J. Math. Kyoto Univ. (JMKYAZ)
33-2 (1993) 315-328

Range characterization of Radon transforms on $\mathbf{S}^n$ and $\mathbf{P}^n\mathbf{R}$

By

Tomoyuki KAKEHI

0. Introduction

It is one of the most important problems in integral geometry to characterize the ranges of Radon transforms. F. John [9] gave the first answer to this problem. His result is that the range of the X-ray transform on $\mathbf{R}^3$ is characterized by a second order ultrahyperbolic differential operator. Gelfand, Graev, and Gindikin [1] extended John's result; they characterized the ranges of d-plane Radon transforms on $\mathbf{R}^n$ and $\mathbf{C}^n$ by a system of second order differential operators on an affine Grassmann manifold. Farthermore, Gonzalez [4] gave a simple characterization of it by an invariant differential operator on an affine Grassmann manifold. Grinberg [5] characterized the range of the projective k-plane Radon transform on the n-dimensional real projective space $\mathbf{P}^n\mathbf{R}$ and the n-dimensional complex projective space $\mathbf{P}^n\mathbf{C}$ by a system of second order differential operators, and in [10], we gave another type of range characterization for the Radon transform on a complex projective space; we characterized the range by a single differential operator which is a fourth order invariant differential operator on a complex Grassmann manifold and which is ultrahyperbolic type of differential operator.

In this paper, we examine mainly the range of the Radon transform $R = R_l$ on the n-dimensional sphere $\mathbf{S}^n$ for $1 \leq l \leq n-2$, which we define by integrating a function f on $\mathbf{S}^n$ over an oriented l-dimensional totally geodesic sphere ξ, that is, we define R as follows

$$Rf(\xi) = \frac{1}{\mathrm{Vol}(\mathbf{S}^l)} \int_{x \in \xi} f(x)\, dv_\xi(x),$$

where $dv_\xi(x)$ is the canonical measure on $\xi \subset \mathbf{S}^n$. This Radon transform R maps smooth functions on $\mathbf{S}^n$ to smooth functions on $\widetilde{Gr}_{l+1,n+1}$, the compact oriented real Grassmann manifold, that is, $R: C^\infty(\mathbf{S}^n) \to C^\infty(\widetilde{Gr}_{l+1,n+1})$.

The main result of this paper is the following:

Theorem. *There exists a fourth order invariant differential operator P on $\widetilde{Gr}_{l+1,n+1}$ such that the range* $\operatorname{Im} R$ *of R is identical with its kernel* $\operatorname{Ker} P$, *i.e.*,

Communicated by Prof. N. Iwasaki, March 30, 1991

Im R = Ker P.

Taking account of John's result and the results in [10] or [11], it is expected that the above P can be represented as an ultrahyperbolic type of differential operator and, in fact, we will construct explicitly the above range-characterizing operator P as an ultrahyperbolic type of operator. The main tools are the same as those in [10]; we use the inversion formula and the method of radial part.

The author would like to thank Professor C. Tsukamoto for suggesting this problem and for many helpful discussions.

1. The range-characterizing operator P

Let M be the set of all l-dimensional oriented totally geodesic spheres of $\mathbf{S}^n$. The oriented Grassmann manifold M is a compact symmetric space $SO(n+1)/SO(l+1)\times SO(n-l)$ of rank $\min\{l+1, n-l\}$. We assume that $r := \operatorname{rank} M \geq 2$, that is, $1 \leq l \leq n-2$.

For a Lie group G and its closed subgroup H, we identify the subspace $C^\infty(G, H)$ of $C^\infty(G)$ defined by $C^\infty(G, H) = \{f \in C^\infty(G); f(gh) = f(g)\ \forall g \in G$ and $h \in H\}$, with $C^\infty(G/H)$. We define an action L_g of G on $C^\infty(G)$ by $(L_g f)(x) = f(g^{-1}x)$ for $x \in G$, and $f \in C^\infty(G)$. Similarly we define an action R_g of G on $C^\infty(G)$ by $(R_g f)(x) = f(xg)$. A differential operator D is called left-G-invariant if $L_g D = D L_g$ for all $g \in G$. Similarly, D is called right-H-invariant if $R_h D = D R_h$ for all $h \in H$. These notations are the same as those of the previous paper [10].

Let G, K, K' be the groups $SO(n+1)$, $SO(l+1)\times SO(n-l)$, $SO(n)$, respectively. Then we have $M = G/K$, $\mathbf{S}^n = G/K'$, and we identify $C^\infty(G, K)$ with $C^\infty(M)$, $C^\infty(G, K')$ with $C^\infty(\mathbf{S}^n)$ respectively. We define metrics on G, K, K', M, and $\mathbf{S}^n$, by the metrics induced from the Killing form metric on G, respectively. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of G and K, respectively.

$$\mathfrak{g} = \{X \in M_{n+1}(\mathbf{R});\ X + {}^tX = 0,\},$$

$$\mathfrak{k} = \left\{\begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \in \mathfrak{g};\ X_1 \in M_{l+1}(\mathbf{R}),\ X_2 \in M_{n-l}(\mathbf{R})\right\}.$$

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be the Cartan decomposition, where $\mathfrak{m}$ is the set of all the matrices of the form

$$X = \begin{pmatrix} 0 & \cdots & 0 & -x_{l+2,1} & \cdots & -x_{n+1,1} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & -x_{l+2,l+1} & \cdots & -x_{n+1,l+1} \\ x_{l+2,1} & \cdots & x_{l+2,l+1} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ x_{n+1,1} & \cdots & x_{n+1,l+1} & 0 & \cdots & 0 \end{pmatrix}.$$

We define differential operators $L_{ij,\alpha\beta}(l+2 \leq i < j \leq n+1,\ 1 \leq \alpha < \beta \leq l+1)$ on G by

$$(1.1) \qquad L_{ij,\alpha\beta} = \left(\frac{\partial^2}{\partial x_{i\alpha} \partial x_{j\beta}} - \frac{\partial^2}{\partial x_{i\beta} \partial x_{j\alpha}} \right) f(g \exp X)|_{X=0}, \qquad f \in C^\infty(G).$$

Using this, we define a differential operator P on G by

$$(1.2) \qquad P = \begin{cases} L_{34,12} & \text{if } n=3,\ l=1, \\ \sum_{\substack{l+2 \leq i < j \leq n+1 \\ 1 \leq \alpha < \beta \leq l+1}} (L_{ij,\alpha\beta})^2 & \text{otherwise.} \end{cases}$$

Then P is right-K-invariant. Thus P is well-defined as a differential operator on M. Its proof is the same as that of Lemma 1.1 in [10], and is reduced to the fact that the polynomial $F(X)$ on $\mathfrak{m}$ is Ad-K-invariant. Here

$$F(X) = \begin{cases} x_{31}x_{42} - x_{32}x_{41} & \text{if } n=3,\ l=1, \\ \sum_{\substack{l+2 \leq i < j \leq n+1 \\ 1 \leq \alpha < \beta \leq l+1}} (x_{i\alpha}x_{j\beta} - x_{i\beta}x_{j\alpha})^2 & \text{otherwise.} \end{cases}$$

We identify the principal symbol of P with $F(X)$.

By definition, P is left-G-invariant. Therefore, P is well-defined as an invariant differential operator on M. The main theorem of this paper is the following:

Theorem 1.1. *The range of R is identical with the kernel of P, that is,*

$$\operatorname{Ker} P = \operatorname{Im} R.$$

Remark 1.2. The differential operator $L_{ij,\alpha\beta}$ in (1.1) is ultrahyperbolic and of the form similar to the range-characterizing operator in [9] or similar to the operator $L_{ij,\alpha\beta}$ defined in [10]. Moreover the operator P defined by (1.2) is almost of the same form as the range-characterizing operator P in [10]. From this point of view, we can say that the range of the Radon transform R on $\mathbf{S}^n$ can be also characterized by an ultrahyperbolic type of differential operator.

Since we gave the proof for the case $l = 1$ in [11], we consider the other case in this paper.

2. Proof that $\operatorname{Im} R \subset \operatorname{Ker} P$

We first prove that $\operatorname{Im} R \subset \operatorname{Ker} P$. It is proved in the same way as the complex case (see [10]).

By the identification $C^\infty(G, K) = C^\infty(M)$ and $C^\infty(G, K') = C^\infty(\mathbf{S}^n)$, we consider the Radon transform R to be a map from $C^\infty(G, K)$ to $C^\infty(G, K')$. Then R is given by

$$(2.1)\qquad (Rf)(g) = \frac{1}{\mathrm{Vol}(K)}\int_{k\in K} f(gk)\,dk, \qquad f \in C^\infty(G, K').$$

From this section, we use the representation of the form (2.1).

We define a bilinear form $\langle \cdot , \cdot \rangle$ on $\mathbf{C}^{n+1} \times \mathbf{C}^{n+1}$ by $\langle u, v\rangle = \sum_{j=1}^{n+1} u_j v_j$ for $u = (u_1, \dots, u_{n+1})$, $v = (v_1, \dots, v_{n+1})$, and a smooth function $h_a^m \in C^\infty(G)$ by $h_a^m(g) = \langle a, g\mathbf{e}_1\rangle^m$, where $a \in \mathbf{C}^{n+1}$, $\mathbf{e}_1 = (1, 0, \dots, 0)$ and m is a non-negative integer. It is easily checked that $h_a^m \in C^\infty(G, K')$, that is, $h_a^m \in C^\infty(\mathbf{S}^n)$. Moreover, the following lemma holds.

Lemma 2.1. *Let V_m denote the subspace of $C^\infty(\mathbf{S}^n)$ generated by the set $\{h_a^m; \langle a, a\rangle = 0\}$. Then V_m is the eigenspace of $\Delta_{\mathbf{S}^n}$, the Laplacian of $\mathbf{S}^n$, corresponding to the m-th eigenvalue and V_m is irreducible under the action of G.*

For the proof, see [12].

We notice that we always consider the Laplacian on a compact manifold to be a non-negative operator.

We will use the following proposition to calculate the eigenvalue of P in Section 6.

Proposition 2.2. Im $R \subset$ Ker P.

Proof. By Lemma 2.1 and by the same argument as in that of Proposition 2.2 in [10], we have only to prove that

$$\begin{aligned} &L_{ij,\alpha\beta}(R(h_a^m))(I) \\ &= \frac{1}{\mathrm{Vol}(K)}\left(\frac{\partial^2}{\partial x_{i\alpha}\partial x_{j\beta}} - \frac{\partial^2}{\partial x_{i\beta}\partial x_{j\alpha}}\right)\int_{k\in K} h_a^m((\exp X)k)\,dk|_{X=0} \\ &= 0, \end{aligned}$$

where I denotes an identity matrix.

The above result follows from the equation:

$$\begin{aligned} &\left(\frac{\partial^2}{\partial x_{i\alpha}\partial x_{j\beta}} - \frac{\partial^2}{\partial x_{i\beta}\partial x_{j\alpha}}\right)\{\langle a, (\exp X)k\mathbf{e}_1\rangle^m\}|_{X=0} \\ &\quad = m(m-1)(a_i k_{\alpha 1} a_j k_{\beta 1} - a_i k_{\beta 1} a_j k_{\alpha 1})\langle a, k\mathbf{e}_1\rangle^{m-2} = 0, \end{aligned}$$

where $k \in K$ and k_{ij} denotes the (i, j) entry of k. Therefore the assertion is verified.

3. The inversion formula

We construct a continuous linear map $S\colon C^\infty(M) \to C^\infty(\mathbf{S}^n)$ such that $SR = Id$ on $C^\infty_{even}(\mathbf{S}^n)$, using the Helgason's inversion formula. Here Id denotes the identity map and $C^\infty_{even}(\mathbf{S}^n)$ denotes the space of all even functions in $C^\infty(\mathbf{S}^n)$. (The Radon transform R maps odd functions on $\mathbf{S}^n$ to 0.)

In this section, we denote by M_l the oriented Grassmann manifold $SO(n+1)/$

$SO(l+1) \times SO(n-l)$, by K_l the closed subgroup $SO(l+1) \times SO(n-l)$ of G, and by R_l the Radon transform $R: C^\infty(\mathbf{S}^n) \to C^\infty(M_l)$ respectively. We define a dual Radon transform $\tilde{R}_l: C^\infty(M_l) \to C^\infty(\mathbf{S}^n)$ by

$$(\tilde{R}_l f)(g) = \frac{1}{\mathrm{Vol}(K_{n-1})} \int_{k \in K_{n-1}} f(gk)\, dk, \qquad f \in C^\infty(G, K_l).$$

If k is even, we define a polynomial $\Phi_k(x)$ by

$$\Phi_k(x) = \left(x + \frac{(k-1)(n-k)}{2n}\right)\left(x + \frac{(k-3)(n-k+2)}{2n}\right) \cdots \left(x + \frac{1(n-2)}{2n}\right)$$

Theorem 3.1 (Helgason [6], Ch. 1, Theorem 4.5). *If l is even, we have the inversion formula for R_l*

$$c_{n,l}\Phi_l(\Delta_{\mathbf{S}^n})\tilde{R}_l R_l = Id \qquad on \quad C^\infty_{even}(\mathbf{S}^n),$$

where $c_{n,l}$ is a constant depending on n and l.

Proposition 3.2. *There exists an inversion map $S = S_l: C^\infty(M_l) \to C^\infty(\mathbf{S}^n)$ such that $S_l R_l = Id$ on $C^\infty_{even}(\mathbf{S}^n)$*

Proof. If l is even, Proposition 3.2 follows immediately from Theorem 3.1, and we may therefore prove this proposition in the case l is odd. We define $R^l_{l+1}: C^\infty(M_l) \to C^\infty(M_{l+1})$ by

$$(R^l_{l+1} f)(g) = \frac{1}{\mathrm{Vol}(K_{l+1})} \int_{k \in K_{l+1}} f(gk)\, dk, \qquad f \in C^\infty(G, K_l)$$

Then it is easily checked that $R^l_{l+1} R_l = R_{l+1}$. Since l is odd, $l+1$ is even and by Theorem 3.1 there exists an inversion map S_{l+1} such that $S_{l+1} R_{l+1} = Id$ on $C^\infty_{even}(\mathbf{S}^n)$. Therefore, if we put $S_l = S_{l+1} R^l_{l+1}$, we get $S_l R_l = Id$ on $C^\infty_{even}(\mathbf{S}^n)$.

4. Representation of (G, K)

In this section, we describe the root, the weight, and the Weyl group of the symmetric pair (G, K).

Let $\mathfrak{a} \subset \mathfrak{m}$ be the set of all matrices of the form

$$H(t) = H(t_1, \dots, t_r) = \begin{pmatrix} 0 & \cdots & 0 & -t_1 & & & & & \\ \vdots & & \vdots & & \ddots & & & & \\ 0 & \cdots & 0 & & & -t_r & & & \\ t_1 & & & 0 & \cdots & 0 & \cdots & 0 \\ & \ddots & & \vdots & & \vdots & & \vdots \\ & & t_r & 0 & \cdots & 0 & \cdots & 0 \\ & & & \vdots & & \vdots & & \vdots \\ & & & 0 & \cdots & 0 & \cdots & 0 \end{pmatrix},$$

where we put $r = \operatorname{rank} M (= \operatorname{rank} G/K)$ in Section 1 and $t = (t_1, \dots, t_r) \in \mathbf{R}^r$. Then, $\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{m}$. We identify $\mathfrak{a}$ with $\mathbf{R}^r$ by the mapping $H(t) \mapsto t$.

Let $(\cdot, \cdot)$ denote an invariant inner product on $\mathfrak{g}$ defined by

$$(X, Y) = -(n-1)\operatorname{tr}(XY) \qquad X, Y \in \mathfrak{g},$$

which is a minus-signed Killing form on $\mathfrak{g}$.

For $\alpha \in \mathfrak{a}$, let

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g}^{\mathbf{C}}; [H, X] = \sqrt{-1}(\alpha, H)X \quad \text{for all} \quad H \in \mathfrak{a}\}$$

An element $\alpha \in \mathfrak{g}$ is called a root of $(\mathfrak{g}, \mathfrak{a})$ if $\mathfrak{g}_\alpha \neq \{0\}$. We put $m_\alpha = \dim_{\mathbf{C}} \mathfrak{g}_\alpha$ and call it a multiplicity of α.

We put $H_i = H(0, \dots, \overset{i}{\breve{1}}, \dots, 0)$ $(1 \leq i \leq r)$ and we fix a lexicographical order $<$ on $\mathfrak{a}$ such that $H_1 > \cdots H_r > 0$. Then the positive root α of $(\mathfrak{g}, \mathfrak{a})$ and its multiplicity m_α are given by the table:

α		m_α
$\frac{1}{2(n-1)}(H_j \pm H_k)$	$(1 \leq j < k \leq r)$	1
$\frac{1}{2(n-1)}H_j$	$(1 \leq j \leq r)$	$n+1-2r.$

The simple roots α_j $(1 \leq j \leq r)$, are given by the table:

$$(n+1 > 2r): \quad \alpha_j = \frac{1}{2(n-1)}(H_j - H_{j+1}) \quad (1 \leq j \leq r-1),$$

$$\alpha_r = \frac{1}{2(n-1)}H_r.$$

$$(n+1 = 2r): \quad \alpha_j = \frac{1}{2(n-1)}(H_j - H_{j+1}) \quad (1 \leq j \leq r-2),$$

$$\alpha_{r-1} = \frac{1}{2(n-1)}(H_{r-1} + H_r)$$

$$\alpha_r = \frac{1}{2(n-1)}(H_{r-1} - H_r).$$

Let M_j $(1 \leq j \leq r)$ be the fundamental weights of G/K corresponding to the simple roots α_j, $(1 \leq j \leq r)$. Then, M_j $(1 \leq j \leq r)$ are given by the table:

$$(n+1 > 2r+1): \quad M_j = \frac{1}{n-1}\sum_{k=1}^{j} H_k \qquad (1 \leq j \leq r-1),$$

$$M_r = \frac{1}{2(n-1)} \sum_{k=1}^{r} H_k.$$

$$(n+1 = 2r, \text{ or } 2r+1): \quad M_j = \frac{1}{n-1} \sum_{k=1}^{j} H_k.$$

If $n+1 > 2r$, the Weyl group $W(G, K)$ of (G, K) is the set of all maps $s: \mathfrak{a} \to \mathfrak{a}$ such that

$$(4.1) \qquad s: (t_1, \dots, t_r) \longmapsto (\varepsilon_1 t_{\sigma(1)}, \dots, \varepsilon_r t_{\sigma(r)}) \quad \varepsilon_j = \pm 1, \ \sigma \in \mathfrak{S}_r.$$

And if $n+1 = 2r$, $W(G, K)$ is the set of all maps s in (4.1) such that $\varepsilon_1 \cdot \varepsilon_2 \cdots \varepsilon_r = 1$.

Let $Z(G, K)$ be the weight lattice generated by $\frac{1}{2(n-1)} H_j$, $(1 \leq j \leq r)$. The highest weight of a spherical reprsentation of (G, K) is of the form $m_1 M_1 + \cdots + m_r M_r$, where $m_1, \dots, m_r$ are non-negative integers. We denote by $V(m_1, \dots, m_r)$ the eigenspace of Laplacian Δ_M on $M = G/K$ which is an irreducible representation space with the highest weight $m_1 M_1 + \cdots + m_r M_r$.

In the same manner, we can define a fundamental weight M'_1 of $(SO(n+1), SO(n))$, (that is, this is the case $l = 0$,) and we have

$$M'_1 = \frac{1}{2(n-1)} H_1$$

Then mM'_1 is the highest weight of the m-th eigenspace V_m of Laplacian Δ_{S^n}, which we defined in Section 2. It is easily checked that $2M'_1$ corresponds to M_1 by an adjoint action. Therefore, we get the following Lemma by Proposition 3.2.

Lemma 4.1. *The Radon transform R isomorphically maps the subspace V_{2m} of $C^\infty(\mathbf{S}^n)$ to the subspace $V(m, 0, \dots, 0)$ of $C^\infty(M)$.*

5. Radial part of P

We will calculate the eigenvalue of P on $V(m_1, \dots, m_r)$ to prove Theorem 1.1. There are two ways to calculate it. One is a representation theoretical approach, and the other is the method of radial part. We use the latter.

We define a density function Ω on $\mathfrak{a}$ by

$$\Omega(t) = \Big| \prod_{\alpha;\text{positive root}} 2\sin(\alpha, H(t))^{m_\alpha} \Big|$$

Then $\Omega(t)$ is given by

$$(5.1) \qquad \Omega(t) = c_{n,r} |\sigma\omega|,$$

where

$$c_{n,r} = 2^{\frac{1}{2}r(2n+1-3r)}$$

(5.2)
$$\sigma = \prod_{j=1}^{r} \sin^{n+1-2r} t_j$$

$$\omega = \prod_{1 \le j < k \le r} (\cos 2t_j - \cos 2t_k)$$

We choose a connected component $\mathscr{A}^+$ of Weyl chambers such that $\sigma > 0$, $\omega > 0$ on $\mathscr{A}^+$. For example, we choose

$$\mathscr{A}^+ = \left\{(t_1, \dots, t_r) \in \mathbf{R}^r;\ 0 < t_1 < \cdots < t_r < \frac{\pi}{2}\right\} \quad (n+1 > 2r),$$

$$\mathscr{A}^+ = \{(t_1, \dots, t_r) \in \mathbf{R}^r;\ 0 < t_j \pm t_k < \pi, \quad 1 \le j < k \le r\} \quad (n+1 = 2r).$$

To each invariant differential operator D on G/K, there corresponds a unique differential operator on $\mathscr{A}^+$ which is invariant under the action of the Weyl group $W(G, K)$. This operator is called a radial part of D, and we denote it by $\mathrm{rad}(D)$.

The following lemma is well-known.

Lemma 5.1.

$$\mathrm{rad}(\Delta_M) = -\frac{1}{n-1} \sum_{j=1}^{r} \left(\frac{\partial^2}{\partial t_j^2} + \frac{\Omega_{t_j}}{\Omega} \frac{\partial}{\partial t_j} \right),$$

where Ω_{t_j} means a differentiation of Ω by t_j.

For the proof, see [12] ch. 10, Cor. 1.

As in [10], let us consider the following four conditions (A), (B), (C), and (D) on a differential operator Q that is regular in all Weyl chambers.

(A) $Q = \sum_{1 \le j < k \le r} \frac{\partial^4}{\partial t_j^2 \partial t_k^2}$ + lower order terms.

(B) Q is formally self-adjoint with respect to the density $\Omega\, dt$.

(C) Q is $W(G, K)$-invariant.

(D) $[Q, \mathrm{rad}(\Delta_M)] := Q\,\mathrm{rad}(\Delta_M) - \mathrm{rad}(\Delta_M) Q = 0$.

Then it is easily checked that the differential operator $\mathrm{rad}(P)$ satisfies the above four conditions (A), (B), (C), and (D), by the same argument as in [10].

We calculate the radial part of P.

By the conditions (A) and (B), we get

$$\text{the third order terms of } \mathrm{rad}(P) = \sum_{j \ne k} \frac{\Omega_{t_k}}{\Omega} \frac{\partial^3}{\partial t_j^2 \partial t_k}.$$

Thus, we can put

$$\operatorname{rad}(P) = \sum_{j<k} \frac{\partial^4}{\partial t_j^2 \partial t_k^2} + \sum_{j\neq k} \frac{\Omega_{t_k}}{\Omega} \frac{\partial^3}{\partial t_j^2 \partial t_k} \tag{5.3}$$
$$+ \sum_{j=1}^{r} A_j \frac{\partial^2}{\partial t_j^2} + \sum_{j<k} B_{jk} \frac{\partial^2}{\partial t_j \partial t_k} + \sum_{j=1}^{r} C_j \frac{\partial}{\partial t_j},$$

where the coefficients A_j, B_{jk}, and C_j are C^∞ functions on Weyl chambers.

By condition (D), the third order terms of $[\operatorname{rad}(P), \operatorname{rad}(\Delta_M)] = 0$, and we get the equations

$$\frac{\partial}{\partial t_j} A_j = \frac{1}{2} \sum_{\alpha\neq j} ((a_\alpha)_{t_\alpha t_\alpha} + a_\alpha (a_\alpha)_{t_j}), \tag{5.4}$$

$$\frac{\partial}{\partial t_j} \{2B_{jk} - 3(a_j)_{t_k} - 2a_j a_k\} = -a_j (a_j)_{t_k} - 2(A_j)_{t_k}, \tag{5.5}$$

$$\frac{\partial}{\partial t_k} \{2B_{jk} - 3(a_j)_{t_k} - 2a_j a_k\} = -a_k (a_k)_{t_j} - 2(A_k)_{t_j}, \tag{5.6}$$

where we put $a_j = \Omega_{t_j}/\Omega$.

We take

$$A_j = -(n+1-2r) \sum_{\alpha\neq j} \cot t_j \frac{\sin 2t_\alpha}{\cos 2t_\alpha - \cos 2t_j} \tag{5.7}$$
$$+ 2 \sum_{\alpha\neq j} \left(\frac{\cos 2t_\alpha}{\cos 2t_\alpha - \cos 2t_j} - \frac{\sin^2 2t_\alpha}{(\cos 2t_\alpha - \cos 2t_j)^2} \right)$$
$$+ \sum_{\alpha\neq j} \frac{\sin^2 2t_\alpha}{\cos 2t_\alpha - \cos 2t_j}$$
$$- 2 \sum_{\substack{\alpha<\beta \\ \alpha,\beta\neq j}} \left\{ 1 + \frac{\sin^2 2t_\alpha}{(\cos 2t_\alpha - \cos 2t_j)(\cos 2t_\beta - \cos 2t_j)} \right\},$$

$$B_{jk} = \frac{3}{2} a_{j,t_k} + a_j a_k. \tag{5.8}$$

Then, after a tedius but straight forward computation, we find that the functions (5.7) and (5.8) satisfy the equations (5.4), (5.5), and (5.6).

We get by the condition (B),

$$C_j = \frac{1}{\Omega} (A_j \Omega)_{t_j} + \frac{1}{2\Omega} \sum_{j<k} \{(B_{jk}\Omega)_{t_k} + (B_{jk}\Omega)_{t_j}\} - \frac{1}{2\Omega} \sum_{j\neq k} \Omega_{t_j t_k t_k} \tag{5.9}$$

We define a fourth order differential operator Q_1 by the right hand side of the equation (5.3), where the coefficients A_j, B_{jk}, and C_j are given by the functions (5.7), (5.8), and (5.9) respectively. Then, a differential operator $Q_2 := \operatorname{rad}(P) - Q_1$ is a second order differential operator and satisfies the conditions (B), (C), and (D). Thus, we will prove that the operator Q_2 can be written as $c \operatorname{rad}(\Delta_M)$ for a suitable constant c.

We define a subgroup W_0 of $W(G, K)$ by the set of all maps s in (4.1) such that $\varepsilon_1 \cdot \varepsilon_2 \cdots \varepsilon_r = 1$. Then, if $n + 1 > 2r$, W_0 is strictly contained in $W(G, K)$, and if $n + 1 = 2r$, W_0 is identical with $W(G, K)$. We can prove the above fact by the following lemma.

Lemma 5.2. *We assume that* $(n, r) \neq (3, 2)$. *If a second order differential operator* Q *satisfies the conditions* (B) *and* (D), *and if* Q *is* W_0*-invariant, then* $Q = c\,\mathrm{rad}(\Delta_M)$ *for some constant* c.

Proof. We put

$$Q := \sum_{j=1}^{r} A_j \frac{\partial^2}{\partial t_j^2} + \sum_{j<k} B_{jk} \frac{\partial^2}{\partial t_j \partial t_k} + \sum_{j=1}^{r} C_j \frac{\partial}{\partial t_j}$$

By the condition (D), the third order terms of $[Q, \mathrm{rad}(\Delta_M)]$ vanish. Thus we have

$$A_{j,t_j} = 0, \qquad (1 \leq j \leq r); \tag{5–10}$$

$$A_{k,t_j} + B_{jk,t_k} = 0, \quad A_{j,t_k} + B_{jk,t_j} = 0, \qquad (j < k); \tag{5–11}$$

$$B_{ij,t_k} + B_{jk,t_i} + B_{ik,t_j} = 0, \qquad (1 \leq i < j < k \leq r). \tag{5–12}$$

By the equations (5.10–12) and the assumption that Q is W_0-invariant and $(n, r) \neq (3, 2)$, the coefficients A_j and B_{jk} are polynomials of the form

$$A_j = \delta_1 \sum_{k \neq j} t_k^2 + \delta_2, \tag{5.13}$$

$$B_{jk} = -2\delta_1 t_j t_k, \tag{5.14}$$

where δ_1 and δ_2 are some constants.

Using the condition (B), we have

$$C_j = \frac{1}{\Omega}(A_j \Omega)_{t_j} + \frac{1}{2\Omega} \sum_{j<k} (B_{jk}\Omega)_{t_k} + \frac{1}{2\Omega} \sum_{k<j} (B_{kj}\Omega)_{t_k}. \tag{5.15}$$

If $\delta_1 = 0$, then the coefficient $B_{jk} = 0$, and the coefficient $C_j = \delta_2 \Omega_{t_j}/\Omega$ by (5.15). Therefore we obtain $Q = -(n-1)\delta_2\,\mathrm{rad}(\Delta_M)$, and the lemma holds.

Now, we suppose that $\delta_1 \neq 0$. In particular, we may suppose that $\delta_1 = 1$ and $\delta_2 = 0$. By the condition (D), the first order terms of $[Q, \mathrm{rad}(\Delta_M)]$ vanish. Then

$$Q a_1 = -(n-1)\,\mathrm{rad}(\Delta_M) C_1, \tag{5.16}$$

where $a_1 = \Omega_{t_1}/\Omega$.

We extend the both sides of (5.16) to $\mathbf{C}$ as meromorphic functions of $t_1 = \mu_1 + \sqrt{-1}\,\nu_1$.

By the formula (5.1–2), we have

$$a_1 = (n+1-2r)\frac{\cos t_1}{\sin t_1} + \sum_{j=2}^{r} \frac{-2\sin 2t_1}{\cos 2t_1 - \cos 2t_j}. \tag{5.17}$$

Let $v_1 \to \infty$, then $a_{1,t_j} \to 0$, $a_{1,t_jt_k} \to 0$ (rapidly decreasing), and $a_1 = O(1)$. The same fact holds for a_j ($j = 2,\dots,r$). Thus $Qa_1 \to 0$ (rapidly decreasing). Therefore we get $\mathrm{rad}(\Delta_M)C_1 \to 0$ (rapidly decreasing) by (5.16). However, when v_1 tends to $+\infty$, we have

$$\begin{aligned} -(n-1)\,\mathrm{rad}(\Delta_M)C_1 &= \frac{1}{2}\sum_{j,k=2}^{r}\left(\frac{\partial^2}{\partial t_k^2} + a_k\frac{\partial}{\partial t_k}\right)(B_{1j}a_j + B_{1j,t_j}) + O(1) \\ &= -t_1 \sum_{k=2}^{r} a_k^2 + O(1). \end{aligned}$$

(In the above computation, we have used (5.13), (5.14) and the fact that $a_j = O(1)$ and the derivatives of $a_j \to 0$ as $v_1 \to 0$.) Therefore, we have $\mathrm{rad}(\Delta_M)C_1 \to \infty$, for suitable $t_2,\dots,t_r$, and μ_1. It is a contradiction.

Remark 5.4. If $n = 3$ and $r = 2$, there exists a differential operator Q such that Q satisfies the conditions in Lemma 5.3 and linearly independent of $\mathrm{rad}(\Delta_M)$. Indeed, if we define a differential operator Q by

$$Q = \frac{\partial^2}{\partial t_1 \partial t_2} + \frac{\Omega_{t_2}}{2\Omega}\frac{\partial}{\partial t_1} + \frac{\Omega_{t_1}}{2\Omega}\frac{\partial}{\partial t_2}, \tag{5.18}$$

then Q satisfies the conditions (B), (C), and (D). Moreover Q is linearly independent of $\mathrm{rad}(\Delta_M)$. Therefore, it is easily checked that this operator Q is the radial part of $P = L_{34,12}$.

By the above argument, we get a following proposition.

Proposition 5.5. *If* $(n, r) \neq (3, 2)$, *the differential operator* $\mathrm{rad}(P)$ *can be expressed of the form*

$$\mathrm{rad}(P) = Q_1 + c(n-1)\,\mathrm{rad}(\Delta_M),$$

for some constant c.

6. Proof of Theorem 1.1

We calculate the eigenvalue of P on $V(m_1,\dots,m_r)$ to prove Theorem 1.1.

Let $a(m_1,\dots,m_r)$ be the eigenvalue of P on $V(m_1,\dots,m_r)$ and $\phi_{(m_1,\dots,m_r)}$ the zonal spherical function which belongs to $V(m_1,\dots,m_r)$. We denote by $u_{(m_1,\dots,m_r)}$ the restriction of $\phi_{(m_1,\dots,m_r)}$ to the Weyl chamber $\mathscr{A}^+$. Since the procedure is almost the same as that in [10], we omit the proofs of the following lemmas.

Lemma 6.1 ([12], THEOREM 8.1). *The function* $u_{(m_1,\dots,m_r)}$ *has a Fourier series expansion on* $\mathscr{A}^+$ *of the form*

$$u_{(m_1,\dots,m_r)}(t_1,\dots,t_r) = \sum_{\substack{\lambda \le m_1M_1+\cdots+m_rM_r \\ \lambda\in Z(G,K),\text{finite sum}}} \eta_\lambda \exp\sqrt{-1}(\lambda, t_1H_1+\cdots+t_rH_r),$$

where $\eta_{m_1M_1+\cdots+m_rM_r} > 0$.

Let f_1 and f_2 be Fourier series of the form

$$f_1 = \sum_{\lambda\le\Lambda_1,\lambda\in Z(G,K)} \zeta_\lambda \exp\sqrt{-1}(\lambda, t_1H_1+\cdots+t_rH_r)$$

$$f_2 = \sum_{\lambda\le\Lambda_2,\lambda\in Z(G,K)} \tilde{\zeta}_\lambda \exp\sqrt{-1}(\lambda, t_1H_1+\cdots+t_rH_r)$$

We denote $f_1 \sim f_2$ when $\Lambda_1 = \Lambda_2 (>0)$ and $\zeta_{\Lambda_1} = \tilde{\zeta}_{\Lambda_2} (\neq 0)$.

Lemma 6.2.

$$\begin{aligned} \Omega_{t_j} &\sim \sqrt{-1}(n+1-2j)\Omega, \\ A_j\Omega^2 &\sim -(n-j)(j-1)\Omega^2, \\ B_{jk}\Omega^2 &\sim -(n+1-2j)(n+1-2k)\Omega^2, \\ C_j\Omega^3 &\sim -\sqrt{-1}(n-j)(j-1)(n+1-2j)\Omega^3, \end{aligned}$$

where the functions A_j, B_{jk}, *and* C_j *are given by* (5.5), (5.6), *and* (5.7) *respectively.*

Now we can calculate the eigenvalue of P on each irreducible eigenspace, by the following theorem.

Theorem 6.3. *Unless* $n = 3$ *and* $r = 2$, *the eigenvalue* $a(m_1,\dots,m_r)$ *of* P *on* $V(m_1,\dots,m_r)$ *is given by the formulae*

$$\begin{aligned} a(m_1,\dots,m_r) = &\sum_{1\le j<k\le r} l_jl_k(l_j+n+1-2j)(l_k+n+1-2k) \\ &+ \sum_{j=2}^{r} (j-1)(n-j)l_j(l_j+n+1-2j), \end{aligned}$$

where l_j *is given as follows.*

$$(n+1 > 2r+1):\ l_j = 2(m_j+\cdots+m_{r-1}) + m_r$$

$$(n+1 = 2r,\ \text{or}\ 2r+1):\ l_j = 2(m_j+\cdots+m_r).$$

Proof. By definition, we have

$$P\phi_{(m_1,\dots,m_r)} = a(m_1,\dots,m_r)\phi_{(m_1,\dots,m_r)}. \tag{6.1}$$

We restrict both sides of (6.1) to the Weyl chamber $\mathscr{A}^+$, and then we have

$$\operatorname{rad}(P)u_{(m_1,\dots,m_r)} = a(m_1,\dots,m_r)u_{(m_1,\dots,m_r)}.$$

By Proposition 5.3, Lemma 6.1 and Lemma 6.2, it follows that

$$
\begin{aligned}
&\Omega^3 \operatorname{rad}(P)\, u_{(m_1,\dots,m_r)} \\
&= \Omega^3\{Q_1 + c(n-1)\operatorname{rad}(\Delta_M)\} u_{(m_1,\dots,m_r)} \\
&\sim \Big\{ \sum_{1\le j<k\le r} l_j l_k (l_j + n + 1 - 2j)(l_k + n + 1 - 2k) \\
&+ \sum_{j=2}^{r} (j-1)(n-j) l_j (l_j + n + 1 - 2j) \Big\} \Omega^3 u_{(m_1,\dots,m_r)} \\
&+ c \sum_{j=1}^{r} l_j (l_j + n + 1 - 2j) \Omega^3 u_{(m_1,\dots,m_r)}.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
a(m_1,\dots,m_r) = &\sum_{1\le j<k\le r} l_j l_k (l_j + n + 1 - 2j)(l_k + n + 1 - 2k) \\
&+ \sum_{j=2}^{r} (j-1)(n-j) l_j (l_j + n + 1 - 2j) \\
&+ c \sum_{j=1}^{r} l_j (l_j + n + 1 - 2j).
\end{aligned}
$$

Here, by Proposition 2.1 and Lemma 4.1, we have $a(2m, 0,\dots,0) = 0$. Therefore, we get $c = 0$, which completes the proof.

Remark 6.4. For the case $(n, r) = (3, 2)$, the eigenvalue of P can be also computed in the same way as above using (5.18). For detail, see [11].

The following corollary is easily verified.

Corollary 6.5. *$V(m_1,\dots,m_r)$ is contained in* $\operatorname{Ker} P$ *if and only if* $m_2 = \cdots = m_r = 0$.

Proof of Theorem 1.1. Our proof of Theorem 1.1 is almost the same as that of Theorem 1.2 in [10].

Let $V := \bigoplus_{m=0}^{\infty} V(m, 0,\dots,0)$ and $\tilde{V} := \bigoplus_{m=0}^{\infty} V_{2m}$. Then we have $R\colon \tilde{V} \to V$ and $S\colon V \to \tilde{V}$. Moreover we have $SR = Id$ on $\tilde{V}$ and $RS = Id$ on V by Proposition 3.2 and Lemma 4.1.

By Corollary 6.5, V is dense in $\operatorname{Ker} P$ in C^∞-topology. Since $S\colon C^\infty(M) \to C^\infty_{even}(\mathbf{S}^n)$ is continuous, we have $RS = Id$ on $\operatorname{Ker} P$. This proves Theorem 1.1.

Remark 6.6. The differential operator P is of least degree in all the invariant differential operators on $\widetilde{Gr}_{l+1,n+1}$ that characterize the range of R. It follows from the fact that the principal symbol $F(X)$ of P, which we defined in Section 1, is of least degree in all the Ad-K-invariant polynomials on $\mathfrak{m}$ except for the principal symbol of the Laplacian.

7. Radon transforms on $\mathbf{P}^n\mathbf{R}$

The set of all projective l-dimensional planes of $\mathbf{P}^n\mathbf{R}$ is a real Grassmann manifold $Gr_{l+1,n+1}$ and a compact symmetric space $O(n+1)/O(l+1) \times O(n-l)$ of rank $\min\{l+1, n-l\}$. We define a Radon transform $\mathscr{R}: C^\infty(\mathbf{P}^n\mathbf{R}) \to C^\infty(Gr_{l+1,n+1})$ as follows.

$$\mathscr{R}f(\eta) = \frac{1}{\mathrm{Vol}(\mathbf{P}^l\mathbf{R})} \int_{x\in\eta} f(x)\,dv_\eta(x),$$

where $dv_\eta(x)$ is the canonical measure on $\eta(\subset \mathbf{P}^n\mathbf{R})$.

Since we can identify $C^\infty_{even}(\mathbf{S}^n)$ with $C^\infty(\mathbf{P}^n\mathbf{R})$, we have

$$Rf(+\eta) = Rf(-\eta) = \mathscr{R}f(\eta) \qquad \text{for } f \in C^\infty(\mathbf{P}^n\mathbf{R}) \text{ and } \eta \in Gr_{l+1,n+1},$$

where $+\eta$ and $-\eta$ denote orientations of η.

We defined the invariant differential operator P on $Gr_{l+1,n+1}$ in Section 2, but we can easily check that P is also well-defined as an invariant differential operator on $Gr_{l+1,n+1}$. Therefore we obtain the following theorem from Theorem 1.1.

Theorem 7.1. *The range of the Radon transform $\mathscr{R}$ on* $\mathbf{P}^n\mathbf{R}$ *is identical with* Ker P.

INSTITUTE OF MATHEMATICS
UNIVERSITY OF TSUKUBA

References

[1] I. M. Gelfand, S. G. Gindikin, and M. I. Graev, Integral geometry in affine and projective space, J. Soviet Math., **18** (1982), 39–167.

[2] I. M. Gelfand, and M. I. Graev, Complexes of straight lines in the space $\mathbf{C}^n$, Funct. Anal. Appl., **2** (1968), 39–52.

[3] I. I. M. Gelfand, M. I. Graev, and R. Rosu, The probrem of integral geometry and intertwining operator for a pair of real Grassmann manifolds, J. Operator Theory, **12** (1984), 339–383.

[4] F. Gonzalez, Invariant differential operator and the range of the radon D-plane transform, Math. Ann., **287** (1990), 627–635.

[5] E. L. Grinberg, On images of Radon transforms, Duke Math. J., **52** (1985), 935–972.

[6] V. Guillemin, and S. Sternberg, Some problems in integral geometry and some related problems in micro-local analysis, Amer. J. Math., **101** (1979), 915–955.

[7] S. Helgason, Groups and Geometric Analysis, Academic Press, New York, 1984.

[8] S. Helgason, Differential Geometry, Lie Groups, and Symmetric Spaces, Academic Press, New York, 1978.

[9] F. John, The ultrahyperbolic differential equation with four independent variables, Duke. Math. J., **4** (1938), 300–322.

[10] T. Kakehi, Range characterization of Radon transforms on complex projective spaces. J. Math. Kyoto Univ., 32-2 (1992) 387–399

[11] T. Kakehi, and C. Tsukamoto, Characterization of images of Radon transforms. (to appear)

[12] M. Takeuchi, Gendai No Kyukansu, (in Japanese), Iwanami shoten, Tokyo, 1974.