependence on the integer $k\geq 1$ is written down explicitly. We suppose above that

$$\frac{k}{b}<\frac{\log\frac{T}{2\pi}}{\log 2}.$$

By our analysis given in the section 2, we see that for $0<\alpha\ll\log T$,

$$\Xi = \mathrm{Cin}\,(2\pi\alpha\frac{b}{k}) - \mathrm{Cin}\,(\frac{2\pi\alpha}{\log\frac{T}{2\pi}}\log 2) + R\,(Z)\frac{1-\cos(h\log Z)}{Z\log Z} + \frac{1-\cos(h\log 2)}{\log 2}$$

$$-h\int_2^Z \frac{R(t)}{t^2\log t}\sin(h\log t)\,dt + \int_2^Z \frac{R(t)}{t^2\log^2 t}(1-\cos(h\log t))\,(\log t+1)\,dt$$

$$= \log(2\pi\alpha) + C_o - \mathrm{Ci}\,(2\pi\alpha) - \mathrm{Cin}\,(\frac{2\pi\alpha}{\log\frac{T}{2\pi}}\log 2) + O(\log(3k))$$

$$+R\,(Z)\frac{1-\cos(h\log Z)}{Z\log Z} + \frac{1-\cos(h\log 2)}{\log 2}$$

$$-h\int_2^Z \frac{R(t)}{t^2\log t}\sin(h\log t)\,dt + \int_2^Z \frac{R(t)}{t^2\log^2 t}(1-\cos(h\log t))\,(\log t+1)\,dt$$

$$= \log(2\pi\alpha) - \mathrm{Ci}\,(2\pi\alpha) + C_o + O(\log(3k)).$$

When $\log T \ll \alpha \ll T\log T$, then we have

$$\Xi(Z) = \Xi(T) + O(\sum_{\min(T,(\frac{T}{2\pi})^{\frac{k}{b}}) < p \le \max(T,(\frac{T}{2\pi})^{\frac{k}{b}})} \frac{1}{p})$$

$$= \log\log T - \log|\zeta(1+\frac{1}{\log T}+ih)| + O(\log(3k))$$

$$= \log\log T - \log|\zeta(1+ih)| + O(\log(3k)),$$

since we have for fixed b,

$$\log|\zeta(1+\frac{1}{\log T}+ih)| - \log|\zeta(1+ih)| \ll \frac{\log h}{\log T} \ll 1.$$

This implies the following result.

Theorem 2. *Suppose that* $0 < \frac{2\pi\alpha}{\log\frac{T}{2\pi}} \ll T$. *Then we have uniformly for an integer* $1 \le k \ll \frac{\log T}{\log\log T}$,

$$\int_0^T (S(t+\frac{2\pi\alpha}{\log\frac{T}{2\pi}}) - S(t))^{2k}\,dt$$

$$= \begin{cases} \frac{2k!}{(2\pi)^{2k}k!}2^k T(\log(2\pi\alpha) - \mathrm{Ci}(2\pi\alpha) + C_o)^k \\ \quad + O(T(Ak)^k((\log(2\pi\alpha) - \mathrm{Ci}(2\pi\alpha) + C_o)^{k-\frac{1}{2}} + k^k)) \\ \qquad \text{if} \quad 0 < \alpha \ll \log T \\ \frac{2k!}{(2\pi)^{2k}k!}2^k T(\log\log T - \log|\zeta(1+i\frac{2\pi\alpha}{\log\frac{T}{2\pi}})|)^k \\ \quad + O(T(Ak)^k((\log\log T - \log|\zeta(1+i\frac{2\pi\alpha}{\log\frac{T}{2\pi}})|)^{k-\frac{1}{2}} + k^k)) \\ \qquad \text{if} \quad \log T \ll \alpha \ll T\log T. \end{cases}$$

In particular, when $k=1$, we get the following.

Theorem 2′. *Suppose that* $0<\frac{2\pi\alpha}{\log\frac{T}{2\pi}}\ll T$. *Then we have*

$$\int_0^T \left(S\left(t+\frac{2\pi\alpha}{\log\frac{T}{2\pi}}\right)-S(t)\right)^2 dt$$

$$=\begin{cases} \frac{T}{\pi^2}\{\log(2\pi\alpha)-\mathrm{Ci}(2\pi\alpha)+C_o+O(\sqrt{\log(2\pi\alpha+3)})\} \\ \qquad \text{if} \quad 0<\alpha\ll\log T \\ \frac{T}{\pi^2}\{\log\log T-\log\left|\zeta\left(1+i\frac{2\pi\alpha}{\log\frac{T}{2\pi}}\right)\right| \\ \qquad +O\left(\sqrt{\left(\log\log T-\log\left|\zeta\left(1+i\frac{2\pi\alpha}{\log\frac{T}{2\pi}}\right)\right|\right)}\right)\} \\ \qquad \text{if} \quad \log T\ll\alpha\ll T\log T. \end{cases}$$

Thus we have obtained the same main term as $\tilde{V}(\alpha,T,\tau^*)$ as far as α is sufficiently large (cf. Theorem 1 with Remark 1 above).

If we assume the Riemann Hypothesis, then by modifying Selberg's argument in pp.179-203 of Selberg[25], we get for each integer $k\geq1$,

$$\int_0^T \left(S\left(t+\frac{2\pi\alpha}{\log\frac{T}{2\pi}}\right)-S(t)\right)^{2k} dt=\frac{2k!}{(2\pi)^{2k}k!}2^kT\Xi^k+O(T\Xi^{k-1}).$$

This together with our evaluation of Ξ described above implies the following theorem, where we omit writing the dependence on k.

Theorem 3 (Under the Riemann Hypothesis). *Suppose that* $0<\frac{2\pi\alpha}{\log\frac{T}{2\pi}}\ll T$. *Then we have for each integer* $k\geq1$,

$$\int_0^T \left(S\left(t+\frac{2\pi\alpha}{\log\frac{T}{2\pi}}\right)-S(t)\right)^{2k} dt$$

$$=\begin{cases} \frac{2k!}{(2\pi)^{2k}k!}2^kT\{(\log(2\pi\alpha)-\mathrm{Ci}(2\pi\alpha)+C_o)^k \\ \qquad +O((\log(2\pi\alpha)-\mathrm{Ci}(2\pi\alpha)+C_o)^{k-1}+1)\} \\ \qquad \text{if} \quad 0<\alpha\ll\log T \\ \frac{2k!}{(2\pi)^{2k}k!}2^kT\{(\log\log T-\log\left|\zeta\left(1+i\frac{2\pi\alpha}{\log\frac{T}{2\pi}}\right)\right|)^k \\ \qquad +O((\log\log T-\log\left|\zeta\left(1+i\frac{2\pi\alpha}{\log\frac{T}{2\pi}}\right)\right|)^{k-1})\} \\ \qquad \text{if} \quad \log T\ll\alpha\ll T\log T. \end{cases}$$

In particular, when $k=1$, we get

Theorem 3′ (Under the Riemann Hypothesis). *Suppose that* $0<\frac{2\pi\alpha}{\log\frac{T}{2\pi}}\ll T$. *Then we have*

$$\int_0^T \left(S\left(t+\frac{2\pi\alpha}{\log\frac{T}{2\pi}}\right)-S(t)\right)^2 dt=\begin{cases}\frac{T}{\pi^2}\{\log(2\pi\alpha)-\mathrm{Ci}(2\pi\alpha)+C_0+O(1)\} & \text{if } 0<\alpha\ll\log T\\ \frac{T}{\pi^2}\{\log\log T-\log\left|\zeta\left(1+i\frac{2\pi\alpha}{\log\frac{T}{2\pi}}\right)\right|+O(1)\} & \text{if } \log T\ll\alpha\ll T\log T.\end{cases}$$

One sees in Theorem 3′ or also in Theorem 2′ that we do not have an asymptotic formula for the case when $0<\alpha\ll 1$. To recover this case, we pick up some of our results mentioned in the previous subsection, combine it with Theorem 3′ and get the following Theorem 3″ and Theorem 4.

Theorem 3″ (Under the Riemann Hypothesis). *Suppose that* $0<\frac{2\pi\alpha}{\log\frac{T}{2\pi}}\ll T$. *Then we have*

$$\int_0^T \left(S\left(t+\frac{2\pi\alpha}{\log\frac{T}{2\pi}}\right)-S(t)\right)^2 dt$$

$$=\begin{cases}\frac{T}{\pi^2}\left\{\int_0^{2\pi\alpha}\frac{1-\cos a}{a}da+\int_1^\infty\frac{F(a)}{a^2}(1-\cos(2\pi\alpha a))da+o(1)\right\} & \text{if } 0<\alpha=o(\log T)\\ \frac{T}{\pi^2}\{\log\log T-\log\left|\zeta\left(1+i\frac{2\pi\alpha}{\log\frac{T}{2\pi}}\right)\right|+O(1)\} & \text{if } \log T\ll\alpha\ll T\log T,\end{cases}$$

where $F(a)$ *is defined in the subsection* 3-2.

Theorem 4 (Under the Riemann Hypothesis and the Montgomery Conjecture on $F(a)$). *Suppose that* $0<\frac{2\pi\alpha}{\log\frac{T}{2\pi}}\ll T$. *Then we have*

$$\int_0^T \left(S\left(t+\frac{2\pi\alpha}{\log\frac{T}{2\pi}}\right)-S(t)\right)^2 dt=\begin{cases}\frac{T}{\pi^2}\{\log(2\pi\alpha)-\mathrm{Ci}(2\pi\alpha)-2\pi\alpha\cdot\mathrm{Si}(2\pi\alpha)\\ \quad+\pi^2\alpha-\cos(2\pi\alpha)+1+C_0+o(1)\} & \text{if } 0<\alpha=o(\log T)\\ \frac{T}{\pi^2}\{\log\log T-\log\left|\zeta\left(1+i\frac{2\pi\alpha}{\log\frac{T}{2\pi}}\right)\right|+O(1)\} & \text{if } \log T\ll\alpha\ll T\log T.\end{cases}$$

The right hand side is exactly the right hand side of $\widetilde{V}(\alpha,T,\tau^*)$ given in Theorem 1 with Remark 1. We may repeat that even without assuming any unproved hypothesis, the main terms coincide in both our mean value theorem, namely, Theorem 2′, and $\widetilde{V}(\alpha,T,\tau^*)$, as far as α is sufficiently large. For $0<\alpha\ll 1$, we need the Riemann Hypothesis and the Montgomery Conjecture as described just above.

In the section 6, we shall give the graphs of

$$\frac{1}{\pi^2}\{\log(2\pi\alpha)-\mathrm{Ci}(2\pi\alpha)-2\pi\alpha\cdot\mathrm{Si}(2\pi\alpha)+\pi^2\alpha-\cos(2\pi\alpha)+1+C_o\}$$

and

$$\frac{2k!}{(2\pi)^{2k}k!}2^k\{\log\log T-\log|\zeta(1+i\frac{2\pi\alpha}{\log\frac{T}{2\pi}})|\}^k$$

for the various ranges of α, certain T's and for $k=1$. We should compare these with the empirical datas given by Odlyzko [23] and also the graphs in pp.404-406 of Berry [1]. We could say that we have succeeded in giving a theoretical proof to explain the phenomenon shown in the empirical datas given by Odlyzko [23].

4. A discrete version of the Berry Conjecture

4-1. A discrete version of the Berry Conjecture. Here we are concerned with a discrete version of our problem.

In stead of the function $B(t)$ in the Berry Conjecture, we shall deal with the quantity g_x defined by

$$\vartheta(g_x)=x\pi \quad \text{for} \quad x\geq -1,$$

where $\vartheta(t)$ is defined in the introduction. We denote g_x by $G(x)$ and use both notations. In view of the results mentioned in the subsection 3-3, we may state a discrete version of the Berry Conjecture as follows.

A discrete version of the Berry Conjecture. For $M>M_0$ and for $0<\alpha\ll M$, we have

$$\sum_{1\leq m\leq M}(N(G(m+\alpha))-N(G(m))-\alpha)^2$$
$$=\begin{cases}\frac{M}{\pi^2}\{\log(2\pi\alpha)-\mathrm{Ci}(2\pi\alpha)-2\pi\alpha\cdot\mathrm{Si}(2\pi\alpha)+\pi^2\alpha-\cos(2\pi\alpha)+1+C_o+o(1)\} \\ \qquad \text{if} \quad 0<\alpha=o(\log M) \\ \frac{M}{\pi^2}\{\log\log M-\log|\zeta(1+i\frac{2\pi\alpha}{\log\frac{M}{2\pi}})|+O(1)\} \\ \qquad \text{if} \quad \log M\ll\alpha\ll M.\end{cases}$$

We might replace $N(\cdot)$ in the above conjecture by $N_+(\cdot)$ without changing the other parts. We shall see below that we have a satisfactory result for a sufficiently large α.

4-2. A discrete number variance for a bounded α and the random matrix theory. In this subsection we shall recall some investigations which are connected with a discrete number variance for a bounded α. It will reveal again a connection of the distribution of the zeros with the random matrix theory.

Special attentions have been paid to the case for $\alpha=1$ long before. g_m or $G(m)$ is called the Gram point. Gram [15] observed that the zeros of $\zeta(\frac{1}{2}+it)$ appears exactly once in the interval $[g_m, g_{m+1})$ up to ≤ 50. In other notations, Gram's observation or (Gram's law) states that

$$N(g_{m+1})-N(g_m)=1 \quad \text{for any integer } m \text{ in } \quad -1\leq m\leq 8.$$

From Haselgrove's table [16], one sees that

$$g_{-1}=9.666908\cdots<\gamma_1=14.134725\cdots<g_0<\gamma_2<g_1<\gamma_3<g_2<\gamma_4<g_3<\gamma_5<g_4<\gamma_6 <g_5<\gamma_7<g_6<\gamma_8<g_7<\gamma_9<g_8<\gamma_{10}<g_9=51.733843\cdots,$$

where γ_n is the n-th positive imaginary part of the zeros of $\zeta(s)$. It appears at first sight that this might continue to hold for $m>9$. However as we know at present that many counterexamples have been found since Hutchinson [17]. In fact, we know that for positive proportion of m

$$N(g_{m+1})-N(g_m)\geq 2$$

and for positive proportion of m

$$N(g_{m+1})-N(g_m)=0$$

(cf. p.353 of Selberg [25] and p.393 of Fujii [5]). These lead to the following problem (cf. Problem in Fujii [5]): *to study the quantity*

$$\lim_{M\to\infty}\frac{1}{M}G_M(k,\alpha)$$

for each $k=0,1,2,\cdots$ *and for any positive* α, *where we put*

$$G_M(k,\alpha)=|\{-1\leq m\leq M;\ N(g_{m+\alpha})-N(g_m)=k\}|.$$

In fact, the problem and the conjecture were proposed by Kosambi [18] for a slightly different choice of the sequence $g_{m+\alpha}$ and g_m. He was concerned with the distribution of the number $f_{b,a}(m)$ defined by

$$f_{b,a}(m)=|\{\gamma_n;\ L(\gamma_n,a)\in((m-1)b,mb]\}|$$

for any positive constants a and b, where $L(x,a)$ for $x\geq a$ is defined by

$$L(x,a)=\frac{1}{2\pi}\int_a^x\log\frac{t}{2\pi}dt.$$

His conjecture states that the distribution is of Poisson type. More precisely, for some $b>0$ and $a>0$,

$$\lim_{M\to\infty}\frac{1}{M}|\{1\le m<M;\, f_{b,a}(m)=k\}|=\frac{e^{-b}b^k}{k!} \quad \text{for} \quad k=0,1,2,\cdots.$$

In pp.124-128 of [3], the author has shown that this is not correct for $b>b_0>0$ and for any $a>0$ (Cf. also Gallagher-Mueller [12]). Similarly, we [5] can show that

$$\lim_{M\to\infty}\frac{1}{M}G_M^+(k,\alpha)\neq\frac{e^{-\alpha}\alpha^k}{k!}$$

for some $k\ge 0$ if $\alpha>\alpha_0>0$, where we put

$$G_M^+(k,\alpha)=|\{-1\le m\le M;\, N_+(g_{m+\alpha})-N_+(g_m)=k\}|.$$

The computer calculations by van de Lune, te Riele and Winter [19] tells us that for $M=1500000000$,

$$\frac{1}{M}G_M(0,1)=0.1378...$$

$$\frac{1}{M}G_M(1,1)=0.7261...$$

$$\frac{1}{M}G_M(2,1)=0.1342...$$

and

$$\frac{1}{M}G_M(3,1)=0.0018... \quad .$$

Concerning this problem, we [5] have once given the following conjecture.

Conjecture. For each integer $k\ge 0$ and for $0<\alpha<\alpha_0<\infty$,

$$\lim_{M\to\infty}\frac{1}{M}G_M(k,\alpha)=E(k,\alpha),$$

where $E(k,\alpha)$ is defined below (cf. 2.32 of Mehta-Cloizeaux [21] and Mehta [20].)

For $0<\alpha<\alpha_0$,

$$E(0,\alpha)=\prod_j(1-\lambda_j)$$

and for each integer $k\ge 1$,

$$E(k,\alpha)=\prod_j(1-\lambda_j)\sum_{\lambda_{j1}<\cdots<\lambda_{jk}}\frac{\lambda_{j_1}}{1-\lambda_{j_1}}\cdots\frac{\lambda_{j_k}}{1-\lambda_{j_k}},$$

where λ_j's for $j\ge 0$ run over the eigen values of the integral operator

$$\lambda f(y)=\int_{-1}^{1}\frac{\sin((y-x)\pi\alpha)}{(y-x)\pi}f(x)\,dx.$$

Numerical computations in p.350 of Mehta-Cloizeaux [21] suggests that

$$E(0,1)=0.17\cdots,\quad E(1,1)=0.74\cdots\quad\text{and}\quad E(2,1)=0.13\cdots.$$

These should be compared with the datas given by van de Lune, te Riele and Winter [19] mentioned above.

Moreover, it seems to be known to the physicists (cf. References in Mehta [20]) that

$$\sum_{k=0}^{\infty}kE(k,\alpha)=\alpha$$

and

$$\sum_{k=0}^{\infty}k^2E(k,\alpha)=\sum_{k=0}^{\infty}(k-\alpha)^2E(k,\alpha)+2\alpha\sum_{k=0}^{\infty}kE(k,\alpha)-\alpha^2\sum_{k=0}^{\infty}E(k,\alpha)$$
$$=\alpha-2\int_0^{\alpha}(\alpha-x)\left(\frac{\sin\pi x}{\pi x}\right)^2dx+\alpha^2.$$

To understand, more easily, the connection of the discrete number variance with the above conjecture, we shall first modify the above conjecture as follows.

Conjecture (+). For each integer $k\geq 0$ and for $0<\alpha<\alpha_0<\infty$,

$$\lim_{M\to\infty}\frac{1}{M}G_M^{+}(k,\alpha)=E(k,\alpha).$$

Then we shall show in the subsection 4-5 under the Conjecture (+) that for any bounded $\alpha(>0)$, we have

$$\sum_{m\leq M}(S_+(g_{m+\alpha})-S_+(g_m))^2$$
$$\sim M\frac{1}{\pi^2}\{\log(2\pi\alpha)-\mathrm{Ci}(2\pi\alpha)-2\pi\alpha\cdot\mathrm{Si}(2\pi\alpha)+\pi^2\alpha-\cos(2\pi\alpha)+1+C_0\}.$$

The right hand side is nothing but the GUE part of the discrete version of the Berry Conjecture for a bounded α.

On the other hand, our unconditional Theorem 5 in the subsection 4-3 implies, as a special case, that

$$\frac{1}{\frac{M}{2}}\sum_{\frac{M}{2}\leq m\leq M}(S(g_{m+\alpha})-S(g_m))^2\sim\frac{1}{\pi^2}\log(2\pi\alpha)$$

as $\alpha\to\infty$ and $\alpha\ll\log M$. This certainly supports the discrete version of the

Berry Conjecture for a sufficiently large α.

We understand that there is a gap between

$$\sum_{m\le M}(S_+(g_{m+\alpha})-S_+(g_m))^2$$

and

$$\sum_{m\le M}(S(g_{m+\alpha})-S(g_m))^2.$$

Furthermore, we understand also that to get the following asymptotic formula, which is a consequence of Conjecture (+) and the formula $\sum_{k=0}^{\infty}kE(k,\alpha)=\alpha$,

$$\sum_{m\le M}(N_+(g_{m+\alpha})-N_+(g_m))\sim\alpha M$$

even for $0<\alpha<1$, without assuming any unproved hypothesis, seems to be very difficult. Since it says that

$$\sum_{m\le M}(N_+(g_{m+\alpha})-N_+(g_m))=\sum_{m\le M}\sum_{m<\frac{1}{\pi}\vartheta(\gamma)\le m+\alpha}\cdot 1\sim\alpha M,$$

namely that

the sequence $\frac{1}{\pi}\vartheta(\gamma_n)$, $n=1,2,3,\cdots$ *is uniformly distributed mod one.*

In fact, the last statement has been conjectured several times (, for example, p.219 of Fujii [11]), although we [4] have proved, among others, that a slightly less fast increasing sequence like

$$\frac{b\gamma_n\log\gamma_n}{\log\log\log\log\log\log\gamma_n},\quad n=n_0+1,\ n_0+2,\ n_0+3,\cdots$$

is uniformly distributed mod one for any positive constant b.

4-3. The evaluation of a discrete number variance. To study the problem mentioned in the previous subsection theoretically, we need to evaluate the mean values

$$\sum_{m\le M}(N(g_{m+\alpha})-N(g_m))^j$$

for each $j=1,2,3,\cdots$ and for any positive α. It is exactly to evaluate the following mean values.

$$\sum_{m\le M}(S(g_{m+\alpha})-S(g_m))^j$$

for each $j=1,2,3,\cdots$ and for any positive α.

We have announced the following theorem in [3] and [5]: for any integer $k \geq 1$, for positive $\alpha \ll \log M$ and for $M > M_0$, we have

$$\sum_{m \leq M} (S(g_{m+\alpha}) - S(g_m))^{2k}$$
$$= \frac{2k!}{(2\pi)^{2k} k!} M(2\log(\alpha+1))^k + O(M(Ak)^k(k^k + (\log(\alpha+1))^{k-\frac{1}{2}})).$$

We can replace $S(\cdot)$ in the above result by $S_+(\cdot)$ without changing the other parts.

These results are strong enough to conclude, among others, three results concerning

$$N(g_{m+1}) - N(g_m)$$

and

$$\lim_{M\to\infty} \frac{1}{M} G_M^+(k,\alpha),$$

which have been mentioned above (cf. p.393 of Fujii[5]).

When $k=1$, $\alpha \to \infty$ and $\alpha \ll \log M$, then the main term of the above theorem coincides with the main term of a discrete version of the Berry Conjecture.

When $0 < \alpha \ll M^{1-\eta}$ with a positive constant $\frac{1}{2} \leq \eta < 1$, we can extend our proof in [3] and [5] and prove, in fact, the following theorems.

Theorem 5. *Suppose that* $M > M_0$ *and* $0 < \alpha \ll M^{1-\eta}$ *with a positive constant* $\frac{1}{2} \leq \eta < 1$. *Then we have*

$$\sum_{\frac{M}{2} \leq m \leq M} (S(G(m+\alpha)) - S(G(m)))^2 = \begin{cases} \frac{M}{2}\frac{1}{\pi^2}\{\log(2\pi\alpha+1) + O(\sqrt{\log(\alpha+1)})\} \\ \qquad \text{if} \quad 0 < \alpha \ll \log M \\ \frac{M}{2}\frac{1}{\pi^2}\{\log\log M + O(\sqrt{\log\log M})\} \\ \qquad \text{if} \quad \log M \ll \alpha \ll M^{1-\eta}. \end{cases}$$

When we treat the separated case $G(m) + \frac{2\pi\alpha}{\log\frac{M}{2\pi}}$ in stead of $G(m+\alpha)$, then the problem becomes simpler and we have the following finer result.

Theorem 5′. *Suppose that* $M > M_0$ *and* $0 < \alpha \ll M$. *Then we have*

$$\sum_{\frac{M}{2} \leq m \leq M} (S(G(m) + \frac{2\pi\alpha}{\log\frac{M}{2\pi}}) - S(G(m)))^2$$

$$
=\begin{cases}
\frac{M}{2}\frac{1}{\pi^2}\{\log(2\pi\alpha+1)+O(\sqrt{\log(2\pi\alpha+1)})\} \\
\qquad\qquad \textit{if} \quad 0<\alpha\ll\log M \\
\frac{M}{2}\frac{1}{\pi^2}\{\log\log M-\log|\zeta(1+i\frac{2\pi\alpha}{\log\frac{M}{2\pi}})|+O(\sqrt{\log\log M-\log|\zeta(1+i\frac{2\pi\alpha}{\log\frac{M}{2\pi}})|})\} \\
\qquad\qquad \textit{if} \quad \log M\ll\alpha\ll M.
\end{cases}
$$

Theorem 5′ corresponds to Theorem 2′ in the subsection 3-3. Theorem 5 will correspond to Theorems 6 and 6′ in the section 5.

We can certainly extend our theorems to the higher moments, namely,

$$\sum_{\frac{M}{2}\le m\le M}(S(G(m+\alpha))-S(G(m)))^{2k}$$

or

$$\sum_{\frac{M}{2}\le m\le M}(S(G(m)+\frac{2\pi\alpha}{\log\frac{M}{2\pi}})-S(G(m)))^{2k},$$

although we shall omit writing them. We can also replace $S(\cdot)$ in Theorems 5 and 5′ by $S_+(\cdot)$ without changing the other parts.

In the subsection 4-4, we shall give the details of the proofs of Theorems 5 and 5′ as much as possible for completeness.

4-4. Proof of Theorems 5 and 5′. We shall prove Theorem 5 first.

We start with the following Selberg's explicit formula for $S(t)$ (cf. p.250 of Selberg[25].)

Lemma. *Suppose that $M>M_0$, $X=M^b$ with a sufficiently small positive constant b. Then for any $M^{\frac{b}{2}}\le t$, we have*

$$
\begin{aligned}
S(t)=&\Im\{\frac{1}{\pi}\sum_{p<X^3}\frac{1}{p^{\frac{1}{2}+it}}\}+O(|\sum_{p<X^3}\frac{\Lambda(p)-\Lambda_X(p)}{\sqrt{p}\log p\cdot p^{it}}|)\\
&+O(|\sum_{p<X^{\frac{3}{2}}}\frac{\Lambda_X(p^2)}{p\log p\cdot p^{2it}}|)+O((\sigma_{X,t}-\frac{1}{2})\log M)\\
&+O((\sigma_{X,t}-\frac{1}{2})X^{(\sigma_{X,t}-\frac{1}{2})}\int_{\frac{1}{2}}^{\infty}X^{\frac{1}{2}-\sigma}|\sum_{p<X^3}\frac{\Lambda_X(p)\log(Xp)}{p^{\sigma+it}}|d\sigma),
\end{aligned}
$$

where we put

$$\Lambda_X(n)=\begin{cases}\Lambda(n) & \text{for } 1\leq n\leq X\\ \Lambda(n)\dfrac{(\log\frac{X^3}{n})^2-2(\log\frac{X^2}{n})^2}{2(\log X)^2} & \text{for } X\leq n\leq X^2\\ \Lambda(n)\dfrac{(\log\frac{X^3}{n})^2}{2(\log X)^2} & \text{for } X^2\leq n\leq X^3\end{cases}$$

and

$$\sigma_{X,t}=\frac{1}{2}+2\max_{\rho}(\beta-\frac{1}{2},\frac{2}{\log X}),$$

ρ running here through all zeros $\beta+i\gamma$ of $\zeta(s)$ for which

$$|t-\gamma|\leq\frac{X^{3|\beta-\frac{1}{2}|}}{\log X}.$$

In stead of g_x, we write $G(x)$. By the above lemma, to evaluate the sum

$$S\equiv\sum_{\frac{M}{2}\leq m\leq M}(S(G(m+\alpha))-S(G(m)))^2,$$

we need to evaluate or estimate the following sums.

$$S_1=\sum_{\frac{M}{2}\leq m\leq M}|\sum_{p<X^3}\frac{a_m(p)}{\sqrt{p}}|^2,\quad S_2=\sum_{\frac{M}{2}\leq m\leq M}(\sum_{p<X^3}\frac{a_m(p)}{\sqrt{p}})^2,$$

$$S_3=\sum_{\frac{M}{2}\leq m\leq M}|\sum_{p<X^3}\frac{a(p)}{\sqrt{p}\cdot p^{iG(m)}}|^{2k},\quad S_3'=\sum_{\frac{M}{2}\leq m\leq M}|\sum_{p<X^3}\frac{a(p)}{\sqrt{p}\cdot p^{iG(m+\alpha)}}|^{2k},$$

$$S_4=\sum_{\frac{M}{2}\leq m\leq M}|\sum_{p<X^{\frac{3}{2}}}\frac{a'(p)}{p^{1+2iG(m)}}|^{2k},\quad S_4'=\sum_{\frac{M}{2}\leq m\leq M}|\sum_{p<X^{\frac{3}{2}}}\frac{a'(p)}{p^{1+2iG(m+\alpha)}}|^{2k},$$

$$S_5=\sum_{\frac{M}{2}\leq m\leq M}(\sigma_{X,G(m)}-\frac{1}{2})^{\nu}\xi^{(\sigma_{X,G(m)}-\frac{1}{2})}$$

and

$$S_5'=\sum_{\frac{M}{2}\leq m\leq M}(\sigma_{X,G(m+\alpha)}-\frac{1}{2})^{\nu}\xi^{(\sigma_{X,G(m+\alpha)}-\frac{1}{2})},$$

where we put

$$a_m(p)=\frac{1}{p^{iG(m+\alpha)}}-\frac{1}{p^{iG(m)}},$$

$$|a(p)| \ll \frac{\log p}{\log X} \quad \text{for} \quad p<X^3,$$

$$|a'(p)| \ll 1 \quad \text{for} \quad p<X^{\frac{3}{2}},$$

$k=1$ and 2, $\nu=2$ and 4, $1 \leq \xi \leq X^4$ and $X^3\xi^2 \leq (\frac{M}{\log M})^{\frac{1}{8}}$.

We shall estimate S_1 first.

$$S_1 = \sum_{\frac{M}{2}\leq m\leq M} \sum_{p_1, p_2<X^3} \frac{a_m(p_1)\bar{a}_m(p_2)}{\sqrt{p_1p_2}}$$

$$= \sum_{\frac{M}{2}\leq m\leq M} \sum_{p<X^3} \frac{|a_m(p)|^2}{p} + \sum_{\frac{M}{2}\leq m\leq M} \sum_{p_1\neq p_2<X^3} \frac{a_m(p_1)\bar{a}_m(p_2)}{\sqrt{p_1p_2}} = S_6+S_7, \quad \text{say.}$$

Since for $0<\alpha\ll\sqrt{M}$

$$G(m+\alpha) = G(m)+\alpha G'(m)+\frac{\alpha^2}{2}G''(m)+\cdots = G(m)+\alpha G'(m)+O(\frac{\alpha^2}{M\log^2 M})$$

$$= G(m)+\frac{2\pi\alpha}{\log\frac{G(m)}{2\pi}}+O(\frac{\alpha^2}{M\log^2 M}+\frac{\alpha}{M^2}),$$

we get

$$S_6 = 2\sum_{\frac{M}{2}\leq m\leq M} \sum_{p<X^3} \frac{1}{p}(1-\cos((G(m+\alpha)-G(m))\log p))$$

$$= 2\sum_{\frac{M}{2}\leq m\leq M} \sum_{p<X^3} \frac{1}{p}(1-\cos(\frac{2\pi\alpha}{\log\frac{G(m)}{2\pi}}\log p))$$

$$+O(\sum_{\frac{M}{2}\leq m\leq M} \sum_{p<X^3} \frac{\log p}{p}(\frac{\alpha^2}{M\log^2 M}+\frac{\alpha}{M^2})).$$

The last remainder term is $O(M)$ under the condition $0<\alpha\ll\sqrt{M}$.

To evaluate the first sum, we shall use the same analysis as in the section 2. When

$$0<\alpha\ll\log M,$$

we get

$$\sum_{p<X^3} \frac{1}{p}(1-\cos(\frac{2\pi\alpha}{\log\frac{G(m)}{2\pi}}\log p))$$

$$= \int_2^{X^3} \frac{1-\cos(h(m)\log t)}{t\log t}dt + R(X^3)\frac{1-\cos(h(m)\log X^3)}{X^3\log X^3}$$

$$+\frac{1-\cos(h(m)\log 2)}{\log 2} - h(m)\int_2^{X^3} \frac{R(t)}{t^2\log t}\sin(h(m)\log t)\,dt$$

$$+\int_2^{X^3}\frac{R(t)}{t^2\log^2 t}(1-\cos(h(m)\log t))(\log t+1)dt,$$

where we put

$$h(m)=\frac{2\pi\alpha}{\log\frac{G(m)}{2\pi}}.$$

We have

$$\int_2^{X^3}\frac{1-\cos(h(m)\log t)}{t\log t}dt=\mathrm{Cin}(h(m)\log X^3)-\mathrm{Cin}(h(m)\log 2)$$

and

$$\sum_{\frac{M}{2}\leq m\leq M}\mathrm{Cin}(h(m)\log X^3)=\frac{M}{2}\log(2\pi\alpha+1)+O(M).$$

Hence, we get

$$S_6=\frac{M}{2}2\log(2\pi\alpha+1)+O(M).$$

When

$$\log M\ll\alpha\ll\sqrt{M},$$

we have

$$\sum_{p<X^3}\frac{1}{p}(1-\cos(\frac{2\pi\alpha}{\log\frac{G(m)}{2\pi}}\log p))=\sum_{p<M}\frac{1}{p}(1-\cos(h(m)\log p))+O(1)$$
$$=\sum_{p<M}\frac{1}{p}-\Re(\sum_{p<M}\frac{1}{p^{1+ih(m)}})+O(1).$$

When $\log M\ll\alpha\ll\log^2 M$, then

$$\sum_{p<M}\frac{1}{p}-\Re(\sum_{p<M}\frac{1}{p^{1+ih(m)}})=\log\log M-\log|\zeta(1+\frac{1}{\log M}+ih(m))|+O(1)$$
$$=\log\log M-\log|\zeta(1+ih(m))|+O(1).$$

Since $h(m)\ll\log M$, we have

$$\log|\zeta(1+ih(m))|\ll\log\log\log M.$$

Hence, we get

$$S_6=\frac{M}{2}2\log\log M+O(M\log\log\log M).$$

When $\log^2 M\ll\alpha\ll\sqrt{M}$, then since

$$\frac{d^2\left(\dfrac{2\pi\alpha\log p}{\log\frac{G(m)}{2\pi}}\right)}{dm^2} \cong \frac{\alpha\log p}{M^2\log^2 M},$$

we get by Theorem 5.9 in p.104 of Titchmarsh [26]

$$\sum_{p<M}\frac{1}{p}\sum_{\frac{M}{2}\le m\le M} e^{-ih(m)\log p} \ll \sum_{p<M}\frac{1}{p}\left(\sqrt{M}\frac{\sqrt{\alpha\log p}}{M\log M}+\frac{M\log M}{\sqrt{\alpha\log p}}\right) \ll M.$$

Thus in this case we get

$$S_6=\frac{M}{2}2\log\log M+O(M).$$

Hence, we get

$$S_6=\begin{cases}\frac{M}{2}\{2\log(2\pi\alpha+1)+O(1)\} \\ \qquad \text{if} \quad 0<\alpha\ll\log M \\ \frac{M}{2}\{2\log\log M+O(\log\log\log M)\} \\ \qquad \text{if} \quad \log M\ll\alpha\ll\sqrt{M}.\end{cases}$$

We shall next estimate S_7.

$$S_7=\sum_{p_1\neq p_2<X^3}\frac{1}{\sqrt{p_1p_2}}\sum_{\frac{M}{2}\le m\le M} a_m(p_1)\overline{a}_m(p_2).$$

The inner sum is

$$=\sum_{\frac{M}{2}\le m\le M} e\left(-G(m+\alpha)\log\frac{p_1}{p_2}\right)-\sum_{\frac{M}{2}\le m\le M} e\left(-G(m)\log p_1+G(m+\alpha)\log p_2\right)$$
$$-\sum_{\frac{M}{2}\le m\le M} e\left(-G(m+\alpha)\log p_1+G(m)\log p_2\right)+\sum_{\frac{M}{2}\le m\le M} e\left(-G(m)\log\frac{p_1}{p_2}\right).$$

Each sum on the right hand side is of the form

$$\sum_{\frac{M}{2}\le m\le M} e(f(m)),$$

where $e(x)=e^{2\pi ix}$,

$$f(x)=G(x+\alpha)A_1+G(x)A_2 \quad \text{and} \quad |A_1+A_2|=\left|\log\frac{p_1}{p_2}\right|.$$

Since

$$f''(m) = -\frac{G'(m)^3}{G(m)}\left(1+\frac{1}{24\pi G^2(m)}+\cdots\right)(A_1+A_2)$$
$$-\left\{\frac{G'(m+\alpha)^3}{G(m+\alpha)}\left(1+\frac{1}{24\pi G^2(m+\alpha)}+\cdots\right)\right.$$
$$\left.-\frac{G'(m)^3}{G(m)}\left(1+\frac{1}{24\pi G^2(m)}+\cdots\right)\right\}A_1 \cong \frac{|A_1+A_2|}{M\log^2 M}+\frac{\alpha|A_1|}{M^2\log^2 M},$$

we get

$$f''(m) \cong \frac{|\log\frac{p_1}{p_2}|}{M\log^2 M},$$

provided that

$$\alpha \ll \frac{M}{X^3 \log M},$$

namely that

$$\alpha \ll M^{1-\eta}$$

with some positive constant $\frac{1}{2} \leq \eta < 1$. Hence by Theorem 5.9 in p.104 of Titchmarsh [26] again, we get

$$S_7 \ll \sum_{p_1 \neq p_2 < X^3} \frac{1}{\sqrt{p_1 p_2}}\left(\sqrt{\frac{M}{\log M}}+\sqrt{\frac{M\log^2 M}{|\log\frac{p_1}{p_2}|}}\right) \ll M^{\theta},$$

where θ denotes some positive constant <1.

Hence, we get

$$S_1 = \begin{cases} \frac{M}{2}\{2\log(2\pi\alpha+1)+O(1)\} \\ \qquad \text{if} \quad 0<\alpha\ll\log M \\ \frac{M}{2}\{2\log\log M + O(\log\log\log M)\} \\ \qquad \text{if} \quad \log M \ll \alpha \ll M^{1-\eta}. \end{cases}$$

S_2 can be estimated in the same manner as S_7 and get

$$S_2 \ll M^{\theta}.$$

Next, we shall estimate S_3.

$$S_3 = \sum_m \sum_{p_1,\cdots,p_{2k}} \frac{a(p_1)\cdots a(p_k)\overline{a}(p_{k+1})\cdots\overline{a}(p_{2k})}{\sqrt{p_1\cdots p_{2k}}}\left(\frac{p_{k+1}\cdots p_{2k}}{p_1\cdots p_k}\right)^{iG(m)}$$
$$\ll \sum_m \sum_{p_1\cdots p_k = p_{k+1}\cdots p_{2k}} \frac{|a(p_1)|^2\cdots|a(p_k)|^2}{p_1\cdots p_k}$$

$$+\sum_{p_1\cdots p_k\neq p_{k+1}\cdots p_{2k}}\frac{1}{\sqrt{p_1\cdots p_{2k}}}\left|\sum_m\left(\frac{p_{k+1}\cdots p_{2k}}{p_1\cdots p_k}\right)^{iG(m)}\right|$$

$=S_8+S_9,$ say.

$$S_8\ll M\Big(\sum_{p<X^3}\frac{|a(p)|^2}{p}\Big)^k\ll M.$$

S_9 can be estimated as S_7 and we get

$$S_9\ll\sum_{p_1\cdots p_k\neq p_{k+1}\cdots p_{2k}}\frac{1}{\sqrt{p_1\cdots p_{2k}}}\left(\frac{\sqrt{M\left|\log\frac{p_{k+1}\cdots p_{2k}}{p_1\cdots p_k}\right|}}{\log M}+\frac{\sqrt{M}\log M}{\sqrt{\left|\log\frac{p_{k+1}\cdots p_{2k}}{p_1\cdots p_k}\right|}}\right)\ll M^{\theta}.$$

Hence, we get

$$S_3\ll M.$$

Similarly, we get

$$S_3',\ S_4,\ S_4'\ll M.$$

Finally, we shall estimate S_5. By the definition of $\sigma_{X,G(m)}$, we get

$$S_5\ll\frac{M}{(\log X)^{\nu}}+S_{10},$$

where we put

$$S_{10}=\sum_{\frac{M}{2}\leq m\leq M}{}'\left(\sigma_{X,G(m)}-\frac{1}{2}\right)^{\nu}\xi^{(\sigma_{X,G(m)}-\frac{1}{2})}$$

and the dash indicates that we sum over all m which satsfy

$$\sigma_{X,G(m)}-\frac{1}{2}>\frac{4}{\log X}.$$

Now

$$S_{10}\ll\sum_{\beta+i\gamma}{}''\left(\beta-\frac{1}{2}\right)^{\nu}\xi^{2(\beta-\frac{1}{2})}\left|\left\{\frac{M}{2}\leq m\leq M;\ |G(m)-\gamma|<\frac{X^{3(\beta-\frac{1}{2})}}{\log X}\right\}\right|$$

$$\ll\sum_{\beta+i\gamma}{}''\left(\beta-\frac{1}{2}\right)^{\nu}\xi^{2(\beta-\frac{1}{2})}\left(\frac{X^{3(\beta-\frac{1}{2})}\log M}{\log X}+O(1)\right)$$

$$\ll\sum_{\beta+i\gamma}{}''\left(\beta-\frac{1}{2}\right)^{\nu}(X^3\xi^2)^{(\beta-\frac{1}{2})}\frac{\log M}{\log X}+\sum_{\beta+i\gamma}{}''\left(\beta-\frac{1}{2}\right)^{\nu}\xi^{2(\beta-\frac{1}{2})}$$

$=S_{11}+S_{12},$ say,

where the double dash indicates that we sum over all $\beta+i\gamma$ for which

$$\beta > \frac{1}{2} + \frac{2}{\log X} \quad \text{and} \quad \gamma \cong \frac{M}{\log M}.$$

$$S_{12} = \sum_{\beta+i\gamma}'' \int_{\frac{1}{2}}^{\beta} ((\log \xi^2)(\sigma - \frac{1}{2})^{\nu} + \nu(\sigma - \frac{1}{2})^{\nu-1}) \xi^{2(\sigma-\frac{1}{2})} d\sigma = S_{13} + S_{14}, \quad \text{say,}$$

where in S_{13} we integrate over the interval $(\frac{1}{2}+\frac{2}{\log X}, \beta]$ and in S_{14} we integrate over the interval $(\frac{1}{2}, \frac{1}{2}+\frac{2}{\log X}]$.

We have, by Selberg's Theorem 1 in p.232 of [25],

$$S_{14} \ll \frac{1}{(\log X)^{\nu}} N(\frac{1}{2}+\frac{2}{\log X}, \frac{AM}{\log M}) \ll \frac{M}{(\log X)^{\nu}},$$

where $N(\sigma, T)$ denotes the number of the zeros $\beta + i\gamma$ for which $\beta > \sigma$ and $0 < \gamma < T$.

In the same manner, we get

$$S_{13} \ll \int_{\frac{1}{2}+\frac{2}{\log X}}^{\infty} ((\log \xi^2)(\sigma - \frac{1}{2})^{\nu} + \nu(\sigma - \frac{1}{2})^{\nu-1}) \xi^{2(\sigma-\frac{1}{2})} (\sum_{\beta > \sigma, \gamma \cong \frac{M}{\log M}} \cdot 1) d\sigma$$

$$\ll M \int_{\frac{1}{2}+\frac{2}{\log X}}^{\infty} ((\log \xi^2)(\sigma - \frac{1}{2})^{\nu} + \nu(\sigma - \frac{1}{2})^{\nu-1}) (\frac{M}{\log M})^{-\frac{1}{8}(\sigma-\frac{1}{2})} d\sigma$$

$$\ll \frac{M}{(\log X)^{\nu}}.$$

Hence, we get

$$S_{12} \ll \frac{M}{(\log X)^{\nu}}.$$

Similarly, we get

$$S_{11} \ll \frac{M}{(\log X)^{\nu}}.$$

Hence, we get

$$S_5, S_5' \ll \frac{M}{(\log X)^{\nu}}.$$

Combining all of our estimates, we get

$$S = \begin{cases} \dfrac{M}{2} \dfrac{1}{\pi^2} \{\log(2\pi\alpha+1) + O(\sqrt{\log(\alpha+1)})\} \\ \qquad \text{if} \quad 0 < \alpha \ll \log M \\ \dfrac{M}{2} \dfrac{1}{\pi^2} \{\log \log M + O(\sqrt{\log \log M})\} \\ \qquad \text{if} \quad \log M \ll \alpha \ll M^{1-\eta}. \end{cases}$$

This proves Theorem 5.

We shall next prove Theorem 5′. In the present case, the treatement of S_6 and S_7 becomes simpler. Now since

$$a_m(p) = \frac{1}{p^{iG(m)}}(p^{-i\frac{2\pi\alpha}{\log\frac{M}{2\pi}}}-1),$$

$$h(m) = \frac{2\pi\alpha}{\log\frac{M}{2\pi}}$$

in the evaluation of S_6.

Thus we get, as in the section 2, when $0<\alpha\ll\log M$,

$$S_6 = \sum_{\frac{M}{2}\leq m\leq M}\sum_{p<M}\frac{1}{p}(1-\cos(h(m)\log p)) + O(M) = \frac{M}{2}2\log(2\pi\alpha+1) + O(M).$$

When $\log M\ll\alpha\ll M$, since

$$\sum_{p<M}\frac{1}{p}(1-\cos(h(m)\log p)) = \log\log M - \log|\zeta(1+i\frac{2\pi\alpha}{\log\frac{M}{2\pi}})| + O(1),$$

we get

$$S_6 = \frac{M}{2}2(\log\log M - \log|\zeta(1+i\frac{2\pi\alpha}{\log\frac{M}{2\pi}})| + O(1)).$$

Hence, we get

$$S_6 = \begin{cases} \frac{M}{2}\{2\log(2\pi\alpha+1) + O(1)\} \\ \qquad\qquad \text{if} \quad 0<\alpha\ll\log M \\ \frac{M}{2}2\{\log\log M - \log|\zeta(1+i\frac{2\pi\alpha}{\log\frac{M}{2\pi}})| + O(1)\} \\ \qquad\qquad \text{if} \quad \log M\ll\alpha\ll M. \end{cases}$$

Furthermore, in the estimate of S_7,

$$f(x) = G(x)(A_1+A_2) \quad \text{with} \quad |A_1+A_2| = |\log\frac{p_1}{p_2}|.$$

Hence we do not have to assume that $\alpha\ll M^{1-\eta}$ and get

$$S_7 \ll M^{\theta}.$$

Consequently, we get

$$S_1=\begin{cases}\frac{M}{2}\{2\log(2\pi\alpha+1)+o(1)\} \\ \qquad\qquad \text{if} \quad 0<\alpha\ll\log M \\ \frac{M}{2}2\{\log\log M-\log|\zeta(1+i\frac{2\pi\alpha}{\log\frac{M}{2\pi}})|+o(1)\} \\ \qquad\qquad \text{if} \quad \log M\ll\alpha\ll M.\end{cases}$$

The rest is similar and we get our Theorem 5′ as described above.

4-5. A supplement to the subsection 4-2. Here we shall give a proof to the statement announced in the subsection 4-2. First of all, we assume the Conjecture(+) in the following form.

$$G_M^+(k,\alpha)=ME(k,\alpha)+O(A_k\frac{M}{\Phi(M)}) \quad \text{uniformly for a bounded } \alpha,$$

where A_k is a constant depending only on k and $\Phi(M)\to\infty$ as $M\to\infty$.

Let α be any bounded positive number. Using the Riemann-von Mangoldt formula, we get first

$$\begin{aligned}&\sum_{m\leq M}(S_+(g_{m+\alpha})-S_+(g_m))^2\\ &\qquad=\sum_{m\leq M}(N_+(g_{m+\alpha})-N_+(g_m)-\alpha)^2\\ &\qquad=\sum_{m\leq M}(N_+(g_{m+\alpha})-N_+(g_m))^2+\alpha^2M-2\alpha\sum_{m\leq M}(N_+(g_{m+\alpha})-N_+(g_m))\\ &\qquad=U_1+\alpha^2M-2\alpha U_2, \quad \text{say.}\end{aligned}$$

Let L be a sufficiently large constant. Then we have

$$U_1=\sum_{1\leq k\ll\log M}k^2G_M^+(k,\alpha)=\sum_{1\leq k\leq L}k^2G_M^+(k,\alpha)+\sum_{L\leq k\ll\log M}k^2G_M^+(k,\alpha)=U_3+U_4, \quad \text{say.}$$

Applying the above conjecture to U_3, we get first

$$U_3=M\sum_{1\leq k<L}k^2E(k,\alpha)+O(\frac{M}{\Phi(M)}\sum_{1\leq k<L}k^2A_k).$$

Using the mean value theorem which has been noticed in the subsection 4-3, we have

$$\sum_{1\leq m\leq M}(N(G(m+\alpha))-N(G(m)))^{2j}\ll MA^j(\alpha+\sqrt{\log(\alpha+3)}+j)^{2j}$$

uniformly for an integer $j\geq 1$ and $0<\alpha\ll\log M$. At the same time, we get also

$$\sum_{1\leq m\leq M}(N_+(G(m+\alpha))-N_+(G(m)))^{2j}\ll MA^j(\alpha+\sqrt{\log(\alpha+3)}+j)^{2j}$$

uniformly for an integer $j\geq 1$ and $0<\alpha\ll\log M$.

This implies that for any $Y\gg 1$,

$$Y^{2j}\sum_{Y\leq k\ll\log M}G_M^{+}(k,\alpha)\leq\sum_{1\leq m\leq M}(N_+(G(m+\alpha))-N_+(G(m)))^{2j}$$
$$\ll MA^j(\alpha+\sqrt{\log(\alpha+3)}+j)^{2j}.$$

Namely, we have for any $Y\gg 1$,

$$\sum_{Y\leq k\ll\log M}G_M^{+}(k,\alpha)\ll Y^{-2j}MA^j(\alpha+\sqrt{\log(\alpha+3)}+j)^{2j}.$$

Now

$$U_4=\sum_{\substack{2^bL\ll\log M\\ b\geq 0}}\ \sum_{2^bL\leq k<2^{b+1}L}k^2G_M^{+}(k,\alpha)$$
$$\ll\sum_{0\leq b\ll\log(\frac{\log M}{L})}2^{2b+2}L^2\sum_{2^bL\leq k\ll\log M}G_M^{+}(k,\alpha)$$
$$\ll\sum_{0\leq b\ll\log(\frac{\log M}{L})}\frac{2^{2b+2}L^2}{(2^bL)^{2j}}MA^j(\alpha+\sqrt{\log(\alpha+3)}+j)^{2j}.$$

Here we take

$$J_1=\max(\alpha,\sqrt{\log(\alpha+3)}),\quad J_2=\max(\alpha^2,\log(\alpha+3))$$
$$j_1=\max([J_1],2)\quad\text{and}\quad L=CJ_2$$

with an arbitrarily large constant C and $[J_1]$ denotes the largest integer $\leq J_1$.

By these choices, we get

$$U_4\ll\frac{MA^{j_1}J_1^{2j_1}C^2J_2^2}{(CJ_2)^{2j_1}}\sum_{b=0}^{\infty}\left(\frac{4}{4^{j_1}}\right)^b\ll M\left(\frac{A}{C^2J_2}\right)^{j_1}C^2J_2^2\ll\frac{M}{C^2}.$$

On the other hand, since

$$\alpha-2\int_0^{\alpha}(\alpha-x)\left(\frac{\sin\pi x}{\pi x}\right)^2dx=\int_{-\infty}^{\infty}\min(|a|,\alpha)\left(\frac{\sin\pi a}{\pi a}\right)^2da$$
$$=\frac{1}{\pi^2}\{\log(2\pi\alpha)-\mathrm{Ci}(2\pi\alpha)-2\pi\alpha\cdot\mathrm{Si}(2\pi\alpha)+\pi^2\alpha-\cos(2\pi\alpha)+1+C_o\},$$

we have

$$\sum_{k=1}^{\infty}k^2E(k,\alpha)=\frac{1}{\pi^2}\{\log(2\pi\alpha)-\mathrm{Ci}(2\pi\alpha)-2\pi\alpha\cdot\mathrm{Si}(2\pi\alpha)$$
$$+\pi^2\alpha-\cos(2\pi\alpha)+1+C_o\}+\alpha^2$$

for any positive α. Thus for any positive ε, we choose C sufficiently large satisfying

$$\Big|\sum_{0\le k<CJ_2} k^2E(k,\alpha)-\frac{1}{\pi^2}\{\log(2\pi\alpha)-\mathrm{Ci}(2\pi\alpha)-2\pi\alpha\cdot\mathrm{Si}(2\pi\alpha)$$
$$+\pi^2\alpha-\cos(2\pi\alpha)+1+C_o\}-\alpha^2\Big|<\varepsilon$$

Since C can be arbitrarily large and ε can be arbitrarily small, we get

$$U_3=M(\frac{1}{\pi^2}\{\log(2\pi\alpha)-\mathrm{Ci}(2\pi\alpha)-2\pi\alpha\cdot\mathrm{Si}(2\pi\alpha)$$
$$+\pi^2\alpha-\cos(2\pi\alpha)+1+C_o\}+\alpha^2)+o(M)$$

and

$$U_4=o(M).$$

Thus we get

$$U_1=M(\frac{1}{\pi^2}\{\log(2\pi\alpha)-\mathrm{Ci}(2\pi\alpha)-2\pi\alpha\cdot\mathrm{Si}(2\pi\alpha)$$
$$+\pi^2\alpha-\cos(2\pi\alpha)+1+C_o\}+\alpha^2)+o(M).$$

In the same manner, using $\sum_{k=0}^{\infty}kE(k,\alpha)=\alpha$, we get

$$U_2=\alpha M+o(M).$$

Consequently, we get

$$\sum_{m\le M}(S_+(g_{m+\alpha})-S_+(g_m))^2$$
$$\sim M\frac{1}{\pi^2}\{\log(2\pi\alpha)-\mathrm{Ci}(2\pi\alpha)-2\pi\alpha\cdot\mathrm{Si}(2\pi\alpha)+\pi^2\alpha-\cos(2\pi\alpha)+1+C_o\}.$$

5. The evaluation of the number variance $V(\alpha,T)$.

In stead of $V(\alpha,T)$, we may evaluate the number variance

$$\int_{\frac{T}{2}}^{T}(N(G(t+\alpha))-N(G(t))-\alpha)^2dt,$$

namely,

$$\int_{\frac{T}{2}}^{T}(S(G(t+\alpha))-S(G(t)))^2dt,$$

where $G(t)$ is the same as in the section 4. The same analysis applies to

$$\int_{\frac{T}{2}}^{T}(S(B(x+\frac{\alpha}{2}))-S(B(x-\frac{\alpha}{2})))^2dt$$

as will be seen below.

We shall indicate only how to modify and how to apply the same method as in the previous section.

For this purpose it is enough to evaluate the following integral.

$$\int_{\frac{T}{2}}^{T} \Big| \sum_{p<X^3} \frac{1}{\sqrt{p}} \Big(\frac{1}{p^{iG(t+\alpha)}} - \frac{1}{p^{iG(t)}} \Big) \Big|^2 dt,$$

where we put $X=T^b$ with some positive constant b. This is

$$\begin{aligned} &= \sum_{p<X^3} \frac{1}{p} \int_{\frac{T}{2}}^{T} (2 - e^{i(G(t+\alpha)-G(t))\log p} - e^{-i(G(t+\alpha)-G(t))\log p})\, dt \\ &\quad + \sum_{p\neq q<X^3} \frac{1}{\sqrt{pq}} \int_{\frac{T}{2}}^{T} \{ e^{-iG(t+\alpha)\log\frac{p}{q}} + e^{-iG(t)\log\frac{p}{q}} \\ &\quad - e^{-i(G(t+\alpha)\log p - G(t)\log q)} - e^{i(G(t+\alpha)\log q - G(t)\log p)} \} dt \\ &= V + V' \quad \text{say.} \end{aligned}$$

Moreover, it is enough to evaluate only the following integral, since the other integrals can be treated in the same manner.

$$V(p) \equiv \int_{\frac{T}{2}}^{T} e^{i(G(t+\alpha)-G(t))\log p} dt$$

Now we have

$$\begin{aligned} (G(t+\alpha) - G(t))\log p &= \alpha G'(t)\log p + O\Big(\frac{\alpha^2 \log p}{T\log^2 T}\Big) \\ &= \frac{2\pi\alpha}{\log\frac{G(t)}{2\pi}} \log p + O\Big(\Big(\frac{\alpha^2}{T\log^2 T} + \frac{\alpha}{T^2}\Big)\log p\Big). \end{aligned}$$

Hence under the condition $0<\alpha \ll \sqrt{T}$, we get

$$V(p) = \int_{\frac{T}{2}}^{T} e^{i\frac{2\pi\alpha}{\log\frac{G(t)}{2\pi}}\log p} dt + O\Big(\Big(\frac{\alpha^2}{\log^2 T} + \frac{\alpha}{T}\Big)\log p\Big).$$

Since

$$\frac{1}{\log\frac{G(t)}{2\pi}} - \frac{1}{\log\frac{G(T)}{2\pi}} \ll \frac{|T-t| G'(T)}{G(T)\log^2 T} \ll \frac{1}{\log^2 T}.$$

we get further under the condition $0<\alpha \ll \log T$,

$$\begin{aligned} V(p) &= \int_{\frac{T}{2}}^{T} e^{i\frac{2\pi\alpha}{\log\frac{G(T)}{2\pi}}\log p} dt + O\Big(\Big(\frac{\alpha^2}{\log^2 T} + \frac{\alpha}{T}\Big)\log p\Big) + O\Big(\frac{T\alpha\log p}{\log^2 T}\Big) \\ &= \frac{T}{2} e^{i\frac{2\pi\alpha}{\log\frac{G(T)}{2\pi}}\log p} + O\Big(\frac{T\alpha\log p}{\log^2 T}\Big). \end{aligned}$$

Hence, we get if $0<\alpha\ll\log T$, then

$$V=\frac{T}{2}2\sum_{p<X^3}\frac{1}{p}\left(1-\cos\left(\frac{2\pi\alpha}{\log\frac{G(T)}{2\pi}}\log p\right)\right)+O(T)=\frac{T}{2}2\log(2\pi\alpha+1)+O(T),$$

where we have used the same argument as in the section 2.

Suppose next that $\log^2 T\ll\alpha\ll\sqrt{T}$. Then since

$$\frac{d^2}{dt^2}\left(\frac{\alpha\log p}{\log\frac{G(t)}{2\pi}}\right)\cong\frac{\alpha\log p}{T^2\log^2 T},$$

we see, by Lemma 4.4 in p.71 of Titchmarsh [26], that

$$V(p)\ll\frac{T\log T}{\sqrt{\alpha\log p}}.$$

Hence, we get

$$\sum_{p<X^3}\frac{1}{p}V(p)=O\left(\sum_{p<X^3}\frac{1}{p}\frac{T\log T}{\sqrt{\alpha\log p}}\right)=O(T).$$

Thus we get when $\log^2 T\ll\alpha\ll\sqrt{T}$

$$V=\frac{T}{2}2\sum_{p<X^3}\frac{1}{p}+O(T)=\frac{T}{2}2\log\log T+O(T).$$

Finally, if $\log T\ll\alpha\ll\log^2 T$, then

$$V=\frac{T}{2}2\sum_{p<X^3}\frac{1}{p}-2\int_{\frac{T}{2}}^{T}\log\left|\zeta\left(1+i\frac{2\pi\alpha}{\log\frac{G(t)}{2\pi}}\right)\right|dt+O(T)$$

$$=\frac{T}{2}2\log\log T+O(T\log\log\log T).$$

Consequently, we get

$$V=\begin{cases}\frac{T}{2}\{2\log(2\pi\alpha+1)+O(1)\} & \text{if}\quad 0<\alpha\ll\log T\\ \frac{T}{2}2\{\log\log T+O(\log\log\log T)\} & \text{if}\quad \log T\ll\alpha\ll\sqrt{T}.\end{cases}$$

The other integrals or the sums can be treated in the same manner and we get the following theorem.

Theorem 6. *Suppose that* $T>T_0$ *and* $0<\alpha\ll T^{1-\eta}$ *with a positive constant* $\frac{1}{2}\leq\eta<1$. *Then we have*

$$\int_{\frac{T}{2}}^{T}(S(G(t+\alpha))-S(G(t)))^2dt=\begin{cases}\frac{T}{2}\frac{1}{\pi^2}\{\log(2\pi\alpha+1)+O(\sqrt{\log(\alpha+1)})\} \\ \qquad if \quad 0<\alpha\ll\log T \\ \frac{T}{2}\frac{1}{\pi^2}\{\log\log T+O(\sqrt{\log\log T})\} \\ \qquad if \quad \log T\ll\alpha\ll T^{1-\eta}.\end{cases}$$

Similarly, we get

Theorem 6′. *Suppose that* $T>T_0$ *and* $0<\alpha\ll T^{1-\eta}$ *with a positive constant* $\frac{1}{2}\leq\eta<1$. *Then we have*

$$\int_{\frac{T}{2}}^{T}(S(B(x+\frac{\alpha}{2}))-S(B(x-\frac{\alpha}{2})))^2dx$$

$$=\begin{cases}\frac{T}{2}\frac{1}{\pi^2}\{\log(\alpha+1)+O(\sqrt{\log(2\pi\alpha+1)})\} \\ \qquad if \quad 0<\alpha\ll\log T \\ \frac{T}{2}\frac{1}{\pi^2}\{\log\log T+O(\sqrt{\log\log T})\} \\ \qquad if \quad \log T\ll\alpha\ll T^{1-\eta}.\end{cases}$$

It is clear that we can obtain the higher moments of the above theorems. It is also clear that such modified conjectures as

$$\sum_{\substack{0<\gamma,\gamma'\leq T,\\ 0<\frac{1}{\pi}\vartheta(\gamma)-\frac{1}{\pi}\vartheta(\gamma')\leq\alpha}}\cdot 1=\frac{T}{2\pi}\log T\{\int_0^{\alpha}(1-(\frac{\sin\pi t}{\pi t})^2)dt+o(1)\}$$

for any $\alpha>0$ *and*

$$\lim_{T\to\infty}\frac{1}{T}|\{0<t\leq T;\ N(G(t+\alpha))-N(G(t))=k\}|=E(k,\alpha)$$

for each integer $k\geq 0$ *and for any* $\alpha>0$
give some information on our problem for a bounded α.

The former is consistent with the Montgomery's pair correlation conjecture if

$$\sum_{\substack{0<\gamma,\gamma'\leq T,\\ \frac{2\pi\alpha}{\log\frac{T}{2\pi}}-\frac{A\alpha}{\log^2\frac{T}{2\pi}}<\gamma-\gamma'\leq\frac{2\pi\alpha}{\log\frac{T}{2\pi}}+\frac{A\alpha}{\log^2\frac{T}{2\pi}}}}\cdot 1=o(T\log T).$$

The latter is a continuous version of Conjecture described in the subsection 4-2 and, in fact, has been proposed in p.394 of Fujii [5] for the separated case. It implies immediately as in the subsections 4-2 and 4-5 that for any $\alpha>0$,

$$\int_0^T (S(G(t+\alpha))-S(G(t)))^2 dt=\int_0^T (N(G(t+\alpha))-N(G(t))-\alpha)^2 dt$$

$$=\int_0^T (N(G(t+\alpha))-N(G(t)))^2 dt-2\alpha\int_0^T (N(G(t+\alpha))-N(G(t)))dt+\alpha^2 T$$

$$\sim T\{\sum_{k=0}^{\infty} k^2 E(k,\alpha)-\alpha^2\}$$

$$\sim T\frac{1}{\pi^2}\{\log(2\pi\alpha)-\mathrm{Ci}(2\pi\alpha)-2\pi\alpha\cdot\mathrm{Si}(2\pi\alpha)+\pi^2\alpha-\cos(2\pi\alpha)+1+C_o\}.$$

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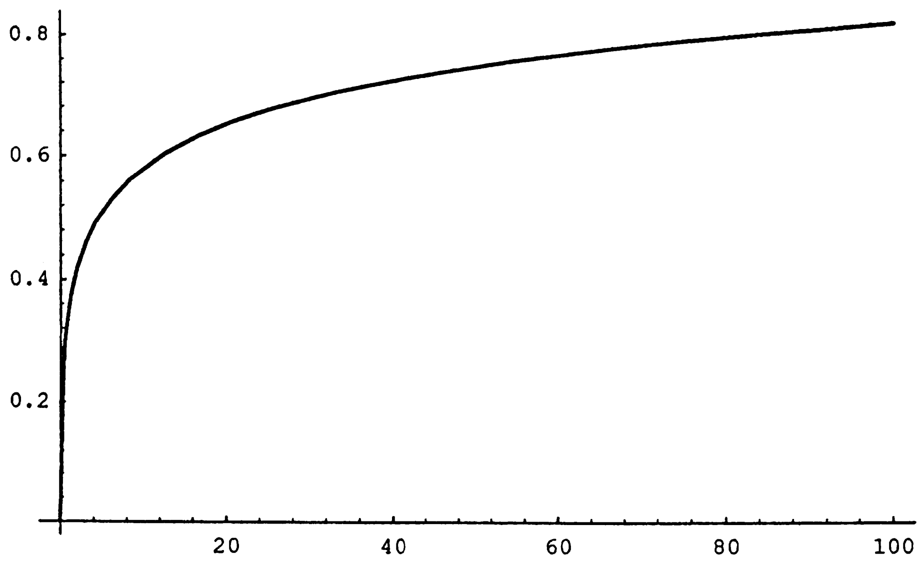
6. Some graphs

6-1. The graph of

$$\frac{1}{\pi^2}\{\log(2\pi\alpha)-\mathrm{Ci}(2\pi\alpha)-2\pi\alpha\cdot\mathrm{Si}(2\pi\alpha)+\pi^2\alpha-\cos(2\pi\alpha)+1+C_o\},$$

for $0\leq\alpha\leq 100$.

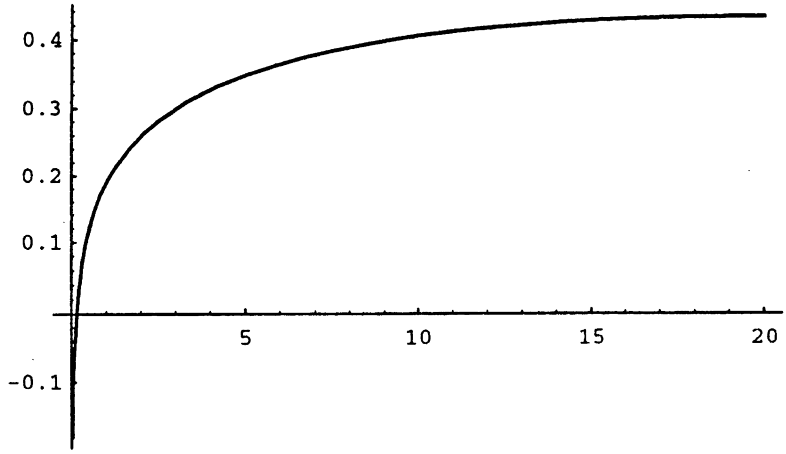


* (Added in proof) It appears in J. Math. Kyoto Univ. 36-4 (1996).

6-2-i. The graph of

$$\frac{2k!}{(2\pi)^{2k}k!}2^k\{\log\log T-\log|\zeta(1+i\frac{2\pi\alpha}{\log\frac{T}{2\pi}})|\}^k$$

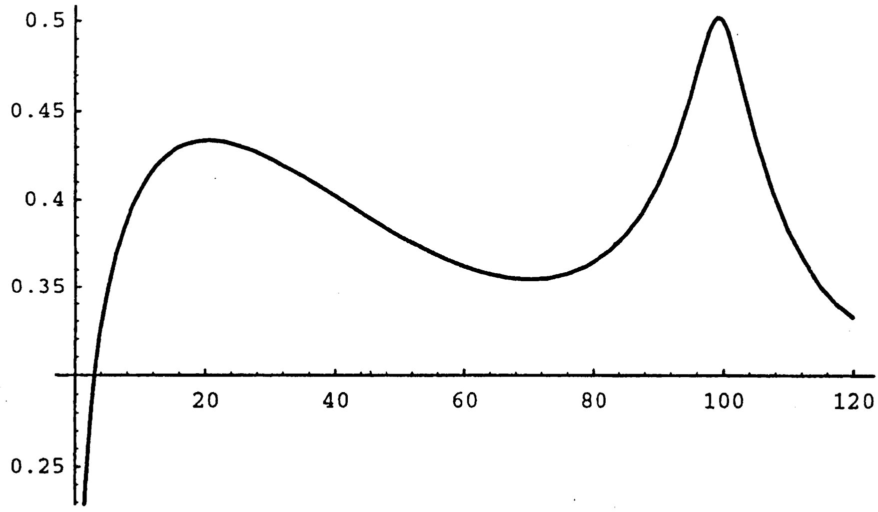
for $k=1$, $T=10^{20}$ **and for** $0\leq\alpha\leq 20$.



6-2-ii. The graph of

$$\frac{2k!}{(2\pi)^{2k}k!}2^k\{\log\log T-\log|\zeta(1+i\frac{2\pi\alpha}{\log\frac{T}{2\pi}})|\}^k$$

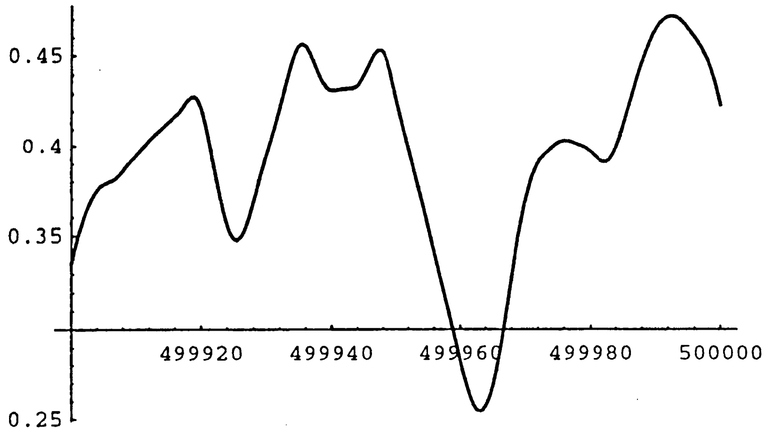
for $k=1$, $T=10^{20}$ **and for** $0\leq\alpha\leq 120$.



6-2-iii. The graph of

$$\frac{2k!}{(2\pi)^{2k}k!}2^k\{\log\log T-\log|\zeta(1+i\frac{2\pi\alpha}{\log\frac{T}{2\pi}})|\}^k$$

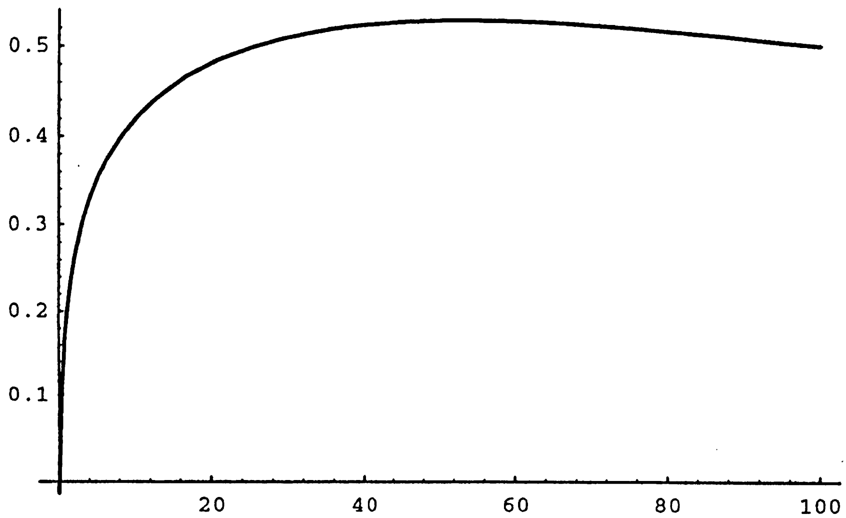
for $k=1$, $T=10^{20}$ **and for** $499900\leq\alpha\leq500000$.



6-3-i. The graph of

$$\frac{2k!}{(2\pi)^{2k}k!}2^k\{\log\log T-\log|\zeta(1+i\frac{2\pi\alpha}{\log\frac{T}{2\pi}})|\}^k$$

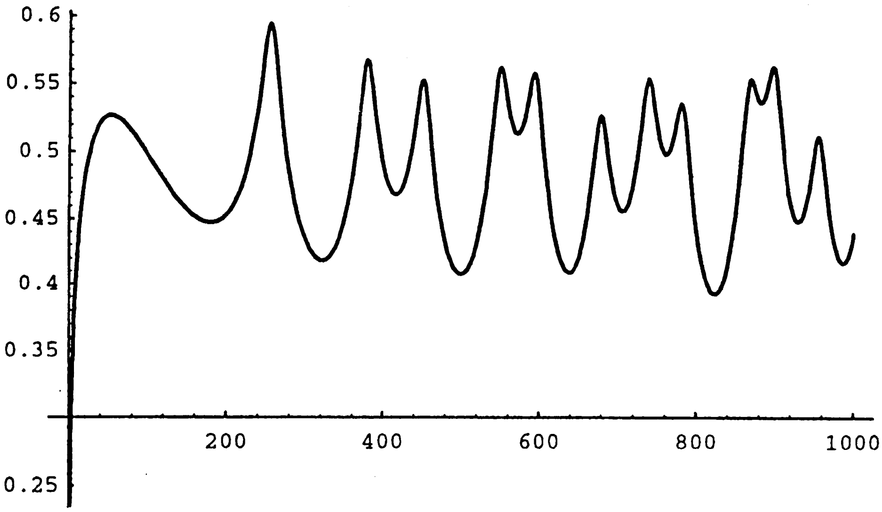
for $k=1$, $T=10^{50}$ **and for** $0\leq\alpha\leq100$.



6-3-ii. The graph of

$$\frac{2k!}{(2\pi)^{2k}k!}2^k\{\log\log T-\log|\zeta(1+i\frac{2\pi\alpha}{\log\frac{T}{2\pi}})|\}^k$$

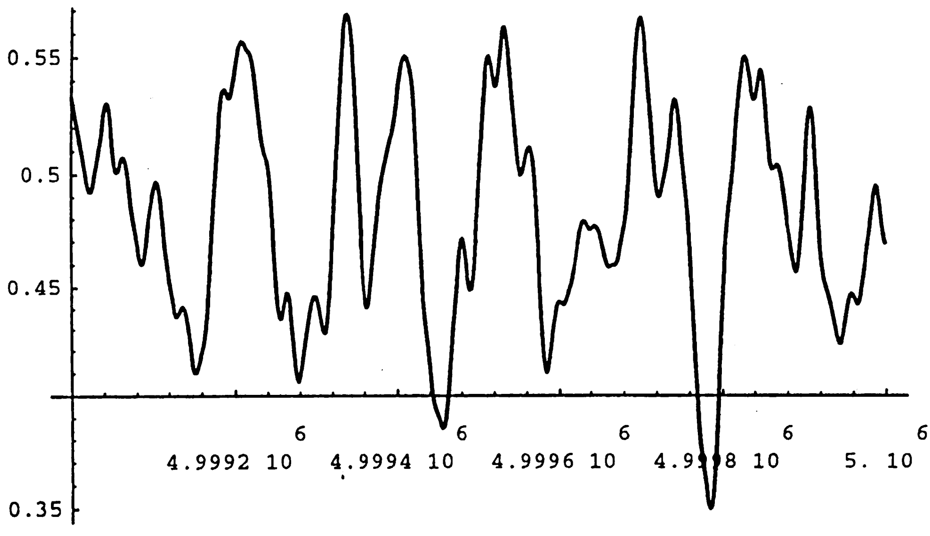
for $k=1$, $T=10^{50}$ **and for** $0\leq\alpha\leq 1000$.



6-3-iii. The graph of

$$\frac{2k!}{(2\pi)^{2k}k!}2^k\{\log\log T-\log|\zeta(1+i\frac{2\pi\alpha}{\log\frac{T}{2\pi}})|\}^k$$

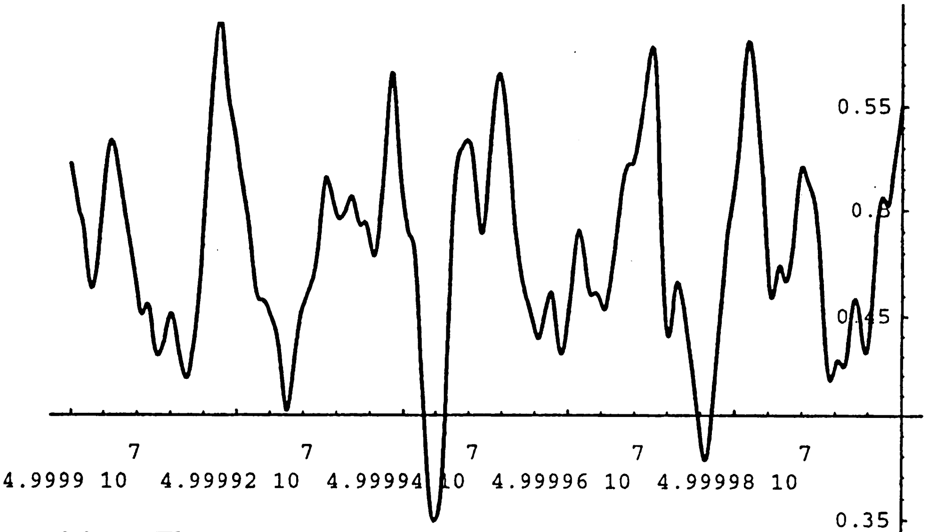
for $k=1$, $T=10^{50}$ **and for** $4999000\leq\alpha\leq 5000000$.



6-3-iv. The graph of

$$\frac{2k!}{(2\pi)^{2k}k!}2^k\{\log\log T-\log|\zeta(1+i\frac{2\pi\alpha}{\log\frac{T}{2\pi}})|\}^k$$

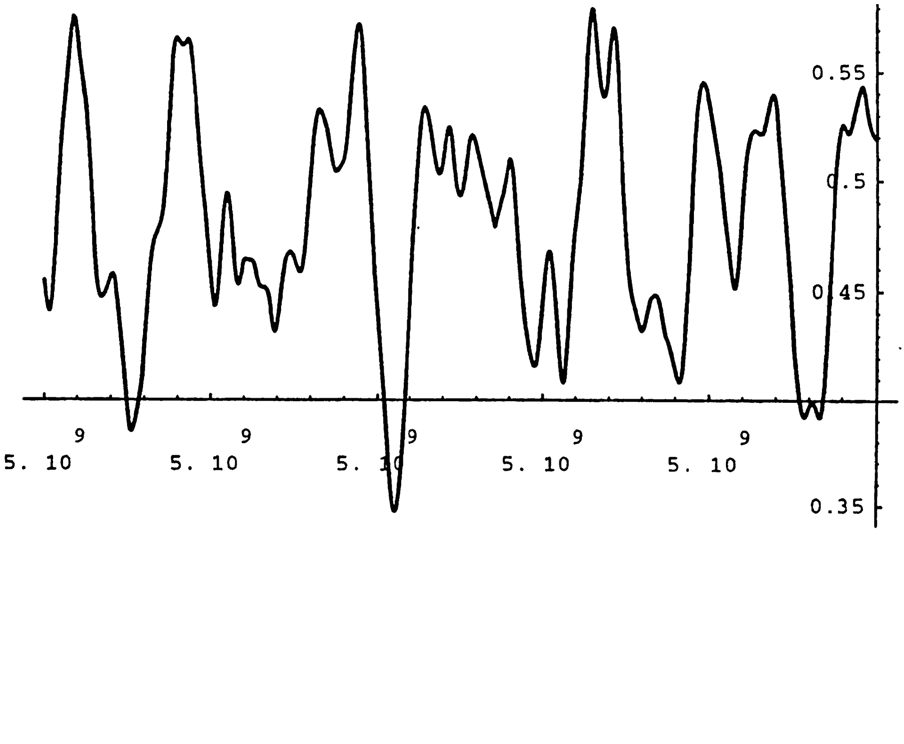
for $k=1$, $T=10^{50}$ **and for** $49999000\leq\alpha\leq 50000000$.



6-3-v. The graph of

$$\frac{2k!}{(2\pi)^{2k}k!}2^k\{\log\log T-\log|\zeta(1+i\frac{2\pi\alpha}{\log\frac{T}{2\pi}})|\}^k$$

for $k=1$, $T=10^{50}$ **and for** $4999999000\leq\alpha\leq 5000000000$.



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