Maximally hyperbolic operators

By

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1. Introduction and statement of results

Let Ω be a domain in $\mathbf{R}^n$ and I be an interval in $\mathbf{R}$. For $(x, t) \in \Omega \times I$, we consider the following differential operator with smooth coefficients:

$$P(x, t, D_x, D_t) = D_t^m + \sum_{|\alpha|+j \leq m, j \leq m-1} a_{\alpha,j}(x, t) D_x^\alpha D_t^j,$$

where $D_x = \sqrt{-1}^{-1} \partial_x$, etc.. We are concerned with the well-posedness for the Cauchy problem for P in the C^∞ class with the initial surface $\{t = t_0\}$.

For real s , we denote the standard Sobolev spaces over $\mathbf{R}^{n+1}$ by H^s and its norm by $\|\cdot\|_s$ respectively. We say that the operator P is strictly hyperbolic if and only if its characteristic roots are real and distinct. It is a classical fact that if P is strictly hyperbolic, then there are some real number γ_0 and a positive constant C such that

$$\gamma \|e^{-\gamma t} u\|_{m-1} \leq C \|e^{-\gamma t} P u\|_0$$

for any $\gamma > \gamma_0$ and $u \in C_0^\infty(K \times I)$, where K is a compact subset of Ω .

The purpose of this paper is to give a generalization of this fact with respect to the regularity property of the solution. In fact, we shall introduce a notion of maximal hyperbolicity and give a sufficient condition for P to be maximally hyperbolic.

To make the statement more precisely, we must recall some class of operators. Let Σ be a conic submanifold of the cotangent space $T^*(\Omega) \setminus \{0\}$ with codimension p . For a real m and a non-negative integer k , we denote by $\mathcal{N}^{m,k}(\Omega, \Sigma)$ a space of pseudo-differential operators $\text{op}(p)$, introduced in [B], [BGH], such that $p(x, \xi) \in S_{1,0}^m$ and

$$p(x, \xi) \sim \sum_{j=0}^{\infty} p_{m-j}(x, \xi),$$

where p_{m-j} is positively homogeneous of degree $m-j$ in ξ and vanishes with order $k-2j$ on Σ if $k-2j \geq 0$: i.e. for some positive constant C and for $k-2j \geq 0$,

$$|p_{m-j}(x, \xi)| \leq C |\xi|^{m-k+j} d_\Sigma(x, \xi)^{k-2j},$$

where $d_{\Sigma}(x, \xi)$ is a positively homogeneous function of degree 1 which is equal to the distance between (x, ξ) and Σ when $|\xi|=1$. When $m=k$, we simply write it by $\mathcal{N}^m$. For $P \in \mathcal{N}^{m,k}(\Omega, \Sigma)$, we define the localization of P by

$$\sigma_{x,\xi}^k(P) = \sum_{|\alpha+\beta|+2j=k} \frac{1}{\alpha! \beta!} \partial_x^\alpha \partial_\xi^\beta p_j(x, \xi) y^\alpha D_y^\beta .$$

We consider the following operators

$$(1.1) \quad P = D_t^m + \sum_{j=1}^m A_j(x, D) D_t^{m-j} ,$$

where $A_j \in \mathcal{N}^j(\Omega, \Sigma)$.

Now, we say that an operator given by (1.1) is maximally hyperbolic in $\Omega \times I$ if and only if for any compact subset K of Ω and $B_j \in \mathcal{N}^j$, there are a real number γ_0 and a positive constant C such that

$$\gamma \sum_{j=0}^{m-1} \|e^{-\gamma t} B_j D_t^{m-1-j} u\|_0 \leq C \|e^{-\gamma t} P u\|_0 ,$$

when $\gamma > \gamma_0$ and $u \in C_0^\infty(K \times I)$.

We make the following assumptions:

(A-1) The restriction to Σ of the canonical 2-form $\sum_{j=1}^n d\xi_j \wedge dx_j$ on $T^*(\Omega)$ has a constant rank p at every point of Σ and $A_m \in \mathcal{N}^m(\Omega, \Sigma)$ is a transversally elliptic: for some positive constant C ,

$$d_{\Sigma}(x, \xi)^m \leq C |a_m(x, \xi)| .$$

(A-2) Denoting the principal symbol of A_j by $a_j(x, \xi)$, we assume that the polynomial in τ

$$p(\tau, x, \xi) \tau^m + \sum_{j=0}^{m-1} a_{m-j}(x, \xi) \tau^j$$

has only real roots which are pairwise distinct if $(x, \xi) \notin \Sigma$.

(A-3) For the operator

$$\begin{aligned} \mathcal{P}_{x,\xi}(\tau) &= \tau^m + \sum_{j=0}^{m-1} \sigma_{x,\xi}^{m-j}(A_{m-j}) \tau^j , \\ \partial_\zeta \mathcal{P}_{x,\xi}^*(\zeta) \mathcal{P}_{x,\xi}(\zeta) - \mathcal{P}_{x,\xi}^*(\zeta) \partial_\zeta \mathcal{P}_{x,\xi}(\zeta) &= 0 , \end{aligned}$$

if $\zeta \in \mathbf{R}$ and $(x, \xi) \in \Sigma$.

(A-4) For any $\zeta \in \mathbf{C}$ with $\text{Im} \zeta < 0$ and any $(x, \xi) \in \Sigma$, $\mathcal{P}_{x,\xi}(\zeta)$ is injective in $\mathcal{B}(\mathbf{R}^n)$: i.e. there is a positive constant C such that

$$\|f\| \leq C \|\mathcal{P}_{x,\xi}(\zeta) f\|$$

for any $f \in \mathcal{B}(\mathbf{R}^n)$ and moreover, for each $\sigma \in \mathbf{R}$ and $(x, \xi) \in \Sigma$, there is a positive constant δ such that for any $f \in \mathcal{B}(\mathbf{R}^n)$,

$$\liminf_{\gamma \rightarrow 0-} \|\gamma^{-1} \mathcal{P}_{x,\xi}(\sigma + i\gamma) f\| \geq \delta \|f\| .$$

Here, $\|\cdot\|$ means $L^2(\mathbf{R}^n)$ -norm.

Theorem 1.1. *For the operator P given by (1.1), we assume (A-1) - (A-4). Then, P is maximally hyperbolic in $\Omega \times I$.*

The proof will be given in the following sections. It is based on the same principle as in [O]₃ but requires some technical complications. Our methods are closed to those of Chap. X in [HN]₂ and [BGH] in which they have investigated the maximal hypoellipticity. In §6, we shall give an example which satisfies our conditions. In §7, we shall discuss the necessary condition for P to be maximally hyperbolic.

2. Notation and classes of symbols

We denote the class of standard pseudo-differential operators of (ρ, δ) type by

$$OPS_{\rho,\delta}^m(\Omega), \text{ or simply, by } OPS_{\rho,\delta}^m.$$

For $a \in S_{\rho,\delta}^m$, we write

$$a_{(\beta)}^{(\alpha)}(x, \xi) = \partial_\xi^\alpha \partial_x^\beta a(x, \xi)$$

and denote by $\text{op}(a)$ the corresponding pseudo-differential operator

$$\text{op}(a)u(x) = (2\pi)^{-n} \int e^{-ix \cdot \xi} a(x, \xi) \hat{u}(\xi) d\xi.$$

Following [BGH], we introduce a system of coordinates (u, v) in $T^*(\Omega)$ and classes of symbols, attached to Σ . Let (x, ξ) be a point of Σ . Then there are a conic neighborhood Θ and C^∞ -functions u_i , $1 \leq i \leq p$, positively homogeneous of degree 0 such that $\{du_j\}$ are linearly independent and

$$\Sigma \cap \Theta = \{u_1 = u_2 = \dots = u_p = 0\}.$$

Also, we can find positively homogeneous C^∞ -functions v_i , $1 \leq i \leq 2n - p$, with degree 1 such that (u, v) is a system of coordinate in Θ (if possibly shrinking it). By $S^{m,k}(\Omega, \Sigma)$ (or simply by $S^{m,k}$), we mean the class of symbols characterized by the following: $a \in S^{m,k}$ is if and only if a is in $S_{1,0}^m(\Omega)$ outside of Σ and every point of Σ has a conic neighborhood in which for any α, β and any $|v| \geq 1$, there is a constant $C_{\alpha,\beta}$ such that

$$|\partial_u^\alpha \partial_v^\beta a| \leq C_{\alpha,\beta} |v|^{m-|\beta|} (|u| + |v|^{-1/2})^{k-|\alpha|}.$$

We set $\mathcal{H}^m = \bigcap_{j=0}^\infty S^{m-j,k-2j}$ and denote the classes of pseudo-differential operators corresponding to these classes by $OPS^{m,k}$ and OPH^m .

For $E = \mathbf{R}^n$, we denote by $S^m(E)$ the space of C^∞ -functions a such that for

any α ,

$$(1 + |\mathcal{E}|)^{-m+|\alpha|} D_{\mathcal{E}}^{\alpha} a(\mathcal{E})$$

is bounded and by $S_{reg}^m(E)$ we mean the space of C^{∞} -functions a such that $a \in S^m(E)$ and

$$a \sim \sum_{k=0}^{\infty} a_k ,$$

where $a_k \in C^{\infty}(E \setminus \{0\})$ is positively homogeneous of degree $m - k$. Here, the asymptotic expansion means that for any non-negative integer N , there is a symbol $b_N \in S^{m-N}(E)$ such that

$$a - \sum_{j=0}^{N-1} b_j = b_N$$

if $|\mathcal{E}| \geq 1$.

We recall some symbol calculus in [BGH]. Let $N = \mathbf{R}^p$ and let L be a linear transformation from $E \times E^*$ to N . We can write

$$L(x, \xi) = Bx + C\xi$$

if $x \in E$ and $\xi \in E^*$. Let

$$A = C({}^t B) \in \mathcal{L}(N^*, N) .$$

For a symbol $a \in S^m(E)$, we define the corresponding pseudo-differential operator a_L by

$$(2.1) \quad a_L f = (2\pi)^{-n} \int e^{ix \cdot \xi} a(L(x, \xi)) \hat{f}(\xi) d\xi .$$

Also, if $E = N$, $B = Id$ and $C = A$, we simply write $a_L = a_A$. Generally, for two symbols $a \in S^m(E)$ and $b \in S^{m'}(E)$,

$$(2.2) \quad a_L b_L = (a_A b)_L \text{ and } a_A b_A = (a_A b)_A .$$

Namely, denoting $c_L = a_L b_L$, we have

$$c(y) = (2\pi)^{-n} \int e^{iy \cdot \eta} a(y + A\eta) \hat{b}(\eta) d\eta$$

and

$$c(\mathcal{E}) \sim \sum_{\alpha \geq 0} \frac{1}{\alpha!} \partial_{\mathcal{E}}^{\alpha} a(\mathcal{E}) (A D_{\mathcal{E}})^{\alpha} b(\mathcal{E}) .$$

Let $N = \mathbf{R}_u^p$. Locally we identify Σ with $\mathbf{R}_v^{2n-p}$. For a domain G of $\mathbf{R}^q$, we say that $a \in C^{\infty}(G, S^m(N))$ belongs to the class $S^m(G \times N)$ if and only if for any compact subset K in G and any multi-index α, β , there is a positive constant $C_{K, \alpha, \beta}$ such that

$$\left| \partial_w^{\alpha} \partial_u^{\beta} a(w, u) \right| \leq C_{K, \alpha, \beta} \langle u \rangle^{m-|\beta|}$$

for $w \in K$. Here, $\langle u \rangle = (1 + |u|^2)^{\frac{1}{2}}$. Also, we assume that

$$L \in C^\infty(G, \mathcal{L}(E \times E^*, N)) .$$

Then, for $a \in S^m(G, N)$, a_L denotes the family of pseudo-differential operators defined by (2.1).

If P is in $\mathcal{N}^{m,k}$, then in a neighborhood Θ of each point (x, ξ) of Σ , there are a unique polynomial $T_k(P)$

$$T_k(P) = \sum_{|\alpha| \leq k} a_\alpha(v) u^\alpha ,$$

and two unique differential operators P_Σ and P'_Σ

$$(2.3) \quad \begin{aligned} P_\Sigma &= \sum a_{\alpha\beta}(v) u^\alpha D_u^\beta , \\ P'_\Sigma &= \sum b_{\alpha\beta}(v) u^\alpha D_u^\beta \end{aligned}$$

such that $P - T_k(P)$ vanishes with order $k+1$ on Σ and for any $Q \in \mathcal{N}^{m',k'}$,

$$T_{k+k'}(P \circ Q) = P_\Sigma T_{k'}(Q) \text{ and } T_{k+k'}(Q \circ P) = P'_\Sigma T_{k'}(Q)$$

where $a_\alpha(v)$ and $a_{\alpha\beta}(v)$ ($b_{\alpha\beta}(v)$) are smooth functions depending only on v and are positively homogeneous of degree $m - \frac{1}{2}(k - |\alpha|)$ and $m - \frac{1}{2}(k - |\alpha| + |\beta|)$, respectively.

Among these localized objects, there are some relations. Let $y \in \mathbf{R}^n$ and η be its dual variable. We define two $p \times n$ -matrices B and C by

$$B = (B_{jk})_{1 \leq j \leq p, 1 \leq k \leq n}, \quad B_{jk}(v) = \left. \frac{\partial u_j}{\partial x_k} \right|_\Sigma$$

and

$$C = (C_{jk})_{1 \leq j \leq p, 1 \leq k \leq n}, \quad C_{jk}(v) = \left. \frac{\partial u_j}{\partial \xi_k} \right|_\Sigma ,$$

respectively. Let

$$L(y, \eta) = By + C\eta \text{ and } A = C({}^t B) .$$

Then, $L \in C^\infty(\Sigma, N)$ and $a = T_k(P) \in S^k(\Sigma, N)$. Proposition 5.14 in [BGH] says that for this symbol a ,

$$(2.4) \quad P_\Sigma = a_A, \quad P'_\Sigma = a_H, \text{ and } \sigma^k(P) = a_L .$$

If $P \in \mathcal{N}^{m,k}$ and $Q \in OPS^{m',k'}$ (resp. $OPH^{m'}$), then

$$PQ \in OPS^{m+m',k+k'} \text{ (resp. } OPH^{m+m',k+k'})$$

and

$$\sigma(PQ) - P_\Sigma \sigma(Q) \in S^{m+m',k+k'+1} \text{ (resp. } \mathcal{H}^{m+m' - \frac{1}{2}(k+1)}) .$$

We shall also use the pseudo-differential operators with a parameter $\lambda \geq 1$. By $S^m(\lambda)$, we denote the set of symbols such that for each (α, β) , there is a positive constant $C_{\alpha\beta}$ such that

$$\left| \partial_x^\alpha \partial_\xi^\beta a(x, \xi, \lambda) \right| \leq C_{\alpha\beta} \lambda^{m-|\beta|}, \quad \forall (x, \xi) \in \mathbf{R}^{2n}, \quad \forall \lambda \geq 1.$$

By $\tilde{S}^m(\lambda)$, we denote the set of symbols such that for each (α, β) , there is a positive constant $C_{\alpha\beta}$ such that

$$\left| \partial_x^\alpha \partial_\xi^\beta a(x, \xi, \lambda) \right| \leq C_{\alpha\beta} \lambda^{m-\frac{1}{2}|\alpha+\beta|}, \quad \forall (x, \xi) \in \mathbf{R}^{2n}, \quad \forall \lambda \geq 1.$$

By $S^{m,k}(\lambda)$, we mean the class of symbols characterized by the following: $a \in S^{m,k}(\lambda)$ if and only if a is in $S^m(\lambda)$ outside of Σ and every point of Σ has a conic neighborhood in which for any α, β and any $|v| \geq 1$, there is a constant $C_{\alpha,\beta}$ such that

$$(2.5) \quad \left| \partial_u^\alpha \partial_v^\beta a \right| \leq C_{\alpha,\beta} |\lambda|^{m-|\beta|} (|u| + |\lambda|^{-1/2})^{k-|\alpha|}.$$

We set $\mathcal{H}^m(\lambda) = \bigcap_{j=0}^\infty S^{m-j,k-2j}(\lambda)$. Then, we see that $a \in \mathcal{H}^m(\lambda)$ if and only if outside of Σ , $a \in S^{-\infty}(\lambda)$ and for any point $(x, \xi) \in \Sigma$, there is a conic neighborhood Θ such that $\forall \alpha, \beta, \gamma, \exists C_{\alpha\beta\gamma}$

$$\left| u^\alpha \partial_u^\beta \partial_v^\gamma a \right| \leq C_{\alpha\beta\gamma} \lambda^{m-|\gamma|+\frac{1}{2}|\beta|-\frac{1}{2}|\alpha|}$$

for any $(v, u) \in \Theta$ and any $\lambda \geq 1$. We denote the classes of pseudo-differential operators corresponding to these classes by $OPS^{m,k}(\lambda)$ and $OPH^m(\lambda)$.

Let

$$N^{m,k}(\lambda) = S^m(\lambda) \cap S^{m,k}(\lambda).$$

By the similar argument to Proposition 3.2 and 3.5 in [B], we can prove the following two lemmata.

Lemma 2.1. *If $P \in OPS^{m,k}(\lambda)$ and $Q \in OPS^{m',k'}(\lambda)$, then*

$$PQ \in OPS^{m+m',k+k'}(\lambda),$$

and for any N ,

$$\sigma(PQ) - \sum_{|\alpha| < N} \frac{1}{\alpha!} \partial_\xi^\alpha \sigma(P) D_x^\alpha \sigma(Q) \in S^{m+m'-N, k+k'-2N}(\lambda).$$

Lemma 2.2. *If $P \in OPN^{m,k}(\lambda)$ and $Q \in OPS^{m',k'}(\lambda)$, then*

$$\sigma(PQ) - \sum_{|\alpha| \leq k} \frac{1}{\alpha!} \partial_\xi^\alpha \sigma(P) D_x^\alpha \sigma(Q) \in S^{m+m'-\frac{1}{2}, k+k'}(\lambda)$$

and

$$\sigma(QP) - \sum_{|\alpha| \leq k} \frac{1}{\alpha!} \partial_{\xi}^{\alpha} \sigma(Q) D_x^{\alpha} \sigma(P) \in S^{m+m'-\frac{1}{2}, k+k'}(\lambda) .$$

By the symbol calculus in Lemma 2.1, we can easily show that the following result holds.

Lemma 2.3. *Let $a \in S^{m,k}(\lambda)$ such that*

$$|a| \geq c \lambda^m d_{\Sigma}^k, \quad (c > 0)$$

where $d_{\Sigma}(\lambda) = (|u|^2 + \lambda^{-1})^{\frac{1}{2}}$. Then, there is a symbol $b \in S^{-m,-k}(\lambda)$ such that

$$\text{op}(a) \text{op}(b) - Id \in OPS^{-\frac{1}{2}, -1}(\lambda)$$

and

$$\text{op}(b) \text{op}(a) - Id \in OPS^{-\frac{1}{2}, -1}(\lambda) .$$

Also, we shall use the following convention. For a family of operators $T = (T_1, \dots, T_l)$ and a finite sequence of positive integers $\alpha = (\alpha_1, \dots, \alpha_N)$, where $1 \leq \alpha_j \leq l$,

$$T^{\alpha} = T_{\alpha_1} \cdots T_{\alpha_N} \text{ and } |\alpha| = N .$$

We denote the L^2 -norm by $\|\cdot\|$ and its inner product by $(\cdot, \cdot)$ respectively. Finally, for two Banach spaces E and F , we denote the space of bounded linear operators from E to F by $\mathcal{L}(E, F)$.

3. Reduction to an L^2 -inequality depending on a parameter

We remark that if $P \in \mathcal{N}^{m,k}$, P can be written as follows :

$$P = \sum_{|\alpha| \leq k} A_{\alpha} U_p^{\alpha_1} \cdots U_p^{\alpha_p} ,$$

where $A_{\alpha} \in OPS_{1,0}^{m-\frac{1}{2}(k-|\alpha|)}(\Omega)$ and $U_j = u_j(x, D) \in \mathcal{N}^{0,1}$. Let K be a compact subset of Ω . Using a partition of unity such that

$$\sum_j \phi_j(x, D) = Id \text{ on } K ,$$

we can reduce Theorem 1.1 to the following result.

Theorem 3.1. *Let (x_0, ξ_0) be a point of Σ . For a bounded neighborhood V of x_0 , there are an operator $\phi(x, D) \in OPS_{1,0}^0$ which is elliptic at (x_0, ξ_0) , positive numbers R and C such that for any $|\text{Im} \zeta| > R$ and $f \in C_0^{\infty}(V)$*

$$\begin{aligned} & \|\operatorname{Im} \zeta \sum_{|\alpha|+j \leq m-1} \|\zeta^j \phi(x, D) U^\alpha f\|_{|\alpha|+\frac{1}{2}(m-1-j-|\alpha|)} \\ & \leq C \|P_\zeta f\|_0 + C \sum_{|\alpha|+j \leq m-1} \|\zeta^j U^\alpha f\|_{|\alpha|+\frac{1}{2}(m-1-j-|\alpha|)} , \end{aligned}$$

where

$$P_\zeta = \zeta^m + \sum_{j=1}^m A_j(x, D) \zeta^{m-j} .$$

The proof shall be given in the following several sections. In this section, we shall reduce it to an L^2 -estimate depending on a parameter λ . Let $\varphi \in C_0^\infty(\mathbf{R}^n)$ be a non-negative function such that

$$\operatorname{supp} \varphi \subset \{\xi \in \mathbf{R}^n; 3/4 \leq |\xi| \leq 3\}$$

and

$$1 = \chi(\xi)^2 + \sum_{j \geq 0} \varphi(\xi/2^j)^2 ,$$

where $\chi \in C_0^\infty(\mathbf{R}^n)$ is another function with its support contained in $\{|\xi| \leq 3/2\}$. Also, let $\tilde{\varphi}$ be another function having the same property as φ such that $\tilde{\varphi} = 1$ on the support of φ . Let Θ be a conic neighborhood of (x_0, ξ_0) . We choose a C^∞ and non-negative function $\psi(x, \xi)$ having its support in Θ such that $\psi(x_0, \xi_0) = 1$ and when $|\xi| \geq 1$, ψ is homogeneous in ξ of degree zero. Let $\tilde{\psi}$ be another function having the same property as ψ such that $\tilde{\psi} \supset \supset \psi$, which means that $\tilde{\psi} = 1$ on the support of ψ .

We can write

$$P_\zeta = \zeta^m + \sum_{j=1}^m \sum_{|\alpha| \leq j} \zeta^{m-j} A_{\alpha,j}(x, D) U^\alpha ,$$

where the degree of $A_{\alpha,j} \leq \frac{1}{2}(j + |\alpha|)$. Let us denote the principal symbol of $A_{\alpha,j}$ by $a_{\alpha,j}$. For each $j=1, \dots, p$, let X_j be a pseudo-differential operator having its symbol $\sigma(X_j)$ such that

$$\sigma(X_j)(x, \xi) = \tilde{\varphi}(x, \xi) |\xi| u_j(x, \xi) ,$$

and let $X_j(\lambda)$ be a pseudo-differential operator with a parameter $\lambda > 1$ having its symbol $\sigma(X_j(\lambda))(x, \xi)$ given by

$$X_j(x, \xi, \lambda) = \tilde{\varphi}(\xi/\lambda) \sigma(X_j)(x, \xi) .$$

Then, we can see that $X_j(\lambda) \in OPS^1(\lambda)$.

We pick up the dominant part of P_ζ :

$$P_\zeta(\lambda) = \zeta^m + \sum_{j=1}^m \sum_{|\alpha|=j} \zeta^{m-j} C_{\alpha,j} X(\lambda)^\alpha ,$$

where $C_{\alpha,j}$ is a pseudo-differential operator with its symbol $c_{\alpha,j}$

$$c_{\alpha,j}(x, \xi) = a_{\alpha,j}(x, \xi) |\xi|^{-j}$$

if $|\xi| \geq 1$. Let $P'_\zeta(\lambda) = \partial_\zeta P_\zeta(\lambda)$.

Theorem 3.2. For any $\varepsilon > 0$, there are positive numbers R_0 and C such that for any $|\operatorname{Im} \zeta| > R_0$, $f \in \mathcal{D}$ and $\lambda \geq 1$,

$$\begin{aligned} & |\operatorname{Im} \zeta| \sum_{|\alpha|+j=m-1} \|\zeta^j X(\lambda)^\alpha f\| \\ & \leq C \|P_\zeta(\lambda) f\|_0 + \sum_{|\alpha|+j \leq m-2} (\varepsilon |\operatorname{Im} \zeta| + C) \|\zeta^j \lambda^{\frac{1}{2}(m-1-j-|\alpha|)} X(\lambda)^\alpha f\|. \end{aligned}$$

Granting this result, we can prove Theorem 3.1. The rest of this section is devoted to its proof.

Let

$$\varphi_\nu(D) = \varphi\left(\frac{D_x}{2^\nu}\right), \quad \lambda_\nu = 2^\nu \text{ and } f_\nu = \varphi(x, D) \varphi_\nu(D) f.$$

We apply Theorem 3.2 to f_ν . To estimate the commutator terms, we use the following basic inequalities. (c.f. [HN]) For each $l \leq m$ and α , $|\alpha| \leq l$,

$$\begin{aligned} & \sum_{\nu \geq 0} \lambda_\nu^{l-|\alpha|} \|\{\varphi_\nu(D) \varphi(x, D) X^\alpha - X(\lambda_\nu)^\alpha \varphi(x, D) \varphi_\nu(D)\} f\|^2 \\ & \leq C \sum_{|\beta| \leq |\alpha|-1} \|X^\beta f\|_{(l-1-|\beta|)/2}^2, \end{aligned}$$

and

$$C^{-1} \|f\|_{s/2}^2 \leq \sum_{\nu \geq 0} \|\lambda_\nu^s \varphi_\nu(D) f\|^2 \leq C \|f\|_{s/2}^2$$

for any $f \in \mathcal{D}$.

Then,

$$\sum_{\nu \geq 0} \lambda_\nu^{l-|\alpha|} \|X(\lambda_\nu)^\alpha f_\nu\|^2 \leq C \left\{ \|\varphi(x, D) X^\alpha f\|_{(l-|\alpha|)/2}^2 + \sum_{|\alpha| \leq l-1} \|X^\alpha f\|_{(l-1-|\alpha|)/2}^2 \right\}.$$

On the other hand, since the system of vector fields $\{X_j\}_{j=1}^p$ and their commutators with length less than or equal to two span the tangent space at each point in Σ , we have

$$\|\varphi(x, D) f\|_{s+\frac{1}{2}}^2 \leq C_s \left(\sum_{j=1}^p \|\varphi(x, D) X_j f\|_s^2 + \|f\|_s^2 \right)$$

for any $f \in \mathcal{D}$. (c.f. [BCN])

$$\begin{aligned} & \sum_{|\alpha| \leq m-2} \|\varphi(x, D) X^\alpha f\|_{(m-1-|\alpha|)/2}^2 \\ & \leq C \sum_{|\alpha|=m-1} \|\varphi(x, D) X^\alpha f\|^2 + C \sum_{|\alpha| \leq m-2} \|X^\alpha f\|_{(m-2-|\alpha|)/2}^2. \end{aligned}$$

If $|\zeta| \geq |\operatorname{Im} \zeta| \geq R_0$ and we take R_0 as a large enough number, we get

$$|\operatorname{Im} \zeta| \sum_{j+|\alpha| \leq m-1} \|\varphi X^\alpha \zeta^j f\|_{\frac{1}{2}(m-1-|\alpha|-j)}$$

$$\leq C\|\phi Pf\| + C\sum_{j+|\alpha|\leq m-1}\|\zeta^j X^\alpha f\|_{\frac{1}{2}(m-1-|\alpha|-j)}.$$

This completes the proof of Theorem 3.1.

The proof of Theorem 3.2 consists of two parts. One of them is to handle the case that $|\operatorname{Im}\zeta| \geq R_1\lambda^{\frac{1}{2}}$ or $|\operatorname{Re}\zeta| \geq R_2\lambda^{\frac{1}{2}}$ holds. This is carried out in the next section in which the assumptions (A-2) and (A-3) play an important role. The other is the consideration devoted to the remaining case, i.e. $|\zeta| \leq R_3\lambda^{\frac{1}{2}}$. This is given in §5 in which the assumptions (A-1) and (A-4) play an essential role.

4. Estimates in the hyperbolic zone

In this section, we shall give the proof of Theorem 3.2 when $|\zeta| \geq R_1\lambda^{\frac{1}{2}}$ holds. Let $\tilde{X}_j(\lambda)$ be an operator having its symbol

$$\tilde{X}_j(x, \xi, \lambda) = X_j(x/\lambda^{\frac{1}{2}}, \lambda^{\frac{1}{2}}\xi, \lambda)$$

and similarly, $\tilde{C}_{\alpha,j}(\lambda)$ be an operator with symbol

$$\begin{aligned} \tilde{C}_{\alpha,j}(x, \xi, \lambda) &= c_{\alpha,j}(x/\lambda^{\frac{1}{2}}, \lambda^{\frac{1}{2}}\xi) \quad , \text{ and put} \\ \tilde{P}_\zeta(\lambda) &= (\lambda^{\frac{1}{2}}\zeta)^m + \sum_{j=1}^m \sum_{|\alpha|=j} \tilde{C}_{\alpha,j}(\lambda) (\lambda^{\frac{1}{2}}\zeta)^{m-j} \tilde{X}(\lambda)^\alpha. \end{aligned}$$

The proof of Theorem 3.2 consists of the following two parts.

Proposition 4.1. *Suppose that (A-2) holds. Let $\tilde{\zeta} = \lambda^{\frac{1}{2}}\zeta$. For any $\varepsilon > 0$, there are a positive number R_1 and a positive constant C such that if $|\operatorname{Im}\zeta| \geq R_1$,*

$$\begin{aligned} (4.1) \quad & |\operatorname{Im}\zeta| \lambda^{\frac{1}{2}} \sum_{|\alpha|+j=m-1} \|\zeta^j \tilde{X}(\lambda)^\alpha f\| \\ & \leq C \|\tilde{P}_\zeta(\lambda) f\| + \sum_{|\alpha|+j \leq m-2} (\varepsilon |\operatorname{Im}\zeta| + C) \lambda^{\frac{1}{2}} \|\lambda^{\frac{1}{2}(m-1-|\alpha|-j)} \tilde{\zeta}^j \tilde{X}(\lambda)^\alpha f\| \end{aligned}$$

for any $f \in \mathcal{S}(\mathbf{R}^n)$.

Proposition 4.2. *Suppose that (A-2) and (A-3) hold. For any $\varepsilon > 0$, there are positive numbers R_2 and R_3 such that if $|\operatorname{Im}\zeta| \leq R_1$, $|\operatorname{Re}\zeta| \geq R_2$ and $|\operatorname{Im}\zeta| \lambda^{\frac{1}{2}} \geq R_3$,*

$$\begin{aligned} (4.2) \quad & |\operatorname{Im}\zeta| \lambda^{\frac{1}{2}} \sum_{|\alpha|+j=m-1} \|\zeta^j \tilde{X}(\lambda)^\alpha f\| \\ & \leq C \|\tilde{P}_\zeta(\lambda) f\| + \sum_{|\alpha|+j \leq m-2} (\varepsilon |\operatorname{Im}\zeta| \lambda^{\frac{1}{2}} + C) \|\lambda^{\frac{1}{2}(m-1-|\alpha|-j)} \zeta^j \tilde{X}(\lambda)^\alpha f\| \end{aligned}$$

for any $f \in \mathcal{S}(\mathbf{R}^n)$.

Let us begin with the proof of Proposition 4.1. We are going to estimate

$$(\tilde{Q}_\zeta(\lambda)f, f) = 2i\text{Im}(\tilde{P}_\zeta(\lambda)f, \tilde{P}'_\zeta(\lambda)f) .$$

Let us denote the principal symbol of $\tilde{P}_\zeta$ by $\tilde{p}_\zeta$:

$$\tilde{p}_\zeta(x, \xi, \lambda, \zeta) = (\lambda^{\frac{1}{2}}\zeta)^m + \sum_{j=1}^m \sum_{|\alpha|=j} \tilde{C}_{\alpha,j}(x, \xi, \lambda) (\lambda^{\frac{1}{2}}\zeta)^{m-j} \tilde{X}^\alpha(x, \xi, \lambda) .$$

Let

$$\tilde{Q}_\zeta(\lambda) = \tilde{P}'_\zeta(\lambda) * \tilde{P}_\zeta(\lambda) - \tilde{P}_\zeta(\lambda) * \tilde{P}'_\zeta(\lambda)$$

and

$$d_\zeta(x, \xi, \lambda) = \sum_{j=1}^p |\tilde{X}_j(x, \xi, \lambda)| + \lambda^{\frac{1}{2}}|\zeta| .$$

Lemma 4.3. Denoting

$$-\bar{q}_\zeta(x, \xi, \lambda) = 2\text{Im}\{\overline{\partial_\zeta \tilde{p}'_\zeta}(x, \xi, \lambda) \tilde{p}_\zeta(x, \xi, \lambda)\} ,$$

then we can find positive constants C_j , $j=1,2$ such that

$$-\text{Im}\zeta\lambda^{\frac{1}{2}}d_\zeta(x, \xi, \lambda)^{2m-2} \leq C_1\bar{q}_\zeta(x, \xi, \lambda) + C_2|\zeta|\lambda^{\frac{1}{2}}$$

for any (x, ξ, λ, ζ) , $\lambda > 1$.

Proof.

$$\frac{\tilde{p}'_\zeta}{\tilde{p}_\zeta} = \sum_{j=1}^m \frac{1}{\zeta - \lambda_j} = \sum_{j=1}^m \frac{\bar{\zeta} - \lambda_j}{|\zeta - \lambda_j|^2}$$

and

$$q_\zeta = 2\text{Im}\left(\frac{\tilde{p}'_\zeta}{\tilde{p}_\zeta} |p_\zeta|^2\right) .$$

The characteristic roots λ_j are homogeneous of degree 1 in $\mathfrak{X}$, where

$$\mathfrak{X}_j = \tilde{X}_j(x, \xi, \lambda)$$

and if $j \neq k$, for some $\delta > 0$, $|\lambda_j - \lambda_k| \geq \delta|\mathfrak{X}|$. Hence, we can find positive constants C_j such that

$$\begin{aligned} q_\zeta &= -\text{Im}\zeta \sum_{j=1}^m \prod_{k \neq j} |\zeta - \lambda_k|^2 \\ &\geq C_1 |\text{Im}\zeta| \langle \zeta; \mathfrak{X} \rangle^{2m-2} - C_2 |\text{Im}\zeta| , \end{aligned}$$

where $\langle \zeta; \mathfrak{X} \rangle = (1 + |\zeta|^2 + |\mathfrak{X}|^2)^{\frac{1}{2}}$.

Let

$$\tilde{A}_R = \{(x, \xi, \lambda, \zeta) \in \mathbf{R}^{2n} \times \mathbf{R}_+ \times \mathbf{C}; d_\zeta(x, \xi, \lambda) \geq R\lambda^{\frac{1}{2}}, \lambda \geq 1\}$$

and

$$\tilde{B}_R = \{ (x, \xi, \lambda, \zeta) \in \mathbf{R}^{2n} \times \mathbf{R}_+ \times \mathbf{C}; d_\zeta(x, \xi, \lambda) \leq R\lambda^{\frac{1}{2}}, \lambda \geq 1 \} .$$

Also, let

$$A_R = \{ (x, \xi, \lambda) \in \mathbf{R}^{2n} \times \mathbf{R}_+; d_0(x, \xi, \lambda) \geq R\lambda^{\frac{1}{2}}, \lambda \geq 1 \}$$

and

$$B_R = \{ (x, \xi, \lambda) \in \mathbf{R}^{2n} \times \mathbf{R}_+; d_0(x, \xi, \lambda) \leq R\lambda^{\frac{1}{2}}, \lambda \geq 1 \} .$$

For each non-negative integer k , let

$$N_k(f) = \sum_{|\alpha+\beta| \leq k} \|y^\alpha D_y^\beta f\|^2$$

and

$$N_{k,x,\xi}(f) = \sum_{|\alpha+\beta| \leq k} \|(y-x)^\alpha (D_y - \xi)^\beta f\|^2 .$$

Then, we note that any $f \in \mathcal{D}(\mathbf{R}^n)$,

$$N_{k,x,\xi}(U_{x,\xi}f) = N_k(f) ,$$

where

$$U_{x,\xi}f = e^{iy \cdot \xi} f(y-x) .$$

Proposition 4.4. *For any $\varepsilon > 0$, there are a positive real number R_ε and a positive constant C such that if $|\operatorname{Im} \zeta| \geq R_\varepsilon$,*

$$\begin{aligned} & |(\tilde{Q}(\lambda) - \tilde{q}_\zeta(x, \xi, \lambda))f, f| \\ & \leq \varepsilon \lambda^{\frac{1}{2}} |\operatorname{Im} \zeta| (d_\zeta(x, \xi, \lambda)^{2m-2} \|f\|^2 + \lambda^{m-1} N_{2m-1,x,\xi}(f)^{\frac{1}{2}} \|f\|) \\ & \quad + \lambda^{\frac{1}{2}} (Cd_\zeta(x, \xi, \lambda)^{2m-2} \|f\| + C\lambda^{m-1} N_{2m-1,x,\xi}(f)^{\frac{1}{2}}) N_{4m-2,x,\xi}(f)^{\frac{1}{2}} \end{aligned}$$

for any $f \in \mathcal{D}$.

Proof. Taylor's theorem says that for $f \in C^{k+1}([0,1])$,

$$f(1) = \sum_0^k \frac{f^{(j)}(0)}{j!} + \frac{1}{k!} \int_0^1 f^{(k+1)}(t) (1-t)^k dt .$$

Applying this formula to the function

$$f(t) = a(x + t(y-x), \xi + t(\eta - \xi)) ,$$

we have

$$a(y, \eta) = \sum_{|\alpha+\beta| \leq k} \frac{1}{\alpha! \beta!} \partial_x^\alpha \partial_\xi^\beta a(x, \xi) (y-x)^\alpha (\eta - \xi)^\beta$$

$$+ (k+1) \sum_{|\alpha+\beta|=k+1} \frac{1}{\alpha! \beta!} (y-x)^\alpha (\eta-\xi)^\beta \\ \times \int_0^1 \partial_x^\alpha \partial_\xi^\beta a(x+t(y-x), \xi+t(\eta-\xi)) (1-t)^k dt .$$

Let

$$\mathfrak{X}_j^{(0)}(x, \xi, \lambda) = \tilde{X}(x, \xi, \lambda) , \\ \mathfrak{X}_j^{(1)} = \sum_{k=1}^n \left(\frac{\partial \tilde{X}_j}{\partial \xi_k}(x, \xi, \lambda) (D_{y_k} - \xi_k) + \frac{\partial \tilde{X}_j}{\partial x_k}(x, \xi, \lambda) (y_k - x_k) \right)$$

and

$$\tilde{X}_j(\lambda) = \mathfrak{X}_j^{(0)}(x, \xi, \lambda) + \mathfrak{X}_j^{(1)}(x, \xi, \lambda) + \mathfrak{X}_j^{(2)}(x, \xi, \lambda) .$$

Also,

$$\tilde{C}_{\alpha,j} = \mathfrak{X}_{0,\alpha,j}^{(0)}(x, \xi, \lambda) + \mathfrak{X}_{0,\alpha,j}^{(2)}(x, \xi, \lambda) ,$$

where

$$\mathfrak{X}_{0,\alpha,j}^{(0)}(x, \xi, \lambda) = \tilde{C}_{\alpha,j}(x, \xi, \lambda) .$$

Then, Taylor's theorem and the estimate of the remainder terms imply that

$$\mathfrak{X}_j^{(0)} \in \tilde{S}^1(\lambda)$$

and the coefficients of the operator $\mathfrak{X}_j^{(k)}(x, \xi, \lambda)$ are in $\tilde{S}^{\frac{1}{2}(2-k)}(\lambda)$ if $k=1,2$.
Since

$$\tilde{X}(\lambda) \in \tilde{S}^1(\lambda) ,$$

the L^2 -continuity theorem by Calderón-Vaillancourt enables us to write

$$\|(\tilde{X}_j(\lambda) - \mathfrak{X}_j^{(0)}(x, \xi, \lambda))f\| \leq C \lambda^{\frac{1}{2}} \sum_{|\alpha+\beta| \leq 1} \|(y-x)^\alpha (D_y - \xi)^\beta f\| , \\ \|\mathfrak{X}_j^{(2)}(x, \xi, \lambda)f\| \leq C \sum_{|\alpha+\beta| \leq 2} \|(y-x)^\alpha (D_y - \xi)^\beta f\| ,$$

and

$$\|(\tilde{C}_{\alpha,j}(\lambda) - \mathfrak{X}_{0,\alpha,j}(x, \xi, \lambda))f\| \leq C \lambda^{-\frac{1}{2}} \sum_{|\alpha+\beta| \leq 1} \|(y-x)^\alpha (D_y - \xi)^\beta f\| .$$

The operator $\tilde{Q}(\lambda)$ is a sum of terms such that

$$\tilde{\zeta}^k \tilde{\zeta}^j (\tilde{X}(\lambda)^\alpha * \tilde{X}_{0,\alpha,j}(\lambda) * \tilde{X}_{0,\beta,k}(\lambda) \tilde{X}(\lambda)^\beta - \tilde{\zeta}^j \tilde{\zeta}^k (\tilde{X}(\lambda)^\beta * \tilde{X}_{0,\beta,k}(\lambda) * \tilde{X}_{0,\alpha,j}(\lambda) \tilde{X}(\lambda)^\alpha) .$$

It is seen that for $|\alpha|=l$,

$$\prod_{k=1}^l \tilde{X}_{i_k}(\lambda) = \prod_{k=1}^l \mathfrak{X}_{i_k}^{(0)}$$

$$+ \sum_{k=1}^l (\prod_{j \neq k} \mathfrak{X}_{ij}^{(0)}) \mathfrak{X}_{ik}^{(1)} \\ + \sum_{k,i=1}^l (\prod_{j \neq i,k} \mathfrak{X}_{ij}^{(0)}) \mathfrak{X}_{ii}^{(1)} \mathfrak{X}_{ik}^{(1)} + \mathfrak{R} ,$$

where $\mathfrak{R}$ represents the sum of the remainder terms. After expanding each terms of $\tilde{Q}(\lambda) - \tilde{q}_\zeta(x, \xi, \lambda)$ as above, we can classify them into the following three groups:

1. the sum of terms having a factor $\text{Im} \tilde{\zeta}$ but having no factor $\mathfrak{X}^{(2)}$,
2. the sum of terms having no factor $\text{Im} \tilde{\zeta}$ and having no factor $\mathfrak{X}^{(2)}$,
3. the sum of remaining terms.

Using the fact that

$$\tilde{X}^*(\lambda) - \tilde{X}(\lambda) \in OPS^0(\lambda)$$

and

$$[\mathfrak{X}_j^{(1)}, \mathfrak{X}_k^{(1)}] \in OPS^1(\lambda) ,$$

we can estimate each group and see that for some positive constant C ,

$$(4.3) \quad \begin{aligned} & |(\tilde{Q}(\lambda) - \tilde{q}_\zeta(x, \xi, \lambda))f, f| \\ & \leq C |\text{Im} \tilde{\zeta}| \lambda^{\frac{1}{2}} \sum_{j=0}^{2m-3} \sum_{|\gamma| \leq 2m-3-j} \sum_{|\alpha+\beta| \leq 2m-2-j-|\gamma|} |\tilde{\zeta}|^j |\tilde{X}^r(x, \xi, \lambda)| \\ & \quad \times \lambda^{\frac{1}{2}|\alpha+\beta|} |((y-x)^\alpha (D_y - \xi)^\beta f, f)| \\ & + C \sum_{j=0}^{2m-3} \sum_{|\gamma| \leq 2m-3-j} \sum_{|\alpha+\beta| \leq 2m-3-j-|\gamma|} |\tilde{\zeta}|^j |\tilde{X}^r(x, \xi, \lambda)| \\ & \quad \times \lambda^{\frac{1}{2}|\alpha+\beta|+1} |((y-x)^\alpha (D_y - \xi)^\beta f, f)| \\ & + C \sum_{j=0}^{2m-3} \sum_{|\gamma| \leq 2m-2-j} \sum_{|\alpha+\beta| \leq 2m-2-j-|\gamma|} |\tilde{\zeta}|^j |\tilde{X}^r(x, \xi, \lambda)| \lambda^{\frac{1}{2}|\alpha+\beta|} \\ & \quad \times \sum_{|\alpha'+\beta'| \leq 2(2m-2-j-|\gamma|-|\alpha+\beta|)+2} |((y-x)^{\alpha+\alpha'} (D_y - \xi)^{\beta+\beta'} f, f)| \end{aligned}$$

for any $(x, \xi) \in \mathbf{R}^{2n}$, any $\lambda \geq 1$ and any $f \in \mathcal{S}(\mathbf{R}^n)$. Here, we have remarked the fact that for $j+|\gamma|=2m-2$, the terms in the second group

$$(\text{Re} \tilde{\zeta})^j (\tilde{X}^r(x, \xi, \lambda) \mathfrak{X}_j^{(1)} f, f)$$

were cancelled out.

The following is a well-known result. (c.f. [HN]₂)

Lemma 4.5. *For each $l \in \mathbf{N}$ and any $\varepsilon > 0$, there is a positive constant C_ε such that we have*

$$\sum_{j=1}^l \delta^j N_j(f) \leq \varepsilon \delta^l N_{l+1}(f) + C_\varepsilon \delta^{1/l} N_0(f)$$

for any $\delta \in (0, 1)$ and any $f \in \mathcal{S}(\mathbf{R}^n)$.

Proof. For any $\varepsilon > 0$ there is a positive constant C_ε such that for any $1 \leq j \leq l$, any $\varepsilon \in \mathbf{R}_+$ and any $\delta \in (0, 1)$,

$$\delta^j \Xi^j \leq \varepsilon \delta^l \Xi^{l+1} + C_\varepsilon \delta^{1/l}.$$

Let

$$H = (-\Delta_y + |y|^2)^{\frac{1}{2}}.$$

Then, for some positive constant C such that

$$C^{-1} N_j(f) \leq \|H^j f\|^2 \leq C N_j(f).$$

Using the spectral resolution of the self-adjoint operator H on $L^2(\mathbf{R}^n)$

$$(Hf, f) = \int_0^\infty t d(E_t f, f),$$

we have

$$(F(H)f, f) \leq (G(H)f, f)$$

if $F(t) \leq G(t)$ for any $t \in \mathbf{R}_+$. Therefore,

$$(a(H) * a(H)f, f) \leq (b(H) * b(H)f, f),$$

if $a(t) \leq b(t)$ on $\mathbf{R}_+$. This completes the proof.

It then follows from (4.3) that

$$\begin{aligned} & |(\{\tilde{Q}(\lambda) - \tilde{q}_\zeta(x, \xi, \lambda)\}f, f)| \\ & \leq |\operatorname{Im} \zeta| \lambda^{\frac{1}{2}} \sum_{k=0}^{2m-3} d_\zeta(x, \xi, \lambda)^k \lambda^{(2m-2-k)/2} N_{2m-2-k, x, \xi}^{1/2}(f) \|f\| \\ & + C \lambda^{\frac{1}{2}} \sum_{k=0}^{2m-2} d_\zeta(x, \xi, \lambda)^k \lambda^{(2m-2-k)/2} N_{2m-2-k, x, \xi}^{1/2}(f) N_{4m-2, x, \xi}^{1/2}(f). \end{aligned}$$

Taking $\delta = (\lambda^{\frac{1}{2}}/d_\zeta(x, \xi, \lambda))^2$ in Lemma 4.5, we have

$$\begin{aligned} & \sum_{k=0}^{2m-3} d_\zeta(x, \xi, \lambda)^k \lambda^{\frac{1}{2}(2m-2-k)} N_{2m-2-k, x, \xi}(f) \\ & \leq \varepsilon \lambda^{\frac{1}{2}(2m-2)} N_{2m-1, x, \xi}(f) + \varepsilon d_\zeta(x, \xi, \lambda)^{2m-2} N_0(f), \end{aligned}$$

for $(x, \xi, \lambda, \zeta) \in \tilde{A}_{R(\varepsilon)}$ if $R(\varepsilon)$ is chosen large enough such that

$$C_\varepsilon \delta^{1/l} \leq \varepsilon.$$

This completes the proof of Proposition 4.4.

Lemma 4.6. *Let $l \in \mathbf{N}$. For any $\varepsilon > 0$, there are a positive real number R and a positive constant C such that if $(x, \xi, \lambda, \zeta) \in \tilde{A}_R$,*

$$\begin{aligned} d_\zeta(x, \xi, \lambda)^{2l} \|f\|^2 & \leq C \sum_{j+|\alpha|=l} \|\tilde{\zeta}^j \tilde{X}(\lambda)^\alpha f\|^2 + \varepsilon \lambda^l N_{l+1, x, \xi}(f) \\ \sum_{j+|\alpha|=l} \|\tilde{\zeta}^j \tilde{X}(\lambda)^\alpha f\|^2 & \leq C d_\zeta(x, \xi, \lambda)^{2l} \|f\|^2 + \varepsilon \lambda^l N_{l+1, x, \xi}(f) \end{aligned}$$

for any $f \in \mathcal{S}$.

Proof. As in Chap.X in [HN]₂, for any $\varepsilon > 0$, there is a positive number R_ε such that if $(x, \xi, \lambda) \in A_{R_\varepsilon}$,

$$\begin{aligned} & \|\{\sum_{|\alpha|=l} X(\lambda)^\alpha - d_0(x, \xi, \lambda)^{2l}\}_f\|^2 \\ & \leq \varepsilon \lambda^l N_{l+1, x, \xi}(f) + \varepsilon d_0(x, \xi, \lambda)^{2l} N_{0, x, \xi}(f) . \end{aligned}$$

On the other hand, if $(x, \xi, \lambda) \in B_R$,

$$\|(\tilde{X}_j(\lambda) - \mathfrak{X}_j^{(0)})_f\|^2 \leq \lambda N_{1, x, \xi}(f) .$$

Hence, if $(x, \xi, \lambda) \in B_R$,

$$\left\| \left\{ \sum_{|\alpha|=l} X(\lambda)^\alpha - d_0(x, \xi, \lambda)^{2l} \right\}_f \right\|^2 \leq C \lambda^l N_{l, x, \xi}(f) .$$

Using Lemma 4.5 again, we see that for any $\varepsilon > 0$, there is a positive number R_ε such that

$$\sum_{j+k \leq l-1} |\tilde{\zeta}|^{2j} \lambda^k N_{k, x, \xi}(f) \leq \varepsilon |\tilde{\zeta}|^{2l-2} N_0(f) + \varepsilon \lambda^{l-1} N_{l, x, \xi}(f)$$

if $|\tilde{\zeta}| \geq R_\varepsilon \lambda^{\frac{1}{2}}$. This completes the proof of Lemma 4.6.

From Lemma 4.4, Proposition 4.3 and Lemma 4.6, it follows that

$$\begin{aligned} & |\operatorname{Im} \tilde{\zeta}| \lambda^{\frac{1}{2}} \sum_{j+|\alpha|=m-1} \|\tilde{\zeta}^j \tilde{X}(\lambda)^\alpha f\|^2 \\ & \leq C |\tilde{Q}(\tilde{\lambda})f, f| + \varepsilon |\operatorname{Im} \tilde{\zeta}| \lambda^{\frac{1}{2}+m-1} N_{2m-1, x, \xi}(f) + C \lambda^{\frac{1}{2}+m-1} N_{4m-2, x, \xi}(f) \end{aligned}$$

for any $f \in \mathcal{S}(\mathbf{R}^n)$ if $|\operatorname{Im} \tilde{\zeta}| \geq R_\varepsilon \lambda^{\frac{1}{2}}$.

To complete the proof of Proposition 4.1, we use a partition of unity as follows:

Let $\phi \in C_0^\infty(\mathbf{R}^n)$ such that

$$\sum_{j \in \mathbf{Z}^n} \phi(x-j)^2 = 1 .$$

For $\mu = (p, q) \in \mathbf{Z}^{2n}$ and $\delta > 0$, let

$$(x_\mu, \xi_\mu) = (p/\delta, q/\delta)$$

and

$$\phi_\mu^\delta(y, \eta) = \phi(\delta y - p) \phi(\delta \eta - q) .$$

Denoting the corresponding pseudo-differential operator by T_μ^δ , we have the following fundamental facts. (c.f. Lemma 4.9 in Chap.X, [HN])

1. For $f \in \mathcal{S}(\mathbf{R}^n)$, $\sum_\mu \|T_\mu^\delta f\|^2 = \|f\|^2$.
2. For any (α, β) , there is a positive constants $C_{\alpha, \beta}(\delta)$ such that

$$\sum_\mu \|(y - x_\mu)^\alpha (D_y - \xi_\mu)^\beta T_\mu^\delta f\|^2 \leq C_{\alpha, \beta}(\delta) \|f\|^2 .$$

for any $f \in \mathcal{S}$

3. If $\tilde{Y}_j(\lambda) \in OPS^1$, $j=1, \dots, q$, for some constant C we have

$$\sum_{\mu} \|(\text{ad } \tilde{Y}_1(\lambda)) \circ \dots \circ (\text{ad } \tilde{Y}_q(\lambda)) T_{\mu}^{\delta} f\|^2 \leq C \delta \lambda^q \|f\|^2$$

for $\lambda > 1$, $\delta \in (0, 1]$, and $f \in \mathcal{S}$, where $(\text{ad } Y) T = YT - TY$.

For $|\alpha| = m$ and two operators Y, T , there are positive constants $C_{\alpha\beta\gamma}$ such that

$$[Y^{\alpha}, T] = \sum_{|\beta|+|\gamma|=m} C_{\alpha\beta\gamma} \{(\text{ad } Y)^{\beta} T\} Y^{\gamma},$$

so that

$$\sum_{\mu} \|[\tilde{X}(\lambda)^{\alpha}, T_{\mu}^{\delta}] f\|^2 \leq C \delta \sum_{|\gamma| \leq m-1} \lambda^{m-|\gamma|} \|\tilde{X}(\lambda)^{\gamma} f\|^2.$$

Summing up, taking δ small enough and letting ε small enough such that

$$C_{\alpha\beta}(\delta) \varepsilon < 1,$$

we have

$$\begin{aligned} & |\text{Im } \zeta| \lambda^{\frac{1}{2}} \sum_{j+|\alpha|=m-1} \|\tilde{\zeta}^j \tilde{X}(\lambda)^{\alpha} f\|^2 \\ & \leq |(\tilde{Q}(\lambda)f, f)| \\ & \leq C \|\tilde{P}_{\zeta}(\lambda)f\| \|\tilde{P}'_{\zeta}(\lambda)f\| \\ & + (\varepsilon |\text{Im } \zeta| + C) \lambda^{\frac{1}{2}} \sum_{|\alpha|+j \leq m-1} \lambda^{\frac{1}{2}(m-1-|\alpha|-j)} \|\tilde{X}(\lambda)^{\alpha} \tilde{\zeta}^j f\|^2. \end{aligned}$$

Since

$$\begin{aligned} \|\tilde{P}_{\zeta}(\lambda)f\| \|\tilde{P}'_{\zeta}(\lambda)f\| & \leq \frac{1}{2} \left\{ (\varepsilon |\text{Im } \zeta| \lambda^{\frac{1}{2}})^{-1} \|\tilde{P}_{\zeta}(\lambda)f\|^2 \right. \\ & \left. + \varepsilon |\text{Im } \zeta| \lambda^{\frac{1}{2}} \sum_{|\alpha|+j \leq m-1} \|\tilde{X}(\lambda)^{\alpha} \tilde{\zeta}^j f\|^2 \right\}, \end{aligned}$$

we get

$$\begin{aligned} & |\text{Im } \zeta| \lambda^{\frac{1}{2}} \sum_{|\alpha|+j=m-1} \|\tilde{\zeta}^j \tilde{X}(\lambda)^{\alpha} f\| \\ & \leq C \|\tilde{P}_{\zeta}(\lambda)f\| + \sum_{|\alpha|+j \leq m-2} (C + \varepsilon |\text{Im } \zeta|) \lambda^{\frac{1}{2}} \|\lambda^{\frac{1}{2}(m-1-|\alpha|-j)} \tilde{\zeta}^j \tilde{X}(\lambda)^{\alpha} f\| \end{aligned}$$

for any $f \in \mathcal{S}(\mathbf{R}^n)$. Therefore, we get the desired inequality (4.1).

Now, we are going to prove Proposition 4.2. Let

$$\mathfrak{X}_j^{(1)}(x, \xi, \lambda)(y, D_y) = S_j(x, \xi, \lambda)(y - x) + T_j(x, \xi, \lambda)(D_y - \xi).$$

Then, S and T belong to the class $\tilde{S}^{\frac{1}{2}}(\lambda)$. By Taylor's theorem in the (u, v) -coordinate, we can write

$$S_j(x, \xi, \lambda) = S_j(0, v, \lambda) + \sum_{j=0}^p R_j(u, v, \lambda)$$

and

$$T_j(x, \xi, \lambda) = T_j(0, v, \lambda) + \sum_{j=0}^p R'_j(u, v, \lambda) .$$

Here, R_j and R'_j are in $\tilde{S}^0(\lambda)$. Since $u_j \sim \tilde{X}_j(\lambda) |\xi|^{-1} \lambda^{-\frac{1}{2}}$, the commutativity assumption in (A-3) gives that the second group in $\tilde{Q}(\lambda) - \tilde{q}_\zeta(x, \xi, \lambda)$ is estimated by

$$\begin{aligned} & \leq C d_0(x, \xi, \lambda) \sum_{j=0}^{2m-3} \sum_{|r| \leq 2m-3-j} \sum_{|\alpha+\beta|=2m-2-j-|r|} |\zeta|^j |\tilde{X}^r(x, \xi, \lambda)| \\ & \times \lambda^{\frac{1}{2}|\alpha+\beta|} |(y-x)^\alpha (D_y - \xi)^\beta f, f| . \end{aligned}$$

Therefore, if $|\operatorname{Im} \zeta| \leq R$, $|\zeta| \geq R_2$ and $|\zeta| \lambda^{\frac{1}{2}} \geq R_3$, then

$$\begin{aligned} & |(\tilde{Q}(\lambda) - \tilde{q}_\zeta(x, \xi, \lambda))f, f| \\ & \leq \{(\varepsilon \lambda^{\frac{1}{2}} |\operatorname{Im} \zeta| + C) \lambda^{m-1} N_{2m-1, x, \xi}(f)^{\frac{1}{2}} + \varepsilon d_\zeta(x, \xi, \lambda)^{2m-2} \|f\|\} \\ & \times N_{4m-2, x, \xi}(f)^{\frac{1}{2}} . \end{aligned}$$

Hence,

$$\begin{aligned} & |\operatorname{Im} \zeta| \lambda^{\frac{1}{2}} \sum_{l+|\alpha|=m-1} \|\tilde{\zeta}^j \tilde{X}(\lambda)^\alpha f\|^2 \\ & \leq |(\tilde{Q}(\lambda)f, f)| + (\varepsilon |\operatorname{Im} \zeta| \lambda^{\frac{1}{2}} + C) \lambda^{m-1} N_{4m-2, x, \xi}(f) . \end{aligned}$$

Let $\tilde{\phi} \in C_0^\infty(\mathbf{R}^n)$ be another function such that $\tilde{\phi} = 1$ on the support of ϕ . Let us define $\tilde{T}_\mu^\delta$ by similar way. Estimating carefully the commutator of $\tilde{Q}(\lambda)$ with T_μ^δ by using the assumption (A-3) and Lemma 4.6 for $f = \tilde{T}_\mu^\delta f$, we have

$$\begin{aligned} & \sum_{\delta} ([\tilde{Q}(\lambda), T_\mu^\delta] \tilde{T}_\mu^\delta f, \tilde{T}_\mu^\delta f) \leq C(\delta) (\varepsilon \lambda^{1/2} |\operatorname{Im} \zeta| + C) \sum_{j+|\alpha| \leq m-1} \lambda^{m-1-|\alpha|-j} \\ & \times \|\tilde{\zeta}^j \tilde{X}(\lambda)^\alpha f\|^2 , \end{aligned}$$

where the constant C is independent of ε . Then, the same argument as in the Propositions 4.1 gives the desired result (4.2).

5. Estimates in the elliptic zone

In this section we shall prove the following result.

Theorem 5.1. *Suppose that (A-1) and (A-4) hold. Let $\tilde{\zeta} = \lambda^{\frac{1}{2}} \zeta$. For any positive number R_3 , there is a positive number R_4 such that if $|\zeta| \leq R_3$, $\operatorname{Im} \zeta < 0$ and $|\operatorname{Im} \zeta| \lambda^{\frac{1}{2}} \geq R_4$,*

$$\begin{aligned} (5.1) \quad & |\operatorname{Im} \zeta| \lambda^{\frac{1}{2}} \sum_{|\alpha|+j=m-1} \|\tilde{\zeta}^j \tilde{X}(\lambda)^\alpha f\| \\ & \leq C (\|\tilde{P}_\zeta(\lambda)f\| + \sum_{|\alpha|+j \leq m-2} \|\lambda^{\frac{1}{2}(m-1-|\alpha|-j)} \tilde{\zeta}^j \tilde{X}(\lambda)^\alpha f\|) \end{aligned}$$

for any $f \in \mathcal{O}$.

We say that $a(\zeta, x, \xi) \in S_{\zeta}^{m,k}(\lambda)$ if and only if a is smooth in (x, ξ) and holomorphic in ζ such that $a_{\zeta} = a(\zeta, x, \xi) \in S^{m,k}(\lambda)$ uniformly with respect to $\zeta \in B(\lambda^{\frac{1}{2}}R_3)$, where $B(r)$ denotes the ball in $\mathbf{C}$ having the center 0 with radius r . Let us denote by $\mathcal{H}_{\zeta}^m(\lambda)$ the set of smooth functions $a(\zeta, v, u)$ which are holomorphic in ζ and smooth in v and u on

$$B(R\lambda^{\frac{1}{2}}) \times \Sigma \times N$$

such that for any α, β and γ , there is a positive constant $C_{\alpha\beta\gamma}$ such that

$$|u^{\alpha} \partial_u^{\beta} \partial_v^{\gamma} a| \leq C_{\alpha\beta\gamma} \lambda^{m-|\gamma|+\frac{1}{2}|\beta|-\frac{1}{2}|\alpha|}$$

for any $(\zeta, v, u) \in B(R\lambda^{\frac{1}{2}}) \times \Sigma \times N$.

Let φ and ψ be two cutoff functions having the similar property to those of ϕ and ψ , respectively such that

$$\varphi \subset \subset \tilde{\varphi} \text{ and } \psi \subset \subset \tilde{\psi}.$$

Proposition 5.2. *Suppose that (A-1) holds. Then, there is an operator $Q_{1,\zeta}(\lambda) \in OPS_{\zeta}^{-m,-m}(\lambda)$ such that*

$$\{Q_{1,\zeta}(\lambda)P(\lambda) - Id\} \phi(x, D) \varphi(D/\lambda^{\frac{1}{2}}) = R_{1,\zeta}(\lambda) \in OP\mathcal{H}_{\zeta}^0(\lambda).$$

Proof. We follow the similar argument to that in [O]₂. In a neighborhood of Σ , the principal symbol $p_{\zeta}(x, \xi, \lambda)$ of $P_{\zeta}(\lambda)$ satisfies

$$|p_{\zeta}(x, \xi, \lambda)| \geq C\lambda^m d_{\Sigma}(\lambda)^m$$

if $\zeta \in B(\lambda^{\frac{1}{2}}R_3)$ and $|\xi||u| \geq R\lambda^{\frac{1}{2}}$ for a large enough $R > 0$. Set

$$q_{3,\zeta}(x, \xi, \lambda) = \chi(x, \xi, \lambda) / p_{\zeta}(x, \xi, \lambda),$$

where χ is a smooth function having its support contained in the set $\{|\xi||u| \geq R\lambda^{\frac{1}{2}}\}$ and taking value 1 if $|\xi||u| \geq 2R\lambda^{\frac{1}{2}}$. Then, by the symbol calculus, we see that

$$\{\text{op}(q_{3,\zeta})P(\lambda) - Id\} \phi(x, D) \varphi(D/\lambda) = R_{3,\zeta}(\lambda) \in OPS_{\zeta}^{-\frac{1}{2},-1}(\lambda)$$

and there exists a symbol $q_{4,\zeta} \in S_{\zeta}^{0,0}(\lambda)$ such that

$$\text{op}(q_{4,\zeta}) - \sum_{j < l} (r_{3,\zeta})^j \in OPS_{\zeta}^{-l/2,-l}$$

for any positive integer l . Let

$$Q_{1,\zeta}(\lambda) = \text{op}(q_{4,\zeta}) \text{op}(q_{3,\zeta}) \quad .$$

Then,

$$Q_{1,\zeta}(\lambda)P_\zeta(\lambda) - Id = R_\zeta(\lambda) \in \cap OPS_\zeta^{-1/2,l} = OP\mathcal{H}_\zeta^0(\lambda) \quad .$$

Let

$$\begin{aligned} \mathfrak{R}_\zeta^m = \{ & a(x, \xi, \lambda, \zeta); \forall \alpha, \beta, \gamma, \exists C, |u^\alpha \partial_u^\beta \partial_v^\gamma a| \leq C \lambda^m (|\text{Im} \zeta| \lambda^{\frac{1}{2}})^{-1-|\gamma|} \\ & \times \lambda^{-\frac{1}{2}|\gamma| + \frac{1}{2}|\beta| - \frac{1}{2}|\alpha|} \} \quad . \end{aligned}$$

Then, we have

Proposition 5.3. *Suppose that (A-1) and (A-4) hold. Then, for any $R_{1,\zeta}(\lambda) \in OP\mathcal{H}_\zeta^0$, there is an operator $Q_{2,\zeta}(\lambda) \in OP\mathfrak{R}_\zeta^{-m}(\lambda)$ such that*

$$Q_{2,\zeta}(\lambda)P_\zeta(\lambda) - R_{1,\zeta}(\lambda) \in OP\mathfrak{R}_\zeta^{-1}(\lambda) \quad .$$

Granting this result, we have

$$\{(Q_{1,\zeta}(\lambda) + Q_{2,\zeta}(\lambda))P_\zeta(\lambda) - Id\} \phi(x, D) \phi(D/\lambda) = R_{2,\zeta}(\lambda) \in OP\mathfrak{R}_\zeta^{-1}(\lambda) \quad .$$

Therefore, the L^2 -boundedness theorem due to Calderón-Vaillancourt, we can show that the inequality (5.1) is valid if R_4 is large enough. This finishes the proof of Theorem 5.1.

The rest of this section is devoted to the proof of Proposition 5.3. First, we have

Lemma 5.4. *For $Q(\lambda) \in OP\mathfrak{R}_\zeta^{-m}(\lambda)$,*

$$\{\sigma(Q(\lambda)P(\lambda)) - P'_\Sigma \sigma(Q(\lambda))\} \phi(\varepsilon/\lambda) \phi(x, \xi) \in \mathfrak{R}_\zeta^{-1}(\lambda) \quad ,$$

where

$$P'_\Sigma = \zeta^m + \sum_{j=0}^{m-1} \zeta^j (A_{m-j})'_\Sigma$$

and $(A_{m-j})'_\Sigma$ are the operators having the form (2.3).

Proof. This can be shown by the similar argument to that in Proposition 5.5 in [B].

To complete the proof of Proposition 5.3, it suffices to show the following result.

Proposition 5.5. *Suppose that (A-1) and (A-4) hold. Then, for any*

$r_{1,\zeta} \in \mathcal{H}_\zeta^0(\lambda)$, there is a symbol $q_{2,\zeta}(\lambda) \in \mathfrak{R}_\zeta^{-m}(\lambda)$ such that

$$P'_{\Sigma} q_{2,\zeta}(\lambda) = r_{1,\zeta}(\lambda) .$$

When $|\zeta| \leq R_3$, we may consider the operator $\tilde{\mathcal{P}}_{x,\zeta}$ as a globally elliptic operator in a sense of Grushin. Let us recall some fundamental facts from [BGH]. Let ω be an open subset of $\mathbf{C}$ and $\Gamma = \omega \times \Sigma \ni \tau = (\zeta, v)$. Modifying the definition in §2, we say that $a(\zeta, v, u) \in C^\infty(\Gamma, S^m(N))$ belongs to the class $S^\zeta_\tau(\Gamma \times N)$ if and only if a is holomorphic in ζ and for any compact subset K in Γ , any multi-index α, β , there is a positive constant $C_{K,\alpha,\beta}$ such that

$$|\partial_v^\alpha \partial_u^\beta a(\zeta, v, u)| \leq C_{K,\alpha,\beta} \langle u \rangle^{m-|\beta|}$$

for $(\zeta, v) \in K$. Also, for $a \in S^\zeta_\tau(\Gamma, N)$ and $\tau = (\zeta, v) \in \Gamma$, set

$$a_\tau(y) = a(\tau, y) \text{ and } L_\tau(y) = L(v, y)$$

and denote its corresponding operator by a_L^τ . Then, we have the analogous result of Theorem 4.1 in [BGH].

Lemma 5.6. *Let $a \in S^\zeta_{\text{reg}}(\Gamma \times N)$ is elliptic: i.e. for some positive number δ ,*

$$|a(\tau, u)| \geq \delta |u|^m$$

for τ in a compact set if $|u|$ is large enough. If a_L^τ is invertible in $\mathcal{L}(E)$ for any $\tau \in \Gamma$, then there is a symbol $b \in S^\zeta_{\text{reg}}(\Gamma \times N)$ such that

$$a_L \cdot b_L = b_L \cdot a_L = Id.$$

Proof. By $a \circ b = a_A b$, we can define a structure of algebra on $S^\infty(N)$. We denote it by $\mathcal{L}_A$. Since a is elliptic, it has a parametrix $b' \in S^\zeta^{-m}(\Gamma \times N)$ and

$$b' \circ a - 1 \in S^\zeta^{-\infty}(\Gamma \times N) .$$

Hence, by the same argument in the proof of Theorem 4.1 in [BGH], we can reduce our assertion to the following result.

Lemma 5.7. *Let $r \in S^\zeta^{-\infty}(\Gamma \times N)$. If $1 + r_\tau$ is invertible for any $\tau \in \Gamma$, then the inverse has the form $(1 + s_\tau)$ with $s \in S^\zeta^{-\infty}(\Gamma \times N)$.*

Proof. Let k be an integer such that $k > \frac{1}{2} \dim N$. For $M \in \mathbf{N}$ and a compact set K of Γ , we define a family of semi-norms on $S^\zeta^{-\infty}(\Gamma \times N)$ by

$$\|a\|_M^K = \sum_{2|\alpha|+j=m} \sup_{(\zeta,v) \in K} \|D_v^\alpha a\|_{E_{k+j}} ,$$

where

$$\|f\|_{E_s} = \|(|D_u|^2 + |u|^2)^{s/2} f\|_{L^2(\mathbf{R})}.$$

Then, by the same method as in the proof of Corollary 4.19 in [BGH], we can show that for a relatively compact open subset U of Γ , there is a positive constant C_0 such that if

$$\|a\|_0^U < 1/C_0,$$

then the sum $\sum_{j=1}^{\infty} a^j$ converges in $S_{\zeta}^{-\infty}(U \times N)$. Therefore, the same reasoning as the last part of §4 in [BGH], we can prove our assertion of Lemma 5.7.

From (2.4), it follows that for $(x, \xi) \in \Sigma$ and for

$$\begin{aligned} a(\zeta, u, v) &= \zeta^m + \sum_{j=0}^{m-1} T_{m-j}(A_{m-j}) \zeta^j, \\ \mathcal{P}_{x,\zeta}(\zeta) &= a_L, \text{ and } P'_\Sigma = a_{t_A}. \end{aligned}$$

We note that a_L is a quasi-homogeneous operator;

$$(a_L)(\zeta \lambda^{\frac{1}{2}}, v \lambda, y/\lambda^{\frac{1}{2}}, \lambda^{\frac{1}{2}} D_y) = \lambda^{\frac{1}{2}m} (a_L)(\zeta, v, y, D_y)$$

and also

$$(a_{t_A})(\zeta \lambda^{\frac{1}{2}}, v \lambda, u/\lambda^{\frac{1}{2}}, \lambda^{\frac{1}{2}} D_u) = \lambda^{\frac{1}{2}m} (a_{t_A})(\zeta, v, u, D_u).$$

The difficulty in the proof of Proposition 5.5 is to get a uniform estimate when ζ approaches to the real line. To do this, we introduce a special coordinate. Let v be fixed. By a metaplectic transformation, we may assume that

$$L(x, \xi) = (\tilde{y}_1, \dots, \tilde{y}_s, \tilde{\eta}_1, \dots, \tilde{\eta}_s) = (\tilde{y}, \tilde{\eta}),$$

where $s = p/2$. Then, we can view a_L as a pseudo-differential operator depending parameter $\tau = (\zeta, v)$:

$$a_L = a_{\zeta,v}(\tilde{y}, D_{\tilde{y}}) = a_\tau(\tilde{y}, D_{\tilde{y}})$$

and the same is valid for a_{t_A} .

From the hypothesis (A-1), it follows that $a_\tau(\tilde{y}, \tilde{\eta})$ is an elliptic symbol and $a_\tau(\tilde{y}, \tilde{D})$ is a Fredholm operator with index 0 because the set of values of the principal symbol of $A_m(x, \xi)$ is contained in a half line of $\mathbf{R}$. In fact, the operator $a_{0,v} - \mu$ has a bounded inverse from $L^2(\mathbf{R}^s)$ to the space $H^{(m)}$ except for the discrete values of μ and the imbedding from $H^{(s)}$ to $H^{(s')}$ is compact if $s > s'$, where

$$H^{(m)} = \{f \in \mathcal{D}'(\mathbf{R}^s); \forall |\alpha| \leq m, (D_y^2 + |\tilde{y}|^2)^{m/2} f \in L^2(\mathbf{R}^s)\}.$$

By the analytic Fredholm theorem (c.f. Theorem VI.14 in [RS]), we know that $\tilde{a}_\tau^{-1}(\tilde{y}, \tilde{D})$ is a meromorphic function in $\zeta \in \mathbf{C}$ with range in $\mathcal{L}(L^2(\mathbf{R}^s), H^{(m)})$.

Since

$$a_\tau = \mathcal{P}_{x,\xi}$$

if $(u, v) = (x, \xi)$ and $u = (y, \eta)$, and the hypothesis (A-4) is invariant under the metaplectic transformation, we see that a_L is injective in $\mathcal{A}(\mathbf{R}^s)$ if $\text{Im} \zeta < 0$. We note that P_Σ is left-inversible and P'_Σ is right-inversible for such ζ . Hence, a_τ^{-1} is also a meromorphic function in $\zeta \in \mathbf{C}$ with range in $\mathcal{L}(L^2(\mathbf{R}^s), L^2(\mathbf{R}^s))$ and its poles are only in the set $\{\text{Im} \zeta \geq 0\}$. We enumerate the poles $\{\zeta_v^{(j)}\}_{j=-\infty}^\infty$ lying on the real line:

$$\zeta_v^{(j)} < \zeta_v^{(k)} \text{ if } j < k.$$

Let $\mathcal{C}_j$ be a sufficiently small circle with center $\zeta_v^{(j)}$. Then, we see that for some positive integer q ,

$$\bar{a}_v^{-1}(\zeta) = \frac{1}{2\pi i} \frac{1}{(\zeta - \zeta_v^{(j)})^q} \oint_{\mathcal{C}_j} (\zeta - \zeta_v^{(j)})^{q-1} \bar{a}_v^{-1}(\zeta') d\zeta' + \mathcal{O}((\zeta - \zeta_v^{(j)})^{-q+1})$$

if ζ is near $\zeta_v^{(j)}$. By the quasi-homogeneity, we can see that for each j , $\zeta_v^{(j)}$ is homogeneous of degree $1/2$ in v and depends continuously on $v/|v|$ since in a locally, it is also a pole of a scalar-valued meromorphic function $(\bar{a}_\tau^{-1}f, g)$ with some f and $g \in \mathcal{A}(\mathbf{R}^s)$. By the second assumption of (A-4), we can conclude that the order of every poles is one. Locally, we can view a pole of a scalar-valued meromorphic function F as a zero point of the holomorphic function $1/F$. Therefore, from the implicit function theorem, it follows that the poles $\zeta_v^{(j)}$ are smooth in $v/|v|$.

By Lemma 5.6, we can write

$$a_\tau^{-1} = (b(\zeta, v, u))_L,$$

where $b(\zeta, v, u)$ is holomorphic in ζ if $\text{Im} \zeta < 0$ and has an analytic continuation near the points $\zeta_v^{(j)}$ which are isolated singular points of $b(\zeta, v, u)$. Using the inequality (1.2) again, we can conclude that the isolated singular points are poles with order 1. We note that the quasi-homogeneity of b

$$b(\zeta \lambda^{\frac{1}{2}}, v \lambda, \lambda^{-\frac{1}{2}} u) = \lambda^{-m/2} b(\zeta, v, u)$$

is important. Let v_0 be a fixed point such that $|v_0| = 1$. The only N poles $\zeta_{v_0}^{(j)}$ are in the domain $\{\zeta \in \mathbf{C}; |\zeta| \leq R_3\}$. Let us consider the closed contours $\mathcal{C}_j, j = 1, \dots, N$

$$\mathcal{C}_j = \{|\zeta - \zeta_{v_0}^{(j)}| = \delta\},$$

where δ is a positive number. If $\delta > 0$ is chosen as small enough, then the in-

terior domains $\text{Int } \mathcal{C}_j$ are disjoint with each other and contain only one pole.

By the continuity of poles with respect to v , there is a small conic neighborhood $\mathcal{U}$ of v_0 such that if $v \in \mathcal{U}$, then

$$|\zeta_{v/|v|}^{(j)} - \zeta_{v_0}^{(j)}| \leq \frac{1}{3}\delta.$$

From Lemma 5.6, it follows that for any α and β , there is a positive constant $C_{\alpha,\beta}$ such that if $v \in \mathcal{U}$ and

$$\begin{aligned} & \zeta \in \cap_{j=1}^N \{|\zeta - \zeta_{v/|v|}^{(j)}| \geq \delta/3\} \cap \{|\zeta| \leq R_3, \text{Im } \zeta < 0\} \\ & \text{or} \\ & \zeta \in \cup_{j=1}^N \{2\delta/3 \geq |\zeta - \zeta_{v/|v|}^{(j)}| \geq \delta/3\} \cap \{|\zeta| \leq R_3, \text{Im } \zeta \geq 0\}, \end{aligned}$$

then

$$|\partial_v^\alpha \partial_u^\beta b(\zeta, v, u)| \leq C_{\alpha,\beta} |v|^{-m} (|u|^2 + |v|)^{-m/2}$$

for $|v| \geq 1$. We note that if $v \in \mathcal{U}$ and $|\zeta - \zeta_{v/|v|}^{(j)}| \leq \delta/3$,

$$\text{dist}(\mathcal{C}_j, \zeta_{v/|v|}^{(j)}) \geq \frac{1}{3}\delta.$$

Hence, using the formula

$$b(\zeta, v/|v|, u) = \frac{1}{2\pi i (\zeta - \zeta_{v/|v|}^{(j)})} \oint_{\mathcal{C}_j} b(\zeta', v/|v|, u) d\zeta' + \Re_j(\zeta),$$

where $\Re_j(\zeta)$ is a holomorphic part near $\zeta_{v/|v|}^{(j)}$ and the quasi-homogeneous property, we see that $\forall \alpha$ and β , $\exists C_{\alpha,\beta}$ such that if $\zeta \in \cup_{j=1}^N \{|\zeta - \zeta_{v/|v|}^{(j)}| \leq \delta/3\}$, $\text{Im } \zeta < 0$,

$$(5.2) \quad |\partial_v^\alpha \partial_u^\beta b(\zeta, v, u)| \leq C_{\alpha,\beta} \lambda^{-m} (|\text{Im } \zeta| \lambda^{\frac{1}{2}})^{-1-|\alpha|} |v|^{-\frac{1}{2}|\alpha|} (|u|^2 + |v|)^{-m/2-|\beta|}$$

because

$$|\partial_v^\alpha \zeta_{v/|v|}^{(j)}| \leq C_\alpha |v|^{\frac{1}{2}-|\alpha|}.$$

By a finite covering of $\{\mathcal{U}\}$ on $B(\lambda^{\frac{1}{2}}R_3) \times \Sigma$, we can conclude that the estimate (5.2) is valid for any $(\zeta, v, u) \in B(\lambda^{\frac{1}{2}}R_3) \times \Sigma \times E$.

Set $q_{2,\zeta} = b_{t_A} r_{1,\zeta}$. Then

$$P'_\Sigma q_{2,\zeta} = a_{t_A} b_{t_A} r_{1,\zeta} = r_{1,\zeta}$$

and we can show that $q_{2,\zeta} \in \mathfrak{R}^m(\lambda)$. This completes the proof of Proposition 5.5.

6. Examples

We say that the Cauchy problem for P is well-posed at $(x_0, t_0) \in \Omega \times I$ if

there is a neighborhood ω of (x_0, t_0) such that for any $f \in C^\infty(\omega)$, $g_j \in C^\infty(\omega \cap \{t=t_0\})$, $0 \leq j \leq m-1$, the equation

$$\begin{cases} Pu = f & \text{in } \omega \cap \{t > t_0\} \\ \partial_t^j u = g_j & \text{on } \omega \cap \{t = t_0\} \end{cases}$$

has a unique solution $u(x, t) \in C^\infty(\omega \cap \{t \geq t_0\})$. Then, it is easy to see that if P and P^* are maximally hyperbolic in $\Omega \times I$, then the Cauchy problem for P is well-posed at each point of $\Omega \times I$. Conversely, if P given by (1.1) is maximally hyperbolic, then the localized operator $\mathcal{P}_{x, \xi}(\zeta)$ is injective $\mathcal{A}(\mathbf{R}^n)$ if $\text{Im} \zeta < 0$ and $(x, \xi, t) \in \Sigma \times I$. This result shall be shown in the next section.

Is our injectivity condition satisfied by any operators? This is an interesting but a difficult problem. Few operators which satisfy it are known. In this section we consider this problem.

Let $t \in \mathbf{R}$ be a time variable and $x \in \mathbf{R}^{n+1}$ be a space variable. We consider a system $\{X_1, \dots, X_{2n}\}$ of $2n$ vector fields on $\mathbf{R}^{n+1}$, given by

$$X_i = D_i = (\sqrt{-1})^{-1} \partial / \partial x_i, \quad 1 \leq i \leq n, \quad X_{n+i} = x_i D_{n+1}, \quad 1 \leq i \leq n.$$

The operators with which we are concerned have the following form

$$\begin{aligned} P(D_t, X) &= D_t^m + \sum_{j=0}^{m-1} a_j(X) D_t^j, \\ a_j(X) &= \sum_{|I|=m-j} a_{j,I} X^I, \end{aligned}$$

where for $I = (i_1, \dots, i_p) \in \{1, \dots, 2n\}^p$,

$$X^I = X_{i_1} \cdots X_{i_p}, \quad a_{j,I} \in \mathbf{C}$$

and

$$\{a_0(\mathcal{E})\}_{\mathcal{E} \in \mathbf{R}^{2n}} \subset \Gamma,$$

where Γ is a half-line in the complex plane. For this operator, we can show the following negative result.

Theorem 6.1. *Let P be an operator as above. Suppose that m is even and*

$$a_0(\mathcal{E}) \not\equiv 0$$

for any $\mathcal{E} \in \mathbf{R}^{2n} \setminus 0$. Then, there exists a non-empty open set U of $\mathbf{C}$ such that the Cauchy problem for the operator

$$P + \alpha D_{n+1}^{m/2}$$

is not well-posed at the origin if $\alpha \in U$.

Proof. We suppose that for any $z \in \mathbf{C}$, the Cauchy problem for $Q(z) = P + z D_{n+1}^{m/2}$ is well-posed at the origin. Let

$$\mathcal{Q}_{\zeta,+}(z) = \zeta^m + \sum_{j+|\alpha|=m, j < m} a_{j,\alpha} \zeta^j \tilde{X}^\alpha + z,$$

where

$$\tilde{X} = \begin{cases} D_i, & \text{if } 1 \leq i \leq n, \\ x_{i-n}, & \text{if } n+1 \leq i \leq 2n. \end{cases}$$

Then, it is easy to see that for any z and ζ , $\text{Im} \zeta < 0$,

$$\ker \mathcal{Q}_{\zeta,+}(z) = \{0\}.$$

Let ζ , $\text{Im} \zeta < 0$ be fixed. From the argument as in the previous section, it follows that the inverse of $\mathcal{Q}_{\zeta,+}(z)$ is an entire function with range in $\mathcal{L}(L^2)$. Moreover, it is classical that the asymptotic behavior of the eigenvalues μ_j of the operator $(\mathcal{Q}_{0,+})^* \mathcal{Q}_{0,+}$ is as follows:

$$\mu_j \sim c_0 j^{m/n} \text{ as } j \rightarrow \infty.$$

(c.f.[S]) Therefore, we can conclude that the operator $\mathcal{Q}_{\zeta,+}^{-1}$ belongs to so called C_p class. (c.f.[DS]) This fact implies that for any $\varepsilon > 0$ there is a constant C such that,

$$\|\mathcal{Q}_{\zeta,+}^{-1}(z)\| \leq C e^{|z|^{\varepsilon}}$$

for any $z \in \mathbf{C}$.

Since the principal symbol $\sigma_0(a_0)$ of $a_0(X)$ satisfies

$$|\sigma_0(a_0)(x, \xi)| \geq \delta \langle x, \xi \rangle^m$$

with $\delta > 0$, $\mathcal{Q}_{\zeta,+}$ has a minimal growth on any ray contained in $\mathbf{C} \setminus \Gamma$: i.e. there are a positive constant C and a positive number ρ_0 such that

$$\|\mathcal{Q}_{\zeta,+}(z)f\| \geq C \left\{ \sum_{|\alpha| \leq m} \|\tilde{X}^\alpha f\| + \|zf\| \right\}$$

if $z \in \Gamma$ and $|z| \geq \rho_0$. In fact, this is easily verified as in §4 in [O]₂. Therefore we can apply the principle of Phragmén-Lindelöf to our operator and we see that $\|\mathcal{Q}_{\zeta,+}^{-1}(z)\|$ is bounded on $\mathbf{C}_z$, which means that $\mathcal{Q}_{\zeta,+}^{-1}(z)$ is independent of z . This is a contradiction.

Let us define a subset $\mathcal{U}$ of $\mathbf{C}$ by

$$\mathcal{U} = \{z \in \mathbf{C}; \exists \zeta, \text{Im} \zeta < 0, \ker \mathcal{Q}_{\zeta,+}(z) \neq \{0\}\}.$$

Now, we are going to prove that $\mathcal{U}$ is open. Let z_0 be a point in $\mathcal{U}$. Then, there is a point ζ_0 such that $\text{Im} \zeta_0 < 0$ and

$$\ker \mathcal{Q}_{\zeta_0,+}(z_0) \neq \{0\}.$$

Since ζ_0 is a pole of the operator $\mathcal{Q}_{\zeta,+}^{-1}(z_0)$, for some positive integer p , we have

$$\int_{\mathcal{C}} \mathcal{Q}_{\zeta,+}^{-1}(z) (\zeta - \zeta_0)^p d\zeta \neq 0$$

at $z=z_0$, where $\mathcal{C}=\{|\zeta-\zeta_0|=\varepsilon\}$ is a contour sufficiently near ζ_0 and contained in $\{\operatorname{Im}\zeta<0\}$. On the other hand, for any z , $\mathcal{Q}_{\zeta,+}(z)^{-1}$ is also a meromorphic function in ζ . Since for $\zeta\in\mathcal{C}$, $\mathcal{Q}_{\zeta,+}(z)$ is uniformly continuous in z near z_0 , we see that for some $p\geq 0$

$$\int_{\mathcal{C}} \mathcal{Q}_{\zeta,+}^{-1}(z) (\zeta-\zeta_0)^p d\zeta \neq 0$$

if $|z-z_0|$ is sufficiently small. This means that the inner domain of $\mathcal{C}$ has a pole ζ of $\mathcal{Q}_{\zeta,+}^{-1}(z)$. Hence,

$$\ker \mathcal{Q}_{\zeta,+}(z) \neq 0$$

if $|z-z_0|$ is sufficiently small. This means that $z\in\mathcal{U}$.

Corollary 6.2. *Suppose that the same condition as in Theorem 6.1 holds and for any j , $a_j(X)$ is formally self-adjoint. If the Cauchy problem for the operator*

$$P+\alpha D_{n+1}^{m/2}$$

is well-posed, then $\alpha\in\mathbf{R}$.

Proof. Let

$$\mathcal{Q}(\lambda) = \zeta^m + \sum_{j=0}^{m-1} a_j(\lambda) \zeta^j + \alpha \lambda^{m/2} ,$$

where

$$a_j(\lambda) = \sum_{|I|=m-j} a_{j,I} X^I(\lambda)$$

and

$$\begin{aligned} X_i &= D_i & \text{if } 1 \leq i \leq n \\ X_{n+i}(\lambda) &= x_i \lambda & \text{if } 1 \leq i \leq n . \end{aligned}$$

If $\zeta\in\mathbf{R}$, the formal self-adjointness of a_j gives

$$|(\mathcal{Q}(\lambda)f, f)| \geq |\operatorname{Im}\alpha| |\lambda|^k \|f\|^2$$

for any $f\in\mathcal{S}(\mathbf{R})$. Therefore, if $\operatorname{Im}\alpha\neq 0$, then any real ζ is not eigenvalue of $\mathcal{Q}(\pm 1)$ because the index of $\mathcal{Q}(\lambda)$ is zero. Since $\mathcal{Q}^{-1}$ is a meromorphic function on the whole space $\mathbf{C}_{\zeta}$, it has at least one pole which is a non-real point. This means that the injectivity breaks down at that point.

Let us denote by H the following operator on $\mathbf{R}^2$

$$D_x^2 + x^2 D_y^2 .$$

For a positive integer k and a complex number c , we consider the operator P

on $\mathbf{R}_t \times \mathbf{R}_x \times \mathbf{R}_y$

$$P = \prod_{j=1}^k \{-D_t^2 + a_j H\} - c D_y^k,$$

where a_j are positive numbers such that

$$a_1 < a_2 < \cdots < a_k.$$

It is easily seen that this operator P satisfies the conditions (A-1) - (A-3).

Let $\Delta_j = [a_j, a_{j+1}]$ if $j > 0$, $\Delta_0 = [0, a_1]$ and

$$f(\tau) = \prod_{j=1}^k (-\tau + a_j).$$

Then, we have

Proposition 6.3. When k is odd, P satisfies (A-4) if $c \in \mathbf{R}$ and

$$\max_{0 \leq j \leq k-1, j=0 \bmod 2} \{\min_{\Delta_j} f(t)\} < \varepsilon^k c < \min_{0 \leq j \leq k-1, j=1 \bmod 2} \{\max_{\Delta_j} f(t)\},$$

for any $\varepsilon = \pm 1$. When k is even, P satisfies (A-4) if and only if $c \in \mathbf{R}$ and

$$\max_{0 \leq j \leq k-1, j=1 \bmod 2} \{\min_{\Delta_j} f(t)\} < c < \min_{0 \leq j \leq k-1, j=0 \bmod 2} \{\max_{\Delta_j} f(t)\}.$$

Proof. By Corollary 6.2, we may assume that $c \in \mathbf{R}$. Let

$$\mathcal{P}_\zeta(\lambda) = \prod_{j=1}^k \{-\zeta^2 + a_j \tilde{H}(\lambda)\} - c \lambda^k,$$

where

$$H(\lambda) = D_x^2 + x^2 \lambda^2.$$

We take a family of Hermite functions $\{h_k(x)\}_{k=0}^\infty$ as an orthonormal basis of $L^2(\mathbf{R})$. Then,

$$\tilde{H}(1) h_k = (2k+1) h_k.$$

Especially,

$$h_0 = \pi^{-\frac{1}{2}} e^{-\frac{1}{2}x^2}.$$

Since $h_k \in \mathcal{B}(\mathbf{R})$ and for $f \in \mathcal{B}(\mathbf{R})$,

$$f = \sum_{k=0}^{\infty} (f, h_k) h_k,$$

$\mathcal{P}_\zeta(\lambda)f=0$ is equivalent to the system of equations

$$(6.1) \quad \forall n, f_n(\zeta^2) - c \varepsilon^k = 0,$$

where $\varepsilon \lambda / |\lambda| = \pm 1$ and

$$f_n(t) = \prod_{j=1}^k (t - a_j(2n+1)) .$$

Let us denote the roots of the following equations

$$f_n(\zeta) - c\varepsilon^k = 0$$

by $\lambda_{n,j}$, $1 \leq j \leq k$. Then, we have

Lemma 6.4. *If the condition in Proposition 6.3 holds, then for any n, j , $\lambda_{n,j}$ are positive and there is a positive constant δ such that for any n ,*

$$|\lambda_{n,j} - \lambda_{n,j'}| > \delta$$

if $j \neq j'$.

Proof. We prove the result when k is odd. The other case can be proved in the same manner. Renaming the roots $\{a_j\}$ of $f(t) = 0$ as follows:

$$a_1 < b_1 < a_2 < b_2 < \cdots < a_{(k+1)/2} .$$

Then, we have

$$\begin{aligned} f(t) &\geq 0 \text{ if } t \in [a_j, b_j] \\ &\leq 0 \text{ if } t \in [b_j, a_{j+1}] . \end{aligned}$$

Let $a_n^{(j)} = (2n+1)a_j$ and $b_n^{(j)} = (2n+1)b_j$. For each $t \in I_n^{(j)} = [a_n^{(j)}, b_n^{(j)}]$, we assign a real number $\alpha \in [0, 1]$ such that

$$t = \alpha a_n^{(j)} + (1 - \alpha) b_n^{(j)} = t_{\alpha,n}^{(j)} .$$

Since

$$\begin{aligned} t_{\alpha,n}^{(j)} - a_n^{(j)} &= (1 - \alpha) (b_j - a_j) (2n+1) , \\ b_n^{(j)} - t_{\alpha,n}^{(j)} &= \alpha (b_j - a_j) (2n+1) \end{aligned}$$

and

$$b_{n+1}^{(j)} - a_{n+1}^{(j)} = (2n+1) (b_j - a_j) , \quad a_{n+1}^{(j)} - b_{n+1}^{(j-1)} = (2n+1) (a_j - b_{j-1}) ,$$

we have

$$f_n(t_{\alpha,n}^{(j)}) \geq f_{n-1}(t_{\alpha,n-1}^{(j)}) .$$

Hence,

$$\max_{I_n^{(j)}} f_n \geq \max_{I_{n-1}^{(j)}} f_n .$$

Similar argument gives

$$\max_{J_n^{(j)}} f_n \leq \max_{J_{n-1}^{(j)}} f_n .$$

where $J_n^{(j)} = [b_n^{(j-1)}, a_n^{(j)}]$ with $b_n^{(0)} = 0$. Therefore, it suffices to show the result

when $n=0$. Taking the shape of the graph $(t, f_0(t))_{t \in \mathbf{R}}$ into consideration, we can get the conclusion.

Now, we can finish to prove Proposition 6.3. In fact, let $\zeta_j^{(n)}$ be the roots of

$$f_n(\zeta^2) = c\epsilon^k.$$

Then, $\zeta_j^{(n)} \in \mathbf{R}$ and there is a positive constant δ such that for any n ,

$$|\zeta_j^{(n)} - \zeta_l^{(n)}| \geq 2\delta > 0$$

if $j \neq l$. Therefore,

$$\begin{aligned} \|\mathcal{P}_\zeta(\epsilon)g\|^2 &= \sum_{n=1}^{\infty} |(f_n(\zeta^2) - c\epsilon^k)c_n(g)|^2 \\ &= \sum_{n=1}^{\infty} \prod_{j=1}^{2k} |\operatorname{Re} \zeta - \zeta_j^{(n)} + i\operatorname{Im} \zeta|^2 |c_n(g)|^2 \\ &\geq |\operatorname{Im} \zeta|^2 \delta^{2(2k-1)} \|g\|^2 \end{aligned}$$

for any $g \in \mathcal{D}(\mathbf{R})$, where $c_n = (g, h_n)$.

Remark 6.5. From the result on the quasi-homogeneous operators in [O]₁, it follows that the well-posedness for the Cauchy problem for P in Proposition 6.3 gives the injectivity of $\mathcal{P}_\zeta(\pm 1)$ if $\operatorname{Im} \zeta < 0$. Hence we can see that if the Cauchy problem for P is well-posed, then $\epsilon^k c$ must be in the closed interval with the same end points as those of the interval in Proposition 6.3.

Finally, we note that the condition (A-4) can be replaced by a simple but stronger condition that for any $(x, \xi) \in \Sigma$, the polynomial

$$p_f(\zeta) = (\mathcal{P}_{x,\xi}(\zeta)f, f)$$

is strictly hyperbolic uniformly in $f \in \mathcal{D}(\mathbf{R}^n) \setminus \{0\}$: i.e. all roots $\zeta_j(f)$ of $p_f(\zeta) = 0$ are real and there is a positive number δ such that for any $f \in \mathcal{D}(\mathbf{R}^n) \setminus \{0\}$,

$$|\zeta_j(f) - \zeta_k(f)| \geq \delta > 0$$

if $j \neq k$. But we do not know what operators satisfy this condition except for $m=2$.

7. Normally hyperbolic operators

In this section, we shall prove the injectivity condition is necessary for P to be maximally hyperbolic. In fact, we can prove a stronger result. To do this, we define a weaker notion as follows. We say that P is normally hyperbolic if and only if for any compact subset K of Ω , there are a real number γ_0 and a positive constant C such that

$$(7.1) \quad \gamma^m \|e^{-\tau t} u\| \leq C \|e^{-\tau t} Pu\|,$$

if $\gamma > \gamma_0$ and $u \in C_0^\infty(K \times I)$. We note that if P is maximally hyperbolic, then P is normally hyperbolic. In fact, it is well-known that there is a real number γ_0 such that

$$\gamma \|e^{-\gamma t} u\| \leq C \|e^{-\gamma t} D_t u\|$$

for $u \in C_0^\infty(\Omega \times I)$ if $\gamma > \gamma_0$. Now, we can show the following result.

Theorem 7.1. *Let P be an operator given by (1.1). If P is normally hyperbolic, then for any point $(x, \xi) \in \Sigma$, the operator $\mathcal{P}_{x, \xi}(\zeta)$ is injective in $\mathcal{S}(\mathbf{R}^n)$ for any $\zeta, \text{Im} \zeta < 0$.*

Proof. The proof is carried out by the localization method of [H]. Let $x_0 = t$, $\xi_0 = \tau$ and

$$x = (x_0, \dots, x_n) = (x_0, x'), \quad \xi = (\xi_0, \dots, \xi_n) = (\xi_0, \xi').$$

Let $N = (1, 0, \dots, 0) \in \mathbf{R}^{n+1}$. For a compact subset K of Ω , let $\varphi \in C_0^\infty(\Omega)$ be equal to 1 in a neighborhood of K . For a fixed point $(x, \xi) \in \Sigma$, let

$$\begin{aligned} f(y) &= \varphi(y) \phi((y-x)|\xi|^{\frac{1}{2}}) e^{i y \cdot \xi + \gamma y_0} \\ &= \varphi(y) v(y) e^{\gamma y_0}. \end{aligned}$$

Putting this function into the inequality (7.1), we have

$$\gamma^m \|\varphi v\| \leq C \|P_\gamma(y, D) \varphi v\|,$$

where

$$P_\gamma = P(y, D_0 + i\gamma N, D').$$

We can write

$$P_\gamma \varphi v = q(y, D + i\gamma N) v,$$

where $q(y, \xi + i\gamma N) \sim p(y, \xi + i\gamma N)$ when $y \in K$ and

$$\|q_\gamma(y, D) v\| = \|q(y, D + \tilde{\xi}) w\|,$$

where $\tilde{\xi} = (\xi_0 + i\gamma, \xi')$ and $w(y) = \phi((y-x)|\xi|^{\frac{1}{2}})$. The Taylor expansion at $\tilde{\xi}$ up to order $m+1$ gives

$$\begin{aligned} \|q_\gamma(y, D) v\| &\leq \|\sum_{|\alpha| \leq m+1} q^{(\alpha)}(y, \tilde{\xi}) D^\alpha w / \alpha!\| \\ &+ C |\xi|^{-m+1} \sum_{|\alpha| = m+1} \|D^\alpha w\|. \end{aligned}$$

Using the Taylor expansion at x

$$\begin{aligned} q^{(\alpha)}(y, \tilde{\xi}) &= \sum_{|\beta| \leq m} q^{(\alpha)}(x, \tilde{\xi}) (y-x)^\beta D^\alpha w / \beta! \\ &+ (m+1) \sum_{|\beta| = m+1} (y-x)^\beta \int_0^1 q^{(\alpha)}(x+t(y-x), \tilde{\xi}) (1-t)^m dt / \beta! \end{aligned}$$

and the fact that

$$\|(y-x)^{\beta} D^{\alpha} w\| = |\xi|^{-(n+1)/4 + \frac{1}{2}(|\alpha| - |\beta|)} \|y^{\beta} D^{\alpha} \phi\| ,$$

we have

$$\begin{aligned} (7.2) \quad & \gamma^{2m} \|\varphi v\|^2 \leq C \|q_{\gamma} v\|^2 \\ & \leq C \int |\sum_{|\alpha+\beta| \leq m} \frac{1}{\alpha! \beta!} q_{(\beta)}^{(\alpha)}(x, \xi) |\xi|^{\frac{1}{2}(|\alpha| - |\beta|)} y^{\beta} D^{\alpha} \phi(y)|^2 dy |\xi|^{(n+1)/2} \\ & + C (1 + |\xi|)^{-1} N_{m+1}(\phi) |\xi|^{(n+1)/2} \end{aligned}$$

for any $\phi \in C_0^{\infty}(\mathbf{R}^{n+1})$. Let

$$q = \sum_{j=0}^m \zeta^j q_{m-j}(x, \xi')$$

and

$$q_k(x, \xi') \sim \sum_{j=0}^{\infty} q_{k,j}(x, \xi') ,$$

where $q_{k,j} \in S_{1,0}^{k-1}$. Then, from

$$(q_{k,j})_{(\beta)}^{(\alpha)}(x, \xi') \neq 0 \text{ if } |\alpha + \beta| + 2j = k$$

it follows that

$$(q_{k,j})_{(\beta)}^{(\alpha)}(x, \xi') |\xi|^{\frac{1}{2}(|\alpha| - |\beta|)} \sim |\xi|^{\frac{1}{2}k} .$$

We note that

$$|\xi|^{\frac{1}{2}(n+1)} \|\varphi v\|^2 \rightarrow \|\phi\|^2$$

as $|\xi| \rightarrow \infty$. Let $\gamma = \tilde{\gamma} |\xi|^{\frac{1}{2}}$, $\text{Im} \gamma_0 < 0$. Dividing (7.2) by $|\xi|^{\frac{1}{2}(m-n-1)}$ and letting $|\xi| \rightarrow \infty$, we have

$$\tilde{\gamma}^{2m} \|\tilde{\phi}\|^2 \leq C \|\mathcal{P}_{x, \xi'}(\eta_0 + i\tilde{\gamma}) \tilde{\phi}\|^2$$

for any $\phi \in C_0^{\infty}(\mathbf{R}^{n+1})$ if $(x, \xi') \in \Sigma$, $|\xi| = 1$. Here,

$$\tilde{\phi} = \tilde{\phi}(\eta_0, x')$$

denotes the Fourier transform of ϕ with respect to the variable x_0 . For $\chi \in C_0^{\infty}(\mathbf{R})$ with $\|\hat{\chi}\| = 1$, we set

$$\hat{\chi}_{\varepsilon}(\eta_0) = \varepsilon^{\frac{1}{2}} \hat{\chi}(\varepsilon(\eta_0 - \eta_0^{(0)})) .$$

For any $f \in \mathcal{S}(\mathbf{R}^n)$, let

$$\tilde{\phi}_{\varepsilon}(x) = \hat{\chi}_{\varepsilon}(\eta_0) \phi_q(x') ,$$

where

$$\phi_q(x') = \tau(x'/q) f(x')$$

with $\tau \in C_0^\infty(\mathbf{R}^n)$ satisfying

$$\tau(x') = 1 \text{ on } |x'| \leq 1.$$

Using

$$\|\{\mathcal{P}_{x,\xi'}(\eta_0 + i\tilde{\gamma}) - \mathcal{P}_{x,\xi'}(\eta_0^{(0)} + i\tilde{\gamma})\}\tilde{\phi}_\varepsilon\| \rightarrow 0 \text{ as } \varepsilon \rightarrow 0 \quad \text{and}$$

taking $\varepsilon \rightarrow +0$, we have

$$\tilde{\gamma}^{2m} \|\phi_q\|^2 \leq C \|\mathcal{P}_{x,\xi'}(\eta_0^{(0)} + i\tilde{\gamma})\phi_q\|^2.$$

Letting $q \rightarrow \infty$ for the obtained inequality, we can conclude that the injectivity in $\mathcal{S}$ of the operator $\mathcal{P}_{x,\xi'}(\zeta)$ holds if $\text{Im}\zeta < 0$ and $(x, \xi') \in \Sigma$.

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