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On the irreducible very cuspidal representations II

By

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Introduction

Let F be a non-archimedean local field and $G=\mathrm{GL}_n(F)$. Carayol [C] introduced the notion of very cuspidal representation of the maximal compact modulo center subgroup of G and showed the compact-induction of an irreducible very cuspidal representation to G is irreducible and supercuspidal. If the irreducible very cuspidal representation has an even level, it is monomial i.e. induced from a one-dimensional representation. But if the level is odd, it is not monomial and the construction of the representation is much more difficult and complicated. We remark that such phenomena occurs whenever one consider the construction of supercuspidal representation. (See e.g. [M], [B-K].)

The aim of this paper is to express the irreducible supercuspidal representation induced from a very cuspidal representation with an odd level as a $\boldsymbol{Q}$-linear combination of monomial representations. To explain more precisely, we use some notation. Let ρ be an irreducible very cuspidal representation of Z_sK_s of level N. (See Definition 1.4 and 1.7.) Then the restriction of ρ to $K_S^{[(N+1)/2]}$ contains a character ψ_u (cf. Definition 1.7.) When $N=2m$, the normalizer of ψ_u in Z_sK_s is $E^\times K_S^m$ where $E=F(u)$. Thus $\rho=\mathrm{Ind}_{F(u)^\times K_S^m}^{Z_sK_s}(\theta\cdot\psi_u)$ where θ is an appropriate quasi-character of $E^\times$. (See Proposition 1.10.) When $N=2m-1$, the normalizer of ψ_u in Z_sK_s is $E^\times K_S^{m-1}$ and the irreducible component of $\mathrm{Ind}_{K_S^m}^{E^\times K_S^{m-1}}\psi_u$ is not one-dimensional. Moreover if E/F is widly ramified, the construction of the irreducible component is not easy. In [T], the author gave the irreducible representation $\eta_{u,\theta}$ of $E^\times K_S^{m-1}$. Our main work is to calculate the character of $\eta_{u,\theta}$. Let $C=E^\times/F^\times(1+P_E)$ and $\hat{C}$ is the character group of C. We can put

$$\mathrm{Ind}_{K_S^m}^{E^\times K_S^{m-1}}((\theta\otimes\lambda)\cdot\psi_u)=\sum_{\tau\in\hat{C}}a_{\lambda\tau}\eta_{u,\theta\otimes\tau}.$$

From the character formula of $\eta_{u,\theta}$, we can calculate the multiplicity $a_{\lambda\tau}$. Thus if we can calculate the inverse of the matrix $M=(a_{\lambda\tau})_{\lambda,\tau\in\hat{C}}$, $\eta_{u,\theta}$ is expressed as a linear combination of monomial respresentations. We can calculate the M^{-1} under some assumption (See Proposition 3.7 and Theorem 3.8.)

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In section 1, we review the result of [C] and [T]. Theorem 1.8 is the main result of [C] and Proposition 1.12 is our starting point to calculate the character of $\eta_{u,\theta}$. Section 2 is devoted to calculate the character of $\eta_{u,\theta}$. In section 3, we get the explicit from of M^{-1}. At first, we treat the case E/F is totally ramified and the ramification degree is a power of p. In this case, M^{-1} can be written in the simple form. But this case contains the essential part of the calculation. For the general case, we assume there exists a uniformizer ϖ_E such that $\varpi_E^s \in F^{\times}(1+P_E)$ where s is the ramification index of E/F. This assumption is not essential, but it simplify the result. Theorem 3.8 is the main theorem of this paper.

The monomial supercuspidal representation is very easy to treat. For example, it is easy to calculate the ε-factor, to give the matrix coefficient and so on. We hope this result is useful for the calculation of the ε-factor of the representation of $\mathrm{GL}_m(F)\times\mathrm{GL}_n(F)$.

Notation. Let F be a non-archimedean local field. We denote by $\mathcal{O}_F, P_F$, ϖ_F, k_F and v_F the maximal order of F, the maximal ideal of $\mathcal{O}_F$, a prime element of P_F, the residue field of F and the valuation of F normalized by $v_F(\varpi_F)=1$. We set $q=p^f$ be the number of elements in k_F. For $x\in\boldsymbol{R}$, we denote by $[x]$ the greatest integer $\leq x$ and set $\boldsymbol{e}(x)=\exp(2\pi\sqrt{-1}x)$. For integers a, b, we denote by (a,b) the greatest common divisor of a and b. The Jacobi symbol is denoted by $\left(\frac{}{}\right)$. We fix an additive character ψ of F whose conductor is P_F i.e. $\psi(P_F)=\{1\}$ and $\psi(\mathcal{O}_F)\neq\{1\}$. Let G be a totally disconnected, locally compact group. We denote by $\widehat{G}$ the set of (equivalence classes of) irreducible admissible representations of G. For a closed subgroup H of G and a representation ρ of H, we denote by $\mathrm{Ind}_H^G\rho$ (resp. $\mathrm{ind}_H^G\rho$) the induced (resp. the compactly induced) representation of ρ to G. For a representation π of G, we denote by $\pi|_H$ the restriction of π to H. The $n\times n$ zero and identity matrice are denoted by $\mathbf{0}_n$ and $\mathbf{1}_n$ respectively. The usual matrix trace is denoted by tr.

1. Review of the very cuspidal representation

In this section, we review the results of [C] and [T]. At first we recall some definitions in order to define very cuspidal representation.

Let F be a non-achimedean local field of residual characteristic p and $G=\mathrm{GL}_n(F)$. We set $V_F=F^n$ so that $\mathrm{M}_n(F)=\mathrm{End}_F(V_F)$ and $G=\mathrm{Aut}_F(V_F)$. Let s be a divisor of n and put $r=n/s$.

Definition 1.1. Let $\{L_i\}_{i\in Z}$ be the set of $\mathcal{O}_F$-lattices in V_F. $\{L_i\}_{i\in Z}$ is said to be a uniform lattice chain of period s if the following conditions hold for all $i\in\boldsymbol{Z}$:

1. $L_{i+1}\subset L_i$.
2. $P_F L_i=L_{i+s}$.
3. $\dim_{k_F}(L_i/L_{i+1})=r$.

Definition 1.2. Let $\{L_i\}_{i\in Z}$ be a uniform lattice chain of period s.

1. For integers m, we set $A_s^m=\{f\in \mathrm{M}_n(F)\,|\,fL_i\subset L_{i+m}$ for all $i\}$.
2. We set $K_s=\{g\in G\,|\,gL_i=$ for all $i\}$ and $K_s^m=1+A_s^m$ for positive integers m. K_s is a compact open subgroup of G and K_s^m is a normal subgroup of K_s for any $m\geq 1$.
3. Let z_s be an element in G such that $(z_s)^s=\varpi_F \mathbf{1}_n$ and Z_s be a cyclic group generated by z_s.

Remark. By taking an approprite $\mathcal{O}_F$-basis of L_0, we can express K_s, z_s, A_s^0 and A_s^1 by the following matrix form :

$$(1.1)\qquad K_s=\left\{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1s}\\ a_{21} & a_{22} & \cdots & a_{2s}\\ \cdots & \cdots & \cdots & \cdots\\ a_{s1} & a_{s2} & \cdots & a_{ss}\end{pmatrix}\middle|\ \begin{array}{ll} a_{ij}\in M_r(\mathcal{O}_F) & \text{if } i<j\\ a_{ii}\in \mathrm{GL}_r(\mathcal{O}_F) & \\ a_{ij}\in \mathrm{M}_r(P_F) & \text{if } i>j,\end{array}\right\}$$

$$(1.2)\qquad z_s=\begin{pmatrix} \mathbf{0}_r & \mathbf{1}_r & \mathbf{0}_r & \cdots & \mathbf{0}_r\\ \mathbf{0}_r & \mathbf{0}_r & \mathbf{1}_r & \cdots & \mathbf{0}_r\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ \varpi_F\mathbf{1}_r & \mathbf{0}_r & \mathbf{0}_r & \cdots & \mathbf{0}_r\end{pmatrix}$$

$$(1.3)\qquad A_s^0=\left\{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1s}\\ a_{21} & a_{22} & \cdots & a_{2s}\\ \cdots & \cdots & \cdots & \cdots\\ a_{s1} & a_{s2} & \cdots & a_{ss}\end{pmatrix}\middle|\ \begin{array}{ll} a_{ij}\in \mathrm{M}_r(\mathcal{O}_F) & \text{if } i\leq j\\ a_{ij}\in \mathrm{M}_r(P_F) & \text{if } i>j,\end{array}\right\},$$

$$(1.4)\qquad A_s^1=\left\{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1s}\\ a_{21} & a_{22} & \cdots & a_{2s}\\ \cdots & \cdots & \cdots & \cdots\\ a_{s1} & a_{s2} & \cdots & a_{ss}\end{pmatrix}\middle|\ \begin{array}{ll} a_{ij}\in \mathrm{M}_r(\mathcal{O}_F) & \text{if } i<j\\ a_{ij}\in \mathrm{M}_r(P_F) & \text{if } i\geq j,\end{array}\right\}.$$

Since the multiplication by ϖ_F induces a k_F-isomorphism between L_i/L_{i+1} and L_{i+s}/L_{i+s+1} and an element of A_s^0 induces an endomorphism of L_i/L_{i+1}, we have a natural ring homomorphism :

$$R : A_s^0\to \prod_{i\in Z/sZ}\mathrm{End}_{k_F}(L_i/L_{i+1}).$$

Since

$$(1.5)\qquad R : A_s^0/A_s^1\to \mathrm{M}_r(k_F)^{Z/sZ}$$

and the kernel of R is A_s^1 we have a ring isomorphism :

$$(1.6)\qquad A_s^0/A_s^1\to \mathrm{M}_r(k_F)^{Z/sZ}.$$

We use same symbol R for this isomorphism.

Lemma 1.3. *Let $a\in A_s^0$ and $R(a)=(\alpha_0, \alpha_1, \cdots, \alpha_{s-1})$ $(\alpha_i\in \mathrm{M}_r(k_F))$.*

1. $R(z_s a z_s^{-1})=(\alpha_{s-1}, \alpha_0, \cdots, \alpha_{s-2})$.
2. *Let $u=z_s^m a$. Then $R(\varpi_F^{-m}u^s)=(\beta_0, \beta_1, \cdots, \beta_{s-1})$ where*

$$\beta_i=\alpha_{i+(s-1)m}\cdots\alpha_{i+m}\alpha_i.$$

Proof. Since $z_s^s=\varpi_F \mathbf{1}_n$, it is easily checked by the definition of R.

Now we define the very cuspidal element.

Definition 1.4. An element $u\in A_s^m/A_s^{m+1}$ is said to be very cuspidal if the following conditions hold:

1. $(m, s)=1$.
2. If $R(\varpi_F^{-m}u^s)=(\beta_0, \beta_1, \cdots, \beta_{s-1})$, then the fields $k_F(\beta_i)$ are extensions of k_F of degree r.

We also say that an element $u\in \mathrm{M}_n(F)$ is very cuspidal of level m if $u\in A_s^m$ and u modA_s^{m+1} is very cuspidal. The very cuspidal element has good properties as follows.

Proposition 1.5. *Let u be very cuspidal of level m.*

1. *$E=F[u]$ is a fild extension of F of degree n and its ramification index over F is s.*
2. *$E^\times\subset Z_sK_s$ and $E^\times\cap K_s=\mathcal{O}_E^\times$.*
3. *$E\cap A_s^m=P_E^m$ for all integers m and $E^\times\cap K_s^m=1+P_E^m$ for all integers $m\geq 1$.*
4. *Let $x\in A_s^l$. If $ux-xu\in A_s^{m+l+1}$. then $x\in E+A_s^{l+1}$.*

Proof. See 3.3 and 3.5 in [C].

Remark Let E be a field extension of F in $\mathrm{M}_n(F)$ and e ramification index of E/F. An element $u\in E$ is called E/F-minimal if $(v_E(u), e)=1$ and k_F $(\varpi_F^{-v/E(u)}u^e \mathrm{mod} P_E)=k_E$.
Thus u is very cuspidal if and only if $F(u)/F$ is a field extension of degree n and u is $F(u)/F$-minimal. The notion of E/F-minimal plays an important role in the work of Bushnell-Kutzko ([B-K]).

Now we start the representation theory of Z_sK_s.

Lemma 1.6. *Let l, m be integers such that $m\leq l\leq 2m$ and $m\geq 1$. For $u\in \mathrm{M}_n(F)$, we define a function ψ_u on K_s^m by*

$$\psi_u(1+x)=\psi(\mathrm{tr}ux). \tag{1.7}$$

Then the map $u\to\psi_u$ induces an isomorphism between A_s^{-l+1}/A_s^{-m+1} and the complex dual, $(K_s^m/K_s^l)^\wedge$, of K_s^m/K_s^l.

Proof. This follows from $\mathrm{tr}(A_s^0)=\mathcal{O}_F$ and $\mathrm{tr}(A_s^1)=P_F$. See 2.7 and 2.8 of

[C].

We call ψ_u very cuspidal if u is very cuspidal.

Definition 1.7. An admissible representation ρ of Z_sK_s is said to be very cuspidal of level $N(N\geq 2)$ if the following conditions hold :

1. ρ is trival on K_s^N.

2. The restriction of ρ to K_s^{N-1} is decomposed into a sum of very cuspidal characters.

Remark. Let u be a very cuspidal element of level $1-N$ and $m=[(N+1)/2]$. Since K_s^m/K_s^N is an abelian group, ψ_u is a character of K_s^m. Therefore we can replace the condition (2) of Definition 1.7 by the condition :

(2′) the restriction of ρ to K_s^m contains a character of the form ψ_u, where u is very cuspidal of level $1-N$.

Theorem 1.8 (Carayol [C]). *Let ρ be an irreducible very cuspidal representation of Z_sK_s. Then the compact induction of ρ to G is an irreducible supercuspidal representation of G. Conversely any irreducible supercuspidal representation of G whose conductor is prime to n is compactly induced from an irreducible very cuspidal representation of Z_sK_s.*

Proof. This is contained in Theorem 4.2 and Theorem 8.1 in [C].

We recall the construction of irreducible very cuspidal representation in [T].

From now on, we fix a very cuspidal element u of level $1-N$ and set $E=F(u)$.

Lemma 1.9. *Let H_u be the normalizer of ψ_u in Z_sK_s, i.e. $H_u=\{g\in Z_sK_s|\psi_u^g=\psi_u\}$ where $\psi_u^g(x)=\psi_u(g^{-1}xg)$ for $x\in K_s^m$. Then $H_u=E^\times K_s^{[N/2]}$.*

Proof. It follows from Proposition 1.5 (4). See 5.5 of [C].

When N is even i.e. $N=2m$, it is easy to treat.

Proposition 1.10. *Let θ be a quasi-character of $E^\times$ such that $\theta(1+x)=\psi_u(1+x)$ for $x\in P_E^m$ and $\eta_{u,\theta}(tk)=\theta(t)\psi_u(k)$ for $t\in E^\times$ and $k\in K_s^m$. Then $\eta_{u,\theta}$ is a quasi-character of H_u and $\sigma(u;\theta)=\mathrm{Ind}_{H_u}^{Z_sK_s}\eta_{u,\theta}$ is an irreducible every cuspidal representation of level $2m$ of Z_sK_s. Conversely every irreducible very cuspidal representation of level $2m$ of Z_sK_s is written in the form $\sigma(u;\theta)$.*

Proof. Obvious. See Proposition 2.1.2 of [T].

When N is odd i.e. $N=2m-1$, the irreducible component of $\mathrm{Ind}_{K_s^m}^{H_u}\psi_u$ is not one-dimensional since $H_u=E^\times K_s^{m-1}$. The construction of the irreducible component of $\mathrm{Ind}_{K_s^m}^{H_u}\psi_u$ is treated in [T]. We need some notations and definitions to state the result.

Let $W=A_s^{m-1}/A_s^m$ and $pr(x)=\bar{x}$ for the natural projection from A_s^{m-1} to W. We denote $I\in \mathrm{End}_{k_F} W$ by the conjugate action of u on W i.e. $I(\bar{x})=\overline{uxu^{-1}}$. Let $s=p^l t$ and $(t, p)=1$. We note p is an odd prime since $(1-N, s)=1$. Set $h=t(q^r-1)/(q-1)$, $J=I^h\in \mathrm{End}_{k_F} W$ and $T=I^{h-1}+I^{h-2}+\cdots+I+1\in \mathrm{End}_{k_F} W$. Then $T(I-1)=(I-1)T=J-1$. We use the same symbols I, J, T on the same actions on A_s^0/A_s^1, i.e. $I(x)=uxu^{-1}$ for $x\in A_s^0/A_s^1$, $J=I^h$ and $T=I^{h-1}+\cdots+I+1$. The next lemma is proved in the proof of Lemma 2.2.1 of [T].

Lemma 1.11. *Let $x\in A_s^0/A_s^1$ and $R(x)=(\gamma_0, \gamma_1, \cdots, \gamma_{s-1})$. Put $u=z_s^{1-N}u_0$ and $R(u_0)=(\alpha_0, \alpha_1, \cdots, \alpha_{s-1})$. Then $u^h x u^{-h}=x$ for $x\in A_s^0/A_s^1$ is equivalent to*

$$\gamma_{i+1(1-N)}=C_i\gamma_i C^{-1}{}_i \quad (i=0, 1, \cdots, s-1)$$

where $C_i=\alpha_{i+(h-1)(1-N)}\alpha_{i+(h-2)(1-N)}\cdots\alpha_{i+(1-N)}\alpha_i$

Set $W_0=(J-1)^{(p^l-1)/2}W$, $W_1=(J-1)^{(p^l-1)/2}TW$, $A_s^{m,0}=pr^{-1}(W_0)$, $A_s^{m,1}=pr^{-1}(W_1)$. and $K_s^{m,0}=1+A_s^{m,1}$ and $K_s^{m,1}=1+A_s^{m,1}$. Put $U=F^\times\langle u^h\rangle(1+P_E)$, $L=E^\times/U$ and $X=UK_s^{m,0}/UK_s^{m,1}$. We remark that L is an abelian group of order relatively prime to p and the conjugate action of U on X is trivial. We denote by σ the conjugate action of L on X and regard X as an $\boldsymbol{F}_q[\mathrm{L}]$-module. For M a subgroup of L, let $\Omega_M=\{x\in X|\sigma(m)x=x\}$. Let X_M be the L-complement in Ω_M of the $\boldsymbol{F}_q[\mathrm{L}]$-module

$$\sum_{M\subset M'\subset L}\Omega_M$$

where the sum is over those subgroups of L which properly contain M. We define $D(M)=\frac{1}{2}\dim_{k_F}X_M$ and $S(M)\in\{\pm 1\}$ by

$$(q^{D(M)}-S(M))/|L/M| \text{ is an integer.} \tag{1.8}$$

Proposition 1.12. *Let $\tilde{\psi}_u$ be a character of $K_s^{m,1}$ defined by $\psi_u(1+x)=\psi(\mathrm{tr}u(x-x^2/2))$ for $x\in A_s^{m,1}$ and θ be a quasi-character of $E^\times$ with the property that $\theta(1+x)=\psi(\mathrm{tr}ux)$ for $x\in P_E^m$.*

1. *There exists an irreducible representation $\kappa_{u,\theta}$ of $E^\times K_s^{m,0}$ which is determined by its charater formula :*

$$\chi_{\kappa_{u,\theta}}(ag)=q^{\Sigma_{a\in M}(D(M))}(\prod_{a\notin M}S(M))\theta(a)\tilde{\psi}_u(g) \tag{1.9}$$

for $a\in E^\times$ and $g\in K_s^{m,1}$;

$$\chi_{\kappa_{u,\theta}}(\gamma)=0$$

if γ is not conjugate to an element of $E^\times K_s^{m,1}$.

2. *Set $\eta_{u,\theta}=\mathrm{Ind}_{E^\times K_s^{m,0}}^{E^\times K_s^{m-1}}\kappa_{u,\theta}$. Then $\eta_{u,\theta}$ is an irreducible component of $\mathrm{Ind}_{K_s^m}^{H_u}\psi_u$ and every irreducible representation of H_u whose restriction to K_s^m contains ψ_u is*

written in the form $\eta_{u,\theta}$.

3. *Put* $\sigma(u;\theta)=\mathrm{Ind}_{E^\times K_s^{m,0}}^{Z_sK_s}\kappa_{u,\theta}$ *Then* $\sigma(u;\theta)$ *is an irreducible very cuspidal representation of level* $2m-1$ *of* Z_sK_s *and every irreducible very cuspidal representation of level* $2m-1$ *of* Z_sK_s *is equivalent to some representation* $\sigma(u;\theta)$ *for some very cuspidal element* u *of level* $2-2m$ *and quasi-character* θ *of* $E^\times$.

Proof. See 2.2-2.6 of [T].

2. Character formula of $\eta_{u,\theta}$

In this section we compute the character formula of $\eta_{u,\theta}$, which is an irreducible component of $\mathrm{Ind}_{K_s^m}^{H_m}\psi_u$. First we compute terms $\sum_{a\in M}D(M)$ and $\prod_{a\notin M}S(M)$ in Proposition 1.12. It gives the character formula of $\kappa_{u,\theta}$.

Proposition 2.1. *For* $a\in E^\times$,

$$\sum_{a\in M}D(M)=r(r(a)(v_E(a),t)-1) \text{ and } \prod_{a\notin M}S(M)=(-1)^{r-r(a)}\left(\frac{q}{t/(v_E(a),t)}\right).$$

Proof. Since A_s^{m-1}/A_s^m is isomorphic to A_s^0/A_s^1 as $E^\times$-module, we may assume $m=1$. First we treat the term $\sum_{a\in M}D(M)$. From the definition of $D(M)$, $\sum_{a\in M}D(M)=\frac{1}{2}\dim_{k_F}X^a$ where $X=W_0/W_1$ and $X^a=\{x\in X|a^{-1}xa=x\}$. We note $(J-1)^{(pl-1)/2}$ induces an L-module isomorphism between X and Ker $(J-1)/\mathrm{Ker}(I-1)$. Let F_t be the maximal tamely ramified extension of F in E. By virtue of the fact $E^\times/F^\times<u^h>(1+P_E)\simeq F_t^\times(1+P_E)/F^\times(1+P_E)$, we may assume $a\in F^\times$. Let $v_E(a)=c, a=z_s^c a_0$ and $R(a_0)=(\delta_0,\cdots,\delta_{s-1})$. Since $a\in F_t$, $p^l|c$. It follows from Lemma 1.11 that $a\in X^a$ is equivalent to $\gamma_{i+c}=\delta_i\gamma_i\delta_i^{-1}$ for $0\le i<t-1$ where $R(a)=(\gamma_0,\cdots,\gamma_{s-1})$. By virtue of $p^l|c$, it is equivalent to

$$\gamma_i=D_i\gamma_iD_i^{-1}$$

where $D_i=\delta_{i+cs/c_0}\cdots\delta_{i+c}\delta_i$ and $c_0=(c,s)$. Since $(D_0,\cdots,D_{s-1})=R(\varpi_F^{-c/c_0}a^{s/c_0})$, $k_F(D_i)=k_F(\varpi_F^{-c/c_0}a^{s/c_0}\bmod P_E)$. Thus have

$$\dim_{k_F}X^a=r(a)^2\frac{r}{r(a)}(v_E(a),t)-r. \tag{2.1}$$

Next we consider the term $\prod_{a\notin M}S(M)$. From Proposition 2.6.8 of [T], $\prod_{a\notin M}S(M)=(-1)^{r-1}\left(\frac{q}{t}\right)^r/\prod_{a\in M}S(M)$. For $a\in E^\times$, we set $\bar{a}=a \bmod U$. Let u_1 be an element of $\mathcal{O}_E^\times$ such that $u_1\bmod(1+P_E)$ generates the cyclic group $k_E^\times$. We shall omit the symbol — when there is no fear of confusion. By the same way of the proof of Proposition 2.6.8 of [T], $\prod_{a\in M}S(M)$ is calculated as follows.

When r is odd,

$$\prod_{a\in M} S(M) = \prod_{c_0|j|t} S(<u_1, u^j>)$$
$$=\left(\frac{q}{t}\right)^r/\left(\frac{q}{c_0}\right)^r.$$

Now we assume r is even. Let $u_2=u_1^{(q^r-1)/(q^{r/2}-1)}$. If $<u_2, a^{t/c_0}> = <u_1>$, $\prod_{a\in M} S(M)=1$. If $<u_2, a^{t/c_0}> = <u^2>$, i.e. $r(a)$ is even, $\prod_{a\in M}S(M) = (-1)^{B(a)}$ where $B(a)$ is the cardinality of the set :

$$\left\{M\subset F_t^\times(1+P_E)/F^\times(1+P_E) \;\middle|\; \begin{array}{l} <u^2,a>\subset M \\ \dim_{k_F(u_2)}X_M\equiv 2 \bmod 4\end{array}\right\}.$$

Since $\Omega_{<u_2,a>}=\oplus_{<u_2,a>\subset M}X_M$, we have

$$\dim_{k_F(u_2)}\Omega_{<u_2,a>}=\frac{r}{2}([k_E : k_F(u_2,a^{s/c_0})]c_0r-r)$$
$$=2(2c_0-1)$$
$$\equiv 2 \bmod 4.$$

Therefore $B(a)$ is odd and $\prod_{a\in M}S(M)=-1$. Hence our proposition.

We can state the character formula of $\kappa_{u,\theta}$.

Crollary 2.2. *For $a\in E^\times$ and $g\in K_S^{m,1}$,*

$$\chi_{\kappa_{u,\theta}}(ag)=q^{(rr(a)c_0-r)/2}\left(\frac{q}{t/c_0}\right)^r(-1)^{r-r(a)}\theta(a)\tilde{\psi}_u(g)$$

where $c_0=(v_E(a),t)$.

Now we start the calculation of the character of $\eta_{u,\theta}=\mathrm{Ind}_{E^\times K_S^{m,0}}^{E^\times K_S^{m-1}}\kappa_{u,\theta}$. It takes many steps.

Lemma 2.3. *For $a\in E^\times$,*

$$\chi_{\eta_{u,\theta}}(a)=\chi_{\kappa_{u,\theta}}(a)\sum_{x\in(a-1)^{-1}W_1+W_0/W_0}\psi\left(\mathrm{tr}\frac{1}{2}u(xx^a-x^ax)\right)$$

where $(a-1)x=axa^{-1}-a$ and $x^a=a^{-1}xa$.

Proof. From the definition of the induced representation

$$\chi_{\eta_{u,\theta}}(a)=\sum_{g\in E^\times K_S^{m-1}/E^\times K_S^{m,0}}\dot{\chi}_{\kappa_{u,\theta}}(g^{-1}ag)$$

where

$$\dot{\chi}_{\kappa_{u,\theta}}(x)=\begin{cases}\chi_{\kappa_{u,\theta}}(x) & \text{if } x\in E^\times K_S^{m,0}\\ 0 & \text{otherwise.}\end{cases}$$

Since the support of $\chi_{\kappa_{u,\theta}}$ is contained in the conjugate class of $E^\times K_S^{m,1}$, $\dot{\chi}_{\kappa_{u,\theta}}(g^{-1}ag)\neq 0$ is equivalent to $g\in 1+A_S^{m,0}+pr^{-1}((a-1)^{-1}W_1)$. Under this condition, $\chi_{\kappa_{u,\theta}}(g^{-1}ag)=\chi_{\kappa_{u,\theta}}(a)\psi_u(a^{-1}gag^{-1})$. Put $g=1+x$. Then

$$\psi_u(a^{-1}gag^{-1})=\psi\Big(\mathrm{tr}u\,((x-x^a+(x^2)^a-x^ax)-\frac{1}{2}(x^2+(x^a)^2-xx^a-x^ax))\Big)$$
$$=\psi\Big(\mathrm{tr}\frac{1}{2}u\,(xx^a-x^ax)\Big).$$

This implies our lemma.

We set

$$\Lambda(a)=\sum_{x\in(a-1)^{-1}W_1+W_0/W_0}\psi\Big(\mathrm{tr}\frac{1}{2}u\,(xx^a-x^ax)\Big). \tag{2.2}$$

Now we have only to compute $\Lambda(a)$ for $a\in E^\times$. We remark

$$E^\times/F^\times(1+P_E)\simeq F_l^\times(1+P_E)/F^\times(1+P_E)\times F^\times<u^h>(1+P_E)/F^\times(1+P_E).$$

First we calculate $\Lambda(a)$ for $a\in F_l^\times$.

Lemma 2.4. *For $a\in F_l^\times(1+P_E)$,*

$$\Lambda(a)=\frac{2}{(p^l+1)}p^{(v_E(a),t)rr(a)}$$

Proof. Since $(a-1)^{-1}W_1+W_0=\mathrm{Ker}\,(a-1)+W_0$, $\Lambda(a)=|\mathrm{Ker}\,(a-1)+W_0/W_0|$. From (2.1), we have $\dim_{k_F}(\mathrm{Ker}\,(a-1)\cap\mathrm{Ker}\,(J-1))=\frac{1}{p^l}\dim_{k_F}\mathrm{Ker}\,(a-1)$. Since $W_0=(J-1)^{(p^l-1)/2}W=\mathrm{Ker}\,(J-1)^{(p^l+1)/2}$, we get our lemma.

Proposition 2.5. *For $a=(u^h)^ia_1$, $(i,p^l)=p^j$ and $a_1\in F_l^\times(1+P_E)$,*

$$\Lambda(a)=q^{rr(a_1)(v_E(a_1),t)(p^j-1)}\Big(\frac{i/p_j}{p}\Big)^fG$$

where $p^f=q$ and

$$G=\sum_{x\in k_E}\psi\Big(\mathrm{tr}_{k_E/k_F}\frac{1}{2}u\,\varpi_E^{2(m-1)}(-1)^{(p^l+1)/2}x^2\Big). \tag{2.3}$$

Proof. First we treat the case $a=(u^h)^{p^j}$ $(j=0,\cdots,l-1)$. Then $(a-1)\in\mathrm{End}_{k_F}(A_s^{m-1}/A_s^m)$ is $(J^{p^j}-1)=(J-1)^{p^j}$. Thus $(a-1)^{-1}W_1+W_0=\mathrm{Im}\,(J-1)^{(p^l-1-2p^j)/2}T$. From the definition of $\Lambda(a)$, we have

$$\Lambda((u^h)^{p^j})=\sum_{x\in\mathrm{Im}(J-1)^{(p^l-1-2p^j)/2}T/\mathrm{Im}(J-1)^{(p^l-1)/2}}\psi\Big(\mathrm{tr}\frac{1}{2}u\,(xJ^{-p^j}(x)-J^{-p^j}(x)x)\Big).$$

Define an alternating from $<,>$ on W by

$$<x,y>=\mathrm{tr}\frac{1}{2}u\,(xy-yx). \tag{2.4}$$

Since $<x,J^{-1}x>=<(J-1)x,x>$, we get

$$\Lambda((u^h)^{p^j}) = \sum_{x\in \mathrm{Im}(J-1)^{(p^l-1-2p^j)/2}T/\mathrm{Im}(J-1)^{(p^l-1)/2}} \psi(<(J-1)^{p^j}x, x>)$$
$$= \sum_{x\in \mathrm{Im}(J-1)^{(p^l+1-2p^j)/2}T/\mathrm{Im}(J-1)^{(p^l+1-2p^j)/2}}$$
$$\sum_{x\in \mathrm{Im}(J-1)^{(p^l+1-2p^j)/2}/\mathrm{Im}(J-1)^{(p^l-1)/2}} \psi<(J-1)^{p^j}(x+y), x+y>).$$

In this expression,

$$\psi(<(J-1)^{p^j}(x+y), x+y>) = \psi(<(J-1)^{p^j}x,x>)\psi(J-1)^{p^j}y, y>)$$
$$\psi(<(J-1)^{p^j}x, y>+<(J-1)^{p^j}y, x>).$$

We prepare two lemmas.

Lemma 2.6. *For* $y\in \mathrm{Im}(J-1)^{(p^l+1-2p^j)/2}$,

$$\psi(<(J-1)^{p^j}y, y>)=1.$$

Proof. By the map $y\mapsto (J-1)^{(p^l+1-2p^j)/2}(\varpi_E^{m-1}y)$,

$$(A_s^0/A_s^1)/\mathrm{Im}(J-1)^{p^j-1}\simeq \mathrm{Im}(J-1)^{(p^l+1-2p^j)/2}/\mathrm{Im}(J-1)^{(p^l-1)/2}$$

as $k_F[E^\times]$-module. For $k_0\notin \mathrm{Im}(J-1)$, $\{k_0, (J-1)k_0, \cdots, (J-1)^{p^j-2}k_0\}$ is a basis of $(A_s^0/A_s^1)/\mathrm{Im}(J-1)^{p^j-1}$ as $\mathrm{Ker}(J-1)$-module. Thus $y\in \mathrm{Im}(J-1)^{(p^l+1-2p^j)/2}$ can be written in the form

$$y=\varpi_E^{m-1}(J-1)^{(p^l+1-2p^j)/2}\sum_{\mu=0}^{p^j-2}a_\mu(J-1)^\mu k_0 \quad (a_\mu\in \mathrm{Ker}(J-1)).$$

Thus it suffices to say $<\varpi_E^{m-1}z_1(J-1)^a k_0, \varpi_E^{m-1}z_2(J-1)^b k_0>=0$ for $z_1,z_2\in \mathrm{Ker}(J-1)$. Since $(2-2m, s)=1$,

$$y\varpi_E^{m-1}=\varpi_E^{m-1}J^{(s-1)/2}y \tag{2.5}$$

for $y\in A_s^0$. Thus we have :

$$<\varpi_E^{m-1}z_1(J-1)^a k_0,\varpi_E^{m-1}z_2(J-1)^b k_0>$$
$$=<\varpi_E^{m-1}z_1(-J^{-1})b(J-1)^{a+b}k_0, \varpi_E^{m-1}z_2k_0$$
$$=\mathrm{tr}\frac{1}{2}u\varpi_E^{2(m-1)}(J^{(s-1)/2}z_1\cdot J^{(s-1)/2}(-J^{-1})^bJ^{(s-1)/2}(J-1)^{(a+b)}k_0\cdot z_0k_0$$
$$-z_2k_0\cdot J^{(s-1)/2}z_1\cdot J^{(s-1)/2}(-J^{-1})^bJ^{(s-1)/2}(J-1)^{(a+b)}k_0).$$

We can take k_0 satisfying $R(k_0)=(\mathbf{1}_r, \mathbf{0}_r, \cdots, \mathbf{0}_r)$. Then $J^{(s-1)/2}J^a k_0\cdot k_0=0$ for $a\in \boldsymbol{Z}$ since $(s-1)/2+ha\not\equiv 0 \bmod s$. Thus we get $<\varpi_E^{m-1}z_1(J-1)^a k_0, \varpi_E^{m-1}z_2(J-1)^b k_0>=0$.

Lemma 2.7. *For* $y\in (J-1)^{(p^l+1-2p^j)/2}W$ *and* $x\in (J-1)^{(p^l+1-2p^j)/2}TW$,

$$\psi(<(J-1)^{p^j}x, y>+<x, (J-1)^{p^j}y>)=1$$

Proof. Put $C=<(J-1)^{p^j}x,\ y>+<x,\ (J-1)^{p^j}y>$, $x=\varpi_E^{m-1}(J-1)^{(p^l+1-2p^j)/2}Tx'$ and $y=\varpi_E^{m-1}(J-1)^{(p^l+1-2p^j)/2}y'$. Then

$$
\begin{aligned}
C&=<\omega_E^{m-1}(J^{-p^j}+1)(J-1)^{(p^l+1)/2}Tx',\ \varpi_E^{m-1}(J-1)^{(p^l+1-2p^j)/2}>\\
&=<\varpi_E^{m-1}B(J-1)^{p^l-p^j}Tx',\ \varpi_E^{m-1}y'>
\end{aligned}
$$

where $B=(J^{-p^j}+1)(-J^{-1})^{(p^l+1-2p^j)/2}$. It follows from (2) that

$$
\begin{aligned}
C&=\mathrm{tr}\frac{1}{2}u\,\varpi_E^{2(m-1)}(B(J-1)^{p^l-p^j}TI^{(s-1)/2}x'y'-I^{(s-1)/2}y'B(J-1)^{p^l-p^j}Tx'\\
&=\mathrm{tr}\frac{1}{2}u\,\varpi_E^{2(m-1)}x''y'BI^{(s-1)/2}(J-1)^{p^l-p^j+1}k_0
\end{aligned}
$$

where $x'=x''k_0$ for $x''\in k_E$ and $R(k_0)=(\mathbf{1}_r,\mathbf{0}_r,\cdots,\mathbf{0}_r)$. Since we can write $y=\sum_{\nu=0}^{p^j-2}a_\nu(J-1)^\nu k_0$ where $a_\nu\in k_E$, it suffices to say

$$
\mathrm{tr}\frac{1}{2}u\omega_E^{2(m-1)}x''a_\nu k_0BI^{(s-1)/2}(-J^{-1})^\nu(J-1)^{p^l-p^j+1+\nu}k_0=0.
$$

This follows from $I^{(s-1)/2}J^ak_0\cdot k_0=0$ for $a\in\boldsymbol{Z}$.

Now we go back to the proof of Proposition 2.5. From the above two lemmas, we have

$$
\Lambda(u^{hp^j})=q^{r^2t(p^j-1)}\sum_{x\in\mathrm{Im}(j-1)^{(p^l-1-2p^j)/2}T/\mathrm{Im}(J-1)^{(p^l+1-2p^j)/2}}\psi(<(J-1)^{p^j}x,x>).
$$

Since $\mathrm{Im}\,(J-1)^{(p^l-1-2p^j)/2}T/\mathrm{Im}\,(J-1)^{(p^l+1-2p^j)/2}$ is one-dimensional k_E-vector space,

$$
\Lambda(u^{hp^j})=q^{r^2t(p^j-1)}\sum_{x\in k_E}\psi(<\varpi_E^{m-1}(J-1)^{(p^j-1)}Txk_0,\ \varpi_E^{m-1}(J-1)^{(p^l+1-2p^j)/2}Txk_0>)
$$

where $k_0\in A_s^0$ satisfying $k_0\notin\mathrm{Im}\,(J-1)$. By the same argument in the proof of the above two lemmas, we can easily show

$$
\begin{aligned}
&<\varpi_E^{m-1}(J-1)^{(p^l-1-2p^j)/2}Txk_0,\ \varpi_E^{m-1}(J-1)^{(p^l-1-2p^j)/2}Txk_0>\\
&=\mathrm{tr}\frac{1}{2}u\,\varpi_E^{2m-1}(-1)^{(p^l+1)/2}x^2(I^{(s-1)/2}J^{p^j-(p^l+1)/2}(J-1)^{p^l-p^j}Tk_0\cdot k_0).
\end{aligned}
$$

Since $J^{p^j-(p^l+1)/2}(J-1)^{p^l-p^j}=\sum_{\mu=0}^{p^{l-j}}J^{(p^l-1)/2-\mu p^j}$ and $T=I^{h-1}+\cdots+I+1$, it suffices to say that the number of the solutions to the equation :

$$
(s-1)/2+h((p^l-1)/2-up^j)+\nu\equiv0 \bmod s\ \ (1\le\mu\le p^{l-j},0\le\nu\le h-1)
$$

is 1 mod$p$. It is easy to see that the number of the solutions is $[h/s]p^{l-j}+1$. Hence we get our proposition for the case $a=u^{hp^j}$.

Next we treat the case $a=u^{hip^j}$ where $(i,p)=1$. By the same argument as above, we have

$$
\begin{aligned}
\Lambda(a) &= q^{r^2 t(p^j-1)} \sum\nolimits_{x\in \mathrm{Im}(J-1)^{(p^l+1-2p^j)/2} T/\mathrm{Im}(J-1)^{(p^l+1-2p^j)/2}} \psi(<(J^i-1)^{p^j}x, x>) \\
&= q^{r^2 t(p^j-1)} \sum\nolimits_{x\in \mathrm{Im}(J-1)^{(p^l+1-2p^j)/2} T/\mathrm{Im}(J-1)^{(p^l+1-2p^j)/2}} \prod_{a=0}^{i-1} \psi(<J^{ap^j}(J-1)^{p^j}x, x>) \\
&= q^{r^2 t(p^j-1)} \sum_{x\in k_E} \prod_{j=0}^{i-1} \psi\Big(\frac{1}{2}\mathrm{tr}u\varpi_E^{(p^l+1-2p^j)/2}x^2\Big) \\
&= q^{r^2 t(p^j-1)} \sum_{x\in k_E} \psi\Big(\frac{1}{2}\mathrm{tr}u\varpi_E^{(p^l+1-2p^j)/2} i x^2\Big).
\end{aligned}
$$

By virtue of the fact i mod p is square in $\boldsymbol{F}_p$ is equivalent to $\left(\frac{i}{p}\right)^f=1$ $(q=p^f)$, we get

$$
\Lambda(a) = \left(\frac{i}{p}\right)^f q^{r^2 t(p^j-1)} \sum_{x\in k_E} \psi\Big(\frac{1}{2}\mathrm{tr}u\varpi_E^{(p^l+1-2p^j)/2}x^2\Big).
$$

We treat the general case $a=a_1(u^h)^{ip^j}$ where $a_1\in F_t^\times(1+P_E)$. Then $(a-1)^{-1}W_1+W_0/W_0=\mathrm{Ker}(a_1-1)\cap(J^{p^j}-1)^{-1}W_1/\mathrm{Ker}(a_1-1)\cap W_0$. Thus we have only to substitute $\mathrm{Ker}(a_1-1)$ for W. From Lemma 2.4 and the calculation for the case $a\in<u^h>$, we get

$$
\Lambda(a) = q^{rr(a_1)(v_E(a_1),t)(p^j-1)} \left(\frac{i/p^j}{p}\right)^f \sum_{x\in k_E} \psi\Big(\frac{1}{2}\mathrm{tr}u\varpi_E^{(p^l+1-2p^j)/2}x^2\Big).
$$

Hence our proposition.

Here we state the character formula of $\eta_{u,\theta}$.

Theorem 2.8. *Let $\eta_{u,\theta}$ be the irreducible representation of $E^\times K_S^{m-1}$ as in Lemma 1.12. The character formula of $\eta_{u,\theta}$ is given by*

$$
\chi_{\eta_{u,\theta}}(ag) = \begin{cases}
q^{\frac{1}{2}r(r(a)(v_E(a),t)p^l-1)}(-1)^{r-r(a)}\left(\dfrac{q}{t/(v_E(a),t)}\right)^r \theta(a)\psi(\mathrm{tr}u(g-1)) \\
\qquad \text{if } a\in F_t^\times(1+P_E) \\
q^{\frac{1}{2}r(r(a)(v_E(a),t)p^j-1)}(-1)^{r-r(a)}\left(\dfrac{i/p^j}{p}\right)^f\left(\dfrac{q}{t/(v_E(a),t)}\right)^r G\theta(a)\psi(\mathrm{tr}u(g-1)) \\
\qquad \text{if } a\in E^\times - F_t^\times(1+P_E)
\end{cases}
$$

where F_t is a maximal tamely ramified extension of F in E, $a=a_1(u^h)^i\in E^\times$, $a_1\in F_t^\times(1+P_E)$, $g\in K_S^m$, $r(a)=[k_E:k_F(\varpi_F^{-v_E(a)/(v_E(a),s)}a^{s/(v_E(a),s)} \bmod P_E)]$, $(i,p^l)=p^j$ $(j\geq 1)$ and the Gauss sum G is defined by

$$
G=\sum_{x\in k_E}\psi\Big(\mathrm{tr}_{k_E/k_F}\frac{1}{2}u\varpi_E^{2(m-1)}(-1)^{(p^l+1)/2}x^2\Big)\Big) \tag{2.6}
$$

and $\chi_{\eta_{u,\theta}}(g)=0$ if g is not conjugate to an element of $E^\times K_S^m$.

3. An expression of $\eta_{u,\theta}$ as a linear combination of monomial representations

We fix quasi-character θ of $E^\times$ satisfying $\theta(1+x)=\psi(\mathrm{tr}_{E/F}ux)$ for $x\in P_E^m$ and put $C=E^\times/F^\times(1+P_E)$. Let $\theta\cdot\psi_u$ be a quasi-character of $E^\times K_S^m$ defined by $(\theta\cdot\psi_u)(t(1+x))=\theta(t)\psi(\mathrm{tr}ux)$ for $t\in E^\times$ and $x\in A_S^m$. In section 2, we get the character formula of $\eta_{u,\theta}$. From this formula, we can write

$$\mathrm{Ind}_{E^\times K_S^m}^{E^\times K_S^{m-1}}((\theta\otimes\lambda)\cdot\psi_u)=\sum_{\tau\in C}a_{\lambda\tau}\eta_{u,\theta\otimes\tau}$$

where $a_{\lambda\tau}\in\boldsymbol{Z}$. Put

$$M=(a_{\lambda\tau})_{\lambda,\tau\in\hat{C}}\in M_{|C|}(\boldsymbol{Z})$$

Our aim in this section is to compute M and M^{-1}.

Lemma 3.1. *Let the notation be as above. For λ, $\tau\in\hat{C}$,*

$$a_{\lambda\tau}=\frac{1}{|C|}\sum_{x\in C}A(x)(\tau\lambda^{-1})(x)$$

where

$$A(x)=\begin{cases} q^{\frac{1}{2}r(r(x)(v_E(x),t)p^l-1)}\left(\dfrac{q}{t/(v_E(x),t)}\right)^r \\ \qquad \text{if } x\in F_t^\times(1+P_E) \\ q^{\frac{1}{2}r(r(x)(v_E(x),t)(2p^j-1)-1)}(-1)^{r-r(x)}\left(\dfrac{i/p^j}{p}\right)^f\left(\dfrac{q}{t/(v_E(x),t)}\right)^r G \\ \qquad \text{if } x=x_1(v^h)^i\in E^\times-F^\times(1+P_E) \\ x_1\in F_t^\times(1+P_E),\ g\in K_S^m,\ (i,p^l)=p^j\ (j\geq 1)\end{cases}$$

where F_t is a maximal tamely ramified extension of F in E.

Proof. From the definition of $a_{\lambda\tau}$ and Frobenius reciprocity law, we have

$$\begin{aligned}a_{\lambda\tau}&=\langle(\theta\otimes\lambda)\cdot\psi_u,\chi_{\eta_{u,\theta\otimes\tau}}|_{E^\times K_S^m}\rangle_{E^\times K_S^m}\\
&=\frac{1}{\mathrm{vol}(E^\times K_S^m/F^\times)}\int_{E^\times K_S^m/F^\times}\chi_{\eta_{u,\theta\otimes\tau}}(g)((\theta\otimes\lambda)\cdot\psi_u)(g^{-1})dg\\
&=\frac{\mathrm{vol}(F^\times(1+P_E)K_S^m/F^\times)}{\mathrm{vol}(E^\times K_S^m/F^\times)}\sum_{g\in E^\times K_S^m/F^\times(1+P_E)K_S^m}\chi_{\eta_{u,\theta\otimes\tau}}(g)((\theta\otimes\lambda)\cdot\psi_u)(g^{-1})\\
&=\frac{1}{|C|}\sum_{x\in C}A(x)\tau(x)\lambda(x^{-1}).\end{aligned}$$

Now we assume $t=r=1$. Then $C=E^\times/F^\times\mathcal{O}_E^\times\simeq\boldsymbol{Z}/p^l\boldsymbol{Z}$, $\hat{C}=\{\lambda_i|\lambda_i(\varpi_E)=\boldsymbol{e}(i/p^l),\ 0\leq i<p^l\}$ and

$$A(x)=\begin{cases} q^{(p^l-1)/2} & \text{if } x\in F^\times(1+P_E)\\ q^{p^j-1}\left(\dfrac{i/p^j}{p}\right)^f G & \text{if } x\in\varpi_E^iF^\times(1+P_E)\end{cases}$$

where $(i, p^l)=p^j$, $1\leq j\leq l-1$.

Lemma 3.2. *Put* $a_{ij}=a_{\lambda_i\lambda_j}$. *Then*

$$(3.1)\qquad a_{ij}=\begin{cases}\dfrac{1}{p^lq}\left(G\sum_{c=l-c(i-j)}^{l}B(c)p^{l-c}+q^{(p^l+1)/2}-Gq^{p^l}\right) & \text{if } f \text{ even}\\[2ex] \dfrac{1}{p^lq}\left(Gq^{p^{l-c(i-j)-1}}p^{c(i-j)}\left(\dfrac{(j-i)/p^{c(i-j)}}{p}\right)G_0+q^{(p^l+1)/2}\right) & \text{if } f \text{ odd}\end{cases}$$

where $(i-j, p^l)=p^{c(i-j)}$, G *as in* (2.6),

$$(3.2)\qquad G_0=\sum_{a=1}^{l}\left(\frac{a}{p}\right)\boldsymbol{e}(a/p)$$

and

$$B(c)=\begin{cases}q & \text{for } c=0\\ q^{p^c}-q^{p^{c-1}} & \text{for } c>0.\end{cases}$$

Proof. From the definition of a_{ij}, we have

$$(3.3)\qquad a_{ij}=\frac{1}{p^l}\left(\sum_{a=1}^{p^l-1}q^{(a,p^l)-1}\left(\frac{a/(a,p^l)}{p}\right)^fG\lambda_{j-i}(a)+q^{(p^l-1)/2}\right).$$

If f is even, then

$$\begin{aligned}a_{ij}&=\frac{1}{p^lq}\left(G\sum_{a=1}^{p^l}q^{(a,p^l)}\boldsymbol{e}((j-i)a/p^l)+q^{(p^l+1)/2}-Gq^{p^l}\right)\\ &=\frac{1}{p^lq}\left(G\sum_{c=0}^{l}B(c)\sum_{a=1}^{p^{l-c}}\boldsymbol{e}((j-i)p^ca/p^l)+q^{(p^l+1)/2}-Gq^{p^l}\right)\\ &=\frac{1}{p^lq}\left(G\sum_{c=l-c(i-j)}^{l}B(c)p^{l-c}+q^{(p^l+1)/2}-Gq^{p^l}\right)\end{aligned}$$

Now we assume f is odd. It follows from (3.3) that

$$\begin{aligned}a_{ij}&=\frac{1}{p^lq}\left(G\sum_{a=1}^{p^l-1}q^{(a,p^l)}\left(\frac{a/(a,p^l)}{p}\right)\boldsymbol{e}((j-i)a/p^l)+q^{(p^l+1)/2}-Gq^{p^l}\right)\\ &=\frac{1}{p^lq}\left(G\left(\sum_{c=0}^{l-1}\sum_{a\in z/p^{l-c}z)^\times}q^{p^c}\left(\frac{a}{p}\right)\boldsymbol{e}((j-i)a/p^{l-c}\right)+q^{(p^l+1)/2}-Gq^{p^l}\right)\\ &=\frac{1}{p^lq}\left(G\left(\sum_{c=0}^{l}q^{p^c}\sum_{a\in z/p^{l-c}z}\left(\frac{a}{p}\right)\boldsymbol{e}((j-i)a/p^{l-c})\right)+q^{(p^l+1)/2}\right)\end{aligned}$$

where $\left(\frac{a}{p}\right)=0$ if $p|a$. We can easily show

$$\sum_{a\in z/p^{l-c}z}\left(\frac{a}{p}\right)\boldsymbol{e}((j-i)a]p^{l-c})=0$$

if $(i-j,p^l)\neq p^{l-c-1}$. Therefore we get our lemma.

In order to caluulate the inverse of the matrix $M=(a_{ij})$, we introduce some matrices. For $0\leq\mu\leq l$, we define the matrices R_μ, $S_\mu\in M_{p^l}(\boldsymbol{Z})$ by

$$(3.4)\qquad R_\mu=\begin{pmatrix}1_{p^\mu} & \cdots & 1_{p^\mu}\\ \cdots & \cdots & \cdots\\ 1_{p^\mu} & \cdots & 1_{p^\mu}\end{pmatrix},\ S_\mu=\begin{pmatrix}c_{11} & \cdots & c_{1p^{l-\mu}}\\ \cdots & \cdots & \cdots\\ c_{p^{l-\mu}1} & \cdots & c_{p^{l-\mu}p^{l-\mu}}\end{pmatrix}$$

where $c_{ij}=\left(\frac{i-j}{p}\right)\mathbf{1}_{p^\mu}$. The matrix M is a linear combination of R_μ and S_μ $(0\leq\mu\leq l)$. The next lemma enables us to calculate M^{-1}.

Lemma 3.3 *For* $0\leq\mu,\nu\leq l$,

1. $R_\mu R_\nu=p^{l-\nu}R_\mu$ *if* $\mu<\nu$.
2. $R_0S_\mu=S_\mu R_0=0$.
3. *For* μ, $\nu<l$,

$$S_\mu S_\nu=\begin{cases}0 & \textit{if } \mu\neq\nu\\ \left(\frac{-1}{p}\right)p^{l-\mu-1}(pR_{\mu+1}-R_\mu) & \textit{if } \mu=\nu.\end{cases}$$

Proof. They are obvious except $S_\mu^2=\left(\frac{-1}{p}\right)p^{l-\mu-1}(pR_{\mu+1}-R_\mu)$. Let

$$S=\left(\left(\frac{i-j}{p}\right)\right)_{1<i,j<p}$$

and

$$R=\begin{pmatrix}1 & 1 & \cdots & 1\\ 1 & 1 & \cdots & 1\\ \cdots & \cdots & \cdots & \cdots\\ 1 & 1 & \cdots & 1\end{pmatrix}\in M_p(\boldsymbol{Z}).$$

In order to show $S_\mu^2=\left(\frac{-1}{p}\right)p^{l-\mu}(pR_{\mu+1}-R_\mu)$, it suffices to see $S^2=\left(\frac{-1}{p}\right)(p\mathbf{1}_p-R)$. It follows from

$$\sum_{x\in Z/pZ}\left(\frac{(x-a)(x-b)}{p}\right)=-1$$

if $a\neq b$.

Now we can obtain the explicit form of M^{-1}.

Proposition 3.4. *Let* a_{ij} *be as in Lemma* 3.2 *and* $M=(a_{ij})\in M_{p^l}(\boldsymbol{Z})$. *If f is even,*

$$M^{-1}=-\frac{q(q^{(p^l+1-2p^{l-1})/2}-G)}{p^lq^{(p^l+1)/2}G}R_0-\sum_{i=1}^{l-1}\frac{q(q^{p^l-p^{i-1}}-1)}{(pq^p)^iG}R_{l-i}+\frac{1}{G}R_l$$

and if is odd,

$$M^{-1}=\frac{1}{p^lq^{(p^l-1)/2}}R_0+\sum_{i=0}^{l-1}\frac{q}{(pq^p)^{l-i-1}GG_0}S_i.$$

(*See* (2.6), (3.2) *for the definition of Gauss sums G and G and* G_0.)

Proof. First we assume f is even. It follows from (3.1) that

$$M=\alpha_0R_0+\sum_{i=1}^{l}\alpha_iR_i$$

where $\alpha_0=\frac{1}{p^lq}(q^{(p^l+1)/2}-Gq^{p^{l-1}})$, $\alpha_i=\frac{G}{p^{l-i}q}(q^{p^{l-i}}-q^{p^{l-i-1}})$ for $1\leq i<l$ and $\alpha_l=G$. Put $M'=\sum_{i=0}^{l}\beta_iR_i$. Then by viture of Lemma 3.3, we have

$$MM'=\sum_{i=0}^{l}(\alpha_i\beta_ip^{l-i}+\sum_{i<j}(\alpha_i\beta_j+\alpha_j\beta_i)p^{l-j})R_i.$$

Thus the condition $MM'=\mathbf{1}_{p^l}$ is equivalent to

$$p^i\alpha_{l-i}\beta_{l-i}+\sum_{j=1}^{i}p^{i-j}(\alpha_{l-i+j}\beta_{l-i}+\alpha_{l-i}\beta_{l-i+j})=0 \text{ for } 1\leq i<l$$
$$\alpha_l\beta_l=1.$$

Solving these equations with respect to β_i, we get

$$\beta_l=\frac{1}{\alpha_l}$$
$$\beta_i=-\frac{\alpha_i}{(\sum_{j=0}^{l-i}p^{l-j}\alpha_{i+j})(\sum_{j=0}^{l-i-1}p^{l-j}\alpha_{i+j})}.$$

This gives the desired formula of M^{-1} for the case f even.

Now assume f is odd. By (3.1), we have

$$M=aR_0+\sum_{i=0}^{l-1}\gamma_iS_i$$

where $a=\frac{q^{(p^l-1)/2}}{p^l}$ and $\gamma_i=\left(\frac{-1}{p}\right)\frac{q^{l-i-1}GG_0}{qp^{l-i}}$. Put $M'=bR_0+\sum_{i=0}^{l-1}\delta_iR_i$. It follows from Lemma 3.3

$$MM'=abp^lR_0+\sum_{i=0}^{l-i}\gamma_i\delta_i\left(\frac{-1}{p}\right)p^{l-i-1}(pR_{i+1}-R_i).$$

Putting $MM'=R_l$, we can show

$$\delta_{l-1}=\left(\frac{-1}{p}\right)\frac{1}{p\gamma_{l-1}}$$

$$\delta_{i-1}=\frac{\gamma_i\delta_i}{p^2\gamma_{r-1}} \text{ for } 1\leq i\leq l-1$$

$$b=\left(\frac{-1}{p}\right)\frac{\gamma_0\delta_0}{pa}.$$

Hence we get our proposition.

Now we get rid of the assumption $t=r=1$. To save the space, we put

$$a(d)=\frac{q^d-1}{q-1}$$

for $d|r$. If the exact sequence

$$1\to k_E^\times/k_F^\times\to C\to E^\times/F^\times\mathcal{O}_E^\times\to 1$$

does not split, the calculation of a_{ij} is very complicated. Here we assume the exact sequence splits. It is equivalent to that there exists a uniformizer ϖ_{F_t} of F_t such tha $\varpi_{F_t}^t=F$. This assumption holds if $(t,a(r))=1$. Under this assumption,

$$\begin{aligned}C&\simeq<u_0>\times<\varpi_{F_t}>\times<u^h>\\ &\simeq \boldsymbol{Z}/a(r)\boldsymbol{Z}\times\boldsymbol{Z}/t\boldsymbol{Z}\times\boldsymbol{Z}/p^l\boldsymbol{Z}.\end{aligned}$$

For $0\leq i<a(r)s$, take i_1, i_2, i_3 such that $i=si_1+p^li_2+i_3$, $0\leq i_1<\mathrm{a}(i)$, $0\leq i_2<t$ and $0\leq i_3<p^l$. Define an element $\lambda_i\in\widehat{C}$ by

$$\lambda_i(u_0\varpi_{Ft}u^h)=\boldsymbol{e}(i_1/a(r))\boldsymbol{e}(i_3/t)\boldsymbol{e}(i_3/p^l). \tag{3.5}$$

Then $\widehat{C}=\{\lambda_i|0\leq i<a(r)p^lt\}$.

Lemma 3.5. *Put $a_{ij}=a_{\lambda_i\lambda_j}$ and take i_1, i_2, i_3, j_1, j_2, j_3 such that $i=si_1+p^li_2+i_3$, $j=sj_1+p^lj_2+j_3$, $0\leq i_{i,j}<a(r)$, $0\leq i_2, j_2<t$ and $0\leq i_3, j_3<p^l$. If f is even*

$$a_{ij}=\frac{1}{a(r)p^lt}\sum_{\frac{r}{d}|d'}\sum_{d''|d'}\sum_{\frac{t}{w}|w'}\sum_{w''|w'}\frac{1}{q^{r(w''d''+1)/2}}(-1)^{r-d''}\frac{ta(d/d')}{w'}$$
$$\mu(w'/w'')\mu(d'/d'')\left(G\sum_{c'=l-c}^{l}F_{r/d'',w''}(c')p^{l-c'}+q^{\frac{1}{2}rw''d''(pl+1)}-Gq^{rw''d''pl}\right) \tag{3.6}$$

and if is odd,

$$a_{ij}=\frac{1}{a(r)p^lt}\sum_{\frac{r}{d}|d',d''|d'}\sum\sum_{\frac{t}{w}|w',w''|w'}\sum\frac{1}{q^{r(w''d''+1)/2}}(-1)^{r-d''}\left(\frac{q}{t/w''}\right)\frac{ta(r/d')}{w'}$$
$$\mu(w'/w'')\mu(d'/d'')\left(GG_0q^{rw''d''pl-c-1}p^c\left(\frac{(j_3-i_3)/p^c}{p}\right)+q^{\frac{1}{2}rw''(pl+1)d''}\right) \tag{3.7}$$

where $(i_3-j_3,\ p^l)=p^c$, $(i_2-j_2,\ t)=w$, $d=[k_F(u_0^{i_1-j_1}):k_F]$, G (*resp.* G_0) *as in* (2.6) (*resp.* (3.2)), μ *is the Möbius function and*

$$F_{d(x),d(y)}(c)=\begin{cases} q^{r^2w(y)/d(x)} & \text{for } c=0 \\ (q^{r^2w(y)/d(x)})^{p^c}-(q^{r^2w(y)/d(x)})^{p^{c-1}} & \text{for } c>0 \end{cases}.$$

Proof. It follows from the definition of a_{ij} that

$$a_{ij}=\frac{1}{a(r)p^lt}\sum_{x=1}^{a(r)}\boldsymbol{e}((j_1-i_1)x/a(r))\sum_{y=1}^{t}\boldsymbol{e}((j_2-i_2)y/t)\sum_{z=1}^{p^l}A(u_0^x\varpi_{F_t}^y(u^h)^z)\boldsymbol{e}((j_3-i_3)z/p^l)$$

In the above expression,

$$A(u_0^x\varpi_{F_t}^y(u^h)^z)=\begin{cases} q^{\frac{r(rw(y)p^l-d(x,y))}{2d(x,y)}}(-1)^{r-(r/d(x,y))}\left(\dfrac{q}{t/w(y)}\right)^r & \text{if } z=p^l \\ q^{\frac{r(rw(y)(2p^c-1)-d(x,y))}{2d(x,y)}}(-1)^{r-(r/d(x,y))}\left(\dfrac{z/(z,p^l)}{p}\right)^f\left(\dfrac{q}{t/w(y)}\right)^rG & \text{otherwise} \end{cases}$$

where $w(y)=(y,\ t)$ and $d(x,\ y)=r/r(u_0^x\varpi_{F_t}^y)$. Since $k_E=k_F(\varpi_E^{2m-2}u^s \mathrm{mod} P_E)$ and $\varpi_{F_t}^t\in F$, $r(u_0^x\varpi_{F_t}^y)=[k_E:k_F(u_0^x)]$. Thus $d(x,\ y)$ is independent of y. Put $d(x)=d(x,\ 0)=[k_F(u_0^x):k_F]$ and

$$B(d(x),\ w(y))=\sum_{z=1}^{p^l}A(u_0^x\varpi_{F_t}^y(u^h)^z)\boldsymbol{e}((j_3-i_3)z/p^l). \tag{3.8}$$

By the same argument in the proof of Lemma 3.2, we can show

$$B(d(x),\ w(y))=\frac{1}{q^{(r^2w(y)/2d(x))+(r/2)}}(-1)^{r-(r/d(x))}\left(G\sum_{c'=l-c}^{l}F_{d(x),w(y)}(c')p^{l-c'}+q^{\frac{r^2w(y)(p^l+1)}{2d(x)}}-Gq^{\frac{r^2(w(y)p^l}{d(x)}}\right) \tag{3.9}$$

when f is even and

$$B(d(x),\ w(y))=\frac{1}{q^{(r^2w(y)/2d(x))+(r/2)}}(-1)^{r-(r/d(x))}\left(\frac{q}{t/w(y)}\right)\left(GG_0q^{r^2w(y)p^{l-c-1}/d(x)}p^c\left(\frac{(j_3-i_3)/p^c}{p}\right)+q^{\frac{r^2w(y)(p^l+1)}{2d(x)}}\right) \tag{3.10}$$

when f is odd. By Möbius inversion formula, we have

$$\begin{aligned} a_{ij}&=\frac{1}{a(r)p^lt}\sum_{x=1}^{a(r)}\boldsymbol{e}((j_1-i_1)x/a(r))\left(\sum_{w'|t}C_{w'}(d(x))\sum_{y\in w'Z/tZ}\boldsymbol{e}((j_2-i_2)y/t)\right) \\ &=\sum_{x=1}^{a(r)}\boldsymbol{e}((j_1-i_1)x/a(r))\sum_{\frac{t}{w}|w'}C_{w'}(d(x))t/w' \end{aligned}$$

where

$$C_{w'}(d(x))=\sum_{w''|w'}B(d(x),w'')\mu(w'/w'').$$

(μ is the Möbius function.) Using Möbious inversion formula once again, we have

$$a_{ij}=\frac{1}{a(r)p^{l}t}\sum_{\frac{r}{d}|d'}\sum_{d''|d'}\sum_{\frac{t}{w}|w'}\sum_{w''|w'}\frac{B(r/d'',w'')a(r/d')t}{w'}\mu(w'/w'')\mu(d'/d'').$$

Hence our lemma.

As in the case $t=r=1$, we define some matrices to calculate the inverse of $M=(a_{ij})$. For integers a, b such that $a|b$, we define matrices $R_{a,b}, S_{a,b}\in \mathrm{M}_b(\boldsymbol{Z})$ by

$$(3.11)\qquad R_{a,b}=\begin{pmatrix}\mathbf{1}_a & \cdots & \mathbf{1}_a\\ \cdots & \cdots & \cdots\\ \mathbf{1}_a & \cdots & \mathbf{1}_a\end{pmatrix},\quad S_{a,b}=\begin{pmatrix}c_{11} & \cdots & c_{1\frac{b}{a}}\\ \cdots & \cdots & \cdots\\ c_{\frac{b}{a}1} & \cdots & c_{\frac{b}{a}\frac{b}{a}}\end{pmatrix}$$

where $c_{ij}=\left(\frac{i-j}{p}\right)\mathbf{1}_{pa}$. Put

$$(3.12)\quad R_{d,w,c}=R_{a(d),a(r)}\otimes R_{w,t}\otimes R_{p^c,p^l}\quad S_{d,w,c}=R_{a(d),a(r)}\otimes R_{w,t}\otimes S_{p^c,p^l}$$

for $d|r$, $w|t$ and $0\leq c\leq l$. The next lemma is proved in the same way as Lemma 3.3.

Lemma 3.6. *For $d,d'|r,w,w'|t$ and $0\leq c\leq c'\leq l$,*

1. $R_{d,w,c}R_{d',w',c'}=\dfrac{a(r)a((d,d'))t(w,w')p^{l-c'}}{a(d)a(d')ww'}R_{(d,d'),(w,w'),c}.$
2. $R_{d,w,0}S_{d',w',c}=S_{d',w',c}R_{d,w,0}=0.$
3. *For $c,c'<l$,*

$$S_{d,w,c}S_{d',w',c'}=$$

$$\begin{cases}0 & \textit{if } c\neq c'\\ \left(\dfrac{-1}{p}\right)p^{l-c-1}\dfrac{a(r)a((d,d'))t(w,w')}{a(d)a(d')ww'}(pR_{(d,d'),(w,w'),c+1}-R_{(d,d'),(w,w'),c}) & \textit{if } c=c'\end{cases}$$

By Lemma 3.5 and Lemma 3.6, we can caluculate M^{-1}.

Proposition 3.7. *Let a_{ij} be as in Lemma 3.5 and $M=(a_{ij})\in\mathrm{M}_{|C|}(\boldsymbol{Z})$. Let G, G_0 be Gauss sums defined in (2.6), (3.2). Put $a(d)=(q^d-1)/(q-1)$ for $d|r$.*

(1) *If f is esen,*

$$M^{-1}=\sum_{d|r}\sum_{w|t}\sum_{c=0}^{l}\beta_{d,w,c}R_{d,w,c}$$

where $\beta_{d,w,c}$ are determinined inductively by the following relations :

$$\beta_{r,t,l}=\frac{1}{G}$$

$$\beta_{d,w,c}=-\frac{1}{f(d,w,c)}\sum_{\substack{d|d'|r\\ w|w'|t\\ c<c'<l\\ d'w'c'\neq dwc}}\ \sum_{\substack{d''|r\\ (d'',d')=d}}\ \sum_{\substack{w''|t\\ (w'',w')=w}}\ \sum_{\substack{c''<l\\ (pc'',pc')=pc}}\alpha_{d'',w'',c''}\beta_{d',w',c'}$$

where

$$f(d,w,c)=\begin{cases}(-1)^{r-(r/d)}q^{(rtp^l/dw)-(r/2)} & \text{if } c=0\\ (-1)^{r-(r/d)}Gq^{(rtp^{l-c}/dw)-(rt/2dw)-(r/2))} & \text{if } c>0\end{cases},$$

$$\alpha_{d,w,0}=\frac{a(d)w}{a(r)p^lt}\sum_{d'|\frac{r}{d}}\sum_{w'|\frac{t}{w}}\frac{1}{q^{r(d'w'+1)/2}}(-1)^{r-d'}\left(q^{\frac{rd'w'(p^l+1)}{2}}-Gp^{rd'w'p^{l-1}}\right)$$
$$\mu(r/dd')\mu(t/ww')$$

and

$$\alpha_{d,w,c}=\frac{a(d)wp^c}{a(r)p^lt}\sum_{d'\frac{r}{d}}\sum_{w'|\frac{t}{w}}\frac{1}{q^{r(d'w'+1)/2}}(-1)^{r-d'}Gq^{rd'w'p^{c'}}\mu(p^{l-c-c'})$$
$$\mu(r/dd')\mu(t/ww')$$

for $c>0$.

(2) *If f is odd,*

$$M^{-1}=\sum_{d|r}\sum_{w|t}\delta_{d,w}R_{d,w,0}+\sum_{d|r}\sum_{w|t}\sum_{c=0}^{l-1}\delta_{d,w,c}S_{d,w,c}$$

where $\delta_{d,w}$, $\delta_{d,w,c}$ are determinined inductively by the following relations:

$$\delta_{r,t,l-1}=(-1)^{r-1}\left(\frac{q}{t}\right)\frac{1}{GG_0}$$

$$\delta_{d,w,c-1}=\frac{1}{g(d,w,c)}\Bigg(\frac{1}{p^2}\sum_{\substack{d|d'|r\\ w|w'|t}}\ \sum_{\substack{d''|r\\ (d'',d')=d}}\ \sum_{\substack{w''|t\\ (w'',w')=w}}\gamma_{d'',w'',c}\delta_{d',w',c}$$
$$-\sum_{\substack{d|d'|r\\ w|w'|t\\ d'w'\neq dw}}\ \sum_{\substack{d''|r\\ (d'',d')=d}}\ \sum_{\substack{w''|t\\ (w'',w')=w}}\gamma_{d'',w'',c-1}\delta_{d',w',c-1}\Bigg)$$

$$\delta_{d,w}=\frac{1}{g(d,w)}\Bigg(\frac{1}{p}\sum_{\substack{d|d'|r\\ w|w'|t}}\ \sum_{\substack{d''|r\\ (d'',d')=d}}\ \sum_{\substack{w''|t\\ (w'',w')=w}}\gamma_{d'',w'',0}\delta_{d',w',0}$$
$$-\sum_{\substack{d'|r\\ w|w'|t\\ d'w'\neq dw}}\ \sum_{\substack{d''|r\\ (d'',d')=d}}\ \sum_{\substack{w''|t\\ (w'',w')=w}}\gamma_{d'',w''}\delta_{d',w'}\Bigg)$$

where

$$g(d,w,c)=\left(\frac{-1}{p}\right)(-1)^{r-r/d}\left(\frac{q}{w}\right)GG_0\frac{q^{r^2tp^{l-c-1}/dw}p^l}{q^{r(rt+dw)/2dw}},$$

$$g(d,w)=(-1)^{r-r/d}\left(\frac{q}{w}\right)q^{r(rtp^l-dw)/2dw},$$

$$\gamma_{d,w,c}=\frac{a(d)wGG_0}{a(r)tp^{l-c}}\left(\frac{-1}{p}\right)\sum_{d'|\frac{r}{d}}\sum_{w|\frac{t}{w}}(-1)^{r-d'}\left(\frac{q}{t/w}\right)$$

$$q^{rd'w'p'^{l-c-1}-r(d'w'+1)/2}\mu(r/dd')\mu(t/ww')$$

and

$$\gamma_{d,w}=\frac{a(d)w}{a(r)t}\left(\frac{-1}{p}\right)\sum_{d'|\frac{r}{d}}\sum_{w|\frac{t}{w}}(-1)^{r-d'}\left(\frac{q}{t/w}\right)q^{r(d'w'p^{l-c-1}-1)/2}\mu(r/dd')\mu(t/ww').$$

Proof. First we assume f is even. By virtue of Lemma 3.5, we can show

$$M=\sum_{d|r}\sum_{w|t}\sum_{c=0}^{l}\alpha_{d,w,c}R_{d,w,c}.$$

Put $M'=\sum_{d|r}\sum_{w|t}\sum_{c=0}^{l}\beta_{d,w,c}R_{d,w,c}$. It follows from Lemma 3.6 that

$$MM'=\sum_{\substack{d|r\\ w|t\\ 0<c<l}}\sum_{\substack{d|d',r\\ w|w'|t\\ c'<c}}\sum_{\substack{d''|r\\ (d'',d')=d}}\sum_{\substack{w''|t\\ (w'',w')=w}}\sum_{\substack{p^{c''}|p^l\\ (p^{c'},p^{c''})=p^c}}\frac{a(r)a(d)twp^{l-c'}}{a(d')a(d'')w'w''}\alpha_{d'',w'',c''}\beta_{d',w',c'}.$$

Thus by putting $MM'=\mathbf{1}_{|C|}$, we have only to show that the coefficient of $\beta_{d,w,c}$ equals to

$$f(d,w,c)=\sum_{d'|\frac{r}{d}}\sum_{w'|\frac{t}{w}}\sum_{c'<c}\frac{a(r)tp^{l-c}}{a(d')w'}\alpha_{d',w',c'}.$$

Since

$$\frac{a(r)tp^{l-c}}{a(d')w'}\alpha_{d',w',c'}=\sum_{d''|\frac{r}{d'}}\sum_{w''|\frac{t}{w'}}\sum_{c''\leq l-c'}$$

$$f(r/d'',t/w'',l-c'')\mu(r/dd'')\mu(t/ww'')\mu(p^{l-c'-c''}),$$

it follows from the Möbius inversion formula. It is obvious that $f(d,w,c)\neq 0$. Therefore we get the proposition for the case f even.

Now assume f is odd. Using Lemma 3.5 and Lemma 3.6, we can prove the proposition by the same way as in the case f even.

We state the main result of this paper.

Theorem 3.8. *Let u be a very cuspidal element of level $2-2m$ (cf. Definition 1.4), $E=F(u)$, θ be a quasi-character of $E^\times$ such that $\theta(1+x)=\psi(\mathrm{tr}\,ux)$ for $x\in P_E^m$. Define a quasi-character $\theta\cdot\psi_u$ of $E^\times K_S^m$ by $(\theta\cdot\psi_u)(tk)=\theta(t)\psi_u(k)$ for $t\in E^\times$ and $k\in K_S^m$. We assume that there exists a uniformizer ϖ_{F_t} of F_t such that $\varpi_{F_t}^2\in F$ where F_t is a maximal tamely ramified extension of F in E. Let $\theta_i=\theta\otimes\lambda_i$ where $\lambda_i\in(E^\times/F^\times(1+P_E))^\wedge$ is defined in (3.5). Set $a(d)=(a^d-1)/(q-1)$ for $d|r$. For $0\leq r<a(r)tp^l$, take i_1, i_2, i_3 such that $i=tp^li_1+p^li_2+i_3$, $0\leq i_1<a(r)$, $0\leq i_2<t$ and $0\leq i_3<p^l$. Put $w(i)=(i_2,t)$, $p^{c(i)}=(i_3,p^l)$ and $d(i)=[k_F(u_0^{i_1}):k_F]$ where u_0 is a generator of $k_E^\times$. When f is even $(q=p^f)$, put*

$$\pi(u,\theta)=\sum_{0\leq i<a(r)tp^l}\sum_{\substack{d|d(i)\\ w|w(i)\\ c<c(i)}}\beta_{d,w,c}\mathrm{ind}_{E^\times K_S^m}^G(\theta_i\cdot\psi_u)$$

where $\beta_{d,W,c}$ *is as in Proposition* 3.7 *and when* f *is odd, put*

$$\pi(u,\theta)=\sum_{0\leq i<a(r)tp^l}\sum_{\substack{d|d(i)\\ w|w(i)}}\left(\delta_{d,w}+\left(\frac{i_3/p^{c(i)}}{p}\right)\delta_{d,w,c}\right)\mathrm{ind}_{E^\times K_S^m}^G(\theta_i\cdot\psi_u)$$

where $\delta_{d,w},\delta_{d,w,c}$ *are as in Proposition* 3.7.

Then $\pi(u,\theta)$ *is an irreducible supercuspidal representation of* G *and every irreducible supercuspidal representation whose restriction on* K_S^m *contains* ψ_u *is written in the form* $\pi(u,\theta)$ *for some* $\theta\in E^\times$.

Proof. It follows from Theorem 1.8, Proposition 1.12 and Proposition 3.7.

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