J. Math. Kyoto Univ. (JMKYAZ)
33-3 (1993) 827-864

Irreducible unitary representations of the group of diffeomorphisms of a non-compact manifold

By

Takeshi HIRAI

Introduction

Let M be a connected paracompact C^∞-manifold and $G=\mathrm{Diff}_0(M)$ the group of all diffeomorphisms of M with compact supports. In this paper, we construct new series of irreducible unitary representations (=IURs) of G, for a non-compact M, by using a certain kind of product measures on $X=\prod_{i\in N}M_i$, $M_i=M$, and IURs of the infinite symmetric group $\mathfrak{S}_\infty$ of all finite permutations on the set $\boldsymbol{N}$ of natural numbers.

1. The group G acts on X from the left and the group $\mathfrak{S}_\infty$ acts from the right through permutations of the coordinates. The latter produces intertwining operators when we consider representations of G on L^2-spaces of $\mathfrak{S}_\infty$-invariant measures on X or tensor products (with respect to some reference vectors) of natural representations of G on the Hilbert spaces $L^2(M_i)$. If we can decompose these representations into irreducibles, then we will obtain many different IURs of G. Actually we do not proceed in this direction, but our results obtained here can be viewed, at least from the spirit, as an infinite version of the Weyl's beatiful situation in [18] for finite-dimensional (holomorphic) irreducible reprensentations of the full linear group $G'=GL(n, \boldsymbol{C})$: for the tensor product $\otimes_{i=1}^N V_i$, $V_i=V$, of natural representation of G' on $V=\boldsymbol{C}^n$, the symmetric group $\mathfrak{S}_N$ of indices $\{1, 2, \cdots, N\}$ generates the algebra of its intertwining operators, and thanks to this, there exists a natural correspondence between IURs of $\mathfrak{S}_N$ and irreducible representations of G' with Young diagrams of rank N (for any $N\geqq 1$).

2. Let us explain our method and results more exactly. An element $(x_i)_{i\in N}$ of X is called an *ordered configuration* in M, if the underlying set of points $\{x_i\}_{i\in N}$ is a configuration in M or it has no accumuration points and $x_i \neq x_j$ $(i\neq j)$. The set $\tilde{X}$ of all ordered configurations can be viewed as a principal bundle with a base set $\Omega=\tilde{X}/\mathfrak{S}_\infty$ and fibres $\mathfrak{S}_\infty$. We can introduce on Ω no suitable topology but rather good measurable structures consistent to those on $\tilde{X}$, and then, to construct IURs of G, we can apply a version of the standard method of associated vector bundles.

Received May 7, 1992.

We proceed as follows. Take a measure μ on M, locally equivalent to Lebesgue measures, defined on the family $\mathfrak{M}_M$ of all Lebesgue measurable subsets of M. We call a subset of X of the form $E=\prod_{i\in N}E_i$, $E_i\in\mathfrak{M}_M$, *unital* if E_i's are mutually disjoint and $\mu(E_i)>0$, $\sum_{i\in N}|\mu(E_i)-1|<\infty$. Two unital product subsets E and $F=\prod_{i\in N}F_i$ are said to be *cofinal* (Notation: $E\sim F$) if $\sum_{i\in N}\mu(E_i\ominus F_i)<\infty$. For any fixed E, we denote by $\mathfrak{M}(E)$ the σ-ring generated by the set of unital product subsets $\{F; F\sim E\}$. Then $\nu_E(F)=\prod_{i\in N}\mu(F_i)$ can be uniquely extended to a measure on $\mathfrak{M}(E)$ which is $\mathfrak{S}_\infty$-invariant. This measure ν_E is supported by $\tilde{X}$: $\nu_E(A)=\nu_E(A\cap\tilde{X})$ for $A\in\mathfrak{M}(E)$, and we get a quotient measure on $\Omega=\tilde{X}/\mathfrak{S}_\infty$. Using the measure ν_E on $(X, \mathfrak{M}(E))$ and an IUR Π of $\mathfrak{S}_\infty$, we can construct a unitary representation T_Σ of G, attached to such a datum $\Sigma=(\Pi; \mu, E)$, by a measurable version of the method of associated vector bundles.

The irreducibility of T_Σ is proved in Theorem 4.1. The equivalence criterion for any pair of such repesentations is given in Theorem 5.2. On the way of proving the latter, we encounter an interesting problem, Problem 5.8, on a series $(c_{ij})_{i,j\in N}$ of non-negative real numbers satisfying the condition that $d_i=\sum_{j\in N}c_{ij}>0$, $e_j=\sum_{i\in N}c_{ij}>0$, and $\prod_{i\in N}d_i$, $\prod_{j\in N}e_j$ are unconditionally convergent. (This $(c_{ij})_{i,j\in N}$ can be said to be, essentially, a stochastic matrix of infinite size.)

We remark that the case of I. M. Gelfand et al. [17] is nothing but the case of principal bundle $\tilde{X}\to\Gamma=\tilde{X}/\tilde{\mathfrak{S}}_\infty$ with $\tilde{\mathfrak{S}}_\infty$ the group of all permutations on $\boldsymbol{N}$ and the space of all configurations in M. Moreover groups of diffeomorphisms themselves or their homogeneous spaces are also studied by several mathematicians (e.g., [3], [4], [11]~[12]).

3. The paper is organized as follows. Section 1 is devoted to studying product measures on X and measurable principal bundles $\tilde{X}\to\Omega=\tilde{X}/\mathfrak{S}_\infty$ with measurable structures determined by fixed unital product subsets E and also the structure of the group $\mathrm{Diff}_0(M)$. In Section 2, we construct unitary representations T_Σ for $\Sigma=(\Pi; \mu, E)$. In Section 3, we collect several lemmas and propositions which are necessary to study irreducibility and equivalence relation for T_Σ's. In Section 4, the irreducibility is proved. In Section 5, we establish the criterion for $T_\Sigma\cong T_{\Sigma'}$, $\Sigma=(\Pi; \mu, E)$, $\Sigma'=(\Pi'; \mu, E')$. The principal part of its proof is to get $E'\sim Eb$ for some $b\in\tilde{\mathfrak{S}}_\infty$.

Acknowledgements. The author expresses his hearty thanks to Prof. A. Hora for his important suggestions and kind discussions in the measure theory. The author is also grateful to Prof. K. Nishiyama who read carefully the first draft of this paper and gave many valuable comments.

§ 1. Product measures

Let M be a C^∞-manifold, and $\mathrm{Diff}(M)$ the group of all diffeomorphisms of M. For a $g\in\mathrm{Diff}(M)$, $\mathrm{supp}(g)$ is defined as the closure of the set $\{p\in M; gp\neq p\}$, and put

(1.1) $G=\mathrm{Diff}_0(M)=\{g\in\mathrm{Diff}(M);\ \mathrm{supp}(g)$ is compact$\}$.

We equip G with the natural topology: a sequence g_n converges to g if supp (g) and every supp(g_n) are contained in a compact set C and g_n, together with all its derivatives, tends to g uniformly on C. In this paper, we study irreducible unitary representations (=IURs) of G assuming M to be connected and non-compact. To construct IURs of G, we shall use product measures on the infinite product space $X=M^\infty=\prod_{i\in\boldsymbol{N}}M_i$ with $M_i=M$ for $i\in\boldsymbol{N}$, where $\boldsymbol{N}$ denotes the set of natural numbers.

Notation. For an open subset U and a subset D of M, we put

$$G(U)=\mathrm{Diff}_0(U)\ \text{(the group is connected)},$$

$$G_{|D}=\{g\in G;\ \mathrm{supp}(g)\subset D\}.$$

Note that $G_{|U}\cong G(U)$ for an open $U\subset M$. Other important subgroups $G(E')$ and $G((E'))$ of G are defined in § 3.

1.1. Measurable structures on the product space X. Let us fix a measure μ on M which is equivalent locally to Lebesgue measures, that is, in each coordinate neighbourhood $U\subset M$, $d\mu$ is given as

$$d\mu(p)=w_U(p)dp_1dp_2\cdots dp_n,$$

where $(p_1,p_2,\cdots,p_n)$, $n=\dim M$, is the local coordinates of p, and $w_U(p)$ is a positive measurable function. We assume that $\mu(M)=\infty$ and further that w_U is bounded both from below and from above on every compact subset of U so that $\mu(K)<\infty$ for every compact subset K of M. The measure μ is understood as is defined on the family $\mathfrak{M}_M$ of all Lebesgue measurable subsets of M.

In each component $M_i=M$ of $X=\prod_{i\in\boldsymbol{N}}M_i$, take a measurable subset E_i, and put $E=\prod_{i\in\boldsymbol{N}}E_i$. We call such a subset of X *unital product subset* if it satisfies the following two conditions:

(UPS1) $\sum_{i\in\boldsymbol{N}}|\mu(E_i)-1|<\infty$ and $\mu(E_i)>0$ $(i\in\boldsymbol{N})$;

(UPS2) E_i $(i\in\boldsymbol{N})$ are mutually disjoint.

We denote by $\mathfrak{E}$ the family of all the unital product subsets of X.

We introduce two kinds of equivalence relations in $\mathfrak{E}$ as follows. Let $E=\prod_{i\in\boldsymbol{N}}E_i$ and $E'=\prod_{i\in\boldsymbol{N}}E'_i$ be two elements in $\mathfrak{E}$.

Definition 1.1. E is *cofinal* with E' if the condition

(CF) $\sum_{i\in\boldsymbol{N}}\mu(E_i\ominus E'_i)<\infty$

holds, and E is *strongly cofinal* with E' if the condition

(SCF) $\mu(E_i\ominus E'_i)=0$ for $i\gg 0$ (sufficiently large i)

holds. Here $E_i \ominus E'_i = (E_1 \setminus E'_i) \cup (E'_1 \setminus E_i)$, the symmetric difference.

Denote by $E \sim E'$ (resp. $E \approx E'$) the relation "cofinal" (resp. "strongly cofinal"). The equivalence class for "$\sim$" in $\mathfrak{E}$ contaning E is denoted by $\mathfrak{E}(E)$.

Definition 1.2. A family $\mathfrak{A}$ of subsets of X is called *finitely additive* if it satisfies

$$A \in \mathfrak{A},\ B \in \mathfrak{A} \Longrightarrow A \cup B \in \mathfrak{A}\,, \tag{1.1}$$

$$A \in \mathfrak{A},\ B \in \mathfrak{A} \Longrightarrow A \setminus B \in \mathfrak{A}\,. \tag{1.2}$$

$\mathfrak{A}$ is called *countably additive* if it satisfies

$$A_n \in \mathfrak{A}\ (n=1, 2, \cdots) \Longrightarrow \bigcup_{n=1}^{\infty} A_n \in \mathfrak{A}\,, \tag{1.3}$$

together with (1.2).

Let $\mathfrak{M}_X$ be the countably additive family of subsets of X which is generated by the collection of the subsets of X of the form

$$K = \prod_{i \in N} K_i \text{ with measurable } K_i \subset M_i = M\ (i \in \boldsymbol{N})\,. \tag{1.4}$$

We denote by $\mathfrak{F}(E)$ (resp. $\mathfrak{M}(E)$) the finitely (resp. countably) additive family generated by $\mathfrak{E}(E)$. Then we get the following

Lemma 1.1. (*i*) *The family* $\mathfrak{F}(E)$ *consists of finite unions of subsets of the form*

$$E' \setminus (\cup_j E^{(j)}) \tag{1.5}$$

with E' *and a finite number of* $E^{(j)}$ *in* $\mathfrak{E}(E)$.

(*ii*) *The family* $\mathfrak{M}(E)$ *consists of elements in* $\mathfrak{M}_X$ *which can be covered by a countably infinite number of elements in* $\mathfrak{E}(E)$.

Proof. The first assertion is direct from the definition. So we prove the second assertion. As is easily seen, it is enough to prove that a subset K of X in (1.4) belongs to $\mathfrak{M}(E)$ if and only if it can be covered by some $E^{(k)} \in \mathfrak{E}(E)$, $k=1, 2, \cdots$. The "only if" part is clear and so let us prove the "if" part. Put $K^{(k)} = E^{(k)} \cap K$, then

$$K^{(k)} = \prod_{i \in N} (E_i^{(k)} \cap K_i) = \lim_{j \to \infty} \Big(\prod_{1 \leq i \leq j} (E_i^{(k)} \cap K_i) \Big) \times \Big(\prod_{i > j} E_i^{(k)} \Big)$$

This means that $K^{(k)} \in \mathfrak{M}(E)$. Q.E.D.

1.2. Product measures. Take a unital product subset $E = \prod_{i \in N} E_i$ of X. For any $E' = \prod_{i \in N} E'_i$ in $\mathfrak{E}(E)$, we put

(1.6) $\nu_E(E')=\prod_{i\in N}\mu(E'_i)$,

where the (unconditional) convergence of the infinite product is guaranteed by the condition (UPS1) for E'. Moreover we see from an extension theorem for measures due to Hopf that the product measure on E' of $\mu|E'_i$ $(i\in \boldsymbol{N})$ is uniquely defined. Therefore we have a completely additive measure on $\mathfrak{M}(E)$ extending ν_E on $\mathfrak{F}(E)$. We denote it again by the same symbol ν_E. Since every element in $\mathfrak{M}(E)$ is covered by a countably infinite number of elements in $\mathfrak{F}(E)$, with finite measures, the extension is unique and for any $L\in\mathfrak{M}(E)$

(1.7) $\nu_E(L)=\inf\{\sum_{1\leq k<\infty}\nu_E(E^{(k)});\ \bigcup_{1\leq k<\infty}E^{(k)}\supset L,\ E^{(k)}\in\mathfrak{F}(E)\}$.

Note 1.1. The measure ν_E on $\mathfrak{M}(E)$ is σ-finite in the sense that for any $B\in\mathfrak{M}(E)$, there exist $B_n\in\mathfrak{M}(E)$ such that $\nu_E(B_n)<\infty$ and $B=\bigcup_{n\in N}B_n$.

Now let $F=\prod_{i\in N}F_i$ be another unital product subset of X. Assume that $E\not\nearrow F$ (not cofinal) and compare the product measures ν_E on $\mathfrak{M}(E)$ and ν_F on $\mathfrak{M}(F)$. We know that $\mathfrak{E}(E)\cap\mathfrak{E}(F)=\emptyset$ and that, for any $E'=\prod_{i\in N}E'_i\in\mathfrak{E}(E)$ and $F'=\prod_{i\in N}F'_i\in\mathfrak{E}(F)$,

$$\nu_E(E'\cap F')=\prod_{i\in N}\mu(E'_i\cap F'_i)=0$$

and similarly $\nu_F(E'\cap F')=0$. Then, we get from Lemma 1.1 (ii) the following

Lemma 1.2. *Assume $E\not\nearrow F$. Then for any $L\in\mathfrak{M}(E)\cap\mathfrak{M}(F)$,*

$$\nu_E(L)=0 \quad \textit{and} \quad \nu_F(L)=0\,.$$

1.3. Actions of G and $\mathfrak{S}_\infty$ on X. An element $g\in G$ acts naturally on X as, for $x=(x_i)_{i\in N}\in X$,

(1.8) $gx=(gx_i)_{i\in N}$.

Further, let $\mathfrak{S}_\infty$ denote the infinite symmetric group or the group of all finite permutations of $\boldsymbol{N}$. Then $\sigma\in\mathfrak{S}_\infty$ acts on X from the right as

(1.9) $x\sigma=(x'_i)_{i\in N}$ with $x'_i=x_{\sigma(i)}$.

Then, since $(\sigma\tau)\,(i)=\sigma(\tau(i))$ for σ, $\tau\in\mathfrak{S}_\infty$, we have $x(\sigma\tau)=(x\sigma)\tau$.

These actions of G and $\mathfrak{S}_\infty$ on X commute with each other, and induce those on subsets of X. First let $g\in G$. Take a unital product subset $E=\prod_{i\in N}E_i$ of X. Then $gE=\prod_{i\in N}gE_i$ is also a unital product subset and $gE\sim E$. In fact, gE_i's are mutually disjoint, and putting $S_g=\mathrm{supp}(g)$, we have $E_i\ominus gE_i\subset S_g$ and so

$$E_i\ominus gE_i\subset(E_i\cap S_g)\cup(gE_i\cap S_g) \quad \text{for any } i\in\boldsymbol{N}\,.$$

Hence

$$\sum_{i\in N}\mu(E_i\ominus gE_i)\leqq\sum_i\mu(E_i\cap S_g)+\sum_i\mu(gE_i\cap S_g)\leqq 2\mu(S_g)<\infty .$$

Thus we get $E\sim gE$.

For $\sigma\in\mathfrak{S}_\infty$, the situation is much simpler. In fact, $E\sigma=\prod_{i\in N}E_i'$ with $E_i'=E_{\sigma(i)}$ and so $E_i'=E_i$ for $i\gg 0$, whence $E\sigma\approx E$.

Thus we obtain, with some additional discussions, the following.

Lemma 1.3. *Let $E=\prod_{i\in N}E_i$ be a unital product subset of X.*

(i) For any $g\in G$, gE is cofinal with E: $gE\sim E$. And for any $\sigma\in\mathfrak{S}_\infty$, $E\sigma$ is strongly cofinal with E : $E\sigma\approx E$.

(ii) The family $\mathfrak{C}(E)$ is invariant under $g\in G$ and $\sigma\in\mathfrak{S}_\infty$, and consequently so are the families $\mathfrak{F}(E)$ and $\mathfrak{M}(E)$.

1.4. Jacobians for the actions of G and $\mathfrak{S}_\infty$. Let $g\in G$ and $E'=\prod_{i\in N}E_i'\in\mathfrak{C}(E)$. Then, $gE'\in\mathfrak{C}(E)$ and

$$\nu_E(gE')=\prod_{i\in N}\mu(gE_i')=\prod_{i\in N}\int_{E_i'}\rho_M(g;x_i)d\mu(x_i)\,, \tag{1.10}$$

where

$$\rho_M(g;p)=\frac{d\mu(gp)}{d\mu(p)}\qquad\text{for}\quad p\in M\,. \tag{1.11}$$

For $x=(x_i)_{i\in N}\in E'$, let us prove that the infinite product

$$\rho_E(g|x)=\prod_{i\in N}\rho_M(g;x_i) \tag{1.12}$$

converges ν_E-almost everywhere on E'. Since $S_g=\mathrm{supp}(g)$ is compact, there exist two constants C_1, C_2 such that

$$C_1\leqq\rho_M(g;p)\leqq C_2\ (p\in S_g)\,. \tag{1.13}$$

Further $\rho_M(g;\rho)=1$ for $p\notin S_g$. Therefore the infinite product (1.12) converges for $x\in E'$ such that $x_i\notin S_g$ for almost all indices $i\in N$ (or except a finite number of i). Our assertion follows from

Lemma 1.4. *Let $E'\in\mathfrak{C}(E)$ and S be a measurable subset of M with finite measure. Put*

$$E'_{(S)}=\{x=(x_i)_{i\in N}\in E';\ x_i\notin S\ \textit{for}\ i\gg 0\}\,.$$

Then $E'_{(S)}\in\mathfrak{M}(E)$ and $\nu_E(E'\setminus E'_{(S)})=0$.

Proof. The subset $E'_{(S)}$ is the union of

$$\Big(\prod_{1\leqq i\leqq N}E_i'\Big)\times\prod_{i>N}(E_i'\setminus S)$$

over $N=1, 2, \cdots$, each of which is cofinal with E' and so with E. Therefore $E'_{(S)}$ is in $\mathfrak{M}(E)$ and its measure is given by

$$\nu_E(E'_{(S)})=\lim_{N\to\infty}\{\prod_{1\leq i\leq N}\mu(E'_i)\cdot\prod_{i>N}\mu(E'_i\setminus S)\}$$

$$=\prod_{i\in N}\mu(E'_i)=\nu_E(E')\,. \qquad \text{Q.E.D.}$$

Now, as is seen below, we can rewrite (1.10) as

$$\nu_E(gE')=\int_{E'}\rho_E(g|x)d\nu_E(x)$$

so that we get the following theorem.

Theorem 1.5. *Let E be a unital product subset of X and ν_E be the associated product measure on $(X, \mathfrak{M}(E))$. Then, for $g\in G$ and $x=(x_i)\in E'\sim E$,*

$$\text{(1.14)} \qquad \frac{d\nu_E(gx)}{d\nu_E(x)}=\rho_E(g|x)\equiv\prod_{i\in N}\rho_M(g;x_i)\,.$$

Proof. It is enough to prove that the infinite product $\prod_{i\in N}\sqrt{\rho_M(g;x_i)}$ converges in $L^2(E', d\nu_E|E')$, because its L^2-norm is convergent accordingly. Let us evaluate

$$I_{m,n}=\|\prod_{1\leq i\leq m}\sqrt{\rho_M(g;x_i)}-\prod_{1\leq i\leq n}\sqrt{\rho_M(g;x_i)}\|^2_{L^2(E')}$$

for $m<n$. This is equal to

$$(\prod_{1\leq i\leq m}\mu(gE'_i))\cdot(\prod_{i>n}\mu(E'_i))\cdot\prod_{m<i\leq n}\int_{E'_i}|\sqrt{\rho_M(g;x_i)}-1|^2d\mu(x_i)$$

Recall the evaluation (1.13) and put $C=\max\{|\sqrt{C_1}-1|^2, |\sqrt{C_2}-1|^2\}$, then

$$\int_{E'_i}|\sqrt{\rho_M(g;x_i)}-1|^2d\mu(x_i)\leqq C\cdot\mu(E'_i\cap S_g)\,.$$

Note that $\prod_{i\in N}\mu(gE'_i)$ and $\prod_{i\in N}\mu(E'_i)$ are convergent and $\sum_{i\in N}C\cdot\mu(E'_i\cap S_g)\leqq C\cdot\mu(S_g)<\infty$. Then we see $I_{m,n}\to 0$ $(m, n\to\infty)$, as desired. Q.E.D.

Note that the above proof follows the line of Kakutani [10].

Let us now consider the action of $\mathfrak{S}_\infty$. We can easily see that, for $\sigma\in\mathfrak{S}_\infty$, there holds $\nu_E(E'\sigma)=\nu_E(E')$ for any $E'\in\mathfrak{E}(E)$, and therefore that

$$\text{(1.15)} \qquad \frac{d\nu_E(x\sigma)}{d\nu_E(x)}=1 \qquad (x\in E')\,.$$

1.5. Relation to two kinds of configuration spaces. An element $x=(x_i)_{i\in N}$ in $X=\prod_{i\in N}M_i$, $M_i=M$, is called an *ordered configuration* in M if (a) x_i

$\neq x_j$ for $i\neq j$, and (b) the sequence x_i, $i\in \boldsymbol{N}$, has no accumulation points in M. Denote by $\tilde{X}$ the set of all ordered configurations in M. Here let us study the product measure ν_E in relation to $\tilde{X}$ and $\mathfrak{S}_\infty$.

Recall that a locally finite subset of M is called a configuration in M. We denote by Γ_M the set of all infinite configurations in M. Let $\tilde{\mathfrak{S}}_\infty$ be the group of all permutations on $\boldsymbol{N}$: $\tilde{\mathfrak{S}}_\infty\supset\mathfrak{S}_\infty$. Then, naturally $\tilde{X}/\tilde{\mathfrak{S}}_\infty\cong\Gamma_M$ and we have a principal fibre bundle $\tilde{X}\to\Gamma_M$. Using this fibre bundle, I. M. Gelfand and others have constructed irreducible unitary representations (=IURs) of the group G in [17].

Let us begin with the following important fact.

Lemma 1.6. *The subset $\tilde{X}$ of X belongs to $\mathfrak{M}_X$. For any unital product subset E, the product measure ν_E is carried by $\tilde{X}$ in the sense that, for any $L\in\mathfrak{M}(E)$, $L\cap\tilde{X}$ is also in $\mathfrak{M}(E)$ and*

$$\nu_E(L)=\nu_E(L\cap\tilde{X})\,. \tag{1.16}$$

Proof. Take a system of countable open base $\{U_j\}_{j\in N}$ of M such that each U_j is relatively compact, that is, the closure $\mathrm{cl}(U_j)$ is compact. Then $\tilde{X}$ is expressed as

$$\bigcap_{j\in N}\bigcup_{0\leq N<\infty}\{x=(x_i)\in X\,|\,\#\{i\,|\,x_i\in U_j\}\leq N\}\,,$$

where $\#A$ denotes the number of elements of a set A. Therefore $\tilde{X}$ belongs to $\mathfrak{M}_X$.

By Lemma 1.1 (ii), every element in $\mathfrak{M}(E)$ is covered by an infinite number of $E^{(q)}\in\mathfrak{M}(E)$, $q=1,2,\cdots$. Hence, to prove the second assertion, it is enough to prove it for $L=E^{(q)}=\prod_{i\in N}E_i^{(q)}$. Then

$$E^{(q)}\cap\tilde{X}=\bigcap_{j\in N}\bigcup_{1\leq N<\infty}E_{j,N}^{(q)}\qquad\text{with}$$

$$E_{j,N}^{(q)}=\Big(\prod_{1\leq i\leq N}E_i^{(q)}\Big)\times\Big(\prod_{i>N}\Big(E_i^{(q)}\setminus\bigcup_{1\leq k\leq j}U_k\Big)\Big)\,.$$

The subsets $E_{j,N}^{(q)}$ of $E^{(q)}$ are decreasing in j and increasing in N. Since $\nu_E(E^{(q)})<\infty$ and $\mu(U_k)<\infty$ for any k, we get

$$\begin{aligned}\nu_E(E^{(q)}\cap\tilde{X})&=\lim_{j\to\infty}\lim_{N\to\infty}\nu_E(E_{j,N}^{(q)})\\&=\lim_{j\to\infty}\prod_{i\in N}\mu(E_i^{(q)})=\nu_E(E^{(q)})\,.\end{aligned}$$

Q.E.D.

The above lemma suggests that the measure ν_E can be considered on $\tilde{X}$ instead of X. Since the family $\mathfrak{M}(E)$ and measure ν_E are invariant under the action of $\mathfrak{S}_\infty$, we may induce from ν_E a measure on the quotient space $\Omega=\Omega_M=\tilde{X}/\mathfrak{S}_\infty$. Then, it may be possible to consider π_Ω: $\tilde{X}\to\Omega_M$ as a principal fibre bundle for the group $\mathfrak{S}_\infty$, and to follow the standard method employed for $\tilde{X} \to \Gamma_M$ also in this situation to construct IURs of $G=\mathrm{Diff}_0(M)$, which acts from the left on $\tilde{X}$ and so on Γ_M and Ω_M.

Definition 1.3. A subset B of Ω_M is called *E-measurable* if $\pi_\Omega^{-1}(B)$ belongs to $\mathfrak{M}(E)$. The set of all E-measurable sets is denoted by $\mathfrak{M}_\Omega(E)$.

The measure $\nu_{E,\Omega}$ is defined as follows: for a measurable $B\subset\Omega_M$, take a *fundamental* set $L\in\mathfrak{M}(E)$ for B such that

$$\pi_\Omega^{-1}(B)=\coprod_{\sigma\in\mathfrak{S}_\infty}L\sigma \qquad \text{(disjoint union)}, \tag{1.17}$$

and put

$$\nu_{E,\Omega}(B)=\nu_E(L). \tag{1.18}$$

Then this value does not depend on the choice of L. Hence we should prove the existence of a fundamental set L.

Lemma 1.7. *For any $B\in\mathfrak{M}_\Omega(E)$, there exists an $L\in\mathfrak{M}(E)$ satisfying* (1.17).

Proof. Since $\pi_\Omega^{-1}(B)$ is in $\mathfrak{M}(E)$, it is covered by a countably infinite number of $E^{(q)}\in\mathfrak{E}(E)$. Put $L_q=E^{(q)}\cap\pi_\Omega^{-1}(B)$. Then, for any q, $L_q\sigma$ ($\sigma\in\mathfrak{S}_\infty$) are mutually disjoint because $E^{(q)}$ satisfies the condition (UPS2). Now set for $q\geqq1$,

$$L'_q=L_q\setminus\bigcup_{\sigma\in\mathfrak{S}_\infty}\Big(\bigcup_{1\leqq j<q}L_j\Big)\sigma$$

and $L=\coprod_{1\leqq q<\infty}L'_q$. Then, $L\sigma$ ($\sigma\in\mathfrak{S}_\infty$) are mutually disjoint and $\pi_\Omega^{-1}(B)=\coprod_{\sigma\in\mathfrak{S}_\infty}L\sigma$. Q.E.D.

As seen above, we may call π_Ω: $\tilde{X}\to\Omega_M$ a *measurable* principal bundle in the sense that the group $\mathfrak{S}_\infty$ acts mesurably on $\tilde{X}$ and the projection π_Ω is also measurable. Lemma 1.7 says, in other words, that for any $\mathfrak{S}_\infty$-invariant $A\in\mathfrak{M}(E)$ there exists an $F\in\mathfrak{M}(E)$ such that $A=\coprod_{\sigma\in\mathfrak{S}_\infty}F\sigma$ (disjoint union). In a certain degree, this property for π_Ω: $\tilde{X}\to\Omega_M$, replaces the local triviality property in the case of usual fibre bundles. In later sections, in studying the irreducibility and mutual equivalence of the representations of G constructed there, it is convenient for us to keep the presentation in the form of $(X, \mathfrak{M}(E), \nu_E)$ rather than that in the form of $(\Omega_M, \mathfrak{M}_\Omega(E), \nu_{E,\Omega})$. So we will use these two forms according to the situations for the convenience of discussions.

1.6. Remarks on topologies. First note that, in the case when the space of configurations Γ_M and the fibre bundle $\tilde{X}\to\tilde{X}/\mathfrak{S}_\infty\cong\Gamma_M$ are treated, it is natural to consider a metric d on $X\supset\tilde{X}$ given as

$$(1.19) \qquad d(x, y)=\sup_{i\in N} d_M(x_i, y_i)$$

for $x=(x_i)$, $y=(y_i)$ in X, where d_M is a metric on M. Then, denoting the orbit $x\tilde{\mathfrak{S}}_\infty$ by $[x]$, we have a metric on Γ_M as

$$d_\Gamma([x], [y])=\inf_{\sigma,\tau\in\tilde{\mathfrak{S}}_\infty} d(x\sigma, y\tau)=\inf_{\sigma\in\tilde{\mathfrak{S}}_\infty} d(x\sigma, y),$$

for $x, y\in\tilde{X}$.

In contrast to this, as in our present situation, in the case when the quotient space Ω_M and the measurable fibre bundle $\tilde{X}\to\tilde{X}/\mathfrak{S}_\infty\cong\Omega_M$ are treated, any suitable topologies cannot be found. In fact, when we introduce the above metric d in $\tilde{X}$ (or in X), the orbit $x\mathfrak{S}_\infty$ is not necessarily closed. So, we should consider in X a much stronger topology. Let us consider a topology for which an open base is given by the family of subsets of the form

$$(1.20) \qquad U=\prod_{i\in N} U_i \text{ with open } U_i\subset M \; (i\in \boldsymbol{N}).$$

Then, on the one hand, this topology is suitable for the present situation in the following points.

(1) Every element in the open base is in $\mathfrak{M}_X$.

(2) For any $E'=\prod_{i\in N}E'_i$ in $\mathfrak{M}(E)$ and $\varepsilon>0$, there exists an open $U\in\mathfrak{M}(E)$ such that $\nu_E(E'\ominus U)<\varepsilon$.

(3) Every orbit $x\mathfrak{S}_\infty$ is closed in X.

(4) The action of $\mathfrak{S}_\infty$ is discontinuous in the sense that (D1) for any $x\in\tilde{X}$, there exists a neighbourhood U_x such that $U_x\sigma\cap U_x=\emptyset$ for $\sigma\in\mathfrak{S}_\infty$ except $\sigma=1$, and further that (D2) for any $x, y\in\tilde{X}$ such that $x\mathfrak{S}_\infty\neq y\mathfrak{S}_\infty$, there exist neighbourhoods U_x and U_y such that $U_x\sigma\cap U_y=\emptyset$ for any $\sigma\in\mathfrak{S}_\infty$.

On the other hand, this topology is not suitable with the present situation in the points that, for an open set $U=\prod_{i\in N}U_i$, the measure $\nu_E(U\cap E')$, with $E'\in\mathfrak{E}(E)$, is in general zero, and that we even do not know if $V\cap E$ is in $\mathfrak{M}(E)$ for any open V.

Note 1.2. In this strong topology, the connected component of a point $x=(x_i)_{i\in N}$ in $X=\prod_{i\in N}M_i$, $M_i=M$, is equal to the union of $(\prod_{1\leq i\leq N}M_i)\times(x_i)_{i>N}$ over $N\geqq 1$ (cf. Exerc.I.11.8 in [1]). Therefore each orbit $x\mathfrak{S}_\infty$ is contained in the connected component of x.

1.7. Relation to an infinite tensor product of Hilbert spaces. Let us consider two Hilbert spaces

$$\mathfrak{H}_M=L^2(M, \mathfrak{M}_M, \mu) \quad\text{and}\quad \mathfrak{H}(E)=L^2(X, \mathfrak{M}(E), \nu_E),$$

where $E=\prod_{i\in N}E_i$ is a unital product subset of X. Then the characteristic function χ_{E_i} of $E_i\subset M$ belongs to $\mathfrak{H}_M$ and the one χ_E for $E\subset X$ belongs to $\mathfrak{H}(E)$, and we have a formal expression

$$\chi_E=\otimes_{i\in N}\chi_{E_i} \quad \text{or} \quad \chi_E(x)=\otimes_{i\in N}\chi_{E_i}(x_i) \quad \text{for} \quad x=(x_i)\,. \tag{1.21}$$

This expression has a rigorous meaning in terms of infinite tensor product of Hilbert spaces (cf. [16]). Take a unit vector $\varphi_i=\chi_{E_i}/\|\chi_{E_i}\|$ from $\mathfrak{H}_M$ and choose $\varphi=(\varphi_i)_{i\in N}$ as a reference vector to form the tensor product

$$\otimes^{\varphi}_{i\in N}\mathfrak{H}_i\equiv\otimes_{i\in N}\{\mathfrak{H}_i,\ \varphi_i\} \quad \text{with} \quad \mathfrak{H}_i=\mathfrak{H}_M\ (i\in \boldsymbol{N})\,. \tag{1.22}$$

Then we can prove that this tensor product Hilbert space is canonically isomorphic to $\mathfrak{H}(E)$, and (1.21) is an equality through this isomorphism. So that the Hilbert space $\mathfrak{H}(E)$ is seen to be separable.

In this and many other reasons, we can understand that, fixing the unital product subset $E=\prod_{i\in N}E_i$ (at the starting point), we choose a direction in which "tensor products" of many things will be taken. This direction is nothing but a one in which the ordered configurations $x=(x_i)$ remain to follow as $i\to\infty$.

1.8. Normalization of unital product subsets. Let $E=\prod_{i\in N}E_i$ be a unital product subset of $X=\prod_{i\in N}M_i$, $M_i=M$. We wish to choose a good unital product subset $E^{(0)}=\prod_{i\in N}E_i^{(0)}$, cofinal with E, as a representative of the equivalence class $\mathfrak{E}(E)$. Then we will see that it is more convenient to use $E^{(0)}$ instead of E. This replacement of E by $E^{(0)}$ is natural in the sense that $\mathfrak{M}(E)=\mathfrak{M}(E^{(0)})$, $\nu_E=\nu_{E^{(0)}}$ and so on.

Now let us choose $E^{(0)}$. Since the measure μ on M is locally equivalent to Lebesgue measures, there exists, for every i, a relatively compact, open subset $E_i'\subset M$ such that $\mu(E_i\ominus E_i')<2^{-i}$. We may further assume that every E_i' has only a finite number of connected components and that $\mu(\bar{E}_i'\setminus E_i')=0$, where $\bar{E}_i'=\mathrm{cl}(E_i')$. Put $E_i''=E_i'\setminus(\bigcup_{1\leq j<i}\bar{E}_j')$. Then E_i'' are mutualy disjoint and

$$E_i\ominus E_i''\subset(E_i\ominus E_i')\cup\{\textstyle\bigcup_{1\leq j<i}(E_i\cap\bar{E}_j')\}\,.$$

Thus we have

$$\sum_{1\leq i<\infty}\mu(E_i\ominus E_i'')\leqq\sum_{1\leq i<\infty}\mu(E_i\ominus E_i')+\sum_{i>j}\mu(E_i\cap\bar{E}_j')\,,$$

$$\sum_{i>j}\mu(E_i\cap\bar{E}_j')\leqq\sum_{1\leq j<\infty}\mu((\bigcup_{j<i<\infty}E_i)\cap\bar{E}_j')\leqq\sum_{1\leq j<\infty}\mu(\bar{E}_j'\setminus E_j)\,,$$

whence $\sum_i\mu(E_i\ominus E_i'')\leqq 1+1=2$, and so $E=\prod_{i\in N}E_i$ is cofinal with $E''=\prod_{i\in N}E_i''$: $E\sim E''$.

We can replace E'' further by another one to arrive at

Proposition 1.8. *Any unital product subset* $E=\prod_{i\in N}E_i$ *is cofinal with a unital product subset* $E^{(0)}=\prod_{i\in N}E_i^{(0)}$ *having the following properties.*

(UPS3) *The closures* $\mathrm{cl}(E_i^{(0)})$ *and* $\mathrm{cl}(\bigcup_{j\neq i}E_j^{(0)})$ *are mutually disjoint for any* *i. Each* $E_i^{(0)}$ *is relatively compact, open and with only a finite number of connected components.*

(USP4) *In case* $n=\dim M>1$, $E_i^{(0)}$ *is connected, simply connected and diffeomorphic* (*together with its closure*), *by an element in G, to an open ball* (*together with its closed ball*) *in a coordinate neighbourhood. For* $i\neq j$, $E_i^{(0)}$ *and* $E_j^{(0)}$ *can be connected by an open path* P_{ij} *such that* $\mathrm{cl}(P_{ij})\cap\mathrm{cl}(\bigcup_{k\neq i,j}E_k^{(0)})=\emptyset$.

Here an *open path* between connected and simply connected sets A and B means a connected, simply connected open set P such that $P\cap A$ and $P\cap B$ are non-empty and connected, simply connected together with the union $P\cup A\cup B$.

Proof. We construct $E_i^{(0)}$ from E_j'''s inductively according to i. For the condition (UPS3), it is enough to shrink each E_i'' a little if necessary. So we consider the condition (UPS4).

We will define B_i inductively and work on it. For $i=1$, put $B_1=E_1''$.

STEP 1. Assume B_i be given. Let the connected components of B_i be $U_1, U_2, \cdots, U_r$. If some U_s is not simply connected, then we cut it by a finite number of hypersurfaces into simply connected pieces and shrink them so that the closures of pieces are mutually disjoint.

STEP 2. Thus, assuming each U_s is simply connected, we connect U_s to U_{s+1} by an open path in such a way that all such paths do not intersect mutually and they form together with all U_s's a connected, simply connected open set $E_i^{(0)}$. The demand "diffeomorphic (together with its closure) to an open ball (together with its closed ball)" in (USP4) is seen to be satisfied if we shrink $E_i^{(0)}$ a little if necessary. Thus we can get $E_i^{(0)}$ so that the difference $B_i\ominus E_i^{(0)}$ is so small that

$$\mu(B_i\ominus E_i^{(0)})<2^{-i}, \tag{1.23}$$

and that $M\setminus\mathrm{cl}(\bigcup_{1\leq j\leq i}E_j^{(0)})$ is connected (note that $M\setminus\mathrm{cl}(\bigcup_{1\leq j\leq i-1}E_j^{(0)})$ is connected by induction hypopthesis).

STEP 3. For $i>1$, we still choose open paths P_{ij} connecting $E_i^{(0)}$ with $E_j^{(0)}$ for $1\leqq j<i$ so that

$$\mathrm{cl}(P_{ij})\cap\mathrm{cl}(E_k^{(0)})=\emptyset \quad \text{for} \quad k\neq j,\quad 1\leqq k<i, \tag{1.24}$$

$$\textstyle\sum_{1\leq j<i}\mu(P_{ij})<2^{-i},$$

and that the union of $E_j^{(0)}$, $1\leqq j\leqq i$, and P_{jk}, $1\leqq k<j\leqq i$, does not cover any E_l'' for $l>i$ (if necessary, shrink $E_i^{(0)}$ a little).

STEP 4. Now assume that $E_1^{(0)}, E_2^{(0)}, \cdots, E_q^{(0)}$ have already been chosen. To choose $E_{q+1}^{(0)}$, we start with

$$B_{q+1}'=E_{q+1}''\setminus\{\bigcup_{1\leq j\leq q}\mathrm{cl}(E_j^{(0)})\cup\bigcup_{1\leq k<j\leq q}\mathrm{cl}(P_{jk})\} \tag{1.25}$$

and shrink it if necssary to get an open set $B_{q+1}\subset B'_{q+1}$ such that

(1.26) $$\mu(B'_{q+1}\setminus B_{q+1})<2^{-(q+1)},\quad \mathrm{cl}(B_{q+1})\cap\mathrm{cl}(E_j^{(0)})=\emptyset\ (1\leqq j\leqq q).$$

Then we follow Steps 1 to 3 for $i=q+1$. Thus we get $E_i^{(0)}$ and P_{ij}, $1\leqq j<i$. The induction process is now completed.

Let us evaluate the sum $\sum_{i\in N}\mu(E''_i\ominus E_i^{(0)})$. Note that

$$E''_i\ominus E_i^{(0)}\subset(E''_i\ominus B'_i)\cup(B'_i\ominus B_i)\cup(B_i\ominus E_i^{(0)})$$

with $B'_1=B_1$, and take into account $B'_i\ominus B_i=B'_i\setminus B_i$ and (1.23), (1.26), then we get

$$\sum_{i\in N}\mu(E''_i\ominus E_i^{(0)})\leqq\sum_{i\in N}\mu(E''_i\ominus B'_i)+2.$$

On the other hand,

$$E''_i\ominus B'_i=E''_i\setminus B'_i=E''_i\cap(\bigcup_{1\leqq j<i}\mathrm{cl}(E_j^{(0)})\cup\bigcup_{1\leqq k<j<i}\mathrm{cl}(P_{jk}))$$

and so

$$\sum_{i\in N}\mu(E''_i\ominus B'_i)\leqq\sum_{1\leqq j<\infty}\mu((\sum_{j<i<\infty}E''_i)\cap\mathrm{cl}(E_j^{(0)}))+\sum_{k<j}\mu(\mathrm{cl}(P_{jk}))$$
$$\leqq\sum_{1\leqq j<\infty}\mu(\mathrm{cl}(E_j^{(0)})\setminus B_j)+\sum_j 2^{-j}\leqq 2.$$

Here we used the fact that $E''_j\supset B_j$ and so

$$(\bigcup_{j<i<\infty}E''_i)\cap\mathrm{cl}(E_j^{(0)})\subset\mathrm{cl}(E_j^{(0)})\setminus B_j.$$

Thus we come to $\sum_i\mu(E''_i\ominus E_i^{(0)})\leqq 4$, which says that $E''=\prod_{i\in N}E''_i$ is cofinal with $E^{(0)}=\prod_{i\in N}E_i^{(0)}$. Hence $E^{(0)}\sim E''\sim E$.

Thus the proposition is now completely proved. Q.E.D.

§ 2. Representations of $G=\mathrm{Diff}_0(M)$

Let Π be a unitary representation (=UR) of $\mathfrak{S}_\infty$ on a Hilbert space $V(\Pi)$. We construct URs of G starting from $(\Pi, V(\Pi))$ and the product measure $(X, \mathfrak{M}(E), \nu_E)$ of $(M, \mathfrak{M}_M, \mu)$ associated to a unital product subset $E=\prod_{i\in N}E_i$ of $X=\prod_{i\in N}M_i$, $M_i=M$.

2.1. Representation T_Σ on a Hilbert space $\mathfrak{H}(\Sigma)$. With a given datum

(2.1) $$\Sigma=(\Pi;\mu,E),$$

we construct first a Hilbert space $\mathfrak{H}(\Sigma)$ and then define a UR T_Σ of G on it.

Let $E'=\prod_{i\in N}E'_i$ be a unital product subset which is cofinal with E: $\sum_{i\in N}\mu(E_i\ominus E'_i)<\infty$. Consider the measure space $(E', \mathfrak{M}(E)|E', \nu_E|E')$, where

$\mathfrak{M}(E)|E'=\{B\cap E';\, B\in\mathfrak{M}(E)\}$. Then the Hilbert space

$$\mathfrak{H}^{\Pi}_{|E'}=L^2(E',\mathfrak{M}(E)|E',\nu_E|E';\, V(\Pi))$$

of L^2-functions on E' with values in $V(\Pi)$ is defined as the completion of the pre-Hilbert space of step functions of the form

$$\varphi=\sum_k \chi_{B_k}\otimes v_k \qquad \text{(finite sum)} \tag{2.2}$$

with $B_k\in\mathfrak{M}(E)|E'$, $v_k\in V(\Pi)$, for which the inner product is given by

$$\|\varphi\|^2=\int_{E'}\|\varphi(x)\|^2_{V(\Pi)}d\nu_E(x)\,. \tag{2.3}$$

We give a Hilbert space $\mathfrak{H}(\Sigma)$ as the one which is generated by the family of $\mathfrak{H}^{\Pi}_{|E'}$ with $E'\sim E$ (cofinal):

$$\mathfrak{H}(\Sigma)=\bigvee_{E'\in\mathfrak{E}(E)}\mathfrak{H}^{\Pi}_{|E'}\,. \tag{2.4}$$

Here, for two elements φ_1,φ_2 in the union $\bigcup_{E'\sim E}\mathfrak{H}^{\Pi}_{|E'}$, their inner product is defined as follows: let $\varphi_i\in\mathfrak{H}^{\Pi}_{|E^{(i)}}$ with $E^{(i)}\sim E$, then

$$\langle\varphi_1,\varphi_2\rangle=\sum_{\sigma\in\mathfrak{S}_\infty}\int_{E^{(1)}\cap E^{(2)}\sigma}\langle\varphi_1(x),\Pi(\sigma)^{-1}(\varphi_2(x\sigma^{-1}))\rangle_{V(\Pi)}d\nu_E(x)\,. \tag{2.5}$$

A representation of G is defined as, for $\varphi\in\mathfrak{H}(\Sigma)$ and $g\in G$,

$$T_\Sigma(g)\varphi(x)=\rho_E(g^{-1}|x)^{1/2}\varphi(g^{-1}x) \quad\text{with}\quad \rho_E(g|x)=\frac{d\nu_E(gx)}{d\nu_E(x)}. \tag{2.6}$$

To recognize that the inner product (2.5) is well-defined and gives actually a Hilbert space, and that the formula (2.6) defines a unitary representation of G on that Hilbert space, it is natural and convenient to restate the above construction of Hilbert space by imitating the case of L^2-sections of a vector bundle associated to a principal fibre bundle. This will be done below.

2.2. Hilbert space $\mathcal{H}(\Sigma)$ canonically isomorphic to $\mathfrak{H}(\Sigma)$. In our present case, we have a *measurable* $\mathfrak{S}_\infty$-principal bundle π_Ω: $\tilde{X}\to\Omega_M$. The group G acts on $\tilde{X}$ measurably, and we get a *measurable* associated bundle for an $\mathfrak{S}_\infty$-module $(\Pi, V(\Pi))$.

We introduce some notation. For a function f on X with values in $V(\Pi)$, we put, for $\sigma\in\mathfrak{S}_\infty$,

$$(R(\sigma)f)(x)=f(x\sigma)\,, \tag{2.7}$$

$$(\Pi(\sigma)f)(x)=\Pi(\sigma)(f(x))\,.$$

Let $\mathfrak{H}^{\Pi}_{|E'}$ be, for a unital product subset $E'\sim E$, the Hilbert space defined above.

To consider $\mathfrak{H}^{\Pi}_{|E'}$ as a space of L^2-sections of the vector bundle associated to $\tilde{X}\to\Omega_M$ for $(\Pi, V(\Pi))$, we extend every $\varphi\in\mathfrak{H}^{\Pi}_{|E'}$ to a function $f=Q_\Pi\varphi$ on $\tilde{X}$ as

$$Q_\Pi\varphi=\sum_{\sigma\in\mathfrak{S}_\infty}(R(\sigma)\cdot\Pi(\sigma))\varphi \tag{2.8}$$

or more exactly, for $x\in E'$ and $\sigma\in\mathfrak{S}_\infty$,

$$(Q_\Pi\varphi)(x\sigma)=\Pi(\sigma)^{-1}(\varphi(x)), \tag{2.8'}$$

recalling that $E'\sigma$ $(\sigma\in\mathfrak{S}_\infty)$ are mutually disjoint. Then $f=Q_\Pi\varphi$ satisfies

$$f(x\sigma)=\Pi(\sigma)^{-1}f(x)\quad(x\in X,\ \sigma\in\mathfrak{S}_\infty) \tag{2.9}$$

and $f(x)=0$ outside $\coprod_{\sigma\in\mathfrak{S}_\infty}E'\sigma$. Put

$$\|f\|^2=\int_{E'}\|f(x)\|^2_{V(\Pi)}d\nu_E(x),$$

then we get a Hilbert space $\mathcal{H}^{\Pi}_{|E'}$, isomorphic to $\mathfrak{H}^{\Pi}_{|E'}$ through the map Q_Π.

The Hilbert space $\mathcal{H}(\Sigma)$ for $\Sigma=(\Pi;\mu,E)$ is the one generated by the family $\mathcal{H}^{\Pi}_{|E'}$, $E'\sim E$:

$$\mathcal{H}(\Sigma)=\bigvee_{E'\in\mathfrak{E}(E)}\mathcal{H}^{\Pi}_{|E'}. \tag{2.10}$$

Here, for two elements $f_i\in\mathcal{H}^{\Pi}_{|E^{(i)}}$ with $E^{(i)}\sim E$, their inner product is defined as follows: take an $F\in\mathfrak{M}(E)$ such that $F\sigma$ $(\sigma\in\mathfrak{S}_\infty)$ are mutually disjoint and $f_1(x)=0$ or $f_2(x)=0$ outside $\coprod_{\sigma\in\mathfrak{S}_\infty}F\sigma$, then

$$\langle f_1, f_2\rangle_{\mathcal{H}(\Sigma)}=\int_F\langle f_1(x), f_2(x)\rangle_{V(\Pi)}d\nu_E(x). \tag{2.11}$$

We see easily that the map Q_Π gives an isomorphism of Hilbert spaces $\mathfrak{H}(\Sigma)$ and $\mathcal{H}(\Sigma)$.

Remark 2.1. The Hilbert spaces $\mathfrak{H}(\Sigma)$ and $\mathcal{H}(\Sigma)$ are actually generated respectively by $\mathfrak{H}^{\Pi}_{|E'}$ and $\mathcal{H}^{\Pi}_{|E'}=Q_\Pi\mathfrak{H}^{\Pi}_{|E'}$ with $E'\approx E$ (strongly cofinal):

$$\mathfrak{H}(\Sigma)=\bigvee_{E'\approx E}\mathfrak{H}^{\Pi}_{|E'},\quad \mathcal{H}(\Sigma)=\bigvee_{E'\approx E}\mathcal{H}^{\Pi}_{|E'}. \tag{2.12}$$

Therefore the subset A of step functions of the form $\varphi=\chi_B\otimes v$ with $B\in\mathfrak{M}(E)|E'$, for some $E'\approx E$, and $v\in V(\Pi)$, is total in $\mathfrak{H}(\Sigma)$, and so the subset $Q_\Pi A=\{Q_\Pi\varphi;\ \varphi\in A\}$ is total in $\mathcal{H}(\Sigma)$. In particular, the Hilbert spaces $\mathfrak{H}(\Sigma)$ and $\mathcal{H}(\Sigma)$ are separable.

2.3. Unitary representation T_Σ of G. A representation of G is given on $\mathcal{H}(\Sigma)$ as

$$T_\Sigma(g)f(x)=\rho_E(g^{-1}|x)^{1/2}f(g^{-1}x) \tag{2.13}$$

for $f\in\mathcal{H}(\Sigma)$, $g\in G$ and $x\in X$, which we denote by the same symbol T_Σ with that on $\mathfrak{H}(\Sigma)$.

The unitarity of T_Σ is clear if we note that the domain of integration $F\in\mathfrak{M}(E)$ in (2.11) can be replaced by gF if f_1 and $f_2\in\mathcal{H}(\Sigma)$ is replaced respectively by $T_\Sigma(g)f_1$ and $T_\Sigma(g)f_2$.

For the continuity of $G\ni g\mapsto T_\Sigma(g)$, we should first mention the topology of G. The group $G=\mathrm{Diff}_0(M)$ is equipped with the usual D-type topology or the inductive topology according to the family $\mathrm{Diff}_K(M)$, K compacts in M, where $\mathrm{Diff}_K(M)$ denotes the subgroup of G consisting of g with $\mathrm{supp}(g)\subset K$, and is equipped with the topology of uniform convergence of every derivative. Then we have

Proposition 2.1. *The formula* (2.13) *gives a continuous unitary representation of* $G=\mathrm{Diff}_0(M)$ *on the Hilbert space* $\mathcal{H}(\Sigma)$.

Proof. Since the unitarity is already known, it is enough to prove the continuity of $G\ni g\mapsto\langle T_\Sigma(g)f_1, f_2\rangle$ for f_1, f_2 in a fixed total subset of $\mathcal{H}(\Sigma)$. Therefore, by Remark 2.1, we can take as f_1, f_2 two step functions of the form $f_j=Q_\Pi\varphi_j$ with $\varphi_j=\chi_{B^{(j)}}\otimes v_j$, $B^{(j)}\in\mathfrak{M}(E)|E^{(j)}$ for unital $E^{(j)}\approx E$ and $v_j\in V(\Pi)$. Then

$$\langle T_\Sigma(g)f_1, f_2\rangle=\langle v_1, v_2\rangle_{V(\Pi)}\cdot\int_{E^{(2)}}\rho_E(g^{-1}|x)^{1/2}\chi_{B^{(1)}}(g^{-1}x)\chi_{B^{(2)}}(x)d\nu_E(x)$$

$$=\langle v_1, v_2\rangle_{V(\Pi)}\cdot\int_{gB^{(1)}\cap B^{(2)}}\rho_E(g^{-1}|x)^{1/2}d\nu_E(x)\,.$$

Further we can take as $B^{(j)}$ a subset of the product form $B^{(j)}=\prod_{i\in N}B_i^{(j)}$ such that $B_i^{(j)}=E_i$ for $i\gg 0$. Then, by (1.12) and an additional discussion on the interchange of "infinite product" and "integration", we get

$$\langle T_\Sigma(g)f_1, f_2\rangle=\langle v_1, v_2\rangle_{V(\Pi)}\cdot\prod_{i\in N}\int_{(gB_i^{(1)})\cap B_i^{(2)}}\rho_M(g^{-1}; x_i)^{1/2}d\mu(x_i)\,.$$

Now, fix a compact $K\subset M$ and take $g\in\mathrm{Diff}_K(M)$. Then, by assumption on the measure μ, there exist positive constants $c_1(g)$, $c_2(g)$ such that

$$c_1(g)\leqq\rho_M(g^{-1}; p)\leqq c_2(g)\ (p\in K)\,;\ \rho_M(g^{-1}; p)=1\ (p\notin K)\,.$$

Note that we can demand $c_1(g)\geqq c_1>0$ and $c_2(g)\leqq c_2<\infty$ for any g in a certain neighbourhood of e=identity in $\mathrm{Diff}_K(M)$. Thus an evaluation similar to that in § 1.4 proves that, when g tends to e=identity, the infinite product in the above expression of $\langle T_\Sigma(g)f_1, f_2\rangle$ tends to

$$\langle v_1, v_2\rangle_{V(\Pi)}\cdot\prod_{i\in N}\mu(B_i^{(1)}\cap B_i^{(2)})=\langle v_1, v_2\rangle_{V(\Pi)}\cdot\nu_E(B^{(1)}\cap B^{(2)})=\langle f_1, f_2\rangle.$$

The continuity of $g\mapsto T_\Sigma(g)$ is now completely proved. Q.E.D.

Thus, for a datum $\Sigma=(\Pi;\mu,E)$ with a UR Π of $\mathfrak{S}_\infty$ and a unital product subset E of X with respect to μ, we get a UR T_Σ of G. For another datum $\Sigma'=(\Pi';\mu,E')$ but with the same μ, we have $T_{\Sigma'}\cong T_\Sigma$ if $\Pi'\cong\Pi$ (unitary equivalent) and $E'\sim E$ (cofinal).

Remark 2.2. To study irreducibility and equivalence relations for T_Σ's, it is often convenient for us to use the realization of T_Σ on $\mathfrak{H}(\Sigma)=\bigvee_{E'\approx E}\mathfrak{H}^{\Pi}_{|E'}$ and utilize the family of subspaces $\mathfrak{H}^{\Pi}_{|E'}$ for explicit calculations. These subspaces play a role, for the Hilbert space $\mathfrak{H}(\Sigma)$, something like local charts for a manifold. Symbolically speaking, global structure of the manifold M is reflected in the structure of $\mathfrak{H}(\Sigma)$ or $\mathcal{H}(\Sigma)=Q_\Pi\mathfrak{H}(\Sigma)$ at the point how to patch together these subspaces $\mathfrak{H}^{\Pi}_{|E'}$ or $\mathcal{H}^{\Pi}_{|E'}=Q_\Pi\mathfrak{H}^{\Pi}_{|E'}$, whereas in each $\mathfrak{H}^{\Pi}_{|E'}$ only local structure can be reflected. Anyhow we have enough reason to keep two kinds of realizations of T_Σ: the one on $\mathfrak{H}(\Sigma)$ and the other on $\mathcal{H}(\Sigma)=Q_\Pi\mathfrak{H}(\Sigma)$.

§ 3. Some fundamental lemmas

Here we collect some lemmas which will be needed later.

3.1. Elementary representations of diffeomorphism groups. Let E_1, $E_2,\cdots,E_r$ be mutually disjoint open subsets of M, and put

$$\mathfrak{H}_1=\bigotimes_{1\leq i\leq r}L^2(E_i)\quad\text{with}\quad L^2(E_i)=L^2(E_i;\,d\mu|E_i)\,,$$

$$G_1=\prod_{1\leq i\leq r}G_{|E_i}\quad\text{with}\quad G_{|E_i}=\{g\in G;\,\mathrm{supp}(g)\subset E_i\}\,.$$

Then we have a natural representation of G_1 on $\mathfrak{H}_1$ as the tensor product of that on $L^2(E_i)$ of $G_{|E_i}$:

for $f\in\mathfrak{H}_1$, $g=(g_i)\in G_1$ and $y=(y_i)\in\prod_{1\leq i\leq r}E_i$

$$(L_1(g)f)(y)=\prod_{1\leq i\leq r}\rho_M(g_i^{-1};\,y_i)^{1/2}\cdot f(g^{-1}y)\tag{3.1}$$

with the Jacobian $\rho_M(g;\,p)$ for $p\in M$ in (1.11), and $gy=(g_iy_i)$.

Note that $G_{|E_i}=G(E_i)\equiv\mathrm{Diff}_0(E_i)$. Let the connected components of E_i be E_{ij}, $j\in J_i$ (we admit the case $|J_i|=\infty$). Then $G_{|E_{ij}}=G(E_{ij})$ for every E_{ij}, and $G(E_i)$ is equal to the restricted direct product of $G(E_{ij})$'s, and is contained in the direct product of $G(E_{ij})$'s:

$$\textstyle\prod'_{j\in J_i}G(E_{ij})=G_{|E_i}\subset\prod_{j\in J_i}G(E_{ij})\,.\tag{3.2}$$

We get easily the following

Lemma 3.1. *The representation L_1 of G_1 on $\mathfrak{H}_1$ is a direct sum of IURs which are not mutually equivalent.*

Proof. The group $G_{|E_i}$ is equal to $\prod'_{j\in J_i}G(E_{ij})$ and the natural representation of each $G(E_{ij})$ on $L^2(E_{ij})$ is irreducible, and the decomposition $L^2(E_i)=\sum_{j\in J_i}^{\oplus}L^2(E_{ij})$ for each $1\leqq i\leqq r$ gives, through tensor product, a desired direct sum decomposition. Q.E.D.

Now let $F_1, F_2, \cdots, F_s$ be also mutually disjoint open subsets of M, and put

$$\mathfrak{H}_2=\bigotimes_{1\leqq j\leqq s}L^2(F_j)\,,\qquad G_2=\prod_{1\leqq j\leqq s}G(F_j)\,,$$

then we have a natural reprensentation L_2 of G_2 on $\mathfrak{H}_2$ similarly as (3.1). Let us compare two representations L_i of G_i ($i=1, 2$). Put $B_{ij}=E_i\cap F_j$ for $1\leqq i\leqq r$, $1\leqq j\leqq s$,

$$B_{i\infty}=E_i\setminus \mathrm{cl}(\bigcup_{1\leqq j\leqq s}F_j)\,,\quad B_{\infty j}=F_j\setminus \mathrm{cl}(\bigcup_{1\leqq i\leqq r}B_{ij})\,,$$

and $I_\infty=\{1, 2, \cdots, r, \infty\}$, $J_\infty=\{1, 2, \cdots, s, \infty\}$. Assume that

$$\mu(E_i\setminus\bigcup_{j\in J_\infty}B_{ij})=0\,,\qquad \mu(F_j\setminus\bigcup_{i\in I_\infty}B_{ij})=0\,. \tag{3.3}$$

Then $L^2(E_i)=\sum_{j\in J_\infty}^{\oplus}L^2(B_{ij})$, and therefore

$$\mathfrak{H}_1\cong\sum_{\boldsymbol{j}}^{\oplus}\bigotimes_{1\leqq i\leqq r}L^2(B_{ij_i})\,,\qquad \mathfrak{H}_2\cong\sum_{\boldsymbol{i}}^{\oplus}\bigotimes_{1\leqq j\leqq s}L^2(B_{i_jj})\,, \tag{3.4}$$

where $\boldsymbol{j}=(j_1, j_2, \cdots, j_r)$, $j_i\in J_\infty$, and $\boldsymbol{i}=(i_1, i_2, \cdots, i_s)$, $i_j\in I_\infty$. We put further

$$G_{12}=\prod_{i\in I_\infty, j\in J_\infty}G(B_{ij})\subset G_1\cap G_2\,,$$

where $B_{\infty\infty}=\emptyset$ and $G(B_{ij})=\{e\}$ for $B_{ij}=\emptyset$. Since $B_{i\infty}\cap F_j=\emptyset$ for any j, we may understand that each $G(B_{i\infty})$ acts trivially on $\mathfrak{H}_2$. Each $G(B_{ij})$ acts naturally on $L^2(B_{ij})$ and trivially on $L^2(B_{i'j'})$ if $(i', j')\neq(i, j)$.

For $\boldsymbol{j}=(j_1, j_2, \cdots, j_r)$ and $\boldsymbol{i}=(i_1, i_2, \cdots, i_s)$ in (3.4), we put

$$\mathfrak{H}(\boldsymbol{j})=\bigotimes_{1\leqq i\leqq r}L^2(B_{ij_i})\,,\quad \mathfrak{H}(\boldsymbol{i})=\bigotimes_{1\leqq j\leqq s}L^2(B_{i_jj})\,.$$

Further consider the set of pairs $[\boldsymbol{j}]=\{(i, j_i); 1\leqq i\leqq r\}$ and $[\boldsymbol{i}]=\{(i_j, j); 1\leqq j\leqq s\}$. In case $[\boldsymbol{i}]\supset[\boldsymbol{j}]$ (and necessarily $r\leqq s$), we define $\mathfrak{H}([\boldsymbol{i}]\backslash[\boldsymbol{j}])$ as the tensor product of $L^2(B_{i_jj})$ over j such that $(i_j, j)\in[\boldsymbol{i}]\backslash[\boldsymbol{j}]$. Now suppose that $[\boldsymbol{i}]\supset[\boldsymbol{j}]$ and $[\boldsymbol{i}]\backslash[\boldsymbol{j}]\subset\{(\infty, j); 1\leqq j\leqq s\}$, then changing the order of the factors in the tensor product, we have a natural isomorphism

$$\Phi_{ji}\colon \mathfrak{H}(\boldsymbol{i})\to\mathfrak{H}(\boldsymbol{j})\otimes\mathfrak{H}([\boldsymbol{i}]\backslash[\boldsymbol{j}])\,, \tag{3.5}$$

On the other hand, for any $\psi\in\mathfrak{H}([\boldsymbol{i}]\backslash[\boldsymbol{j}])$, $\|\psi\|=1$, we have an isomorphism, under $\prod_{1\leqq i\leqq r}G(B_{ij_i})$, of $\mathfrak{H}(\boldsymbol{j})$ into $\mathfrak{H}(\boldsymbol{j})\otimes\mathfrak{H}([\boldsymbol{i}]\backslash[\boldsymbol{j}])$ given as $Id_{\boldsymbol{j}}\otimes\psi\colon\varphi\to\varphi\otimes\psi$, where $Id_{\boldsymbol{j}}$ is the identity operator on $\mathfrak{H}(\boldsymbol{j})$. Then $\Phi_{ji}{}^{-1}\circ(Id_{\boldsymbol{j}}\otimes\psi)$ gives an isomorphism of $\mathfrak{H}(\boldsymbol{j})$ into $\mathfrak{H}(\boldsymbol{i})$.

Lemma 3.2. *For two representations $(L_i, \mathfrak{H}_i)$ of G_i for $i=1, 2$, let $A: \mathfrak{H}_1 \to \mathfrak{H}_2$, be an intertwining operator of $L_1|G_{12}$ with $L_2|G_{12}$. Assume that the condition* (3.3) *holds. Then A is a sum of scalar multiples of the natural isomorphisms*

$$\Phi_{ji}^{-1}\circ(Id_j\otimes\psi)\,, \tag{3.6}$$

where $\psi\in\mathfrak{H}([\boldsymbol{i}]\backslash[\boldsymbol{j}])$, and the pairs $(\boldsymbol{i}, \boldsymbol{j})$ satisfies the condition

$$[\boldsymbol{i}]\supset[\boldsymbol{j}]\,,\quad [\boldsymbol{i}]\backslash[\boldsymbol{j}]\subset\{(\infty, j); 1\leqq j\leqq s\}\,.$$

A proof can be given by using Lemma 3.1. In summary, we can say that dividing E_i's and F_j's into $B_{ij}=E_i\cap F_j$ and their outsides $B_{i\infty}$, $B_{\infty j}$, it is enough, for intertwining operators, to pick up in (3.4) the factors of $\mathfrak{H}_1$ and $\mathfrak{H}_2$ having the same B_{ij}'s in common.

3.2. Infinite tensor products of representations. Similarly as above, we get the following, for an infinite tensor product. Let $E'=\prod_{i\in N}E'_i$ be a unital product subset in $\mathfrak{E}(E)$ such that each E'_i is open and connected. Put $G(E')=\prod'_{i\in N}G(E'_i)$, the restricted direct product of $G(E'_i)\cong G_{|E'_i}$. Then $G(E')$ is contained in G naturally.

We define a natural representation of $G(E')$ on $\mathfrak{H}_{|E'}=L^2(E', \mathfrak{M}(E)|E', \nu_E|E')$ as

$$L_{E'}(g)f(x)=\rho_E(g^{-1}|x)^{1/2}f(g^{-1}x) \tag{3.7}$$

with $\rho_E(g|x)$ in (1.14), for $g\in G(E')$, $f\in\mathfrak{H}_{|E'}$ and $x\in E'$. Then

Lemma 3.3. *The unitary representation $L_{E'}$ of $G(E')$ on $\mathfrak{H}_{|E'}$ is irreducible.*

Proof. Put $\mathfrak{H}_i=L^2(E'_i, d\mu|E'_i)$, $\varphi_i=\chi_{E'_i}/\|\chi_{E'_i}\|\in\mathfrak{H}_i$ and $\varphi=(\varphi_i)_{i\in N}$. Then $\mathfrak{H}_{|E'}$ is canonically isomorphic to the tensor product $\otimes^{\varphi}_{i\in N}\mathfrak{H}_i=\otimes_{i\in N}\{\mathfrak{H}_i, \varphi_i\}$ of $\mathfrak{H}_i$ with respect to the reference vector φ. Because of (1.14), the representation $L_{E'}$ of $G(E')=\prod'_{i\in N}G(E'_i)$ is equivalent, under the above isomorphism, to the outer tensor product $\otimes^{\varphi}_{i\in N}L_i$ of reprensentation L_i of $G(E'_i)$ on $\mathfrak{H}_i$, where

$$L_i(g)\psi(p)=\rho_M(g^{-1}; p)^{1/2}\psi(g^{-1}p) \tag{3.8}$$

for $g\in G(E'_i)$, $\psi\in\mathfrak{H}_i$ and $p\in E'_i$. Each L_i is irreducible since E'_i is connected. Therefore its tensor product is also irreducible, whence so is $L_{E'}$. Q.E.D.

3.3. Structure of a subgroup $G((E'))$. Assume now $n=\dim M\geqq 2$. For a subset $E'=\prod_{i\in N}E'_i$, $E'_i\subset M$, of X, let $G((E'))$ be a subgroup of G given by

$$G((E'))=\{g\in G; gE'_i=E'_{\sigma(i)}\ (i\in N),\ \exists\,\sigma\in\mathfrak{S}_\infty\}\,. \tag{3.9}$$

Let $E'=\prod_{i\in N}E'_i$ be a unital product subset in $\mathfrak{E}(E)$ satisfying the conditions (UPS3)$-$(UPS4) (replacing $E_i^{(0)}$ there by E'_i), which exists by Proposition 1.8. Let us study the structure of $G((E'))$ for such an E'.

By (UPS4), for any $i\neq j$, $\mathrm{cl}(E'_i)$, $\mathrm{cl}(E'_j)$ and $C_{ij}=\mathrm{cl}(\cup_{k\neq i,j}E'_k)$ are mutually disjoint, and there exists an open path P_{ij} connecting E'_i, E'_j such that $\mathrm{cl}(P_{ij})\cap C_{ij}=\emptyset$. Put $M_{ij}=E'_i\cup E'_j\cup P_{ij}$. Then M_{ij} is a connected open submanifold of M and so $G(M_{ij})=\mathrm{Diff}_0(M_{ij})$ is canonically imbedded into $G=\mathrm{Diff}_0(M)$. We shall construct an element $g_{ij}\in G(M_{ij})$ such that

$$g_{ij}E'_i=E'_j\,,\qquad g_{ij}E'_j=E'_i\,. \tag{3.10}$$

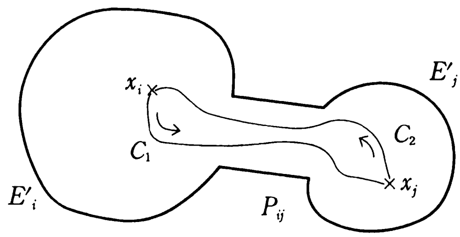
Then $g_{ij}|E'_k$ is the identity map on E'_k for $k\neq i, j$. Thus we get the following

Lemma 3.4. *Let $n=\dim M\geqq 2$ and $E'=\prod_{i\in N}E'_i$ be a unital product subset in $\mathfrak{E}(E)$ satisfying* (UPS3)$-$(UPS4). *Then there exists for any $\sigma\in\mathfrak{S}_\infty$, an element $g_\sigma\in G$ such that $g_\sigma E'_i=E'_{\sigma(i)}$ and $g_\sigma|E'_i=$identity on E'_i if $\sigma(i)=i$.*

Proof. It is enough to construct for each $i\neq j$ an element $g_{ij}\in G(M_{ij})\subset G$ in (3.10), and put $g_{\sigma_{ij}}=g_{ij}$ for the transposition $\sigma_{ij}=(i, j)$.

STEP 1. Inside of P_{ij}, we can choose two non-intersecting simple paths C_1 and C_2 connecting a point $x_i\in E'_i$ with an $x_j\in E'_j$.

Figure 3.1.



We give to the path C_1 (resp. C_2) a direction from x_i to x_j (resp. x_j to x_i). By (UPS4), E'_i and E'_j are respectively diffeomorphic to open balls $B_{i0}\subset E'_i$ with center x_i and $B_{j0}\subset E'_j$ with center x_j by elements h_i, $h_j\in G$ with supports in small neighbourhoods of $\mathrm{cl}(E'_i)$, $\mathrm{cl}(E'_j)$ respectively. Now choose a series of points $x_{i1}, x_{i2}, \cdots, x_{ir}$ on C_1 (resp. $x_{j1}, x_{j2}, \cdots, x_{js}$ on C_2) and small open balls $B_{i1}, B_{i2}, \cdots, B_{ir}$ (resp. $B_{j1}, B_{j2}, \cdots, B_{js}$) with center $x_{i1}, x_{i2}, \cdots, x_{ir}$ (resp. $x_{j1}, x_{j2}, \cdots, x_{js}$) respectively in such a way that

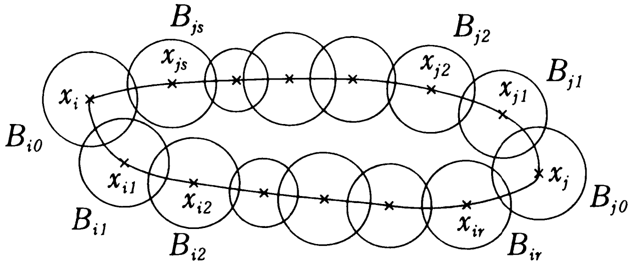
$$\mathrm{cl}(B_{ik})\cap\mathrm{cl}(B_{jl})=\emptyset \quad \text{for any } k, l\geqq 1\,;$$

$$B_{i0}\cap B_{i1}\neq\emptyset\,,\quad B_{i1}\cap B_{i2}\neq\emptyset,\ \cdots,\ B_{ir}\cap B_{j0}\neq\emptyset\,;$$

$$B_{i0}\cap B_{js}\neq 0\,,\quad B_{js}\cap B_{j,s-1}\neq\emptyset,\ \cdots,\ B_{j1}\cap B_{j0}\neq\emptyset\,.$$

STEP 2. Put

Figure 3.2.



$$E_i''=E_i' \setminus \mathrm{cl}\{\bigcup_{1<k\leq r} B_{ik} \cup \bigcup_{1\leq l<s} B_{jl}\},$$

$$E_j''=E_j' \setminus \mathrm{cl}\{\bigcup_{1\leq k<r} B_{ik} \cup \bigcup_{1<l\leq s} B_{jl}\}.$$

Then there exists an $h_{i0}\in G_{|E_i''}$ (resp. $h_{j0}\in G_{|E_j''}$) such that $D_{i0}=h_{i0}B_{i0}$ (resp. $D_{j0}=h_{j0}B_{j0}$) is completely contained in $B_{i0}\cap B_{i1}$ (resp. in $B_{j0}\cap B_{j1}$). Here a set A is *completely contained* in an open B if $\mathrm{cl}(A)\subset B$, by definition. Then there exists an $h_{i1}\in G_{|\bar{B}_{i1}}$ with $\bar{B}_{i1}=\mathrm{cl}(B_{i1})$ (resp. $h_{j1}\in G_{|\bar{B}_{j1}}$) such that $D_{i1}=h_{i1}D_{i0}$ (resp. $D_{j1}=h_{j1}D_{j0}$) is completely contained in $B_{i1}\cap B_{i2}$ (resp. in $B_{j1}\cap B_{j2}$). Repeatedly we find $h_{i2}\in G_{|\bar{B}_{i2}}, \cdots, h_{ir}\in G_{|\bar{B}_{ir}}$ such that $D_{ik}=h_{ik}D_{i,k-1}$ is completely contained in $B_{ik}\cap B_{i,k+1}$ for $0\leqq k\leqq r-1$ and $\bar{D}_{ir}\subset B_{ir}\cap B_{j0}$. Similarly we find $h_{j2}\in G_{|\bar{B}_{j2}}, \cdots, h_{js}\in G_{|\bar{B}_{js}}$ having the corresponding property.

STEP 3. Finally we can find $h_{i\infty}\in G_{|E_j'}$, $h_{j\infty}\in G_{|E_i'}$ such that $h_{i\infty}D_{ir}=D_{j0}$, $h_{j\infty}D_{js}=D_{i0}$. Then, since $h_{j0}^{-1}D_{j0}=B_{j0}$, $h_j^{-1}B_{j0}=E_j'$ and $h_{i0}^{-1}D_{i0}=B_{i0}$, $h_i^{-1}B_{i0}=E_i'$, we see that the element

$$g_{ij}=h_j^{-1}h_{j0}^{-1}h_i^{-1}h_{i0}^{-1}(h_{js}h_{ir})(h_{j,s-1}\cdots h_{j1}h_{j0})(h_{i,r-1}\cdots h_{i1}h_{i0})h_jh_i \tag{3.11}$$

gives a desired element in $G(M_{ij})\subset G$:

$$g_{ij}E_i'=E_j', \quad g_{ij}E_j'=E_i', \quad \mathrm{supp}(g_{ij})\subset E_i'\cup E_j'\cup P_{ij},$$

whence $\mathrm{supp}(g_{ij})\cap C_{ij}=\emptyset$. Q.E.D.

3.4. Representations of $G((E'))$ on $\mathfrak{H}_{|E'}^{\Pi}$. Let $E'=\prod_{i\in N}E_i'$ be as in § 3.2, and Π an IUR of $\mathfrak{S}_\infty$. Put $\mathfrak{H}_{|E'}^{\Pi}=L^2(E', \mathfrak{M}(E)|E', d\nu_E|E'; V(\Pi))$ and $\mathcal{H}_{|E'}^{\Pi}=Q_\Pi\mathfrak{H}_{|E'}^{\Pi}$ as in § 2. For any $g\in G((E'))$, there exists a $\sigma\in\mathfrak{S}_\infty$ such that $gE'=E'\sigma$, whence, for any $x=(x_i)_{i\in N}\in E'$, we have $g^{-1}x=y\sigma^{-1}$ with $y\in E'$. Therefore, for $f=Q_\Pi\varphi$, $\varphi\in\mathfrak{H}_{|E'}^{\Pi}$ and $x\in E'$,

$$f(g^{-1}x)=f(y\sigma^{-1})=\Pi(\sigma)f(y)=\Pi(\sigma)\varphi(y),$$

and so

$$(T_\Sigma(g)f)(x)=\rho_E(g^{-1}|x)^{1/2}f(g^{-1}x)=\rho_E(g^{-1}|x)^{1/2}\Pi(\sigma)\varphi(g^{-1}x\sigma).$$

This means the following: for any $g\in G((E'))$, $T_\Sigma(g)$ leaves the subspace $\mathcal{H}^{\Pi}_{|E'}$ of $\mathcal{H}(\Sigma)$ invariant, and it induces a unitary representation $T_{E'}$ of $G((E'))$ on $\mathfrak{H}^{\Pi}_{|E'}\cong\mathcal{H}^{\Pi}_{|E'}$ as

$$T_{E'}(g)\varphi(x)=\rho_E(g^{-1}|x)^{1/2}\Pi(\sigma)\varphi(g^{-1}x\sigma) \tag{3.12}$$

for $g\in G((E'))$, $\varphi\in\mathfrak{H}^{\Pi}_{|E'}$, $x\in E'$, where $\sigma\in\mathfrak{S}_\infty$ is so chosen that $gE'=E'\sigma$.

Let us prove the irreducibility of $T_{E'}$.

Lemma 3.5. *For a unital product subset E' satisfying* (UPS3)-(UPS4), *the unitary representation $T_{E'}$ on $\mathfrak{H}^{\Pi}_{|E'}$ of $G((E'))$ is irreducible.*

Proof. Note that $\mathcal{H}^{\Pi}_{|E'}\cong\mathfrak{H}_{|E'}\otimes V(\Pi)$ with $\mathfrak{H}_{|E'}=L^2(E',\mathfrak{M}(E)|E',d\nu_E|E')$. When $T_{E'}$ is restrited to the subgroup $G(E')\subset G((E'))$, we have

$$T_{E'}(g)\cong L_{E'}(g)\otimes\mathbf{1}_{V(\Pi)}, \tag{3.13}$$

where $\mathbf{1}_{V(\Pi)}$ denotes the identity operator on $V(\Pi)$. Take an intertwining operator A of $T_{E'}$ with itself. Then, identifying $\mathcal{H}^{\Pi}_{|E'}$ with $\mathfrak{H}_{|E'}\otimes V(\Pi)$, we have

$$A\circ(L_{E'}(g)\otimes\mathbf{1}_{V(\Pi)})=(L_{E'}(g)\otimes\mathbf{1}_{V(\Pi)})\circ A \quad (g\in G(E')).$$

Since $L_{E'}$ on $\mathfrak{H}_{|E'}$ is irreducible by Lemma 3.3, A is of the form $A=\mathbf{1}_{\mathfrak{H}_{|E'}}\otimes A_1$ with a bounded operator A_1 on $V(\Pi)$.

For any $\sigma\in\mathfrak{S}_\infty$, there exists a $g_\sigma\in G((E'))$ such that $g_\sigma E'=E'\sigma$. In (3.12), put $g=g_\sigma$, $\varphi(x)=\psi(x)\otimes v$ with $\psi\in\mathfrak{H}_{|E'}$ and $v\in V(\Pi)$. Then the equation $A\circ T_{E'}(g_\sigma)=T_{E'}(g_\sigma)\circ A$ is written down for φ as

$$\psi'\otimes(A_1\Pi(\sigma)v)=\psi'\otimes(\Pi(\sigma)A_1v)$$

with $\psi'(x)=\rho_E(g_\sigma{}^{-1}|x)^{1/2}\psi(g_\sigma{}^{-1}x\sigma)$. Thus we get $A_1\circ\Pi(\sigma)=\Pi(\sigma)\circ A_1$, $\sigma\in\mathfrak{S}_\infty$. Since Π is irreducible, A_1 should be a scalar multiple of the identity operator. Hence so is A, and the irreducibility of $T_{E'}$ is proved. Q.E.D.

§ 4. Irreducibility of the representation T_Σ of G

Let $\Sigma=(\Pi;\mu,E)$ be a parameter of the unitary representation T_Σ of G on $\mathcal{H}(\Sigma)$. Here Π is an IUR of $\mathfrak{S}_\infty$ and $E=\prod_{i\in N}E_i$ is a unital product subset of X, with respect to a measure μ on M, satisfying the conditions at the beginning of § 1.1. We prove the following theorem, one of our main results. Recall that we have assumed $\dim M\geqq 2$.

Theorem 4.1. *The unitary representation T_Σ of $G=$*Diff$_0(M)$ *on the Hilbert space $\mathcal{H}(\Sigma)$ is always irreducible.*

4.1. Structure of the Hilbert space $\mathcal{H}(\Sigma)$. Before entering into the details of the proof, we give a remark on the structure of the Hilbert space

$\mathcal{H}(\Sigma)$. Let $E^{(0)}=\prod_{i\in N}E_i^{(0)}$ be a unital product subset cofinal with E for which the conditions (UPS3)$-$(UPS4) in Proposition 1.8 hold. Put $\Sigma^{(0)}=(\Pi;\mu, E^{(0)})$, then $\mathcal{H}(\Sigma^{(0)})=\mathcal{H}(\Sigma)$ and $T_{\Sigma^{(0)}}=T_\Sigma$. Hence we can work with $\Sigma^{(0)}$ instead of Σ.

By Remark 2.1, the Hilbert space $\mathcal{H}(\Sigma^{(0)})$ is spanned by the family of subspaces $\mathcal{H}^{\Pi}_{|E'}$, where E''s are strongly cofinal with $E^{(0)}$: $E'\approx E^{(0)}$. Further, as seen below, we can chose $E^{(0)}$ so as to satisfy one more condition (UPS5). For any so chosen $E^{(0)}$, the Hilbert space $\mathcal{H}(\Sigma)=\mathcal{H}(\Sigma^{(0)})$ is spanned by $\mathcal{H}^{\Pi}_{|E'}$'s with $E'\approx E^{(0)}$ which satisfy (UPS3)$-$(UPS4) too.

For a unital product subset $F=\prod_{\in N}F_i$ satisfying (UPS3)$-$(UPS4), we consider the following condition.

(UPS5) For every $N>0$, the complement $M\setminus \mathrm{cl}(\bigcup_{i>N}F_i)$ of $\mathrm{cl}(\bigcup_{i>N}F_i)$ is connected.

Lemma 4.2. *Let $F=\prod_{i\in N}F_i$ be a unital product subset satisfying* (UPS3) $-$(UPS4). *Then, cutting off a small part of each F_i, $i\in \boldsymbol{N}$, we get a unital product subset $F'=\prod_{i\in N}F_i'$, $F_i'\subset F_i$, cofinal to F, which satisfies* (UPS5) *together with* (UPS3)$-$(UPS4).

Proof. Put $A=M\setminus \mathrm{cl}(\bigcup_{i\in N}F_i)$, then A is not empty. In fact, if $A=\emptyset$, then $M=\mathrm{cl}(\bigcup_{i\in N}F_i)=\mathrm{cl}(F_1)\cup\mathrm{cl}(\bigcup_{i\geq 2}F_i)$ on the one hand, and $\mathrm{cl}(F_1)$ and $\mathrm{cl}(\bigcup_{i\geq 2}F_i)$ are mutually disjoint by (UPS3) on the other hand. This contradicts the connectedness of M. Since A is open, its connected components A_1, $A_2,\cdots$ are at most countably infinite. We connect A_j with A_{j+1} inductively as follows, getting F_i' from F_i accordingly.

First connect A_1 with A_2 by a (rectifiable simple) curve C. In case C meets with F_i, we shift continuously the part $C\cap F_i$ of the curve very near to the boundary $\mathrm{cl}(F_i)\setminus F_i$, and pare off a small part of F_i together with the shifted curve inside F_i, thus getting a connected open $F_i^{(1)}\subset F_i$ such that $\mu(F_i\setminus F_i^{(1)})<3^{-i-1}\mu(F_i)$. So that $F^{(1)}=\prod_{i\in N}F_i^{(1)}$ is cofinal to F, and A_1 and A_2 is connected outside $\mathrm{cl}(\bigcup_{i\in N}F_i^{(1)})$ by the shifted curve. Secondly, to connect A_2 with A_3 properly, we work similarly with a curve C connecting them and with $F^{(1)}=\prod_{i\in N}F_i^{(1)}$. Paring a small part of each $F_i^{(1)}$ if necessary, we get a connected open $F_i^{(2)}\subset F_i^{(1)}$ such that $\mu(F_i^{(1)}\setminus F_i^{(2)})<3^{-i-2}\mu(F_i)$ and that a continuously shifted version of C is outside $\mathrm{cl}(\bigcup_{i\in N}F_i^{(2)})$.

Now assume that for each $i\in \boldsymbol{N}$, we have chosen opens $F_i^{(1)}\supset F_i^{(2)}\supset\cdots\supset F_i^{(k-1)}$ such that $\mu(F_i^{(j+1)}\setminus F_i^{(j)})<3^{-i-j}\mu(F_i)$ and that A_j and A_{j+1} is connected by a curve outside $\mathrm{cl}(\bigcup_{i\in N}F_i^{(j)})$ for $j=1,2,\cdots,k-1$. Then, to connect A_k with A_{k+1} properly, we work similarly with a curve C connecting them and $F^{(k-1)}=\prod_{i\in N}F_i^{(k-1)}$ cofinal to F. Thus we get connected opens $F_i^{(k)}$ such that $F_i^{(k)}\subset F_i^{(k-1)}$, $\mu(F_i^{(k-1)}\setminus F_i^{(k)})<3^{-i-k}\mu(F_i)$, and that a shifted version of C is outside $\mathrm{cl}(\bigcup_{i\in N}F_i^{(k)})$.

By induction, we get for each $i\in \boldsymbol{N}$ a series of decreasing connected opens $F_i^{(k)}$, $k=1,2,\cdots$. Note that

$$\sum_{k\in N}\mu(F_i^{(k-1)}\backslash F_i^{(k)})<\sum_{k\in N}3^{-i-k}\mu(F_i)=2^{-1}3^{-i}\mu(F_i)$$

with $F_i^{(0)}=F_i$. Then we see $F_i^{(\infty)}=\bigcap_{k\in N}F_i^{(k)}$ is of measure $>(1-2^{-1}3^{-i})\mu(F_i)$ and so $F^{(\infty)}=\prod_{i\in N}F_i^{(\infty)}$ is cofinal to F. Recall that, by (USP4), every F_i is diffeomorphic (together with its closure) to an open ball (together with its closed ball), under some $g\in G$. Then it is seen that we can manage parings of $F_i^{(k)}$, $k=1, 2, \cdots$, appropriately so that we get as $F_i^{(\infty)}$ a closed or open ball, after the transformation by the same g. So doing, let F_i' be the interior of $F_i^{(\infty)}$. Then $\mu(F_i^{(\infty)}\backslash F_i')=0$ and so $F'=\prod_{i\in N}F_i'$ is unital and cofinal to F. Further, F' satisfies (UPS3)$-$(UPS4) since $F_i'\subset F_i$. Especially $M\backslash \mathrm{cl}(\bigcup_{i\geqq 1}F_i')$ is connected.

Now put $B_N=M\backslash \mathrm{cl}(\bigcup_{i>N}F_i')$ for $N\geqq 0$. Let us prove that B_N are connected for $N\geqq 1$, by induction on N. Note that, by (UPS3), $\mathrm{cl}(\bigcup_{i>N}F_i')$ is a disjoint union of $\mathrm{cl}(F_{N+1}')$ and $\mathrm{cl}(\bigcup_{i>N+1}F_i')$. Then we see $B_{N+1}=B_N\cup \mathrm{cl}(F_{N+1}')$ for $N\geqq 0$. At first, as seen above, B_0 is connected. Since $\mathrm{cl}(F_{N+1}')$ is connected, we have a connected open neighbourhood F'' of it, disjoint with $\mathrm{cl}(\bigcup_{i>N+1}F_i')$. Then $B_{N+1}=B_N\cup F''$, $B_N\cap F''\neq\emptyset$, and therefore B_{N+1} is connected. This completes the induction. So the condition (UPS5) is satisfied.

Q.E.D.

4.2. A lemma for irreducibility. Assume that $E^{(0)}\sim E$ is taken to satisfy (UPS3)$-$(UPS4) and (UPS5), and that $E'\approx E^{(0)}$ are taken to satisfy (UPS3)$-$(UPS4). Then we see, by Lemma 3.5, the restriction $T_{E'}(g)=T_{\Sigma^{(0)}}(g)|\mathcal{H}^{\Pi}_{|E'}$ for $g\in G((E'))$ gives an IUR of $G((E'))$. To obtain the irreducibility of $T_{\Sigma^{(0)}}$ of G from that of the family $(T_{E'}, \mathcal{H}^{\Pi}_{|E'})$, we apply an elementary lemma given below.

Lemma 4.3. *Let H be a group and T its unitary representation on a Hilbert space $\mathfrak{H}$. Assume there exist a family of subgroups $\{H_\delta\}_{\delta\in\Delta}$ and that of subspaces $\{\mathfrak{H}_\delta\}_{\delta\in\Delta}$ such that*

(a) *$\mathfrak{H}_\delta$ is H_δ-invariant and H_δ-irreducible;*

(b) *$\mathfrak{H}$ is spanned by the family $\{\mathfrak{H}_\delta\}_{\delta\in\Delta}$;*

(c) *for any $\delta, \delta'\in\Delta$, there exists a finite sequence $\delta_1=\delta, \delta_2, \cdots, \delta_r=\delta'$ such that $\mathfrak{H}_{\delta_i}\cap\mathfrak{H}_{\delta_{i+1}}\neq(0)$ for $1\leqq i<r$;*

(d) *for some $\delta_0\in\Delta$, the IUR of H_{δ_0} on $\mathfrak{H}_{\delta_0}$ does not appear in the orthogonal complement $(\mathfrak{H}_{\delta_0})^\perp\subset\mathfrak{H}$.*

Then the representation $(T, \mathfrak{H})$ of H is irreducible.

Proof. By the assumption (d), any H-invariant subspace of $\mathfrak{H}$ either contains the H_{δ_0}-irreducible subspace $\mathfrak{H}_{\delta_0}$ or is orthogonal to it. Therefore there exists an H-irreducible subspace $\mathfrak{H}'$ containig $\mathfrak{H}_{\delta_0}$. Now take any $\delta\in\Delta$. We can find $\delta_1, \delta_2, \cdots, \delta_s$ such that $\mathfrak{H}_{\delta_i}\cap\mathfrak{H}_{\delta_{i+1}}\neq(0)$ for $0\leqq i\leqq s$ with $\delta_{s+1}=\delta$. Since $\mathfrak{H}'$ contains $\mathfrak{H}_{\delta_0}\cap\mathfrak{H}_{\delta_1}\neq(0)$ and $\mathfrak{H}_{\delta_1}$ is H_{δ_1}-irreducible, we see that $\mathfrak{H}'$ does contain the whole $\mathfrak{H}_{\delta_1}$. Similar arguments prove inductively that $\mathfrak{H}'$ contains $\mathfrak{H}_{\delta_2}, \cdots, \mathfrak{H}_{\delta_s}$, and $\mathfrak{H}_{\delta_{s+1}}=\mathfrak{H}_\delta$. Thus $\mathfrak{H}'$ contains all of $\mathfrak{H}_\delta$, $\delta\in\Delta$. This proves that

$\mathfrak{H}'=\mathfrak{H}$ by (b). Q.E.D.

4.3. Proof of Theorem 4.1. We prove the theorem by applying Lemma 4.3 for $H=G$, $(T,\mathfrak{H})=(T_{\Sigma^{(0)}},\mathcal{H}(\Sigma^{(0)}))$ and the set of parameters

$$\Delta=\{E';\,E'\approx E^{(0)},\text{ with (UPS3)}-\text{(UPS4)}\}\,.$$

For $\delta=E'\in\Delta$, we take $H_\delta=G((E'))$, $\mathfrak{H}_\delta=\mathcal{H}^{\Pi}_{|E'}$. Then (a) is proved by Lemma 3.5, and also (b) holds. So we prove (c) and (d) here.

First consider (c). Take E', $E''\in\Delta$ and put $\mathfrak{H}_1=\mathcal{H}^{\Pi}_{|E'}$, $\mathfrak{H}_3=\mathcal{H}^{\Pi}_{|E''}$. Let us find $E^{(2)}\in\Delta$ such that $\mathfrak{H}_2=\mathcal{H}^{\Pi}_{|E^{(2)}}$ satisfies $\mathfrak{H}_1\cap\mathfrak{H}_2\neq(0)$, $\mathfrak{H}_2\cap\mathfrak{H}_3\neq(0)$. Since E', E'' are strongly cofinal with $E^{(0)}$, there exists an $N>0$ such that $E'_i=E''_i=E_i^{(0)}$ for $i>N$. We see easily that there exist a $\sigma\in\mathfrak{S}_N$ and an N_1, $0\leqq N_1\leqq N$, such that $E'_i\cap E''_{\sigma(i)}\neq\emptyset$ for $N_1<i\leqq N$, and that $\{E'_j\cup E''_{\sigma(j)};\,1\leqq j\leqq N_1\}$ are mutually disjoint. Put $E^{(1)}=E'$, $E^{(3)}=E''\sigma$ and

$$E^{(2)}=(\prod_{1\leqq i\leqq N}F_i)\times E^{(0)}_{>N}\quad\text{with}\quad E^{(0)}_{>N}=\prod_{i>N}E_i^{(0)}$$

and with $F_i=E_i\cup E''_{\sigma(i)}$ for $1\leqq i\leqq N_1$ and $F_i=E'_i\cap E''_{\sigma(i)}$ for $N_1<i\leqq N$, then $E^{(2)}\in\Delta$. Put $\mathfrak{H}_2=\mathcal{H}^{\Pi}_{|E^{(2)}}$, and note that $\mathfrak{H}_1=\mathcal{H}^{\Pi}_{|E'}=\mathcal{H}^{\Pi}_{|E^{(1)}}$ and $\mathfrak{H}_3=\mathcal{H}^{\Pi}_{|E''}=\mathcal{H}^{\Pi}_{|E^{(3)}}$. Then we have $\mathfrak{H}_i\cap\mathfrak{H}_{i+1}\neq(0)$ for $1\leqq i\leqq 2$ because $\nu_E(E^{(i)}\cap E^{(i+1)})\neq0$. This proves (c) with $r=3$.

Let us now prove (d). Fix a $\delta_0=E'$. Take an arbitrary $\delta=E''$ from Δ and study the decomposition of the subspace $\mathfrak{H}_\delta$ along the orthogonal decomposition $\mathfrak{H}=\mathfrak{H}_{\delta_0}\oplus(\mathfrak{H}_{\delta_0})^\perp$. Since E' and E'' are strongly cofinal with $E^{(0)}$, there exists an $N\gg0$ such that

$$E'=(\prod_{1\leqq i\leqq N}A_i)\times E^{(0)}_{>N}\,,\qquad E''=(\prod_{1\leqq i\leqq N}B_i)\times E^{(0)}_{>N}\,,$$

where A_i's (resp. B_j's) are mutually disjoint open subsets of $M\setminus\mathrm{cl}(\bigcup_{i>N}E_i^{(0)})$. Put

$$C_{ij}=A_i\cap B_j\,,\quad C_{i0}=A_i\setminus\mathrm{cl}(\textstyle\bigcup_{1\leqq i\leqq N}B_j)\,,\quad C_{0j}=B_j\setminus\mathrm{cl}(\textstyle\bigcup_{1\leqq i\leqq N}A_i)\,,$$

then $A_i\setminus\bigcup_{0\leqq j\leqq N}C_{ij}$ and $B_j\setminus\bigcup_{0\leqq i\leqq N}C_{ij}$ are of measure zero. Therefore, modulo null sets, $B=\prod_{1\leqq j\leqq N}B_j$ is a disjoint union of subsets of the form $D=\prod_{1\leqq j\leqq N}D_j$ with $D_j=C_{i_jj}$ for some i_j. Let D_B be the set of all non-empty such $D\subset B$ and put $[D]=D\times E^{(0)}_{>N}$, then $\mathcal{H}^{\Pi}_{|E''}=\sum^{\oplus}_{D\in D_B}\mathcal{H}^{\Pi}_{|[D]}$. Similar statement is also true for $A=\prod_{1\leqq i\leqq N}A_i$ so that $\mathcal{H}^{\Pi}_{|E'}=\sum^{\oplus}_{D\in D_A}\mathcal{H}^{\Pi}_{|[D]}$. Take a $D\in D_B$. Then, each component D_j is either contained in some A_i or disjoint with all A_i. Therefore we have two cases: (i) $\mathcal{H}^{\Pi}_{|[D]}\subset\mathcal{H}^{\Pi}_{|E'}$ in case $D\sigma\subset A$ for some $\sigma\in\mathfrak{S}_N$, and (ii) $\mathcal{H}^{\Pi}_{|[D]}\perp\mathcal{H}^{\Pi}_{|E'}$ in case some D_j is disjoint with all A_i or two D_j, $D_{j'}$ ($j\neq j'$) are contained in the same A_i. Thus we get an orthogonal decomposition of $\mathfrak{H}_\delta$, for each $\delta\in\Delta$, as

(5.1) $\mathfrak{H}_\delta \equiv \mathcal{H}^{\Pi}_{[E''} = \mathfrak{H}_\delta{}^1 \oplus \mathfrak{H}_\delta{}^2, \quad \mathfrak{H}_\delta{}^1 \subset \mathfrak{H}_{\delta_0}, \quad \mathfrak{H}_\delta{}^2 \perp \mathfrak{H}_{\delta_0},$

where $\mathfrak{H}_\delta{}^1$ and $\mathfrak{H}_\delta{}^2$ are respectively the direct sums of $\mathcal{H}^{\Pi}_{[[D]}$ in the cases (i) and (ii).

Now consider a subgroup $H_{\delta_0\delta}$ of $H_{\delta_0} = G((E'))$ defined as

$$H_{\delta_0\delta} = \bigvee_{D\in D_A} G_D \quad \text{with} \quad G_D = \bigvee_{1\le i\le N} G_{|D_i} \cong \prod_{1\le i\le N} G_{|D_i}.$$

The representation of G_D on $\mathcal{H}^{\Pi}_{[[D]}$ is a multiple of the natural IUR on $L^2(D)$, and the one of G_D on other $\mathcal{H}^{\Pi}_{[[D']}$ with $D' \in D_A \cup D_B$, $D' \neq D$, does not contain the IUR on $L^2(D)$ (cf. Lemma 3.1). Therefore we see that the representation of the subgroup $H_{\delta_0\delta}$ of H_{δ_0} on the space $\mathfrak{H}_{\delta_0} = \mathcal{H}^{\Pi}_{[E'}$ is disjoint with that on $\mathfrak{H}_\delta{}^2 \subset (\mathfrak{H}_{\delta_0})^\perp$. On the other hand, the orthogonal complement $(\mathfrak{H}_{\delta_0})^\perp$ is spanned by $\mathfrak{H}_\delta{}^2$, $\delta \in \Delta$, because the whole space $\mathfrak{H}$ is spanned by $\mathfrak{H}_\delta$'s and each $\mathfrak{H}_\delta$ has the decomposition (5.1). Therefore the IUR of H_{δ_0} on $\mathfrak{H}_{\delta_0}$ does not appear in $(\mathfrak{H}_{\delta_0})^\perp = \bigvee_{\delta\in\Delta} \mathfrak{H}_\delta{}^2$. This is exactly the assumption (d).

Thus the proof of the irreducibility of T_Σ is now complete, and so Theorem 4.1 is proved.

§ 5. Equivalence relations among the IURs T_Σ

Let Π_1, Π_2 be two IURs of $\mathfrak{S}_\infty$, and $E = \prod_{i\in N} E_i$, $F = \prod_{i\in N} F_i$ be two unital product subsets of X with respect to a measure μ on M. Put $\Sigma_1 = (\Pi_1; \mu, E)$, $\Sigma_2 = (\Pi_2; \mu, F)$. We study here a criterion for the equivalence $T_{\Sigma_1} \cong T_{\Sigma_2}$.

5.1. Natural equivalence relations. Let $\tilde{\mathfrak{S}}_\infty$ be the group of all permutations of $\boldsymbol{N}$. Then it acts on X from the right: $xa = (x_{a(i)})_{i\in N}$ for $a \in \tilde{\mathfrak{S}}_\infty$ and $x = (x_i)_{i\in N} \in X$. Further $\tilde{\mathfrak{S}}_\infty$ acts on $\mathfrak{S}_\infty$ and accordingly on Π_1 as

$${}^a\sigma = a\sigma a^{-1}, \quad ({}^a\Pi_1)(\sigma) = \Pi_1({}^{a^{-1}}\sigma) = \Pi_1(a^{-1}\sigma a) \quad (\sigma \in \mathfrak{S}_\infty).$$

By the action $x \to xa$ on X, a unital product subset $E = \prod_{i\in N} E_i$ is sent to $Ea = \prod_{i\in N} E_{a(i)}$. Further, if $E' \sim E$, then $E'a \sim Ea$ and

$$\nu_{Ea}(E'a) = \prod_{i\in N} \mu(E'_{a(i)}) = \prod_{j\in N} \mu(E'_j) = \nu_E(E').$$

This means that the action $x \to xa$ on X gives an isomorphism of measure spaces $(X, \mathfrak{M}(E), \nu_E)$ and $(X, \mathfrak{M}(Ea), \nu_{Ea})$.

Since E is cofinal with Ea if and only if a belongs to the subgroup $\mathfrak{S}_\infty$, the measure spaces coincides with each other if and only if $a \in \mathfrak{S}_\infty$. Put, for $\Sigma = (\Pi; \mu, E)$ and $a \in \tilde{\mathfrak{S}}_\infty$,

(5.1) $\quad {}^a\Sigma = ({}^a\Pi; \mu, Ea^{-1}),$

and, for $f \in \mathcal{H}(\Sigma)$,

(5.2) $\quad (R_a f)(x) = f(xa) \quad (x \in X).$

Then, as is easily seen, R_a gives an isomorphism of $\mathcal{H}(\Sigma)$ onto $\mathcal{H}({}^a\Sigma)$. Further we obtain the following

Lemma 5.1. *For* $a\in\tilde{\mathfrak{S}}_\infty$, *the map* R_a *gives an isomorphism of IURs* $(T_\Sigma, \mathcal{H}(\Sigma))$ *and* $(T_{{}^a\Sigma}, \mathcal{H}({}^a\Sigma))$ *of* G.

5.2. Equivalence criterion. As a necessary and sufficient condition for unitary equivalence $T_{\Sigma_1}\cong T_{\Sigma_2}$, we will get the following rather simple criterion.

Theorem 5.2. *Assume* $\dim M\geqq 2$. *Let* $\Sigma_1=(\Pi_1;\mu, E)$ *and* $\Sigma_2=(\Pi_2;\mu, F)$ *be as above two parameters of IURs of* $G=\mathrm{Diff}_0(M)$. *Then,* $(T_{\Sigma_i}, \mathcal{H}(\Sigma_i))$, $i=1,2$, *are mutually equivalent if and only if, for some element* $a\in\tilde{\mathfrak{S}}_\infty$,

$$\Pi_1\cong{}^a\Pi_2 \quad \textit{and} \quad E\sim Fa^{-1}\ (\textit{cofinal})\,. \tag{5.3}$$

The rest of this section is devoted to prove this theorem.

5.3. Relation between E and F. To prove the above criterion, let us begin with the relation between E and F.

Proposition 5.3. *Assume* $T_{\Sigma_1}\cong T_{\Sigma_2}$, *then necessarily* $E\sim Fb$ *for some* $b\in\tilde{\mathfrak{S}}_\infty$.

The proof of this proposition is rather long and is divided into several steps.

STEP 1. As is seen in § 1.8 and in § 4.1, we may assume from the beginning that each E and F satisfy the conditions (UPS3)−(UPS4) and (UPS5). Recall that

$$\mathcal{H}(\Sigma_1)=\bigvee_{E^{(1)}\approx E}\mathcal{H}^{\Pi_1}_{|E^{(1)}}\,,\quad \mathcal{H}(\Sigma_2)=\bigvee_{F^{(1)}\approx F}\mathcal{H}^{\Pi_2}_{|F^{(1)}}\,, \tag{5.4}$$

where we can also take $E^{(1)}$ and $F^{(1)}$ satisfying (UPS3)−(UPS4) too.

Let A: $\mathcal{H}(\Sigma_1)\to\mathcal{H}(\Sigma_2)$, be a non-zero intertwining operator of T_{Σ_1} with T_{Σ_2}. Since both T_{Σ_i} $(i=1,2)$ are irreducible, we may assume A is unitary. Denote by $P^{\Pi_i}_{E^{(1)}}$ the orthogonal projection of $\mathcal{H}(\Sigma_i)$ onto $\mathcal{H}^{\Pi_1}_{|E^{(1)}}$, and put $A_{F^{(1)}E^{(1)}}=P^{\Pi_2}_{|F^{(1)}}\circ A\circ P^{\Pi_1}_{|E^{(1)}}$. Then $A_{F^{(1)}E^{(1)}}$, viewed as a map

$$A_{F^{(1)}E^{(1)}}\colon \mathcal{H}^{\Pi_1}_{|E^{(1)}}=Q_{\Pi_1}\mathfrak{H}^{\Pi_1}_{|E^{(1)}}\longrightarrow\mathcal{H}^{\Pi_2}_{|F^{(1)}}=Q_{\Pi_2}\mathfrak{H}^{\Pi_2}_{|F^{(1)}}\,, \tag{5.5}$$

intertwines $T_{\Sigma_1}|G^{(1)}$ with $T_{\Sigma_2}|G^{(1)}$ for a subgroup $G^{(1)}=G((E^{(1)}))\cap G((F^{(1)}))$ of G.

By definition, for $g\in G^{(1)}$, there exist σ and $\tau\in\mathfrak{S}_\infty$ such that

$$gE_i^{(1)}=E^{(1)}_{\sigma(i)}\ (i\in\boldsymbol{N})\,,\quad gF_j^{(1)}=E^{(1)}_{\tau(j)}\ (j\in\boldsymbol{N})\,.$$

Put $\mathrm{hul}(E^{(1)})=\mathrm{cl}(\bigcup_{i\in\boldsymbol{N}}E_i^{(1)})$ and

(5.6) $$E_{ij}^{(1)}=F_{ji}^{(1)}=E_i^{(1)}\cap F_j^{(1)},$$

$$E_{i\infty}^{(1)}=E_i^{(1)}\setminus \mathrm{hul}(F^{(1)}),\quad F_{j\infty}^{(1)}=F_j^{(1)}\setminus \mathrm{hul}(E^{(1)}),$$

then $\mathrm{hul}(E^{(1)})$ and $\mathrm{hul}(F^{(1)})$ are invariant under g and

(5.7) $$gE_{ij}^{(1)}=E_{\sigma(i)\tau(j)}^{(1)},\quad gE_{i\infty}^{(1)}=E_{\sigma(i)\infty}^{(1)},\quad gF_{j\infty}^{(1)}=F_{\tau(j)\infty}^{(1)}.$$

Put further

(5.8) $$E_i^{(2)}=\sum_{1\leq j\leq\infty}E_{ij}^{(1)},\quad F_j^{(2)}=\sum_{1\leq i\leq\infty}F_{ji}^{(1)}\quad \text{(disjoint unions)}.$$

Then $E_i^{(1)}\equiv E_i^{(2)}$ and $F_j^{(1)}\equiv F_j^{(2)}$ modulo null sets, and therefore, for $E^{(2)}=\prod_{i\in N}E_i^{(2)}$ and $F^{(2)}=\prod_{j\in N}F_j^{(2)}$,

$$\mathcal{H}_{|E^{(1)}}^{\Pi_1}=\mathcal{H}_{|E^{(2)}}^{\Pi_1},\quad \mathcal{H}_{|F^{(1)}}^{\Pi_2}=\mathcal{H}_{|F^{(2)}}^{\Pi_2}.$$

STEP 2. Let C_k, $k\in K$, be all the connected components of $E_{ij}^{(1)}=F_{ji}^{(1)}$, $E_{i'\infty}^{(1)}$ and $F_{j'\infty}^{(1)}$. Then $G^{(1)}$ contains the restricted direct product $G^{(2)}$ of $G_{|C_k}=G(C_k)$: $G^{(2)}=\prod'_{i\in N}G(C_k)\subset G^{(1)}$. To study the action of $G^{(2)}$, it is convenient for us to use $\mathfrak{H}_{|E^{(2)}}^{\Pi_1}\cong\mathfrak{H}_{|E^{(2)}}\otimes V(\Pi_1)$ instead of $\mathcal{H}_{|E^{(2)}}^{\Pi_1}=Q_{\Pi_1}\mathfrak{H}_{|E^{(2)}}^{\Pi_1}$. The Hilbert space $\mathfrak{H}_{|E^{(2)}}=L^2(E^{(2)},\mathfrak{M}(E)|E^{(2)},d\nu_E|E^{(2)})$ is isomorphic to the infinite product $\otimes_{i\in N}^{\psi}L^2(E_i^{(2)})$ of

$$L^2(E_i^{(2)})=\sum_{1\leq j\leq\infty}^{\oplus}L^2(E_{ij}^{(1)})=\sum_{k\in K_{1i}}^{\oplus}L^2(C_k)$$

with respect to the reference vector $\psi=(\psi_i)_{i\in N}$, $\psi_i=\chi_{E_i^{(2)}}/\|\chi_{E_i^{(2)}}\|$, where $L^2(E_i^{(2)})=L^2(E_i^{(2)},d\nu|E_i^{(2)})$ etc. and $K_{1i}=\{k\in K;\ C_k\subset E_i^{(2)}\}$. Fix an $N>0$, and put

(5.9) $$\mathfrak{H}_1(\boldsymbol{k})=\bigotimes_{1\leq i\leq N}L^2(C_{k_i})\quad\text{for}\quad \boldsymbol{k}=(k_1,k_2,\cdots,k_N)\in\prod_{1\leq i\leq N}K_{1i},$$

then we have a natural isomorphism

(5.10) $$\mathfrak{H}_{|E^{(2)}}\cong(\textstyle\sum_{\boldsymbol{k}}^{\oplus}\mathfrak{H}_1(\boldsymbol{k}))\otimes(\bigotimes_{i>N}^{\psi'}L^2(E_i^{(2)}),$$

where $\psi'=(\psi_i)_{i>N}$. By means of this, a consideration similar to that for Lemma 3.2 proves that $A_{F^{(1)}E^{(1)}}=A_{F^{(2)}E^{(2)}}$ kills all the components $\mathfrak{H}_1(\boldsymbol{k})\otimes(\otimes_{i>N}^{\psi'}L^2(E_i^{(2)}))$ with $\boldsymbol{k}=(k_1,k_2,\cdots,k_N)$ for which $k_i=\infty$ or $C_{k_i}\subset E_{i\infty}^{(1)}=E_i^{(1)}\setminus\mathrm{hul}(F^{(1)})$ for some $i\leq N$. Thus $A_{F^{(2)}E^{(2)}}$ kills all the components containing a factor from $L^2(E_{i\infty}^{(1)})$ for some $i\in\boldsymbol{N}$. (To see this, we actually compare the action of $G_{|E_{i\infty}^{(1)}}\subset G$.) From this fact, we get the following

Lemma 5.4. *Assume that there exists a non-zero intertwining operator A of T_{Σ_1} with T_{Σ_2}. Then,*

(5.11) $$\textstyle\sum_{i\in N}\mu(E_{i\infty}^{(1)})<\infty,\quad \sum_{j\in N}\mu(F_{j\infty}^{(1)})<\infty,$$

for any $E^{(1)}\sim E$ *and* $F^{(1)}\sim F$, *where* $E_{i\infty}^{(1)}=E_i^{(1)}\setminus \mathrm{hul}(F^{(1)})$, $F_{j\infty}^{(1)}\setminus \mathrm{hul}(E^{(1)})$ *as in* (5.6).

Proof. Put $E_{if}^{(1)}=\sum_{j\in N}E_{ij}^{(1)}$, then $E_i^{(2)}=E_{if}^{(1)}+E_{i\infty}^{(1)}$ and so

$$L^2(E_i^{(2)})=L^2(E_{if}^{(1)})\oplus L^2(E_{i\infty}^{(1)})\,.$$

Since $A_{F^{(2)}E^{(2)}}$ kills all the components containing $L(E_{i\infty}^{(1)})$-factor for some $i\in N$, we get $A_{F^{(2)}E^{(2)}}(\chi_{E^\wedge})=0$ for any $E^\wedge\approx E^{(2)}$ (strongly cofinal) if $\sum_{i\in N}\mu(E_{i\infty}^{(1)})=\infty$ or equivalently if $\prod_{i\gg 0}\mu(E_{if}^{(1)})=0$. This means that $A_{F^{(1)}E^{(1)}}=A_{F^{(2)}E^{(2)}}=0$.

On the other hand, if $\sum_{i\in N}\mu(E_{i\infty}^{(1)})=\infty$ for some pair $E^{(1)}\sim E$, $F^{(1)}\sim F$, then it holds also for any pair $E^{(1)}\sim E$, $F^{(1)}\sim F$. Hence we get $A=0$, a contradiction.

A similar argument for A^{-1} proves the assertion for $F_{j\infty}^{(1)}$'s. Q.E.D.

STEP 3. Now replace $E_i^{(1)}$ and $F_j^{(1)}$ respectively by $E_i^{(3)}=E_i^{(1)}+F_{i\infty}^{(1)}$, $F_j^{(3)}=F_j^{(1)}+E_{j\infty}^{(1)}$. Then, for $E^{(3)}=\prod_{i\in N}E_i^{(3)}$ and $F^{(3)}=\prod_{j\in N}F_j^{(3)}$, we have

$$E^{(3)}\supset E^{(1)}\,,\quad E^{(3)}\sim E^{(1)}\,;\quad F^{(3)}\supset F^{(1)}\,,\quad F^{(3)}\sim F^{(1)}\,;$$

$$\mathrm{hul}(E^{(3)})=\mathrm{hul}(F^{(3)})\ (=\mathrm{hul}(E^{(1)})\cup\mathrm{hul}(F^{(1)}))\,.$$

Note that $\mathfrak{H}_{|E^{(3)}}\supset\mathfrak{H}_{|E^{(1)}}$, $\mathfrak{H}_{|F^{(3)}}\supset\mathfrak{H}_{|F^{(1)}}$, then we obtain

Lemma 5.5. *Assume that there exists a non-zero intertwining operator A of T_{Σ_1} with T_{Σ_2}. Then A is approximated strongly by the family $A_{F^{(1)}E^{(1)}}=P_{F^{(1)}}^{\Pi_2}\circ A\circ P_{|E^{(1)}}^{\Pi_1}$ with $E^{(1)}\sim E$, $F^{(1)}\sim F$ such that $E_i^{(1)}$ and $F_j^{(1)}$ are open and* $\mathrm{hul}(E^{(1)})=\mathrm{hul}(F^{(1)})$.

Corollary 5.6. *Assume $T_{\Sigma_1}\cong T_{\Sigma_2}$. Then, replacing unital product subsets E in Σ_1 and F in Σ_2 by their cofinal ones, we can assume that* $\mathrm{hul}(E)=\mathrm{hul}(F)$. (*Here the conditions* (UPS3)−(UPS4) *are not necessarily satisfied.*)

5.4. From hul(E)=hul(F) to $E\sim Fb(\exists\, b\in\tilde{\mathfrak{S}}_\infty)$. We continue to study the relation between E and F, and wish to get $E\sim Fb$ for some $b\in\tilde{\mathfrak{S}}_\infty$.

STEP 4. By Lemma 5.5, we may pursue $A_{F^{(1)}E^{(1)}}$ with $\mathrm{hul}(E^{(1)})=\mathrm{hul}(F^{(1)})$. In this case, $E_{i\infty}^{(1)}=\emptyset$, $F_{j\infty}^{(1)}=\emptyset$ and

$$E_i^{(2)}=\sum_{j\in N}E_{ij}^{(1)}=\sum_{k\in K_{1i}}C_k\,,\quad \mu(E_i^{(1)}\setminus E_i^{(2)})=0\,;\tag{5.12}$$

$$F_j^{(2)}=\sum_{i\in N}F_{ji}^{(1)}=\sum_{k\in K_{2j}}C_k\,,\quad \mu(F_j^{(1)}\setminus F_j^{(2)})=0\,,$$

where $K_{2j}=\{k\in K;\ C_k\subset F_j^{(1)}\}$. Consider the action of $G^{(2)}=\prod'_{k\in K}G(C_k)$ on both of $\mathfrak{H}_{|E^{(1)}}^{\Pi_1}\cong\mathfrak{H}_{|E^{(1)}}\otimes V(\Pi_1)$ and $\mathfrak{H}_{|F^{(1)}}^{\Pi_2}\cong\mathfrak{H}_{|F^{(1)}}\otimes V(\Pi_2)$. Note that $G^{(2)}$ acts trivially on the second factors $V(\Pi_i)$, $i=1,2$, and that $\mathfrak{H}_{|E^{(1)}}=\mathfrak{H}_{|E^{(2)}}$ is decomposed as in (5.10), and $\mathfrak{H}_{|F^{(1)}}=\mathfrak{H}_{|F^{(2)}}$ as in

(5.13) $$\mathfrak{H}_{|F^{(2)}}=(\sum_{k'}{}^{\oplus}\mathfrak{H}_2(\boldsymbol{k}'))\otimes(\bigotimes_{j>N'}{}^{\cdot} L^2(F_j^{(2)}))$$

for $N'>0$ (the reference vector is omitted for $\otimes_{j>N'}$), with

(5.14) $$\mathfrak{H}_2(\boldsymbol{k}')=\bigotimes_{1\leq j\leq N'} L^2(C_{k_j}) \quad \text{for} \quad \boldsymbol{k}'=(k_1', k_2', \cdots, k_{N'}')\in\prod_{1\leq j\leq N'} K_{2j}.$$

Considerations similar to that for Lemma 3.2, on the action of $G^{(2)}$, give us the following crucial lemma.

Lemma 5.7. *The image of* $A_{F^{(1)}E^{(1)}}=A_{F^{(2)}E^{(2)}}$ *is contained in the sum of the components*

(5.15) $$\mathfrak{H}_2(\boldsymbol{k}')\otimes(\bigotimes_{j>N'}{}^{\cdot} L^2(F_j^{(2)}))$$

for which the parameter $\boldsymbol{k}'=(k_1', k_2', \cdots, k_{N'}')\in\prod_{1\leq j\leq N'}K_{2j}$ *satisfies the condition*

(NT) *among* $k_1', k_2', \cdots, k_{N'}'$, *no two of them belong to the same* K_{1i}, $i\in\boldsymbol{N}$.

STEP 5. Denote by $P_{N'}$ the orthogonal projection of $\mathfrak{H}_{|F^{(2)}}$ onto the subspace spanned by such subspaces in (5.15) that the condition (NT) holds for them. Then it is enough for us to prove $P_{N'}\to 0$ strongly in case $E^{(2)}\not\sim F^{(2)}b$ or $E\not\sim Fb$ for any $b\in\tilde{\mathfrak{S}}_\infty$. In fact, if so, we have $A_{F^{(1)}E^{(1)}}=A_{F^{(2)}E^{(2)}}=0$. This means $A=0$ by Lemma 5.5, a contradiction.

To prove $P_{N'}\to 0$ is reduced to a problem on series of real numbers as follows, by considering $P_{N'}(\chi_{F'})\to 0$ $(N'\to\infty)$ for any unital product subsets $F'\subset F^{(2)}$ which are strongly cofinal to $F^{(2)}$. Put $c_{ij}=\mu(E_{ij}^{(1)})=\mu(F_{ji}^{(1)})$, and we consider $E_{ij}^{(1)}=F_{ji}^{(1)}$ grouping C_k's, instead of C_k themselves. Put

$$d_i=\mu(E_i^{(2)})=\sum_{j\in N}c_{ij}, \quad e_j=\mu(F_j^{(2)})=\sum_{i\in N}c_{ij},$$

then $\mu(E_i^{(2)}\ominus F_j^{(2)})=d_i+e_j-2c_{ij}$, and we come to the following problem (N' is replaced by N).

Problem 5.8. *Let* $c_{ij}\geq 0$ *for* $i, j\in\boldsymbol{N}$. *Put* $d_i=\sum_{j\in N}c_{ij}$, $e_j=\sum_{i\in N}c_{ij}$, *and assume that*

$$d_i>0, \quad e_j>0 \quad (i, j\in\boldsymbol{N});$$

$\prod_{i\in N}d_i$, $\prod_{j\in N}e_j$ *are (unconditionally) convergent.*

For $N>0$, *expand the product* $\prod_{1\leq j\leq N}e_j$ *(resp.* $\prod_{1\leq i\leq N}d_i$*) in terms of* c_{ij}*'s and let* p_N *(resp.* q_N*) be the sum of monomial terms* $c_{i_1 1}c_{i_2 2}\cdots c_{i_N N}$ *(resp.* $c_{1j_1}c_{2j_2}\cdots c_{Nj_N}$*) such that*

(DE) $i_1, i_2, \cdots, i_N$ *(resp.* $j_1, j_2, \cdots, j_N$*) are different from each other.*

Assume further that

(5.16) $$\sum_{i\in N}(d_i+e_{b(i)}-2c_{i,b(i)})=\infty \text{ for any } b\in\tilde{\mathfrak{S}}_\infty .$$

Then, does there hold $p_N\to 0$ or $q_N\to 0$ as $N\to\infty$?

Note that (5.16) is expressed as

(5.16′) $$\sum_{\substack{i,j\in N\\ j\neq b(i)}} c_{ij}=\infty \quad \text{for any } b\in\tilde{\mathfrak{S}}_\infty.$$

5.5. Solution of Problem 5.8 (Step 1). For a finite subset J of $\boldsymbol{N}$, consider the product

$$\prod_{j\in J}e_j=\prod_{j\in J}(\sum_{i\in N}c_{ij})$$

and expand it in a (possibly infinite) sum of terms

(5.17) $$\prod_{j\in J}c_{i_j j} \quad \text{with} \quad (i_j)_{j\in J}\in\boldsymbol{N}^J .$$

Denote by p_J the sum of all such terms (5.17) that the sets of indices $\{i_j; j\in J\}$ consist of different integers. Then $p_N=p_{J_N}$ for $J_N=\{1, 2, 3, \cdots, N\}$, and we get easily the following

Lemma 5.9. *For two disjoint finite subsets J, J' of $\boldsymbol{N}$,*

$$p_{J\cup J'}\leqq p_J\cdot p_{J'} .$$

To treat p_N or p_J, we can normalize (c_{ij}) as follows. Put $c'_{ij}=c_{ij}/e_j$ with $e_j=\sum_{i\in N}c_{ij}$, then $\sum_{i\in N}c'_{ij}=1$. Let p'_J be the sum for (c'_{ij}) corresponding to p_J for (c_{ij}). Then, $p'_J=p_J/(\prod_{j\in J}e_j)$, and $\prod_{j\in J}e_j$ is bounded from above and below for any finite $J\subset\boldsymbol{N}$. Therefore the assertion on p_N for (c_{ij}) is equivalent to that for (c'_{ij}). Hence, to treat p_N, we may put the following additional condition on (c_{ij}):

(5.18) $$e_j\equiv\sum_{i\in N}c_{ij}=1 \quad (j\in\boldsymbol{N}) ,$$

and in this case we may call $(c_{ij})_{i,j\in N}$ a stochastic matrix of infinite size. Here we should note that $\prod_{i\in N}d'_i$, with $d'_i=\sum_{j\in N}c'_{ij}$, is again convergent. This can be seen from that every term $c'_{1j_1}c'_{2j_2}\cdots c'_{Nj_N}$ in the expansion of $\prod_{1\leq i\leq N}d'_i$ is a multiple of $c_{1i_1}c_{2j_2}\cdots c_{Nj_N}$ by $(\prod_{1\leq i\leq N}e_{j_i})^{-1}$, and that all the products $\prod_{j\in J}e_j$, $J\subset\boldsymbol{N}$, are bounded from below and above.

Under the condition (5.18), put $\delta_J=1-p_J=\prod_{j\in J}e_j-p_J\geqq 0$. Then, for any disjoint subsets $J_1, J_2, \cdots$ of $\boldsymbol{N}$, we have by Lemma 5.9

(5.19) $$\lim_{N\to\infty}p_N\leqq\prod_{n=1}^{\infty}(1-\delta_{J_n}) .$$

Therefore, to verify $p_N\to 0$ $(N\to\infty)$, it is sufficient to see the existence of disjoint subsets J_n such that $\sum_{n=1}^{\infty}\delta_{J_n}=\infty$ or more simply $\delta_{J_n}\geqq c$ $(n=1, 2, \cdots)$ for a constant $c>0$.

Now, for the term in (5.17), put $L_k=\{j\in J;\ i_j=k\}$ for $k\in \boldsymbol{N}$, and $K=\{k\in \boldsymbol{N};\ L_k\neq\emptyset\}$, $K_{\geqq p}=\{k\in K;\ |L_k|\geqq p\}$, then

$$\prod_{j\in J}c_{i_j j}=\prod_{k\in K}\left(\prod_{l\in L_k}c_{kl}\right). \tag{5.20}$$

The factor $\prod_{l\in L_k}c_{kl}$ is the contribution to $\prod_{j\in J}c_{i_j j}$ from the k-th row of (c_{ij}). Therefore

$$p_J=\text{the sum of terms for which } K_{\geqq 2}=\emptyset.$$

Since $e_j=1$ and $\prod_{j\in J}e_j=1$, we have

$$\delta_J=\text{the sum of terms for which } K_{\geqq 2}\neq\emptyset. \tag{5.21}$$

Let us evaluate the sum in the right hand side. For a finite subset Q of $\boldsymbol{N}$ and a family $(R_q)_{q\in Q}$ of disjoint subsets of $J\subset \boldsymbol{N}$, let $\mathcal{A}((q;\ R_q)_{q\in Q})$ be the union of all terms in (5.17) containing the factor $\prod_{q\in Q}(\prod_{r\in R_q}c_{qr})$. Put $J'=J\setminus(\bigcup_{q\in Q}R_q)$, then the total sum of elements in $\mathcal{A}((q;\ R_q)_{q\in Q})$ is equal to

$$\prod_{q\in Q}\left(\prod_{r\in R_q}c_{qr}\right)\cdot\left(\prod_{j\in J'}e_j\right)=\prod_{q\in Q}\left(\prod_{r\in R_q}c_{qr}\right)\qquad(\text{by }(5.18)).$$

We denote it by $[\mathcal{A}((q;\ R_q)_{q\in Q})]$. Note that the term in (5.20) is contained in $\mathcal{A}((q;\ R_q)_{q\in Q})$ if and only if $Q\subset K$ and $R_q\subset L_q$ for any $q\in Q$.

Let us now first take $Q=\{q\}$, one point set, and $R_q=\{r_1,\ r_2\}$, two points set. In the sum s_2^J of all such $[\mathcal{A}(q;\ R_q)]=[\mathcal{A}(q;\ \{r_1,\ r_2\})]$, the monomial (5.20) contributes to s_2^J through several $\mathcal{A}(q;\ R_q)$, and the total number of times of its contribution is equal to

$$n_2=\sum_{q\in K_{\geqq 2}}\binom{l_q}{2}\quad\text{with}\quad l_q=|L_q|.$$

Next take $Q=\{q\}$ and $R_q=\{r_1,\ r_2,\ r_3\}$, three points set, and let s_3^J be the sum of all such $[\mathcal{A}(q;\ R_q)]$. Then the total number of times in which the monomial (5.20) contributes to s_3^J is given by

$$n_3=\sum_{q\in K_{\geqq 3}}\binom{l_q}{3}.$$

Further we take $Q=\{q_1,\ q_2\}$, two points set, and R_{q_1}, R_{q_2} disjoint two points sets. Denote by $s_{2,2}^J$ the sum of all $[\mathcal{A}((q;\ R_q)_{q\in Q})]$ of such type. Then the total number of times in which the monomial (5.20) contributes to $s_{2,2}^J$ is equal to

$$n_{2,2}=\sum_{Q\subset K}\binom{l_{q_1}}{2}\cdot\binom{l_{q_2}}{2},\quad\text{where}\quad Q=\{q_1,\ q_2\}.$$

In particular, $n_{2,2}=0$ if $|K_{\geq 2}|<2$.

Now let us evaluate $m=n_2-n_3-n_{2,2}$ for the monomial (5.20). In case $|K_{\geq 2}|=1$, let $K_{\geq 2}=\{q\}$, then

$$m=\binom{l_q}{2}-\binom{l_q}{3}=\frac{1}{3!}l_q(l_q-1)(5-l_q)\,.$$

Hence $m\leqq 0$ except the cases where $l_q=2, 3, 4$ and $m=1, 2, 2$ correspondingly. In case $|K_{\geq 2}|=2$, we see similarly that $m\leqq 0$ except the cases where $l_q=|L_q|=2$ for $q\in K_{\geq 2}$ and $m=1$ accordingly. If $|K_{\geq 2}|>2$, then $m\leqq 0$ necessarily.

Thus we get the following

Lemma 5.10. *Let $s_2{}^J$, $s_3{}^J$ and $s_{2,2}^J$ be the sums of $[\mathcal{A}((q;\,R_q)_{q\in Q})]$ defined above. Then there holds always the inequality*

$$1-p_J\equiv\delta_J\geqq\frac{1}{2}(s_2{}^J-s_3{}^J-s_{2,2}^J)\,.$$

The above evaluation of δ_J is sometimes not convenient to apply in certain situations. So we give another but similar evaluation as follows. Fix a subset I of $\boldsymbol{N}$. Consider only the terms $\prod_{j\in J}c_{i_j j}$ in (5.17) or (5.20) with

(5.22) $\qquad K_{\geq 2}\subset I\quad$ or $\quad$"if $i_j=i_{j'}$ for $j, j'\in J$, $j\neq j'$, then $i_j\in I$".

Denote by $\mathcal{A}^I((q;\,R_q)_{q\in Q})$ the subset of $\mathcal{A}((q;\,R_q)_{q\in Q})$ consisting of such terms that (5.22) holds, and denote also by $s_2{}^{I,J}$ (resp. $s_3{}^{I,J}$, $s_{2,2}^{I,J}$) the sum of $[\mathcal{A}^I((q;\,R_q)_{q\in Q})]$ analogous to the sum $s_2{}^J$ (resp. $s_3{}^J$, $s_{2,2}^J$) of $[\mathcal{A}((q;\,R_q)_{q\in Q})]$. Since the evaluation of the number of times of contribution of $\prod_{j\in J}c_{i_j j}$ is always true, we get similarly as above the following

Lemma 5.11. *Let I be a subset and J a finite subset of $\boldsymbol{N}$. Then*

$$\delta_J\geqq\frac{1}{2}(s_2{}^{I,J}-s_3{}^{I,J}-s_{2,2}^{I,J})\,.$$

5.6. Solution of Problem 5.8 (Step 2). Put $I=\{i\}$, one point set, then $s_{2,2}^{I,J}=0$ and so $\delta_J\geqq 2^{-1}(s_2{}^{I,J}-s_3{}^{I,J})$ with

$$s_2{}^{I,J}=\sum_{\{j_1,j_2\}\subset J}c_{ij_1}c_{ij_2}\,,\qquad s_3{}^{I,J}=\sum_{\{j_1,j_2,j_3\}\subset J}c_{ij_1}c_{ij_2}c_{ij_3}\,.$$

Since $\prod_{i\in N}d_i$ converges and so $d_i\to 1$, we may assume that $d_i\equiv\sum_{j\in N}c_{ij}\leqq 3/2$ for $i\in\boldsymbol{N}$. Then

$$s_3{}^{I,J}=\frac{1}{3}\sum_{\{j_1,j_2\}\subset J}c_{ij_1}c_{ij_2}\Big(\sum_{j\in J}c_{ij}-c_{ij_1}-c_{ij_2}\Big)\leqq\frac{1}{2}s_2{}^{I,J}\,.$$

Hence $\delta_J\geqq 4^{-1}s_2{}^{I,J}$.

Now we apply the following

Lemma 5.12. *Let $x_1, x_2, \cdots$ be a finite number of non-negative real numbers. Then,*

$$\sum_{j_1<j_2} x_{j_1}x_{j_2} \geqq \frac{1}{2}(\sum_j x_j)(\sum_j x_j - \max_j\{x_j\}) .$$

Thus we get

$$\delta_J \geqq \frac{1}{8}(\sum_{j\in J} c_{ij})(\sum_{j\in J} c_{ij} - \max_{j\in J}\{c_{ij}\}) . \tag{5.23}$$

Using this evaluation of δ_J, we prove

Lemma 5.13. *Let (c_{ij}) be as in* Problem 5.8. *Assume there exists an infinite set $U\subset \boldsymbol{N}$ of indices such that*

$$\max\{c_{ij};\, j\in \boldsymbol{N}\} \leqq 1-\kappa \quad (i\in U)$$

for some $\kappa>0$. Then $p_N\to 0$ as $N\to\infty$.

Proof. Since $d_i\to 1$ as $i\to\infty$, we may assume without loss of generality that $|d_i-1|<\kappa/3$. First take an $i=u_1\in U$. For this i, we can find a finite subset $J_1\subset \boldsymbol{N}$ such that $d_i-\sum_{j\in J_1}c_{ij}\leqq \kappa/3$. Then, $\sum_{j\in J_1}c_{ij}\geqq d_i-\kappa/3\geqq 1-2\kappa/3$, and by (5.23) we have

$$\delta_{J_1} \geqq \frac{1}{8}\left(1-\frac{2}{3}\kappa\right)\left\{\left(1-\frac{2}{3}\kappa\right)-(1-\kappa)\right\} = \frac{1}{24}\left(1-\frac{2}{3}\kappa\right)\kappa \quad (=\kappa' \text{ (put)}) .$$

Assume we have taken mutually disjoint subsets $J_1, J_2, \cdots, J_n$ of $\boldsymbol{N}$ such that $\delta_{J_p}\geqq \kappa'$ $(p=1, 2, \cdots, n)$. Then we can find a $u_{n+1}\in U$ and a finite subset $J_{n+1}\subset \boldsymbol{N}\setminus(\bigcup_{p=1}^n J_p)$ such that, for $i=u_{n+1}$,

$$d_i-\textstyle\sum_{j\in J_{n+1}}c_{ij}\leqq \kappa/3 .$$

By a similar evaluation as above, we get $\delta_{J_{n+1}}\geqq \kappa'$.

Thanks to (5.19), we have $p_N\to 0$ $(N\to\infty)$. Q.E.D.

By this lemma, we see that it rests for us to check the case where $\max\{c_{ij};\, j\in \boldsymbol{N}\}\to 1$ as $i\to\infty$. In this case, there exists an injection u of $\boldsymbol{N}$ into itself such that $c_{i,u(i)}\to 1$ $(i\to\infty)$.

Note that the assumption on c_{ij} in Probelm 5.8 is symmetric in (i, j). Then, except a case similar to Lemma 5.13, where $q_N\to 0$, we come to the case $c_{v(j),j}\to 1$ $(j\to\infty)$ with an injection v: $\boldsymbol{N}\hookrightarrow \boldsymbol{N}$, to be checked. Thus, altogether, taking into account $e_j=1$ and $d_i\to 1$, we come to the following three cases, modulo appropriate permutations of rows and columns of c_{ij} by elements in $\tilde{\mathfrak{S}}_\infty$:

(A) $c_{ii}\to 1$; (B) $c_{i,i+N_1}\to 1$; (C) $c_{i+N_2,i}\to 1$;

as $i\to\infty$, where $N_1>0$, $N_2>0$.

5.7. Solution of Problem 5.8 (Step 3). Let us first treat the case (A). Then the assumption (5.16′) for (c_{ij}) is equivalent to

$$(5.24)\qquad \sum_{\substack{i,j\in N\\ i\neq j}} c_{ij}=\infty\,.$$

On the other hand, we can easily get the inequality $s_{2,2}^{I,J}\leqq 2^{-1}(s_2^{I,J})^2$, and therefore

$$s_2^{I,J}-s_3^{I,J}-s_{2,2}^{I,J}\geqq\Big(1-\frac{1}{3}\max_{i\in I}\{d_i\}\Big)s_2^{I,J}-\frac{1}{2}(s_2^{I,J})^2\,.$$

We may assume without loss of generality $d_i\leqq 3/2$ for $i\in \boldsymbol{N}$. Then, we obtain

$$(5.25)\qquad \delta_J\geqq\frac{1}{4}s_2^{I,J}(1-s_2^{I,J})\,.$$

Solving the inequality $x(1-x)\geqq 8^{-1}$, we get the following

Lemma 5.14. *Assume for a finite subsets J of $\boldsymbol{N}$ there exists an $I\subset \boldsymbol{N}$ such that*

$$(5.26)\qquad \frac{2-\sqrt{2}}{4}\leqq s_2^{I,J}\leqq\frac{2+\sqrt{2}}{4}\,.$$

Then $\delta_J\equiv 1-p_J\geqq 1/32$.

Now we put $I=J$. Then

$$s_2^{J,J}\geqq\sum_{i\in J}c_{ii}\Big(\sum_{\substack{j\in J\\ j\neq i}}c_{ij}\Big)\geqq\min_{i\in J}\{c_{ii}\}\cdot\Big(\sum_{\substack{i,j\in J\\ i\neq j}}c_{ij}\Big)\,.$$

Since $c_{ii}\to 1$, we may assume $c_{ii}\geqq 1/2$. Hence we have

$$(5.27)\qquad s_2^{J,J}\geqq\frac{1}{2}\sum_{\substack{i,j\in J\\ i\neq j}}c_{ij}\,.$$

On the other hand, put $I'=J'=J\cup\{p\}$, one point bigger than $I=J$. Then the difference $s_2^{J',J'}-s_2^{J,J}$ is small as shown by

$$s_2^{J',J'}-s_2^{J,J}\leqq\sum_{\{j_1,j_2\}\subset J}c_{pj_1}c_{pj_2}+\sum_{j\in J}c_{pj}c_{pp}+\sum_{i\in J}\sum_{j\in J}c_{ij}c_{ip}$$

$$\leqq\frac{1}{2}(d_p-c_{pp})^2+(d_p-c_{pp})c_{pp}+\max_{i\in J}\{d_i\}\cdot(e_p-c_{pp})$$

$$\leqq \frac{5}{4}(d_p - c_{pp}) + \frac{3}{2}(e_p - c_{pp}) .$$

Here $d_i \equiv \sum_{j \in N} c_{ij} \leqq 3/2$ and $d_i \to 1$ $(i \to \infty)$, $e_p \equiv \sum_{i \in N} c_{ip} = 1$, and $c_{pp} \to 1$ $(p \to \infty)$. This means that $s_2^{J',J'} - s_2^{J,J}$ is sufficiently small if $p \gg 0$. Thus we see from (5.24) and (5.27) that, when we make $I = J$ increase one by one appropriately, then some of $s_2^{J,J}$ comes into the interval (5.26), and $\delta_J \geqq 1/32$ for such J by Lemma 5.14.

From the above discussion, it is seen by induction that there exists a series of mutually disjoint finite subsets $J_1, J_2, \cdots$ of $\boldsymbol{N}$ such that $\delta_{J_n} \geqq 1/32$ for $n = 1, 2, \cdots$. From this we obtain by (5.19) that $p_N \to 0$ as $N \to \infty$.

Note that, by the symmetry of assumption, we have also $q_N \to 0$ in Case (A).

5.8. Solution of Problem 5.8 (Step 4). The cases (B) and (C) are similar, and so we treat only the former here. In Case (B) the condition (5.16′) is automatically satisfied.

First assume that a similar conditions as (5.24) holds:

$$\sum_{\substack{i,j \in N \\ i+N_1 \neq j}} c_{ij} = \infty . \tag{5.28}$$

Then, we can reduce the situation to Case (A) and get $p_N \to 0$ and $q_N \to 0$. In fact, for p_N, it is sufficient to apply Lemma 5.9 and the result in Case (A). For q_N, we consider a new (c'_{ij}) with $c'_{i1} = \sum_{1 \leq k \leq 1+N_1} c_{ik}$, $c'_{ij} = c_{i,j+N_1}$ $(j \geqq 2)$. Put q'_N for (c'_{ij}) the quantity similar o q_N for (c_{ij}). Then $q_N \leqq q'_N$ clearly, and $q'_N \to 0$ since (c'_{ij}) is in Case (A).

Now assume (5.28) does not hold. Then $\lim_{N \to \infty} q_N > 0$. In fact, $q_N \geqq \prod_{1 \leq i \leq N} c_{i,i+N_1}$, and $\prod_{i \in N} c_{i,i+N_1}$ converges because $\sum_{i \in N}(d_i - c_{i,i+N_1}) < \infty$. So let us prove $p_N \to 0$. Define (c'_{ij}) as $c'_{i1} = \sum_{1 \leq k \leq N_1} c_{ik}$, $c'_{ij} = c_{i,j+N_1-1}$ $(j \geqq 2)$. Then, $q_N \leqq q'_N$ and therefore we can reduce the case to $N_1 = 1$.

Assuming $N_1 = 1$, let us evaluate p_N. Put $C = (c_{ij})_{i,j \in N}$ and let $C_{(N;k)}$ be an $(N-1) \times (N-1)$ matrix obtained from C by cutting off i-th row for $i \geqq N$ and j-th row for $j = k$ and $j > N$. Further let D_N be an $N \times N$ matrix with elements

$$d_{ij} = c_{ij} \quad (i < N, \quad j \leqq N), \quad d_{Nj} = \sum_{N \leq i < \infty} c_{ij} \quad (i \leqq N) .$$

Denote by $p_N(C)$ the quantity p_N for $C = (c_{ij})$. Then we have also $p_{N-1}(C_{(N;k)})$ and $p_N(D_N)$, and get by simple calculations

$$p_N(C) \leqq p_N(D_N) \leqq \sum_{1 \leq k \leq N} d_{Nk} \cdot p_{N-1}(C_{(N;k)}) ,$$

$$p_{N-1}(C_{(N;k)}) \leqq \prod_{\substack{1 \leq j \leq N \\ j \neq k}} \Big(\sum_{1 \leq i < N} c_{ij} \Big) \leqq \prod_{\substack{1 \leq j \leq N \\ j \neq k}} e_j \leqq L$$

with a constant $L>0$. Therefore

$$p_N(C) \leqq L \cdot \sum_{1 \leqq k \leqq N} \sum_{N \leqq i < \infty} c_{ik} .$$

The right hand side tends to zero as $N \to \infty$ because $\sum_{j \neq i+N_1} c_{ij} < \infty$ by assumption. Thus we get $p_N = p_N(C) \to 0$. [END OF STEP 4]

Summalizing §§ 5.5~5.8, we have solved Problem 5.8 affirmatively. So that the proof of Proposition 5.3 is also completed.

5.9. Relation between Π_1 and Π_2. It is now established that, if $T_{\Sigma_1} \cong T_{\Sigma_2}$, then $E \sim Fb$ for some $b \in \tilde{\mathfrak{S}}_\infty$. Therefore, through the natural equivalence by $a = b^{-1} \in \tilde{\mathfrak{S}}_\infty$, we may assume $E \sim F$. Changing the representative unital product subsets with their cofinal ones, we may further assume that $E = F$ and the conditions (UPS3)−(UPS4) in § 1.8 are satisfied.

Let $G((E))$ be the subgroup of G defined in (3.9). Consider the representations of $G((E))$ under T_{Σ_i} $(i=1,2)$ on the subspaces $\mathfrak{H}^{\Pi_i}_{|E} = \mathfrak{H}_{|E} \otimes V(\Pi_i)$ of $\mathfrak{H}(\Sigma_i) \cong \mathcal{H}(\Sigma_i)$. Then the discussions in § 3.4, especially those in the proof of Lemma 3.5, prove that, if $T_{\Sigma_1} \cong T_{\Sigma_2}$, then necessarily $\Pi_1 \cong \Pi_2$.

Thus we have completely proved our equivalence criterion, Theorem 5.2.

DEPARTMENT OF MATHEMATICS
KYOTO UNIVERSITY

References

[1] N. Bourbaki, Topologie générale, Chapitres 1 à 4, Hermann, Paris, 1971.
[2] P. Delorme, Irréductibilité de certaines représentations de $G^{(X)}$, J. Funct. Anal., **30** (1977), 36-47.
[3] M. I. Golenishcheva-Kutuzova, Local classification of moments for groups of diffeomorphisms, Usp. Mat. Nauk, **42** (1987), 181-182 (=Russ. Math. Surv., **42** (1987), 217-218).
[4] M. I. Golenishcheva-Kutuzova, Functional moduli of moments for groups of diffeomorphisms of two-dimensional manifolds, Funct. Anal. Appl., **21** (1987), 69-70 (=Funct. Anal., **21** (1988), 315-316).
[5] T. Hirai, Construction of irreducible unitary representations of the infinite symmetric group $\mathfrak{S}_\infty$, J. Math. Kyoto Univ., **31** (1991), 495-541.
[6] R. S. Ismagilov, Unitary representations of the group of diffeomorphisms of a circle, Funct. Anal. Appl., **5** (1971), 45-53 (=Funct. Anal., **5** (1971), 209-216).
[7] R. S. Ismagilov, On unitary represetations of diffeomorphisms of a compact manifold, Izv. Akad. Nauk SSSR, **36** (1972), 180-208 (=Math. USSR Izv., **6** (1972), 181-209).
[8] R. S. Ismagilov, Unitary representations of the group of diffeomorphisms of the space $\boldsymbol{R}^n$, $n \geqq 2$, Funct. Anal. Appl., **9** (1975), 144-145 (=Funct. Anal., **9** (1975), 154-155).
[9] R. S. Ismagilov, imbedding of a group of measure-preserving diffeomorphisms into a semidirect product and its unitary representations, Mat. Sb., **113** (1980), 81-97 (=Math. USSR Sb., **41** (1982),67-81).
[10] S. Kakutani, On equivalence of infinite product measures, Ann. Math., **49** (1948), 214-224.
[11] A. A. Kirillov, Orbits of the group of diffeomorphisms of a circle and local Lie superalgebras, Funct. Anal. Appl., **15** (1981), 75-76 (=Funct. Anal., **15** (1981), 135-137).

[12] A. A. Kirillov, Kähler structures on K-orbits of the group of diffeomorphisms of a circle, ibid., **21** (1987), 42-45 (=Funct. Anal., **21** (1987), 122-125),.

[13] A. A. Kirillov and D. V. Yu'rev, Kähler geometry of the infinite-dimensional homogeneous space $M=\mathrm{Diff}_+(S^1)/\mathrm{Rot}(S^1)$, ibid., **21** (1987), 35-46 (=Funct. Anal., **21** (1988) 284-294).

[14] Yu. A. Neretin, The complementary representations of the group of diffeomorphisms of the circle, Usp. Mat. Nauk, **37** (1981), 213-214 (=Russ. Math. Surv., **37** (1982), 229-230).

[15] G. B. Segal, Unitary representations of some infinite dimensional groups, Commun. Math. Phys., **80** (1981), 301-342.

[16] M. Takesaki, Theory of operator algebras, Iwanami Shoten, Tokyo, 1983.

[17] A. M. Vershik, I. M. Gelfand and M. I. Graev, Representations of the group of diffeomorphisms, Usp. Mat. Nauk, **3** (1975), 3-50 (=Russ. Math. Surv., **30** (1975), 1-50).

[18] H. Weyl, The Classical Groups, Their Invariants and Representations, 2nd ed., Princeton University Press, Princeton, 1946.