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On the theory of Jacobi forms and Fourier-Jacobi coefficients of Eisenstein series

By

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Introduction

In [1], [2], Böcherer calculated the Fourier-Jacobi expansion of holomorphic Eisenstein series. It was shown that a Fourier-Jacobi coefficient of holomorphic Eisenstein series is a finite sum of products of a theta function and an Eisenstein series. The purpose of this paper is to develop the theory of the Fourier-Jacobi coefficients of Eisenstein series on some quasi-split classical groups. Unlike holomorphic case, a Fourier-Jacobi coefficient of nonholomorphic Eisenstein series is no longer a finite sum of products of a theta function and an Eisenstein series, but can be infinitely approximable by them.

Let k be a global field with char $(k)\neq 2$, and $\mathbf{A}$ be the adele ring of k. §1 is devoted to the theory of automorphic forms on Jacobi groups. A Jacobi group D is a semi-direct product of 2-step-nilpotent unipotent algebraic group V and an algebraic group H whose action on the center Z of V is trivial. For simplicity, we consider the following subgroups of $G=Sp_{m+n}$:

$$Z=\left\{\left(\begin{array}{c|c} \mathbf{1}_{m+n} & \begin{matrix} z & 0 \\ 0 & \mathbf{0}_n \end{matrix} \\ \hline \mathbf{0}_{m+n} & \mathbf{1}_{m+n} \end{array}\right) \middle| z\in \mathrm{Sym}_m(k)\right\},$$

$$V=\left\{\left(\begin{array}{c|c} \begin{matrix} \mathbf{1}_m & x \\ 0 & \mathbf{1}_n \end{matrix} & \begin{matrix} z & y \\ {}^t y & \mathbf{0}_n \end{matrix} \\ \hline \mathbf{0}_{m+n} & \begin{matrix} \mathbf{1}_m & 0 \\ -{}^t x & \mathbf{1}_n \end{matrix} \end{array}\right) \middle| x,\, y\in \mathrm{M}_{mn}(k),\ z-x\,{}^t y\in \mathrm{Sym}_m(k)\right\},$$

$$X=\left\{\left(\begin{array}{c|c} \begin{matrix} \mathbf{1}_m & x \\ 0 & \mathbf{1}_n \end{matrix} & \mathbf{0}_{m+m} \\ \hline \mathbf{0}_{m+n} & \begin{matrix} \mathbf{1}_m & 0 \\ -{}^t x & \mathbf{1}_n \end{matrix} \end{array}\right) \middle| x\in \mathrm{M}_{mn}(k)\right\},$$

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$$H=\left\{\left(\begin{array}{cc|cc} \mathbf{1}_m & 0 & \mathbf{0}_m & 0 \\ 0 & A & 0 & B \\ \hline \mathbf{0}_m & 0 & \mathbf{1}_m & 0 \\ 0 & C & 0 & D \end{array}\right) \middle| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_n \right\} \simeq Sp_n \ .$$

Then $D=VH$ is a Jacobi group. Let S be a non-degenerate symmetric matrix of size m. We regard S as a homomorphism $Z\to k$ by $z\mapsto \mathrm{tr}\,(Sz/2)$. Let ψ be a non-trivial additive character of $\mathbf{A}/k$, and put $\psi_S=\psi\circ S$. Let $\widetilde{G(\mathbf{A})}$ and $\widetilde{H(\mathbf{A})}$ be the metaplectic covering of $G(\mathbf{A})$ and $H(\mathbf{A})$, respectively, and $\widetilde{D(\mathbf{A})}$ be the semi-direct product $V(\mathbf{A})$ and $\widetilde{H(\mathbf{A})}$ of $H(\mathbf{A})$. Put $V_0=V/\mathrm{Ker}S$. Since $V_0(\mathbf{A})$ is a Heisenberg group, it has the Schrödinger representation which is realized on the Schwartz space $S(X(\mathbf{A}))$. It can be naturally extended to the Weil representation ω_S of $\widetilde{D(\mathbf{A})}$. Let $C_S^\infty(D(k)\backslash\widetilde{D(\mathbf{A})})$ be the space of C^∞-functions φ on $D(k)\backslash\widetilde{D(\mathbf{A})}$ such that $\varphi(zvh)=\psi_S(z)\,\varphi(vh)$ for any $z\in Z(\mathbf{A})$. For each $\phi\in S(X(\mathbf{A}))$, we define the theta function Θ^ϕ by:

$$\Theta^\phi(vh)=\sum_{l\in X(k)}\omega_S(vh)\,\phi(l) \ .$$

By the definition, $\Theta^\phi\in C_S^\infty(D(k)\backslash\widetilde{D(\mathbf{A})})$.

In §1, we shall show that any closed subspace W of $C_S^\infty(D(k)\backslash\widetilde{D(\mathbf{A})})$ invariant under the right translation of $V(\mathbf{A})$ is generated by functions of the form:

$$\Theta^{\phi_1}(vh)\int_{V(k)\backslash V(\mathbf{A})}\varphi(uh)\,\overline{\Theta^{\phi_2}(uh)}\,du \ .$$

Here $v\in V(\mathbf{A})$, $h\in\widetilde{H(\mathbf{A})}$, $\varphi\in W$, and ϕ_1, ϕ_2 are Schwartz function on $X(\mathbf{A})$. (Proposition 1.3).

In §3, we shall apply our theory to Fourier-Jacobi coefficients of Eisenstein series. Let ω be a unitary character of $\mathbf{A}^\times/k^\times$. Let $I(\omega,s)$ be the space of functions f on $G(\mathbf{A})$ such that

$$f(pg)=\omega(\det A)\,|\det A|^{s+\frac{m+n+1}{2}}f(g) \ ,$$

for any $g\in G(\mathbf{A})$, $p=\begin{pmatrix} A & B \\ \mathbf{0}_{m+n} & {}^tA^{-1}\end{pmatrix}\in P(\mathbf{A})$. Here,

$$P=\left\{\begin{pmatrix} A & B \\ \mathbf{0}_{m+n} & {}^tA^{-1}\end{pmatrix} \middle| A\in \mathrm{GL}_{m+n},\ A^{-1}B\in \mathrm{Sym}_{m+n}(k)\right\} \ .$$

We define an Eisenstein series $E(g;f)$ by

$$E(g;f)=\sum_{\gamma\in P\backslash G} f(\gamma g) \ ,$$

for $f\in I(\omega,s)$. Then Proposition 1.3 implies the space of S-th Fourier-Jacobi coefficients of $E(g;f)$, $f\in I(\omega,s)$ is generated by the products of a theta function and a function of the form:

$$(1) \qquad \int_{V(k)\backslash V(\mathbf{A})} E(uh; f)\, \Theta^{\phi}(uh)\, du \ .$$

We shall show that (1) is also an Eisenstein series of Siegel type associated to

$$R(h; f, \phi) = \int_{V(\mathbf{A})} f(w_{m+n} v w_n h)\, \overline{\omega_S(v w_n h)\, \phi(0)}\, dv \ .$$

(Theorem 3.2). Here

$$w_i = \begin{pmatrix} \mathbf{0}_i & \mathbf{1}_i \\ -\mathbf{1}_i & \mathbf{0}_i \end{pmatrix}$$

and we think of w_{m+n} (resp. w_n) as an element of G (resp. H). We remark that $R(h; f, \phi)$ is "genuine" when m is odd. Similar results also hold for "genuine" f.

Our theory has some applications to the calculation of the residues of Eisenstein series. The author will treat this problem in forthcoming paper.

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Notation

The space of $n \times n$ and $m \times n$ matrices over k is denoted by $M_n(k)$, and $M_{mn}(k)$, respectively. The space of $n \times n$ symmetric and alternative matrices are denoted by $\mathrm{Sym}_n(k)$ and $\mathrm{Alt}_n(k)$, respectively. The $n \times n$ zero and identity matrices are denoted by $\mathbf{0}_n$ and $\mathbf{1}_n$, respectively. If X is a square matrix, det X and tr X stand for its determinant and trace, respectively. For a function f on a group G and $x \in G$, we denote by $\rho(x)f$ the right translation of f by x , i.e., $\rho(x)f(g) = f(gx)$. When G is locally compact, the Schwartz-Bruhat space of G is denoted by $S(G)$. If G is an algebraic group defined over a field k, the group of k-valued points of G is denoted by $G(k)$ or G. If π is a representation of G, its contragredient is denoted by $\tilde{\pi}$. When k is a global field, the adele ring (resp. the idele group) of k is denoted by $\mathbf{A}_k$ or $\mathbf{A}$ (resp. $\mathbf{A}_k^\times$ or $\mathbf{A}^\times$). We fix a non-trivial additive character ψ of $\mathbf{A}/k$. The volume: $\mathbf{A} \to \mathbf{R}_+^\times$ is denoted by $|\ |$. For a unipotent algebraic group U, we normalize Haar measure du on $U(\mathbf{A})$ so that $\mathrm{Vol}(U(k)\backslash U(\mathbf{A})) = 1$.

§1. Representation theory of Jacobi groups

We shall recall the theory of metaplectic covering and Weil representation. Let Sp_n be the symplectic group of rank n defined over k:

$$Sp_n = \left\{ g \in \mathrm{GL}_{2n} \,\middle|\, g \begin{pmatrix} \mathbf{0}_n & -\mathbf{1}_n \\ \mathbf{1}_n & \mathbf{0}_n \end{pmatrix} {}^t g = \begin{pmatrix} \mathbf{0}_n & -\mathbf{1}_n \\ \mathbf{1}_n & \mathbf{0}_n \end{pmatrix} \right\}$$

$$=\left\{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \middle| A, B, C, D \in \mathrm{M}_n(k)\ , \right.$$

$$\left. A{}^tB=B{}^tA,\ C{}^tD=D{}^tC,\ A{}^tD-B{}^tC=\mathbf{1}_n \right\}\ ,$$

For each place v of k, we define 2-cocycle $c(g_1, g_2)$ on $Sp_n(k_v)$ with values in $\{\pm 1\}$ as in [9]. The metaplectic group $\widetilde{Sp_n(k_v)}$ is by definition the 2-fold covering group of $Sp_n(k_v)$ determined by $c(g_1, g_2)$: An element of $\widetilde{Sp_n(k_v)}$ is a pair (g, ζ), $g \in Sp_n(k_v)$, $\zeta \in \{\pm 1\}$, and the multiplication law is given by $(g_1, \zeta_1)(g_2, \zeta_2) = (g_1g_2, c(g_1, g_2)\zeta_1\zeta_2)$. The Weil representation ω_{ψ_v} of $\widetilde{Sp_n(k_v)}$ on $S(\mathrm{M}_{1n}(k_v))$ is characterized by the following equations:

$$\omega_{\psi_v}\left(\left(\begin{pmatrix} A & \mathbf{0}_n \\ \mathbf{0}_n & {}^tA^{-1} \end{pmatrix}, \zeta\right)\right)\Phi(X) = \zeta\frac{\gamma_v(1)}{\gamma_v(\det A)}|\det A|_v^{\frac{1}{2}}\Phi(XA)\ ,$$

$$\omega_{\psi_v}\left(\left(\begin{pmatrix} \mathbf{1}_n & B \\ \mathbf{0}_n & \mathbf{1}_n \end{pmatrix}, \zeta\right)\right)\Phi(X) = \zeta\psi_v\left(\frac{1}{2}(XB{}^tX)\right)\Phi(X)\ ,$$

$$\omega_{\psi_v}\left(\left(\begin{pmatrix} \mathbf{0}_n & -\mathbf{1}_n \\ \mathbf{1}_n & \mathbf{0}_n \end{pmatrix}, \zeta\right)\right)\Phi(X) = \zeta\gamma_v(1)^{-n}F\Phi(-X)\ ,$$

$\Phi \in S(\mathrm{M}_{1n}(k_v))$, $X \in \mathrm{M}_{1n}(k_v)$, $A \in \mathrm{GL}_n(k_v)$, $B \in \mathrm{Sym}_n(k_v)$. Here $F\Phi$ is the Fourier transform of Φ with respect to ψ_v:

$$F\Phi(X) = \int_{M_{1n}(k_v)} \Phi(Y)\psi_v(X\,{}^tY)dY\ .$$

Here the measure dY is the self-dual measure for the Fourier transform F. $\gamma_v(a)$ is the Weil constant associated to ψ_v. It is defined by the following equation:

$$\int_{k_v}\psi_v\left(\frac{1}{2}ax^2\right)\phi(x)dx = \gamma_v(a)|a|_v^{-\frac{1}{2}}\int_{k_v}\psi_v\left(-\frac{1}{2}a^{-1}x^2\right)\hat{\phi}(x)dx\ ,$$

$$\hat{\phi}(x) = \int_{k_v}\phi(y)\psi_v(xy)dy\ .$$

Here dx, dy are the self-dual measure for the Fourier transform. If $v<\infty$ and $v\nmid 2$, then there is a canonical splitting over the standard maximal compact subgroup K_v. The image of the splitting, which we also denote by K_v, is the stabilizer of the characteristic function of $\mathrm{M}_{1n}(\mathfrak{o}_v)$ for almost all v. The global metaplectic group $\widetilde{Sp_n(\mathbf{A})}$ is the restricted direct product of $\widetilde{Sp_n(k_v)}$ with respect to $\{K_v\}$ divided $\{(t_v) \in \bigoplus_v\{\pm 1\} \mid \prod_v t_v = 1\}$. Then the global Weil representation ω_ψ of $\widetilde{Sp_n(\mathbf{A})}$ on $S(\mathrm{M}_{1n}(\mathbf{A}))$ is well-defined. It is well-known that there is a unique splitting over $Sp_n(k)$, which we identify with $Sp_n(k)$. Since $c(g_1, g_2)$ is identically 1 on $(P_v \cap K_v) \times (P_v \cap K_v)$ for almost all v, the inverse image $\widetilde{P(\mathbf{A})}$ of $P(\mathbf{A})$ is identified with the covering group defined by the 2-cocycle $\prod_v c(g_1, g_2)$, $g_1, g_2 \in P(\mathbf{A})$. Then by (1.1) and (1.2),

$$\omega_\psi\left(\left(\begin{pmatrix} A & \mathbf{0}_n \\ \mathbf{0}_n & {}^tA^{-1} \end{pmatrix}, \zeta\right)\right)\Phi(X) = \zeta\frac{1}{\gamma(\det A)}\left|\det A\right|^{\frac{1}{2}}\Phi(XA) \ ,$$

$$\omega_\psi\left(\left(\begin{pmatrix} \mathbf{1}_n & B \\ \mathbf{0}_n & \mathbf{1}_n \end{pmatrix}, \zeta\right)\right)\Phi(X) = \zeta\psi\left(\frac{1}{2}XB{}^tX\right)\Phi(X) \ ,$$

$X\in \mathrm{M}_{1n}(\mathbf{A})$, $A\in \mathrm{GL}_n(\mathbf{A})$, $B\in \mathrm{Sym}_n(\mathbf{A})$, $\gamma(a)=\prod_v\gamma_v(a_v)$. Put

$$w_n=\begin{pmatrix} \mathbf{0}_n & \mathbf{1}_n \\ -\mathbf{1}_n & \mathbf{0}_n \end{pmatrix} .$$

Then

$$\omega_\psi(w_n)\,\Phi(X)=F\Phi(X) \ .$$

Now we define a Jacobi group and a non-degenerate homomorphism of its center.

Definition. *A Jacobi group D is a semi-direct of 2-step-nilpotent unipotent algebraic group V and an algebraic group H whose action on the center Z of V is trivial. A non-degenerate homomorphism $S:Z\to k$ is a homomorphism such that $V/\mathrm{Ker}(S)$ is a Heisenberg group with center $Z_0=Z/\mathrm{Ker}(S)$.*

Put $V_0=V/\mathrm{Ker}(S)$, and $D_0=D/\mathrm{Ker}(S)$. Z_0 can be identified with k via S. Since $V/Z=V_0/Z_0$ has a natural symplectic structure, the conjugate action gives a homomorphism $H\to Sp_{V/Z}$. Let $\widetilde{Sp_{V/Z}(\mathbf{A})}$ be the metaplectic cover of $Sp_{V/Z}(\mathbf{A})$. Let $\widetilde{H(\mathbf{A})}$ be the covering of $H(\mathbf{A})$ induced by $H(\mathbf{A})\to Sp_{V/Z}(\mathbf{A})$. Put $\widetilde{D(\mathbf{A})}=V(\mathbf{A})\widetilde{H(\mathbf{A})}$. Let J be the semidirect product of V and $Sp_{V/Z}$. Put $\widetilde{J(\mathbf{A})}=V(\mathbf{A})\widetilde{Sp_{V/Z}(\mathbf{A})}$. The homomorphisms $V\to V_0$ and $H\mapsto Sp_{V/Z}$ give homomorphisms $D\to J$ and $\widetilde{D(\mathbf{A})}\to\widetilde{J(\mathbf{A})}$. We denote these homomorphisms by ι.

We will consider representations of $\widetilde{D(\mathbf{A})}$ on which $Z(\mathbf{A})$ acts by ψ_S. Here $\psi_S=\psi\circ S$. Since V_0 is a Heisenberg group, V_0 has a coordinate system

$$V_0=\{v_0=(x,y,z)\,|\,x,y\in k^n,\ z\in k\} \ .$$

such that the composition law of V_0 is given by

$$(x_1,y_1,z_1)\cdot(x_2,y_2,z_2)=\left(x_1+x_2,\ y_1+y_2,\ z_1+z_2+\frac{(x_1{}^ty_2-x_2{}^ty_1)}{2}\right) \ .$$

We define subgroups X,Y of V_0 by

$$X=\{(x,y,z)\,|\,y=0,\ z=0\} \ ,$$
$$Y=\{(x,y,z)\,|\,x=0,\ z=0\} \ .$$

Then X and Y are maximal totally isotropic subspaces of V/Z complementary to each other. $Sp_{V/Z}=Sp_n$ acts on V_0 from the right by

$$(x,y,z)\begin{pmatrix} A & B \\ C & D \end{pmatrix}=(xA+yC,\ xB+yD,\ z) \ .$$

The Schrödinger representation ω_ψ of $V_0(\mathbf{A})$ on $S(X(\mathbf{A}))$ is given by

$$\omega_\psi(v)\phi(t)=\phi(t+x)\psi\Big(z+t^ty+\frac{1}{2}x^ty\Big) \ ,$$

for $v=(x, y, z)\in V_0(\mathbf{A})$, and $\phi\in S(X(\mathbf{A}))$. By Stone von-Neumann theorem, ω_ψ is the unique irreducible representation of $V_0(\mathbf{A})$ on which $Z_0(\mathbf{A})$ acts by ψ, i.e., the unique irreducible representation of $V(\mathbf{A})$ on which $Z(\mathbf{A})$ acts by ψ_S. The Schrödinger representation of $V_0(\mathbf{A})$ extends to the representation of $\widetilde{J(\mathbf{A})}$, the Weil representation ω_ψ. The restriction of ω_ψ to $\widetilde{Sp_n(\mathbf{A})}$ is exactly what we have described before.

For each $\phi\in S(X(\mathbf{A}))$, the theta function $\Theta^\phi(vh)$ is given by

$$\begin{aligned}\Theta^\phi(vh)&=\sum_{l\in X(k)}\omega_\psi(vh)\phi(l)\\&=\sum_{l\in X(k)}\omega_\psi(h)\phi(l+x)\psi\Big(z+l^ty+\frac{1}{2}x^ty\Big) \ ,\end{aligned}$$

for $v\in V_0(\mathbf{A})$, $h\in\widetilde{Sp_n(\mathbf{A})}$.

Let $C^\infty_\psi(V_0(k)\backslash V_0(\mathbf{A}))$ be the space of smooth functions f on $V_0(k)\backslash V_0(\mathbf{A})$ such that $f(zv)=\psi(z)f(v)$ for $z\in Z(\mathbf{A})$ with C^∞-topology. Then the homomorphism

$$\theta\colon S(X(\mathbf{A}))\to C^\infty_\psi(V_0(k)\backslash V_0(\mathbf{A}))$$

given by $\phi\mapsto\Theta^\phi$ is a topological isomorphism.

There is a $\widetilde{J(\mathbf{A})}$ invariant non-degenerate Hermitian inner product on $S(X(\mathbf{A}))$ given by

$$(\phi_1, \phi_2)=\int_{X(\mathbf{A})}\phi_1(t)\overline{\phi_2(t)}dt \ .$$

It is easy to see that

$$(\phi_1, \phi_2)=\int_{Z_0(\mathbf{A})V_0(k)\backslash V_0(\mathbf{A})}\Theta^{\phi_1}(v)\overline{\Theta^{\phi_2}(v)}dv \ .$$

In particular, the contragredient of ω_ψ is $\omega_{\psi^{-1}}=\overline{\omega_\psi}$.

Let $S_\psi(V_0(\mathbf{A}))$ be the space of smooth functions φ on $V_0(\mathbf{A})$ which satisfy 1) and 2):

1) $\varphi(zv)=\psi^{-1}(z)\varphi(v)$.

2) $|\varphi|$ is rapidly decreasing on $Z_0(\mathbf{A})\backslash V_0(\mathbf{A})$.

$S_\psi(V_0(\mathbf{A}))$ is isomorphic to $S((X\oplus Y)(\mathbf{A}))$. We put topology into $S_\psi(V_0(\mathbf{A}))$ by this isomorphism. Let (σ, W) be a representation of $V_0(\mathbf{A})$ on which $Z_0(\mathbf{A})$ acts by ψ. We say that the representation σ extends to $S_\psi(V_0(\mathbf{A}))$ if the following integral

$$\sigma(\varphi)w=\int_{Z_0(\mathbf{A})\backslash V_0(\mathbf{A})}\varphi(v)\sigma(v)w\,dv$$

defines separately continuous map $S_\psi(V_0(\mathbf{A}))\times W\to W$. It is known that the

Schrödinger representation ω_ψ extends to $S_\psi(V_0(\mathbf{A}))$, and $\omega_\psi(\varphi)=0$ if and only if $\varphi=0$. It is easy to see that if we put

$$\varphi(v)=\int_{X(\mathbf{A})}\phi_1\Big(t-\frac{x}{2}\Big)\overline{\phi_2\Big(t+\frac{x}{2}\Big)}\psi(-z-t^ty)\,dt$$

for $\phi_1,\ \phi_2\in S(X(\mathbf{A}))$, then $\varphi\in S_\psi(V_0(\mathbf{A}))$, and

$$\omega_\psi(\varphi)\phi=(\phi,\ \phi_2)\cdot\phi_1\ .$$

Moreover functions of this form generate a dense subspace of $S_\psi(V_0(\mathbf{A}))$.

Lemma 1.1. *Let ϕ_1, ϕ_2, and φ as above. Then*

$$\sum_{l\in Z_0(k)/V_0(k)}\varphi(h^{-1}v^{-1}luh)=\Theta^{\phi_1}(vh)\,\overline{\Theta^{\phi_2}(uh)}\ ,$$

for $h\in\widetilde{Sp_n(\mathbf{A})}$ $u,\ v\in V_0(\mathbf{A})$.

Proof. As a function of u, both sides are elements of $C^\infty_{\psi^{-1}}(V_0(k)\backslash V_0(\mathbf{A}))$. For any $\phi\in S(V_0(\mathbf{A}))$,

$$\int_{Z_0(\mathbf{A})V_0(k)\backslash V_0(\mathbf{A})}\sum_{l\in Z_0(k)\backslash V_0(k)}\varphi(h^{-1}v^{-1}luh)\,\Theta^\phi(uh)\,du$$

$$=\int_{Z_0(\mathbf{A})\backslash V_0(\mathbf{A})}\varphi(h^{-1}v^{-1}uh)\,\Theta^\phi(uh)\,du$$

$$=\int_{Z_0(\mathbf{A})\backslash V_0(\mathbf{A})}\varphi(u)\,\Theta^\phi(vhu)\,du$$

$$=\Theta^{\omega_\psi(\varphi)\phi}(vh)$$

$$=(\phi,\ \phi_2)\,\Theta^{\phi_1}(vh)\ .$$

Since the pairing (,) is non-degenerate, the lemma follows.

Lemma 1.2. *Let $(\sigma,\ W)$ be a representation of $V_0(\mathbf{A})$ on which $Z_0(\mathbf{A})$ acts by ψ. Assume that (σ,W) extends to the represention of $S_\psi(V_0(\mathbf{A}))$. Then $\sigma(S_\psi(V_0(\mathbf{A})))W$ is dense in W.*

Proof. Let $\tilde{w}$ be a linear functional on W such that $<\sigma(\varphi)w,\ \tilde{w}>=0$, for any $\varphi\in S_\psi(V_0(\mathbf{A}))$ and any $w\in W$. Then for any $v_1=(x_1,\ y_1,\ z_1)\in V_0(\mathbf{A})$,

$$<\sigma(v_1)\sigma(\varphi)\sigma(v_1^{-1})w,\ \tilde{w}>=\int_{Z_0(\mathbf{A})\backslash V_0(\mathbf{A})}\varphi(v)<\sigma(v_1vv_1^{-1})w,\ \tilde{w}>dv$$

$$=\int_{V_0(\mathbf{A})}<\sigma(v)w,\ \tilde{w}>\varphi(v)\,\psi(x_1y-xy_1)\,dv\ .$$

If $\mathrm{Supp}(\varphi)$ is compact mod $Z_0(\mathbf{A})$, then this integral is absolutely convergent and equal to the Fourier transform of $<\sigma(v)w,\ \tilde{w}>\varphi(v)$. Therefore $<\sigma(v)w,\ \tilde{w}>$ must be identically zero.

Let $C_S^\infty(D(k)\backslash\widetilde{D(\mathbf{A})})$ be the space of functions f on $D(k)\backslash\widetilde{D(\mathbf{A})}$ such that $f(zvh)=\psi_S(z)f(vh)$ for any $z\in Z(\mathbf{A})$, $v\in V(\mathbf{A})$, $h\in\widetilde{H(\mathbf{A})}$. We regard theta functions as elements of $C_S^\infty(D(k)\backslash\widetilde{D(\mathbf{A})})$ by the embedding ι.

Proposition 1.3. *Let W be a closed subspace of $C_S^\infty(D(k)\backslash\widetilde{D(\mathbf{A})})$ invariant under the right translation of $V(\mathbf{A})$. Then functions of the following form generate a dense subspace of W.*

$$\Theta^{\phi_1}(vh)\int_{V(k)\backslash V(\mathbf{A})}f(uh)\overline{\Theta^{\phi_2}(uh)}\,du\ ,$$

Here $v\in V(\mathbf{A})$, $h\in\widetilde{H(\mathbf{A})}$, $f\in W$, $\phi_1,\phi_2\in S(X(\mathbf{A}))$.

Proof. We regard W as a representation of $V(\mathbf{A})$ by the right translation ρ. Let φ be as in Lemma 1.1. Then

$$\begin{aligned}\rho(\varphi)f(vh)&=\int_{Z(\mathbf{A})\backslash V(\mathbf{A})}\varphi(u)f(uhu)\,du\\&=\int_{Z(\mathbf{A})\backslash V(\mathbf{A})}\varphi(h^{-1}v^{-1}uh)f(uh)\,du\\&=\int_{Z(\mathbf{A})V(k)\backslash V(\mathbf{A})}\sum_{l\in Z(k)\backslash V(k)}\varphi(h^{-1}v^{-1}luh)f(uh)\,du\ .\end{aligned}$$

Since the last integral is absolutely convergent, the assumption of Lemma 1.2 is satisfied. Therefore the proposition follows by Lemma 1.2.

Remark. *Stone von-Neumann theorem implies that any representation π of $\widetilde{D(\mathbf{A})}$ on which $Z(\mathbf{A})$ acts by ψ_S is essentially a tensor product:*

$(\omega_\phi\circ\iota)\otimes\tau$.

Here ω_ϕ is the Weil representation of $\widetilde{J(\mathbf{A})}$ and τ is a representation of $\widetilde{H(\mathbf{A})}$. (cf. Piatetski-Shapiro, [11]) Proposition 1.3 means that when π is realized in $C_S^\infty(D(k)\backslash\widetilde{D(\mathbf{A})})$, τ is the space generated by functions on $H(k)\backslash\widetilde{H(\mathbf{A})}$ of the form:

$$\int_{V_0(k)\backslash V_0(\mathbf{A})}f(uh)\overline{\Theta^{\phi}(uh)}\,du\ ,$$

$h\in\widetilde{H(\mathbf{A})}$, $f\in W$, $\phi\in S(X(\mathbf{A}))$.

§2. Eisenstein series of Siegel type

In this section, we consider Eisenstein series of Siegel type. Let m,n be positive integers. Although our theory works for other groups considered in [16], we will confine ourselves to the following situations for the sake of simplicity.

(**Case 1**): Symplectic or metaplectic case:

$$G=Sp_{m+n}=\left\{g\in\mathrm{GL}_{2m+2n}\;\middle|\;g\begin{pmatrix}\mathbf{0}_{m+n}&\mathbf{1}_{m+n}\\-\mathbf{1}_{m+n}&\mathbf{0}_{m+n}\end{pmatrix}{}^tg=\begin{pmatrix}\mathbf{0}_{m+n}&\mathbf{1}_{m+n}\\-\mathbf{1}_{m+n}&\mathbf{0}_{m+n}\end{pmatrix}\right\}$$

$$=\left\{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \,\middle|\, \begin{array}{l} A, B, C, D \in \mathrm{M}_{m+n}(k), \\ A{}^tB = B{}^tA,\ C{}^tD = D{}^tC,\ A{}^tD - B{}^tC = \mathbf{1}_{m+n} \end{array} \right\},$$

$$P=\left\{\begin{pmatrix} A & B \\ \mathbf{0}_{m+n} & {}^tA^{-1} \end{pmatrix} \,\middle|\, A \in \mathrm{GL}_{m+n},\ A^{-1}B \in \mathrm{Sym}_{m+n}(k) \right\},$$

$$Z=\left\{\left(\begin{array}{c|c} \mathbf{1}_{m+n} & \begin{matrix} z & 0 \\ 0 & \mathbf{0}_n \end{matrix} \\ \hline \mathbf{0}_{m+n} & \mathbf{1}_{m+n} \end{array}\right) \,\middle|\, z \in \mathrm{Sym}_m(k) \right\},$$

$$V=\left\{\left(\begin{array}{c|c} \begin{matrix} \mathbf{1}_m & x \\ 0 & \mathbf{1}_n \end{matrix} & \begin{matrix} z & y \\ {}^ty & \mathbf{0}_n \end{matrix} \\ \hline \mathbf{0}_{m+n} & \begin{matrix} \mathbf{1}_m & 0 \\ -{}^tx & \mathbf{1}_n \end{matrix} \end{array}\right) \,\middle|\, x, y \in \mathrm{M}_{mn}(k),\ z - x{}^ty \in \mathrm{Sym}_m(k) \right\},$$

$$X=\left\{\left(\begin{array}{c|c} \begin{matrix} \mathbf{1}_m & x \\ 0 & \mathbf{1}_n \end{matrix} & \mathbf{0}_{m+m} \\ \hline \mathbf{0}_{m+n} & \begin{matrix} \mathbf{1}_m & 0 \\ -{}^tx & \mathbf{1}_n \end{matrix} \end{array}\right) \,\middle|\, x \in \mathrm{M}_{mn}(k) \right\},$$

$$Y=\left\{\left(\begin{array}{c|c} \mathbf{1}_{m+n} & \begin{matrix} \mathbf{0}_m & y \\ {}^ty & \mathbf{0}_n \end{matrix} \\ \hline \mathbf{0}_{m+n} & \mathbf{1}_{m+n} \end{array}\right) \,\middle|\, y \in \mathrm{M}_{mn}(k) \right\},$$

$$H=\left\{\left(\begin{array}{c|c} \begin{matrix} \mathbf{1}_m & 0 \\ 0 & A \end{matrix} & \begin{matrix} \mathbf{0}_m & 0 \\ 0 & B \end{matrix} \\ \hline \begin{matrix} \mathbf{0}_m & 0 \\ 0 & C \end{matrix} & \begin{matrix} \mathbf{1}_m & 0 \\ 0 & D \end{matrix} \end{array}\right) \,\middle|\, \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_n \right\} \simeq Sp_n .$$

Z can be identified with $\mathrm{Sym}_m(k)$. Any homomorphism $Z \rightarrow k$ is of the form: $z \mapsto \mathrm{tr}(zS/2)$, for some $S \in \mathrm{Sym}_m(k)$. We denote this homomorphism by S, too. One can easily check that $V_0 = V/\mathrm{Ker}(S)$ is a Heisenberg group if and only if $\det S \neq 0$.

(**Case 2**) Unitary case:

Let K be a quadratic extension of k, or $k\oplus k$. Let σ be the non-trivial automorphism of K/k if K/k is a quadratic extension and $(x, y)^{\sigma}=(y, x)$ if $K=k\oplus k$. We fix an element η such that $\eta^{\sigma}=-\eta$. For a matrix A, we denote $A^{*}={}^{t}A^{\sigma}$. We denote the space of Hermitian (resp. skew-Hermitian) matrices of size n by $\mathrm{Her}_{n}(K)$ (resp. $\mathrm{SH}_{n}(K)$).

$$G=SU(m+n, m+n)$$
$$=\left\{g\in\mathrm{SL}_{2m+2n}(K)\ \middle|\ g\begin{pmatrix}\mathbf{0}_{m+n} & \mathbf{1}_{m+n}\\ -\mathbf{1}_{m+n} & \mathbf{0}_{m+n}\end{pmatrix}g^{*}=\begin{pmatrix}\mathbf{0}_{m+n} & \mathbf{1}_{m+n}\\ -\mathbf{1}_{m+n} & \mathbf{0}_{m+n}\end{pmatrix}\right\}$$
$$=\left\{g=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\ \middle|\ \det g=1,\ A, B, C, D\in\mathrm{M}_{m+n}(K),\right.$$
$$\left. AB^{*}=BA^{*},\ CD^{*}=DC^{*},\ AD^{*}-BC^{*}=\mathbf{1}_{m+n}\right\},$$

$$P=\left\{\begin{pmatrix}A & B\\ \mathbf{0}_{m+n} & (A^{*})^{-1}\end{pmatrix}\ \middle|\ \det A\in k,\ A^{-1}B\in\mathrm{Her}_{m+n}(K)\right\},$$

$$Z=\left\{\left(\begin{array}{c|c}\mathbf{1}_{m+n} & \begin{matrix}z & 0\\ 0 & \mathbf{0}_{n}\end{matrix}\\ \hline \mathbf{0}_{m+n} & \mathbf{1}_{m+n}\end{array}\right)\ \middle|\ z\in\mathrm{Her}_{m}(K)\right\},$$

$$V=\left\{\left(\begin{array}{c|c}\begin{matrix}\mathbf{1}_{m} & x\\ 0 & \mathbf{1}_{n}\end{matrix} & \begin{matrix}z & y\\ y^{*} & \mathbf{0}_{n}\end{matrix}\\ \hline \mathbf{0}_{m+n} & \begin{matrix}\mathbf{1}_{m} & 0\\ -x^{*} & \mathbf{1}_{n}\end{matrix}\end{array}\right)\ \middle|\ x, y\in\mathrm{M}_{mn}(K),\ z-xy^{*}\in\mathrm{Her}_{m}(K)\right\},$$

$$X=\left\{\left(\begin{array}{c|c}\begin{matrix}\mathbf{1}_{m} & x\\ 0 & \mathbf{1}_{n}\end{matrix} & \mathbf{0}_{m+m}\\ \hline \mathbf{0}_{m+n} & \begin{matrix}\mathbf{1}_{m} & 0\\ -x^{*} & \mathbf{1}_{n}\end{matrix}\end{array}\right)\ \middle|\ x\in\mathrm{M}_{mn}(K)\right\},$$

$$Y=\left\{\left(\begin{array}{c|c}\mathbf{1}_{m+n} & \begin{matrix}\mathbf{0}_{m} & y\\ y^{*} & \mathbf{0}_{n}\end{matrix}\\ \hline \mathbf{0}_{m+n} & \mathbf{1}_{m+n}\end{array}\right)\ \middle|\ y\in\mathrm{M}_{mn}(K)\right\},$$

$$H=\left\{\left(\begin{array}{cc|cc} \mathbf{1}_m & 0 & \mathbf{0}_m & 0 \\ 0 & A & 0 & B \\ \hline \mathbf{0}_m & 0 & \mathbf{1}_m & 0 \\ 0 & C & 0 & D \end{array}\right) \middle| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SU(n,n)\right\} \simeq SU(n,n) \ .$$

Z can be identified with $\mathrm{Her}_m(K)$. Any homomorphism $Z \rightarrow k$ is of the form: $z \mapsto \mathrm{tr}(zS)$, for some $S \in \mathrm{Her}_m(K)$. We denote this homomorphism by S, too. As in case 1, one can easily check that $V_0 = V/\mathrm{Ker}(S)$ is a Heisenberg group if and only if $\det S \neq 0$.

Let ω be a unitary quasi-character of $\mathbf{A}^\times/k^\times$. Let $s \in \mathbf{C}$. Let $I(\omega, s) = I_G(\omega, s)$ be the space of functions f on $G(\mathbf{A})$ such that

$$f(pq) = \omega(\det A)|\det A|^{s+\rho} f(g) \ ,$$

for any $g \in G(\mathbf{A})$, $p = \begin{pmatrix} A & B \\ \mathbf{0}_{m+n} & (A^*)^{-1} \end{pmatrix} \in P(\mathbf{A})$. Here $\rho = \dfrac{m+n+1}{2}$, or $m+n$, according as $G = Sp_{m+n}$, or $SU(m+n, m+n)$. We also assume f is right finite by standard maximal compact subgroup of $G(\mathbf{A})$.

For (Case 1), we define $I(\omega, s)^\sim = I_G(\omega, s)^\sim$ by the space of functions f on $\widetilde{G(\mathbf{A})} = \widetilde{Sp_{m+n}}(\mathbf{A})$ such that

$$f(pg) = \varepsilon \frac{\gamma(1)}{\gamma(\det A)} \omega(\det A) \left|\det A\right|^{s+\frac{m+n+1}{2}} f(g) \ ,$$

for any $g \in G(\mathbf{A})$, $p = \left(\begin{pmatrix} A & B \\ \mathbf{0}_{m+n} & {}^tA^{-1} \end{pmatrix}, \varepsilon\right) \in \widetilde{P(\mathbf{A})}$, where $\widetilde{P(\mathbf{A})}$ is the inverse image of $P(\mathbf{A})$ in $\widetilde{Sp_{m+n}}(\mathbf{A})$.

We define an Eisenstein series $E(g; f)$ of type (ω, s) (resp. $(\omega, s)^\sim$) by

$$E(g; f) = \sum_{\gamma \in P \backslash G} f(\gamma g) \ ,$$

$f \in I_G(\omega, s)$ (resp. $I_G(\omega, s)^\sim$). This series converges for $\mathrm{Re}(s) \gg 0$, and can be meromorphically continued to whole s-plane if f depends on s holomorphically.

The Weil representation ω_S of $\widetilde{H(\mathbf{A})}$ on $S(X(\mathbf{A}))$ is given as follows:

(**Case 1**): In this case $\widetilde{H(\mathbf{A})}$ is canonically isomorphic to either $\widetilde{Sp_n}(\mathbf{A})$ or $Sp_n(\mathbf{A}) \times \{\pm\}$, according as m is odd or even. We regard ω_S as a representation of $\widetilde{Sp_n}(\mathbf{A})$ by the canonical homomorphism $\widetilde{Sp_n}(\mathbf{A}) \rightarrow \widetilde{H(\mathbf{A})}$. Then we have:

$$\omega_S\left(\left(\begin{pmatrix} A & \mathbf{0}_n \\ \mathbf{0}_n & {}^tA^{-1} \end{pmatrix}, \varepsilon\right)\right)\phi(X) = \varepsilon^m \frac{\gamma_S(1)}{\gamma_S(\det A)} \left|\det A\right|^{\frac{m}{2}} \phi(XA) \ ,$$

$$\omega_S\left(\left(\begin{pmatrix}\mathbf{1}_n & B\\ \mathbf{0}_n & \mathbf{1}_n\end{pmatrix}, \varepsilon\right)\right)\phi(X)=\varepsilon^m\psi_S(XB^tX)\phi(X) ,$$

$$\omega_S(w_n)\phi(X)=F_S\phi(X) ,$$

$$F_S\phi(X)=\int_{X(\mathbf{A})}\phi(Y)\psi(\mathrm{tr}SX^tY)dY .$$

Here $\gamma_S(a)$ is the Weil constant with respect to S. If S is equivalent to $\mathrm{diag}(s_1, s_2,\ldots, s_m)$, then $\gamma_S(a)=\prod_{i=1}^m\gamma(s_ia)$.

(**Case 2**): In this case $\widetilde{H(\mathbf{A})}$ is canonically isomorphic to $H(\mathbf{A})\times\{\pm1\}$. We regard $H(\mathbf{A})$ as a subgroup of $\widetilde{H(\mathbf{A})}$. Then we have

$$\omega_S\left(\begin{pmatrix}A & \mathbf{0}_n\\ \mathbf{0}_n & (A^*)^{-1}\end{pmatrix}\right)\phi(X)=\chi_{K/k}(\det A^m)|\det A|^m\phi(X,A) ,$$

$$\omega_S\left(\begin{pmatrix}\mathbf{1}_n & B\\ \mathbf{0}_n & \mathbf{1}_n\end{pmatrix}\right)\phi(X)=\psi_S(XBX^*)\phi(X) ,$$

$$\omega_S(w_n)\phi(X)=F_S\phi(X) ,$$

$$F_S\phi(X)=\int_{X(\mathbf{A})}\phi(Y)\psi(\mathrm{Tr}_{K/k}\mathrm{tr}(SXY^*))dY .$$

Here $\chi_{K/k}(a)$ is the character of $\mathbf{A}^\times/k^\times$ corresponding to K/k by the class field theory.

§3. Fourier Jacobi coefficients of Eisenstein series

Definition. *Let φ be a C^∞-function on $G(K)\backslash G(\mathbf{A})$. The S-th Fourier-Jacobi coefficient φ_S of φ is a function on $D(k)\backslash\widetilde{D(\mathbf{A})}$ given by*

$$\varphi_S(vh)=\int_{Z(k)\backslash Z(\mathbf{A})}\varphi(zvh)\psi_S^{-1}(z)dz ,$$

$v\in V(\mathbf{A})$, $h\in\widetilde{H(\mathbf{A})}$. *Obviously $\varphi_S\in C_S^\infty(D(k)\backslash\widetilde{D(\mathbf{A})})$.*

As is shown in §1, the representation of $\widetilde{D(\mathbf{A})}$ generated by the Fourier-Jacobi coefficients of Eisenstein series are generated by functions of the form:

$$\Theta^{\phi_1}(vh)\int_{V(k)\backslash V(\mathbf{A})}E_S(uh;f)\overline{\Theta^{\phi_2}(uh)}du , \tag{3.1}$$

where $v\in V(\mathbf{A})$, $h\in\widetilde{H(\mathbf{A})}$, $\phi_1, \phi_2\in S(X(\mathbf{A}))$.

We consider the functions of the form;

(3.2) $$\int_{V(k)V(\mathbf{A})} E_S(vh;f)\,\overline{\Theta^{\phi}(vh)}\,dv \ ,$$

where $\phi\in S(X(\mathbf{A}))$.

Let Q be the normalizer of V in G. The double coset $P\backslash G/Q$ is naturally bijective to the Weyl coset $W_P\backslash W_G/W_Q$. By Casselman [3], each double coset of $W_P\backslash W_G/W_Q$ has unique element of minimal length. From this, one can easily check that a complete set of representatives of $W_P\backslash W_G/W_Q$ is given by

$$\xi_i=\left(\begin{array}{cc|cc} \mathbf{0}_{m-i} & 0 & -\mathbf{1}_{m-i} & 0 \\ 0 & \mathbf{1}_{n+i} & 0 & \mathbf{0}_{n+i} \\ \hline \mathbf{1}_{m-i} & 0 & \mathbf{0}_{m-i} & 0 \\ 0 & \mathbf{0}_{n+1} & 0 & \mathbf{1}_{n+1} \end{array}\right),$$

$i=0, 1\cdots, m$. Note that $P\xi_0 Q$ is the unique open cell.

Lemma 3.1. *If $\gamma\in G$ is not contained in the open cell $P\xi_0Q$, then S is not trivial on $\gamma^{-1}P\gamma\cap Z$.*

Proof. We may assume $\gamma=\xi_i$, $i>0$, since for any $q\in Q$, q normalizes Z and qSq^{-1} is also non-degenerate. It is easily seen that $\gamma^{-1}P_\gamma\cap Z$ contains the subgroup of consisting of the last column and row.

Let $<, >$ be the Hilbert symbol on $\mathbf{A}^\times\times\mathbf{A}^\times$. Put $\chi_a(x)=<a, x>$.

Theorem 3.2. *Let $f\in I(\omega, s)$ or $I(\omega, s)^\sim$. If $\phi\in S(X(\mathbf{A}))$ is right finite by the action of the standard maximal compact subgroup of $\widetilde{H(\mathbf{A})}$, and* $\mathrm{Re}(s)\gg 0$, *then (3.2) is an Eisenstein series associated to:*

$$R(h;f,\phi)=\int_{V(\mathbf{A})} f(w_{m+n}vw_nh)\,\overline{\omega_S(vw_nh)\,\phi(0)}\,dv \ .$$

The type of $R(h;f,\phi)$ is as follows.

(3.3) $$\begin{cases} I_H(\omega\chi_a, s),\ a=(-1)^{\frac{m}{2}}\det S\ , & \text{If } G=Sp_{m+n},\ 2|m,\ f\in I_G(\omega, s) \\ I_H(\omega\chi_a, s)^\sim,\ a=(-1)^{\frac{m}{2}}\det S\ , & \text{If } G=Sp_{m+n},\ 2|m,\ f\in I_G(\omega, s)^\sim \\ I_H(\omega\chi_a, s)^\sim,\ a=(-1)^{\frac{m+1}{2}}\det S\ , & \text{If } G=Sp_{m+n},\ 2\nmid m,\ f\in I_G(\omega, s) \\ I_H(\omega\chi_a, s),\ a=(-1)^{\frac{m-1}{2}}\det S\ , & \text{If } G=Sp_{m+n},\ 2\nmid m,\ f\in I_G(\omega, s)^\sim \\ I_H(\omega\chi^m{}_{K/k}, s)\ , & \text{If } G=SU(m+n, m+n)\ . \end{cases}$$

Proof. We may assume (3.2) is absolutely convergent. We treat only (Case 1), and $f\in I(\omega, s)$. The proof for the remaining cases are similar.

We break up the coset $P\backslash G$ into the following disjoint union:

$$P\backslash G=\bigcup_{i>0}(P\backslash P\xi_i Q)\bigcup(P\backslash P\xi_0 Q)\ .$$

By Lemma 3.1,

$$\int_{V(k)\backslash V(\mathbf{A})}E_S(vh;f)\overline{\Theta^\phi(vh)}dv$$

$$=\int_{V(k)\backslash V(\mathbf{A})}E(vh;f)\overline{\Theta^\phi(vh)}dv$$

$$=\sum_{i>0}\sum_{\gamma\in P\backslash P\xi_i Q}\int_{V(k)\backslash V(\mathbf{A})}f(\gamma vh)\overline{\Theta^\phi(vh)}dv$$

$$+\sum_{\gamma\in P\backslash P\xi_0 Q}\int_{V(k)\backslash V(\mathbf{A})}f(\gamma vh)\overline{\Theta^\phi(vh)}dv\ .$$

By Lemma 3.1, the first integral vanishes. One can easily check

$$P\backslash P\xi_0 Q=\xi_0\cdot(Y\backslash V)\cdot(P_H\backslash H)\ .$$

Here

$$P_H=\left\{\begin{pmatrix}A & * \\ \mathbf{0}_n & {}^tA^{-1}\end{pmatrix}\middle|\, A\in\mathrm{GL}_n\right\}\ ,$$

Since each $\gamma\in H$ normalizes $V(k)$ and $V(\mathbf{A})$,

$$\sum_{\gamma\in P\backslash P\xi_0 Q}\int_{V(k)\backslash V(\mathbf{A})}f(\gamma vh)\overline{\Theta^\phi(vh)}dv$$

$$=\sum_{\gamma_1\in Y\backslash V}\sum_{\gamma\in P_H\backslash H}\int_{V(k)\backslash V(\mathbf{A})}f(\xi_0\gamma_1\gamma vh)\overline{\Theta^\phi(vh)}dv$$

$$=\sum_{\gamma_1\in Y\backslash V}\sum_{\gamma\in P_H\backslash H}\int_{V(k)\backslash V(\mathbf{A})}f(\xi_0\gamma_1 v\gamma h)\overline{\Theta^\phi(v\gamma h)}dv$$

$$=\sum_{\gamma\in P_H\backslash H}\int_{Y(k)\backslash V(\mathbf{A})}f(\xi_0 v\gamma h)\overline{\Theta^\phi(v\gamma h)}dv$$

$$=\sum_{\gamma\in P_H\backslash H}\int_{Y(k)\backslash V(\mathbf{A})}f(\xi_0 v\gamma h)\sum_{l\in Y(k)}\overline{F(\omega_S(lv\gamma h)\phi(0))}dv$$

$$=\sum_{\gamma\in P_H\backslash H}\int_{V(\mathbf{A})}f(\xi_0 v\gamma h)\overline{F(\omega_S(v\gamma h)\phi(0))}dv$$

$$=\sum_{\gamma\in P_H\backslash H}\int_{V(\mathbf{A})}f(\xi_0 v\gamma h)\overline{\omega_S(w_n v\gamma h)\phi(0)}dv$$

$$=\sum_{\gamma\in P_H\backslash H}\int_{V(\mathbf{A})}f(w_{m+n}vw_n\gamma h)\overline{\omega_S(vw_n\gamma h)\phi(0)}dv\ .$$

Put

$$R(h;f,\phi)=\int_{V(\mathbf{A})}f(w_{m+n}vw_nh)\overline{\omega_S(vw_nh)\phi(0)}\,dv\ .$$

The convergence of $R(h;f,\phi)$ will be discussed later. We have shown that

$$(3.1)=\sum_{\gamma\in P_H\backslash H}R(\gamma h;f,\phi)\ .$$

We have to prove $R(h;f,\phi)\in I(\omega,s)^{\sim}$.

$$R(h;f,\phi)=\int_{V(\mathbf{A})}f(w_{m+n}vw_nh)\overline{\omega_S(vw_nh)\phi(0)}\,dv$$

$$=\int_{X(\mathbf{A})}\int_{Y(\mathbf{A})}\int_{Z(\mathbf{A})}f\left(w_{m+n}\left(\begin{array}{cc|cc}\mathbf{1}_m & x & z-x{}^ty & y\\ 0 & \mathbf{1}_n & {}^ty & \mathbf{0}_n\\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{$\mathbf{0}_{m+n}$}} & \mathbf{1}_m & 0\\ & & -{}^tx & \mathbf{1}_n\end{array}\right)w_nh\right)$$

$$\times\overline{\omega_S(w_nh)\phi(x)\psi(\mathrm{tr}(S(z+x{}^ty)/2))}\,dzdydx$$

$$=\int_{X(\mathbf{A})}\int_{Y(\mathbf{A})}\int_{Z(\mathbf{A})}f\left(w_{m+n}\left(\begin{array}{c|cc}\mathbf{1}_{m+n} & z & y\\ & {}^ty & \mathbf{0}_n\\ \hline \mathbf{0}_{m+n} & \multicolumn{2}{c}{\mathbf{1}_{m+n}}\end{array}\right)w_nh\right)$$

$$\times\overline{\omega_S(w_nh)\phi(x)\psi(\mathrm{tr}(S(z+2x{}^ty)/2))}\,dzdydx$$

$$=\int_{Y(\mathbf{A})}\int_{Z(\mathbf{A})}f\left(w_{m+n}\left(\begin{array}{c|cc}\mathbf{1}_{m+n} & z & y\\ & {}^ty & \mathbf{0}_n\\ \hline \mathbf{0}_{m+n} & \multicolumn{2}{c}{\mathbf{1}_{m+n}}\end{array}\right)w_nh\right)$$

$$\times\overline{F_S(\omega_S(w_nh)\phi)\,(y)\,\psi_S(z)}\,dzdy$$

$$=\int_{Y(\mathbf{A})}\int_{Z(\mathbf{A})}f\left(w_{m+n}\left(\begin{array}{c|cc}\mathbf{1}_{m+n} & z & y\\ & {}^ty & \mathbf{0}_n\\ \hline \mathbf{0}_{m+n} & \multicolumn{2}{c}{\mathbf{1}_{m+n}}\end{array}\right)w_nh\right)$$

$$\times\overline{\omega_S(h)\phi(-y)\psi_S(z)}\,dzdy$$

$$=\int_{Y(\mathbf{A})}\int_{Z(\mathbf{A})} f\left(w_{m+n}\left(\begin{array}{c|c} \mathbf{1}_{m+n} & \begin{matrix} z & y \\ {}^t y & \mathbf{0}_n \end{matrix} \\ \hline \mathbf{0}_{m+n} & \mathbf{1}_{m+n} \end{array}\right) w_n h\right)$$

$$\times \overline{\omega_S(h)\phi(-y)\phi_S(z)}\,dzdy$$

If $p=\left(\begin{pmatrix} \mathbf{1}_n & B \\ \mathbf{0}_n & \mathbf{1}_n \end{pmatrix}, \varepsilon\right)$, then

$$w_{m+n}\left(\begin{array}{c|c} \mathbf{1}_{m+n} & \begin{matrix} z & y \\ {}^t y & \mathbf{0}_n \end{matrix} \\ \hline \mathbf{0}_{m+n} & \mathbf{1}_{m+n} \end{array}\right) w_n p$$

$$=\left(\begin{array}{c|c} \begin{matrix} \mathbf{1}_m & 0 \\ B\,{}^t y & \mathbf{1}_n \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & B \end{matrix} \\ \hline \mathbf{0}_{m+n} & \begin{matrix} \mathbf{1}_m & -yB \\ 0 & \mathbf{1}_n \end{matrix} \end{array}\right) w_{m+n} \left(\begin{array}{c|c} \mathbf{1}_{m+n} & \begin{matrix} z+yB\,{}^t y & y \\ {}^t y & \mathbf{0}_n \end{matrix} \\ \hline \mathbf{0}_{m+n} & \mathbf{1}_{m+n} \end{array}\right) w_n \ .$$

We have

$$R(ph; f, \phi)=\int_{Y(\mathbf{A})}\int_{Z(\mathbf{A})} f\left(w_{m+n}\left(\begin{array}{c|c} \mathbf{1}_{m+n} & \begin{matrix} z & y \\ {}^t y & \mathbf{0}_n \end{matrix} \\ \hline \mathbf{0}_{m+n} & \mathbf{1}_{m+n} \end{array}\right) w_n ph\right)$$

$$\times \overline{\omega_S(ph)\phi(-y)\phi_S(z)}\,dzdy$$

$$=\varepsilon^m\int_{Y(\mathbf{A})}\int_{Z(\mathbf{A})} f\left(w_{m+n}\left(\begin{array}{c|c} \mathbf{1}_{m+n} & \begin{matrix} z+yB\,{}^t y & y \\ {}^t y & \mathbf{0}_n \end{matrix} \\ \hline \mathbf{0}_{m+n} & \mathbf{1}_{m+n} \end{array}\right) w_n h\right)$$

$$\times \overline{\omega_S(h)\phi(-y)\phi_S((z+yB\,{}^t y))}\,dzdy$$

$$=\varepsilon^m R(h; f, \phi) \ .$$

If $p=\left(\begin{pmatrix} A & \mathbf{0}_n \\ \mathbf{0}_n & {}^t A^{-1} \end{pmatrix}, \varepsilon\right)$, then

$$w_{m+n}\left(\begin{array}{c|c} \mathbf{1}_{m+n} & \begin{matrix} z & y \\ {}^t y & \mathbf{0}_n \end{matrix} \\ \hline \mathbf{0}_{m+n} & \mathbf{1}_{m+n} \end{array}\right) w_n p = p w_{m+n}\left(\begin{array}{c|c} \mathbf{1}_{m+n} & \begin{matrix} z & yA \\ {}^t(yA) & \mathbf{0}_n \end{matrix} \\ \hline \mathbf{0}_{m+n} & \mathbf{1}_{m+n} \end{array}\right) w_n .$$

We have

$$R(ph; f, \phi) = \int_{Y(\mathbf{A})}\int_{Z(\mathbf{A})} f\left(w_{m+n}\left(\begin{array}{c|c} \mathbf{1}_{m+n} & \begin{matrix} z & y \\ {}^t y & \mathbf{0}_n \end{matrix} \\ \hline \mathbf{0}_{m+n} & \mathbf{1}_{m+n} \end{array}\right) w_n ph\right)$$

$$\times \overline{\omega_S(ph)\,\phi(-y)\,\psi_S(z)}\,dzdy$$

$$= \varepsilon^m \frac{\gamma_{-s}(1)}{\gamma_{-s}(\det A)}\omega(\det A)\,|\det A|^{s+m+\frac{n+1}{2}} \int_{Y(\mathbf{A})}\int_{Z(\mathbf{A})}$$

$$f\left(w_{m+n}\left(\begin{array}{c|c} \mathbf{1}_{m+n} & \begin{matrix} z & yA \\ {}^t(yA) & \mathbf{0}_n \end{matrix} \\ \hline \mathbf{0}_{m+n} & \mathbf{1}_{m+n} \end{array}\right) w_n h\right) \overline{\omega_S(h)\,\phi(-yA)\,\psi_S(z)}\,dzdy$$

$$= \varepsilon^m \frac{\gamma_{-s}(1)}{\gamma_{-s}(\det A)}\omega(\det A)\,\Big|\det A\Big|^{s+\frac{n+1}{2}} R(h; f, \phi) .$$

The theorem follows by the property of the Weil constant:
$\gamma(a)\gamma(b) = \langle a,b\rangle \gamma(1)\gamma(ab)$.

We define holomorphic sections of $I(\omega, s)$ as in Ikeda [6]. Roughly speaking a holomorphic section of $I(\omega, s)$ is a function $f^{(s)}(h)$ which is holomorphic in $s\in\mathbf{C}$, and $f^{(s)}(h)\in I(\omega, s)$ for each $s\in\mathbf{C}$.

Lemma 3.3. *If $f^{(s)}(h)$ is a holomorphic section of $I(\omega, s)$, then $R(m; f^{(s)}, \phi)$ is absolutely convergent for the domain*

$$(3.4)\qquad \mathrm{Re}(s) > \begin{cases} -\dfrac{n-m-1}{2} & \text{(Case 1)} \\ -n+m & \text{(Case 2)} \end{cases}$$

and can be meromorphically continued to the domain

$$(3.5)\qquad \mathrm{Re}(s) > \begin{cases} -\dfrac{n-m+1}{2} & \text{(Case 1)} \\ -n+m-1 & \text{(Case 2)} \end{cases}$$

Moreover, $R(m; f^{(s)}, \phi)$ *is holomorphic on*

$$(3.6)\qquad \mathrm{Re}(s) > \begin{cases} -\dfrac{n-1}{2} & (\text{Case } 1),\ m=1\ , \\ -\dfrac{n-m+1}{2} & (\text{Case } 1),\ m>1\ , \\ -n+m-1 & (\text{Case } 2) \end{cases}$$

Proof. We may assume $f^{(s)}$ and ϕ are decomposable; $f^{(s)} = \prod_v f_v^{(s)}$, $\phi = \prod_v \phi_v$. There is a finite set T of places of k such that

(1) T contains all places above 2, ∞,

(2) T contains all places which ramify in K/k in Case 2.

(3) If $v \notin T$, then ω_v is unramified and ϕ_v is of order 0.

(4) If $v \notin T$, then $f_v^{(s)}|_{K_v} \equiv 1$, and ϕ_v is the characteristic function of $\mathfrak{o}_v^n$.

We have to show the absolute convergence of

$$\prod_v \int_{Y(K_v)} \int_{Z(K_v)} \left| f_v^{(s)} \left(w_{m+n} \left(\begin{array}{c|cc} \mathbf{1}_{m+n} & \begin{matrix} z & y \\ {}^t y & \mathbf{0}_n \end{matrix} \\ \hline \mathbf{0}_{m+n} & \mathbf{1}_{m+n} \end{array} \right) \right) \phi_v(-y) \right| dz dy\ .$$

For $v \notin T$,

$$\int_{Y(K_v)} \int_{Z(K_v)} \left| f_v^{(s)} \left(w_{m+n} \left(\begin{array}{c|cc} \mathbf{1}_{m+n} & \begin{matrix} z & y \\ {}^t y & \mathbf{0}_n \end{matrix} \\ \hline \mathbf{0}_{m+n} & \mathbf{1}_{m+n} \end{array} \right) \right) \phi_v(-y) \right| dz dy$$

$$= \int_{Z(K_v)} \left| f_v^{(s)} \left(w_{m+n} \left(\begin{array}{c|cc} \mathbf{1}_{m+n} & \begin{matrix} z & y \\ {}^t y & \mathbf{0}_n \end{matrix} \\ \hline \mathbf{0}_{m+n} & \mathbf{1}_{m+n} \end{array} \right) \right) \right| dz\ .$$

This can be calculated by the usual Gindikin-Karperevich argument:

$$\begin{cases} \dfrac{\zeta_v\left(s+\dfrac{n-m+1}{2}\right)}{\zeta_v\left(s+\dfrac{n+m+1}{2}\right)} \displaystyle\prod_{r=1}^{[\frac{m}{2}]} \dfrac{\zeta_v(2s+n-m+2r)}{\zeta_v(2s+n+m+1-2r)} & (\text{Case } 1) \\ \displaystyle\prod_{r=1}^{m} \dfrac{L_v(s+n-m+r,\ \chi_{K/k}^{r-1})}{L_v(s+n+m+1-r,\ \chi_{K/k}^{r-1})} & (\text{Case } 2) \end{cases}$$

It follows that the product over $v \notin T$ is absolutely convergent for the domain (3.4). If $v \notin T$,

$$\int_{Z(K_v)} \left| f_v^{(s)}\left(w_{m+n} \left(\begin{array}{c|c} \mathbf{1}_{m+n} & \begin{matrix} z & y \\ {}^t y & \mathbf{0}_n \end{matrix} \\ \hline \mathbf{0}_{m+n} & \mathbf{1}_{m+n} \end{array} \right) \right) \right| dz$$

is absolutely convergent for the domain (3.5). Moreover it is slowly increase function with respect to the variable y. Therefore the local integrals are absolutely convergent for the domain (3.5). We proved the first statement. For the second statement, it will suffice to prove that

$$\prod_{v \notin T} \int_{Y(K_v)} \int_{Z(K_v)} f_v^{(s)} \left(w_{m+n} \left(\begin{array}{c|c} \mathbf{1}_{m+n} & \begin{matrix} z & y \\ {}^t y & \mathbf{0}_n \end{matrix} \\ \hline \mathbf{0}_{m+n} & \mathbf{1}_{m+n} \end{array} \right) \right) \overline{\phi_v(-y)\,\psi_S(z)}\, dz dy$$

$$= \prod_{v \notin T} \int_{Z(K_v)} f_v^{(s)} \left(w_{m+n} \left(\begin{array}{c|c} \mathbf{1}_{m+n} & \begin{matrix} z & 0 \\ 0 & \mathbf{0}_n \end{matrix} \\ \hline \mathbf{0}_{m+n} & \mathbf{1}_{m+n} \end{array} \right) \right) \overline{\psi_S(z)}\, dz dy$$

can be meromorphically continued to the domain (3.5), and is holomorphic on the domain (3.6). In fact this kind of integral is calculated in Shimura [14], [15].

$$\frac{L_T\left(s+\frac{n+1}{2}, \omega\chi_\Delta\right)}{L_T\left(s+\frac{n+m+1}{2}, \omega\right)} \prod_{r=1}^{\frac{m}{2}} \frac{1}{L_T(2s+n+m+1-2r, \omega^2)} \qquad \begin{array}{c}\text{(Case 1)}\\ m\text{: even,}\\ f \in I(\omega, s)\end{array}$$

$$\frac{1}{L_T\left(s+\frac{n+m+1}{2}, \omega\right)} \prod_{r=1}^{\frac{m+1}{2}} \frac{1}{L_T(2s+n+m+1-2r, \omega^2)} \qquad \begin{array}{c}\text{(Case 1)}\\ m\text{: odd,}\\ f \in I(\omega, s)\end{array}$$

$$\prod_{r=1}^{\frac{m}{2}} \frac{1}{L_T(2s+n+m+2-2r, \omega^2)} \qquad \begin{array}{c}\text{(Case 1)}\\ m\text{: even,}\\ f \in I(\omega, s)^{\sim}\end{array}$$

$$L_T\left(s+\frac{n+1}{2},\ \omega\chi_\Delta\right)\prod_{r=1}^{\frac{n+1}{2}}\frac{1}{L_T(2s+n+m+2-2r,\omega^2)}\qquad \begin{matrix}\text{(Case 1)}\\ m\text{: odd,}\\ f\in I(\omega,s)^\sim\end{matrix}$$

$$\prod_{r=1}^{m}\frac{1}{L_T(s+n+m+1-r,\ \omega\chi_{K/k}^{r-1})}\qquad \begin{matrix}\text{(Case 2)}\\ f\in I(\omega,s)\end{matrix}$$

Here $\Delta=(-1)^{[\frac{m}{2}]}\det S$. Thus the second statement is proved.

Corollary 3.4. *Let $f^{(s)}(h)$ be a holomorphic section of $I(\omega, s)$. Put $R^{(s)}(h)=R(h;f^{(s)},\phi)$. Then*

$$\int_{V(k)\backslash V(\mathbf{A})}E_\phi(vh;f^{(s)})\overline{\Theta^\phi(vh)}dv=E(h;R^{(s)}) \quad ,$$

for the domain (3.5).

Lemma 3.5. *Let s be a complex number in the domain* (3.5). *Let $R(h)$ be a function on $\widetilde{H(\mathbf{A})}$ whose type is as in* (3.3). *Then there exist an $f(g)\in I_G(\omega, s)$ (or $I_G(\omega, s)^\sim$ in* (*Case* 1)) *and $\phi\in S(X(\mathbf{A}))$ such that $R(h)=R(h;f,\phi)$.*

Proof. We may assume $R(h)$ is decomposable, so the problem is of local nature: we have to find f_v and ϕ_v such that

$$R(h)=\int_{Y(\mathbf{A})}\int_{Z(\mathbf{A})}f\left(w_{m+n}\left(\begin{array}{c|c}\mathbf{1}_{m+n} & \begin{matrix} z & y\\ {}^ty & \mathbf{0}_n\end{matrix}\\ \hline \mathbf{0}_{m+n} & \mathbf{1}_{m+n}\end{array}\right)w_nh\right)$$
$$\times\overline{\omega_S(h)\phi(-y)\psi_S(z)}dzdy.$$

First we assume v is non-archimeden. For simplicity, we treat (Case 1) and omit v from the notation. Take any non-zero $\phi\in S(X(k))$. Take $\psi\in S(Z(k))$ such that

$$\int_{Z(k)}\varphi(x)\overline{\psi_S(z)}dz=\alpha\neq 0 \ .$$

When $g\in P\cdot w_{n+m}\cdot V\cdot w_n\cdot H$, we put

$$f(g)=\|\phi\|^{-1}\alpha^{-1}\omega(\det A)|\det A|^{s+\frac{n+1}{2}}\omega_S(h)\phi(-y)\varphi(z)R(h) \ .$$

Here

$$\|\phi\|=\int_{Y(k)}|\phi(y)|^2dy \ ,$$

$$g=pw_{m+n}\left(\begin{array}{c|c} \mathbf{1}_{m+n} & \begin{matrix} z & y \\ {}^t y & \mathbf{0}_n \end{matrix} \\ \hline \mathbf{0}_{m+n} & \mathbf{1}_{m+n} \end{array}\right)w_n h\ ,$$

$$p=\begin{pmatrix} A & B \\ \mathbf{0}_{m+n} & {}^tA^{-1} \end{pmatrix}\in P(k)\ .$$

Put $f(g)=0$, if $g\notin P\cdot w_{m+n}\cdot V\cdot w_n\cdot H$. Then one can easily check that this is a well-defined function in $I_G(\omega, s)$, and that $R(h; f, \phi)=R(h)$.

Next we assume v is archimedean. Take ϕ and φ as above, and now we assume ϕ is right K_H-finite under ω_S. Define $f(g)$ as above. Then $f(g)$ is no longer K_G-finite function, but a well-defined continuous function. Put L^1-topology on $I_G(\omega, s)$ by the restriction to $(P\cap K_G)\backslash K_G\simeq P\backslash G$. Put L^∞-topology on $I_H(\omega, s)$, similarly. Then the proof of Lemma 3.3 implies $(f, \phi)\mapsto R(h; f, \phi)$ is continuous with respect to L^1-topology on $I_G(\omega, s)$, Schwartz topology on $S(X(k))$ and L^1-topology on $I_G(\omega, s)$. K_G-finite vectors are dense in the L^1-completion of $I_G(\omega, s)$, so we can find K_G-finite $f(g)$ such that $R(h; f, \phi)$ is arbitrarily close to $R(h)$ in L^∞-topology on $I_H(\omega, s)$. Since the subspace of $I_H(\omega, s)$ of given K_H-type is finite dimensional, this implies there exists an $f(g)\in I_G(\omega, s)$ such that $R(h)=R(h; f, \phi)$.

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