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Compactification of moduli of parabolic sheaves and moduli of parabolic Higgs sheaves

By

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Introduction

Let $f: X \to S$ be a smooth, projective, geometrically integral morphism of locally noetherian schemes, D be an effective relative Cartier divisor on X/S and let $\mathcal{O}_X(1)$ be an f-very ample invertible sheaf. Assume that S is of finite type over a universally Japanese ring Ξ. In the previous paper "Moduli of Parabolic Stable Sheaves" [13], we have extended the notion of parabolic bundles on curves to higher dimensional cases, i.e. parabolic sheaves on a geometric fibre X_s of f is a triple (E, F_*, α_*) consisting of a torsion free coherent sheaf E, a filtration $E = F_1(E) \supset F_2(E) \supset \cdots \supset F_l(E) \supset F_{l+1}(E) = E(-D)$ and a system of weights $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_l < 1$. Moreover, we have constructed a coarse moduli scheme $M_{D/X/S}^{H_*, \alpha_*}$ of parabolic stable sheaves with fixed weights α_* and fixed Hilbert polynomials H_*.

In this article, we shall construct a moduli scheme $\bar{M}_{D/X/S}^{H_*, \alpha_*}$ of equivalence classes of parabolic semi-stable sheaves and show that it is projective over S under some boundedness conditions. We could do more. In fact, the method used in constructing moduli schemes of stable pairs (cf. [26]) leads us to a construction of a moduli scheme of "parabolic pairs". Let Ω be a locally free $\mathcal{O}_X$-module. Combining the notion of parabolic sheaves and that of Ω-pairs, we come to a notion of parabolic Ω-pairs, i.e. a parabolic Ω-pair is a pair (E_*, φ) of a parabolic sheaf E_* and a parabolic homomorphism $\varphi: E_* \to E_* \otimes \Omega$ with $\varphi \wedge \varphi = 0$. The word "parabolic Higgs sheaves" used in our title means $\Omega_X^1(\log D)$-pairs (in the case where $\Omega_X^1(\log D)$ is locally free). In the case of curves, it is in fact equivalent to the notion of Simpson's "filtered regular Higgs bundles" [25]. Simpson [25] gave a natural one-to-one correspondence between stable filtered regular Higgs bundles of degree zero and stable filtered local systems of degree zero. In this paper, since we shall restrict ourselves to the moduli problem for parabolic Ω-pairs, Ω can be any locally free sheaf. Our main theorem is the existence of a moduli scheme of equivalence classes of parabolic semi-stable Ω-pairs. Moreover, we shall define a morphism of the moduli scheme to an affine space of "characteristic polynomials" and prove that it is projective as a

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natural generalization of results of Hitchin [7] in the case of Higgs bundles on curves or Simpson's [24] in higher dimensional cases. Then as a special case, we obtain the moduli scheme $\bar{M}^{H_*,\alpha_*}_{D/X/S}$ which is projective over S.

In § 1, we shall give various definitions on parabolic pairs. Almost all notions are naturally extended to our case. §2 is devoted to the construction of a parameter space R^{ss} of all parabolic semi-stable Ω-pairs with fixed Hilbert polynomials and weights. Moreover, a morphism $\tilde{\Psi}$ of R^{ss} to the product of a Gieseker space $\tilde{Z}$ and some Grassmann schemes G_i is constructed. Then the results in [13] can be generalized to our case. In particular, the existence of a coarse moduli scheme of stable parabolic Ω-pairs is proved. §3 is devoted to the analysis of orbit spaces of $(\tilde{Z} \times \prod G_i)^{ss}$ which is a natural generalization of results in §2 of M. Maruyama [10]. In the case of parabolic sheaves, under the assumption that S is a scheme over a field of characteristic zero, most of these are not needed since the projectivity of the morphism $\tilde{\Psi}$ of R^{ss} to $(\tilde{Z} \times \prod G_i)^{ss}$ is proved in our Appendix. The notion of extensions of points in Gieseker spaces is naturally extended to our case but it does not work well since the set of all extensions of two points in $\tilde{Z} \times \prod G_i$ is not a complete family. Hence, we shall introduce a notion of "quasi-extension". In §4, we shall prove the main theorem, that is, the existence of a moduli scheme of parabolic semi-stable Ω-pairs. The projectivity of the moduli scheme of parabolic semi-stable sheaves is proved in §5. For the moduli scheme of parabolic semi-stable pairs, the properness of the morphism of the moduli scheme to the space of "characteristic polynomials" is proved. We shall derive these results by the method used in S. G. Langton [8]. In appendix, we shall show that the morphism Ψ: $\Gamma^{ss} \to (Z \times \prod G_i)^{ss}$ constructed in [13] is proper in the case of characteristic zero. Strangely, the author could not prove it without an additional condition "$\alpha_1 > 0$" where α_1 is the minimum weight. However, in the case of curves or more generally in the case where μ-(semi-)stability is the same as (semi-)stability, by changing weights, we can also get a moduli scheme which is projective over S.

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Notation and Convention. Let f: $X \to S$ be a smooth, projective, geometrically integral morphism of locally noetherian schemes, D be an effective relative Cartier divisor on X/S and let $\mathcal{O}_X(1)$ be an f-very ample invertible sheaf. For a coherent $\mathcal{O}_X$-module E and a numerical polynomial H, we denote simply by Quot(E, H) the Quot-scheme $\mathrm{Quot}^H_{E/X/S}$. If s is a geometric point of S, then X_s means the geometric fiber of X over s and $E_s = E \otimes_{\mathcal{O}_s} k(s)$. We denote by $S^i(E)$ the i-th symmetric product of E, by $S^*(E)$ the symmetric $\mathcal{O}_X$-algebra. For a coherent $\mathcal{O}_{X_s}$-module F, the degree of F with respect to $\mathcal{O}_X(1)$ is that of the first Chern class of F with respect to $\mathcal{O}_{X_s}(1) = \mathcal{O}_X(1) \otimes \mathcal{O}_{X_s}$ and it is denoted by $\deg_{\mathcal{O}_X(1)} F$ or simply $\deg F$. Moreover, the rank of F is denoted by rk(F), $\mu(F) = \deg F/\mathrm{rk}(F)$ and $h^i(F) = \dim_{k(s)} H^i(X_s, F)$.

For polynomials $f_1(n)$ and $f_2(n)$, $f_1(n) \prec f_2(n)$ (or, $f_1(n) \preceq f_2(n)$) means that

$f_1(n) < f_2(n)$ (or, $f_1(n) \leq f_2(n)$, resp.) for all sufficiently large integers n. For a polynomial H and a number m, $H[m]$ denotes the polynomial such that $H[m](x) = H(m + x)$.

1. Parabolic pairs

Let X be a non-singular, projective variety over an algebraically closed field k and let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X. Fix an effective Cartier divisor $D \subset X$ and a locally free $\mathcal{O}_X$-module Ω of finite rank.

Let us recall some definitions on parabolic sheaves. (For details, see § 1 of [13].)

Definition 1.0. A parabolic sheaf is a triple (E, F_*, α_*) of a torsion free coherent $\mathcal{O}_X$-module E, a filtration

(1.0.1) $$E = F_1(E) \supset F_2(E) \supset \cdots \supset F_l(E) \supset F_{l+1}(E) = E(-D)$$

and a system of weights $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_l < 1$. For a parabolic sheaf (E, F_*, α_*), we have a filtration

(1.0.2) $$\bigcup_{m \in \mathbf{Z}} E(-mD) = \bigcup_{\alpha \in \mathbf{R}} E_\alpha \supset \cdots \supseteq E_\alpha \supseteq E_\beta \supseteq \cdots$$

where $E_\alpha = F_i(E)(-[\alpha]D)$ with i an integer such that $\alpha_{i-1} < \alpha - [\alpha] \leq \alpha_i$ ($\alpha_0 = \alpha_l - 1$ and $\alpha_{l+1} = 1$). We often denote (E, F_*, α_*) by E_*.

A parabolic homomorphism of E_* to F_* is an $\mathcal{O}_X$-homomorphism of E to F which maps E_α into F_α for all $\alpha \geq 0$. In this paper, we change the definition of parabolic subsheaves given in [13], i.e. a parabolic sheaf F_* is said to be a parabolic subsheaf of E_* when if F is a coherent subsheaf of E and $F_\alpha \subseteq E_\alpha$ for all α.

The parabolic Hilbert polynomial of E_* is

(1.0.3) $$\text{par-}\chi(E_*(m)) = \int_0^1 \chi(E_\alpha(m))\, d\alpha.$$

The polynomial $\text{par-}\chi(E_*(m))/\text{rk}(E)$ is denoted by $\text{par-}P_{E_*}(m)$. Moreover, the parabolic degree of E_* is

(1.0.4) $$\text{par-deg}(E_*) = \int_0^1 \deg(E_\alpha)\, d\alpha + \text{rk}(E) \cdot \deg D.$$

$\text{par-}\mu(E_*)$ is $\text{par-deg}(E_*)/\text{rk}(E)$ and $\text{wt}(E_*)$ is $\text{par-deg}(E_*) - \deg E$.

For $0 \leq \alpha \leq 1$, $\deg E(-D) \leq \deg E_\alpha \leq \deg E$. Hence, by (1.0.4), we have the following inequalities.

(1.0.5) $$0 \leq \text{wt}(E_*) \leq \text{rk}(E) \cdot \deg D.$$

We can naturally extend the notion of Ω-pairs (cf. [6], [19], [24] and [26]) to parabolic cases.

Definition 1.1. A pair (E_*, φ) of a parabolic sheaf E_* and a parabolic homomorphism $\varphi: E_* \to E_* \otimes_X \Omega$ is said to be a parabolic Ω-pair if $\varphi \wedge \varphi = 0$, where $\varphi \wedge \varphi$ is the following homomorphism

$$E \xrightarrow{\varphi} E \otimes_X \Omega \xrightarrow{\varphi \otimes 1} E \otimes_X \Omega \otimes_X \Omega \longrightarrow E \otimes_X \wedge^2 \Omega$$

and $E_* \otimes_X \Omega$ is a parabolic sheaf such that $(E \otimes_X \Omega)_\alpha = E_\alpha \otimes_X \Omega$. The polynomial par-$\chi(E_*(m))$ is called the parabolic Hilbert polynomial of (E_*, φ).

A parabolic subsheaf E'_* of E_* is called φ-invariant when for all $0 \leq \alpha \leq 1$, $\varphi(E'_\alpha)$ is contained in $E'_\alpha \otimes_X \Omega$. For parabolic pairs (E_*, φ) and (E'_*, φ'), an $\mathcal{O}_X$-homomorphism f of E to E' is said to be a homomorphism of parabolic pairs when $\varphi' \circ f = (f \otimes id_\Omega) \circ \varphi$ and f is a parabolic homomorphism of E_* to E'_*. (E'_*, φ') is called a (parabolic) sub-pair of (E_*, φ) if E' is a coherent subsheaf of E, $E'_\alpha \subseteq E_\alpha$ for all α and $\varphi|_{E'} = \varphi'$.

Let F be a φ-invariant coherent subsheaf of E such that E/F is torsion free. If we put $F_\alpha = E_\alpha \cap F$, then $(F_*, \varphi|_F)$ is a sub-pair of (E_*, φ). In this case, we call this structure of $(F_*, \varphi|_F)$ the induced (sub-)structure of (E_*, φ). Let G be a torsion free coherent quotient sheaf of E with quotient map $g: E \to G$. Assume that $\ker(g)$ is φ-invariant. Setting $G_\alpha = (E_\alpha + F)/F$ for all $\alpha \geq 0$, we get a quotient pair $(G_*, \bar{\varphi})$ of (E_*, φ) where $\bar{\varphi}$ is the homomorphism of G to $G \otimes_X \Omega$ induced from φ.

Remark 1.2. φ induces an $\mathcal{O}_X$-homomorphism $\check{\varphi}$ of $\Omega^\vee$ to $\mathcal{E}nd^{Par}(E_*)$. The condition "$\varphi \wedge \varphi = 0$" implies that $\check{\varphi}$ is extended to a homomorphism of $\mathcal{O}_X$-algebras of $S^*(\Omega^\vee)$ to $\mathcal{E}nd^{Par}(E_*)$. Thus we have a parabolic homomorphism associated to φ;

$$\varphi^a: E_* \otimes_X S^*(\Omega^\vee) \longrightarrow E_*.$$

Definition 1.3. 1) (E_*, φ) is said to be parabolic stable (or, parabolic semi-stable) if for every φ-invariant parabolic subsheaf F_* of E_* with $0 \neq F \neq E$, we have

$$\text{par-}P_{F_*}(m) \prec \text{par-}P_{E_*}(m) \quad (\text{or, } \preceq, \text{ resp.}).$$

2) (E_*, φ) is said to be parabolic μ-stable (or, parabolic μ-semi-stable) if for every φ-invariant parabolic subsheaf F_* of E_* with $0 \neq F \neq E$, we have

$$\text{par-}\mu(F_*) < \text{par-}\mu(E_*) \qquad (\text{or, } \leq, \text{ resp.}).$$

3) Let e be an integer. (E_*, φ) is said to be of type e if for every φ-invariant parabolic subsheaf F_* of E_* with $0 \neq F \neq E$, we have

$$\text{par-}\mu(F_*) \leq \text{par-}\mu(E_*) + e.$$

Remark 1.4. By Remark 1.11 of [13], we may assume that in the above definitions, F_* has the induced structure.

Let $f: X \to S$ be a smooth, projective, geometrically integral morphism of noetherian schemes, $D \subset X$ be a relative effective Cartier divisor with respect to f and let $\mathcal{O}_X(1)$ be an f-very ample invertible sheaf. Let $H, H_1, \dots H_l$ be polynomials and $\alpha_1, \dots, \alpha_l$ be real numbers such that $0 \leq \alpha_1 < \cdots < \alpha_l < 1$. Set $H_* = \{H_1, \dots H_l\}$ and $\alpha_* = \{\alpha_1, \dots, \alpha_l\}$.

We denote by $\mathscr{F}_\Omega(H, H_*, \alpha_*)$ the family of classes of parabolic Ω-pairs on the fibres of X over S such that (E_*, φ) is contained in $\mathscr{F}_\Omega(H, H_*, \alpha_*)$ if and only if (E_*, φ) is a parabolic semi-stable Ω-pair on a geometric fibre of X over S, $\chi(E(m)) = H(m)$, $\chi((E/F_{i+1}(E))(m)) = H_i(m)$ and the system of weights is α_*.

By the inequality (1.0.5), if (E_*, φ) is of type e, then for all φ-invariant coherent subsheaf F of E, we have that $\mu(F) \leq \mu(E) + \deg D + e$. Hence, by Proposition 1.6 of [26], we have

Lemma 1.5. *If (E_*, φ) is of type e, then there exists an integer e' which depends only on e, D, Ω and* rk(E) *such that E is of type e', i.e. for all coherent subsheaf F of E, $\mu(F) \leq \mu(E) + e'$.*

By virtue of the boundedness results on the families of coherent sheaves (cf. [11]), we have the following.

Corollary 1.6. *The family $\mathscr{F}_\Omega(H, H_*, \alpha_*)$ is bounded if one of the following conditions is satisfied.*

1) *S is a noetherian scheme over a field of characteristic zero.*
2) *The rank is not greater than* 3.
3) *The dimension of X over S is not greater than* 2.

Let us recall that a torsion free coherent sheaf E on a geometric fibre X_k is said to be of c-type e if for general non-singular curves $C = D_1 \cdot \cdots \cdot D_{n-1}$, $D_i \in |\mathcal{O}_{X_k}(1)|$, every subsheaf $E' (\neq 0)$ of $E \otimes_X \mathcal{O}_C$ has a degree $\leq \mathrm{rk}(E')(\mu(E) + e)$.

Definition 1.7. 1) (E_*, φ) is said to be parabolic e-stable (or, parabolic e-semi-stable) if (E_*, φ) is parabolic stable (or, parabolic semi-stable, resp.) and E is of c-type e.

2) (E_*, φ) is said to be strictly parabolic e-semi-stable if it is e-semi-stable and if for every φ-invariant parabolic quotient sheaf F_* of E_* with par-P_{E_*} = par-P_{F_*}, $(F_*, \varphi|_{F_*})$ is parabolic e-semi-stable.

Let $\mathscr{F}^e_\Omega(H, H_*, \alpha_*)$ be the sub-family of $\mathscr{F}_\Omega(H, H_*, \alpha_*)$ such that (E_*, φ) is contained in $\mathscr{F}^e_\Omega(H, H_*, \alpha_*)$ if and only if (E_*, φ) is parabolic e-semi-stable.

By virtue of Lemma 3.3 of [9] and Lemma 1.5, we have

Proposition 1.8. *The family $\mathscr{F}^e_\Omega(H, H_*, \alpha_*)$ is bounded.*

By virtue of Proposition 1.8, Lemma 2.6 of [13] and by a similar proof as Proposition 2.5 of [13], we have

Proposition 1.9. *There exists an integer N_0 such that*

1) *if $(E_*, \varphi) \in \mathscr{F}_\Omega^e(H, H_*, \alpha_*)$ is parabolic stable, then for all $m \geq N_0$ and all φ-invariant parabolic subsheaves F_* of E_* with $0 \neq F \neq E$,*

$$\int_0^1 h^0(F_\alpha(m))\,\mathrm{d}\alpha/\mathrm{rk}(F) < \int_0^1 h^0(E_\alpha(m))\,\mathrm{d}\alpha/\mathrm{rk}(E),$$

2) *if $(E_*, \varphi) \in \mathscr{F}_\Omega^e(H, H_*, \alpha_*)$ is not parabolic stable, then for all $m \geq N_0$ and all φ-invariant parabolic subsheaves F_* of E_* with $0 \neq F \neq E$,*

$$\int_0^1 h^0(F_\alpha(m))\,\mathrm{d}\alpha/\mathrm{rk}(F) \leq \int_0^1 h^0(E_\alpha(m))\,\mathrm{d}\alpha/\mathrm{rk}(E),$$

and there exists a non-trivial φ-invariant parabolic subsheaf E'_* of E_* such that for all $m \geq N$ and $i > 0$,

$$\int_0^1 h^0(E'_\alpha(m))\,\mathrm{d}\alpha/\mathrm{rk}(E') = \int_0^1 h^0(E_\alpha(m))\,\mathrm{d}\alpha/\mathrm{rk}(E),$$

$h^i(E'(m)) = 0$ and $E'(m)$ is generated by its global sections.

Remark 1.10. Recall that a coherent $\mathcal{O}_X$-module E is said to be f-torsion free or relatively torsion free if it is flat over S and for every geometric fibre X_s of f, $E \otimes_X \mathcal{O}_{X_s}$ is a torsion free $\mathcal{O}_{X_s}$-module. By the argument below Definition 1.13 of [13], we know that if E is f-torsion free, then the canonical homomorphism $E \otimes_X \mathcal{O}_X(-D) \to E$ is injective.

Let (Sch/S) be the category of locally noetherian schemes over S. Let T be an object of (Sch/S). A triple (E, F_*, α_*) of a coherent $\mathcal{O}_{X_T}$-module E, a filtration F_* of E as in (1.0.1) and a system of weights α_* is called a *flat family of parabolic sheaves* on X_T/T if E is f_T-torsion free and all $E/F_i(E)$ are flat over T (hence, all $F_i(E)$ are flat over T). Note that a flat family of parabolic sheaves has a filtration as in (1.0.2), hence we denote it simple by E_*. A *flat family of parabolic pairs* is a pair (E_*, φ) of a flat family of parabolic sheaves E_* and an $\mathcal{O}_{X_T}$-homomorphism φ of E to $E \otimes_X \Omega$ such that $\varphi(E_\alpha) \subseteq E_\alpha \otimes_X \Omega$ for all α.

For the openness of parabolic stability of parabolic pairs, we have

Proposition 1.11. *Let $g: Y \to T$ be a smooth, projective, geometrically integral morphism of locally noetherian schemes, $\mathcal{O}_Y(1)$ be a g-very ample invertible sheaf, $D \subset Y$ be a relative effective Cartier divisor and (E_*, φ) be a flat family of parabolic Ω-pairs on Y/T. If $H^i(Y_t, \mathcal{O}_Y(1) \otimes k(t)) = 0$ for all $i > 0$ and $t \in T$, then there exist open sets T^{ss} and T^s of T such that for all algebraically closed fields k,*

$$T^{ss}(k) = \{t \in T(k) \mid (E_*, \varphi) \otimes k(t) \text{ is strictly parabolic e-semi-stable}\}$$

$$T^s(k) = \{t \in T(k) \mid (E_*, \varphi) \otimes k(t) \text{ is parabolic e-stable}\}.$$

Proof. By virtue of Corollary 2.3, there exists a closed subscheme Q^{φ} of $Q = \mathrm{Quot}(E, H)$ such that for every object T' of (Sch/T),

$$Q^{\varphi}(T') = \{G \in Q(T') | G \text{ is } \varphi_{X_{T'}}\text{-invariant}\}.$$

Using the scheme Q^{φ} instead of Q in the proof of Proposition 2.8 of [13], the proof in [13] holds good for our case. □

By the similar proof as in the usual case, we see that every parabolic semi-stable Ω-pair (E_*, φ) has a Jordan-Hölder filtration

$$E = E^0 \supset E^1 \supset \cdots \supset E^{m+1} = 0$$

where for all i, E_i is φ-invariant, $((E^i/E^{i+1})_*, \bar{\varphi})$ with the induced structure is parabolic stable and par-$P_{(E^i/E^{i+1})_*}(m) =$ par-$P_{E_*}(m)$. We denote by $\mathrm{gr}(E_*, \varphi)$ the direct sum $\bigoplus_{i=0}^{m}((E^i/E^{i+1})_*, \bar{\varphi})$. The Jordan-Hölder filtration is not in general unique but $\mathrm{gr}(E_*, \varphi)$ is uniquely determined up to parabolic isomorphisms. It is easy to see that $\mathrm{gr}(E_*, \varphi)$ is also parabolic semi-stable. Moreover, every parabolic pair has a unique (μ-) Harder-Narasimhan filtration of parabolic pairs, see §5 for the proof.

Definition 1.12. For an object T of (Sch/S), set

$$\text{par-}\bar{\Sigma}^{H_*, \alpha_*}_{\Omega/D/X/S}(T) = \left\{ (E_*, \varphi) \;\middle|\; \begin{array}{l} (E_*, \varphi) \text{ is a flat family of} \\ \text{parabolic } \Omega\text{-pairs on } X_T/T \\ \text{with the property (1.12.1)} \end{array} \right\} \Big/ \sim$$

where $\sim$ is the equivalence relation defined by (1.12.2).

(1.12.1) For every geometric point t of T, $((E_t, (F_*)_t, \alpha_*), \varphi_t)$ is parabolic semi-stable, $\chi(E_t(m)) = H(m)$ and $\chi((E_t/F_{i+1}(E)_t)(m)) = H_i(m)$, where $(F_*)_t$ is the filtration consisting of φ-invariant subsheaves

$$E_t = F_1(E)_t \supset F_2(E)_t \supset \cdots \supset F_{l+1}(E)_t = E_t(-D).$$

(1.12.2) $(E_*, \varphi) \sim (E'_*, \varphi')$ if and only if (1) $(E_*, \varphi) \simeq (E'_*, \varphi') \otimes_T L$ or (2) there exist filtrations consisting of φ-invariant subsheaves $E = E^0 \supset E^1 \supset \cdots \supset E^m = 0$ and $E' = E'^0 \supset E'^1 \supset \cdots \supset E'^m = 0$ such that for every geometric point t of T, their restrictions to X_t provide us with Jordan-Hölder filtrations of $((E_t)_*, \varphi)$ and $((E'_t)_*, \varphi')$, respectively, $\mathrm{gr}(E_*, \varphi) = \bigoplus_{i=0}^{m}((E^i/E^{i+1})_*, \varphi_i)$ is T-flat and that $\mathrm{gr}(E_*, \varphi) \cong \mathrm{gr}(E'_*, \varphi') \otimes_T L$, for some invertible sheaf L on T.

For a morphism $g\colon T' \to T$ in (Sch/S), g^* defines a map of par-$\bar{\Sigma}^{H_*, \alpha_*}_{\Omega/D/X/S}(T)$ to par-$\bar{\Sigma}^{H_*, \alpha_*}_{\Omega/D/X/S}(T')$. Then par-$\bar{\Sigma}^{H_*, \alpha_*}_{\Omega/D/X/S}$ is a contravariant functor of (Sch/S) to $(Sets)$. We denote by par-$\Sigma^{H_*, \alpha_*}_{\Omega/D/X/S}$ the sub-functor of par-$\bar{\Sigma}^{H_*, \alpha_*}_{\Omega/D/X/S}$ consisting of all flat families of parabolic stable Ω-pairs. Moreover for each non-negative integer e, par-$\bar{\Sigma}^{H_*, \alpha_*, e}_{\Omega/D/X/S}$ (or, par-$\Sigma^{H_*, \alpha_*, e}_{\Omega/D/X/S}$) denotes a subfunctor of par-$\bar{\Sigma}^{H_*, \alpha_*}_{\Omega/D/X/S}$ (or, par-$\Sigma^{H_*, \alpha_*}_{\Omega/D/X/S}$, resp.) consisting of all flat families of strictly parabolic

e-semi-stable (or, parabolic e-stable, resp.) Ω-pairs. By virtue of Proposition 1.11, if we assume that $H^i(X_s, \mathcal{O}_X(1) \otimes k(s)) = 0$ for all $i > 0$ and all $s \in S$, then these are open sub-functors of par-$\Sigma^{H_*, \alpha_*}_{\Omega/D/X/S}$.

2. Moduli of parabolic stable pairs

In this section, we shall construct a coarse moduli scheme $M^{H_*, \alpha_*}_{\Omega/D/X/S}$ of the functor par-$\Sigma^{H_*, \alpha_*}_{\Omega/D/X/S}$. We shall fix the following situation:

(2.0.1) Let S be a scheme of finite type over a universally Japanese ring Ξ and let $f\colon X \to S$ be a smooth, projective, geometrically integral morphism such that the dimension of each fiber of X over S is n. Let $\mathcal{O}_X(1)$ be an f-very ample invertible sheaf such that for all points s in S and all $i > 0$, $H^i(X_s, \mathcal{O}_X(1) \otimes \mathcal{O}_{X_s}) = 0$. Let $D \subset X$ be a relative effective Cartier divisor and let Ω be a locally free $\mathcal{O}_X$-module.

Fix a non-empty family $\mathscr{F}^e_\Omega(H, H_*, \alpha_*)$. We assume that all α_i are rational numbers. By Proposition 1.8, there exists an integer N_0 such that for every member (E_*, φ) of $\mathscr{F}^e_\Omega(H, H_*, \alpha_*)$, the conditions 1), 2) in Proposition 1.9 and the following conditions are satisfied.

(2.0.2) For all i and all $m \geq N_0$, $F_i(E)(m)$ and $(E/F_i(E))(m)$ are generated by its global sections.

(2.0.3) For all i, all $j \geq 1$ and all $m \geq N_0$, $H^j(F_i(E)(m)) = 0$ and $H^j((E/F_i(E))(m)) = 0$.

(2.0.4) For all $m \geq N_0$, if an invertible sheaf L on a geometric fibre X_s has the same Hilbert polynomial as $\det(E(m))$, then

$$\mathrm{Ext}^j_{\mathcal{O}_{X_s}}(\wedge^r(V \otimes_\Xi S^*_r(\Omega^\vee)_{X_s}), L) = 0$$

for all $j \geq 1$, where r is the rank of E, V is a free Ξ-module of rank r and $S^*_r(\Omega^\vee)$ is the sheaf $\bigoplus_{i=0}^{(r-1)\mathrm{rk}(\Omega)} S^i(\Omega^\vee)$.

Remark 2.1. 1) If (2.0.4) holds, then for all $j \geq 1$, all free Ξ-modules V and all invertible sheaves L on X_s with the same Hilbert polynomial as $\det(E(m))$, we have

$$\mathrm{Ext}^j_{\mathcal{O}_{X_s}}(\wedge^r(V \otimes_\Xi S^*_r(\Omega^\vee)_{X_s}), L) = 0$$

2) In the previous paper [26], for a locally sheaf Ω, we have denoted by $S^*_r(\Omega)$ the sheaf $\bigoplus_{i=0}^{r-1} S^i(\Omega)$. Lemma 1.2 of [26] was wrong. But it becomes correct and hence all results in [26] hold good if we change the definition of $S^*_r(\Omega)$ to $\bigoplus_{i=0}^{(r-1)\mathrm{rk}(\Omega)} S^i(\Omega)$. We must change, in the proof of Lemma 1.2, 1.17 in p. 313 of [26]:

$$\text{(or, } \{\varphi'(x_1)^{i_1} \cdots \varphi'(x_m)^{i_m}(f) \mid f \in F,\ 0 \leq i_1, \ldots, i_m \leq r-1\}\text{, resp.)}$$

to the following.

$$(\text{or, } \{\varphi'(x_1)^{i_1}\cdots\varphi'(x_m)^{i_m}(f) \mid f\in F,\ \sum_{j=1}^{m} i_j \le (r-1)m,\ 0\le i_1,\dots,i_m\},\ \text{resp.})$$

We need the following result of the base change theory. (See for the proof §1 of A. Altman and S. Kleiman [1].)

Proposition 2.2. *Let $f\colon X\to S$ be a proper morphism of noetherian schemes and let I and F be two coherent $\mathcal{O}_X$-modules, with F flat over S. Then there exist a coherent $\mathcal{O}_S$-module $H(I,F)$ and an element $h(I,F)$ of* $\mathrm{Hom}_X(I, F\otimes_S H(I,F))$ *which represents the functor*

$$M \longmapsto \mathrm{Hom}_X(I, F\otimes_S M)$$

defined on the category of quasi-coherent $\mathcal{O}_S$-modules M, and the formation of the pair commutes with base change; in other words, the Yoneda map defined by $h(I,F)$,

$$\text{(2.2.1)} \qquad y\colon \mathrm{Hom}_T(H(I,F)_T, M) \longrightarrow \mathrm{Hom}_{X_T}(I_T, F\otimes_X M)$$

is an isomorphism for every S-scheme T and every quasi-coherent $\mathcal{O}_T$-module M. Moreover if I is flat over S and if $\mathrm{Ext}^1_{X_s}(I\otimes k(s), F\otimes k(s)) = 0$ *for all points s of S, then $H(I,F)$ is locally free.*

Corollary 2.3. *Let $f\colon X\to S$ be a proper morphism of noetherian schemes and let $\varphi\colon I\to F$ be an $\mathcal{O}_X$-homomorphism of coherent $\mathcal{O}_X$-modules with F flat over S. Then there exists a unique closed subscheme Z of S such that for all morphisms $g\colon T\to S$, $g^*(\varphi)=0$ if and only if g factors through Z.*

Proof. By the isomorphism (2.2.1), φ corresponds to an $\mathcal{O}_S$-homomorphism $\psi\colon H(I,F)\to\mathcal{O}_S$. The closed subscheme Z of S defined by the ideal sheaf Image (ψ) is the desired one. □

Fix an integer $m\ge N_0$ and a free Ξ-module V_m of rank $H(m)$. Set $Q=\mathrm{Quot}(V_m\otimes_\Xi\mathcal{O}_X, H[m])$ and $Q_i=\mathrm{Quot}(V_m\otimes_\Xi\mathcal{O}_X, H_i[m])$. Let $\phi\colon V_m\otimes_\Xi \mathcal{O}_{X_Q}\to\tilde{E}(m)$ (or, $\phi_i\colon V_m\otimes_\Xi\mathcal{O}_{X_{Q_i}}\to\tilde{E}_i(m)$) be the universal quotient on X_Q (or, X_{Q_i}, resp.). Let Q^0 be the open subscheme of Q such that for all algebraically closed fields K,

$$Q^0(K)=\{x\in Q(K)\mid \tilde{E}|_{X_x} \text{ is torsion free}\}.$$

Let U_i be the maximal open subscheme of Q_i such that for all points x of U_i and all $j\ge 1$, $H^j(\tilde{E}_i(m)|_{X_x})=0$ and $f_{i*}(\phi_i)\colon V_m\otimes_\Xi\mathcal{O}_{U_i}\to f_{i*}(\tilde{E}_i(m)|_{X_{U_i}})$ is surjective where f_i is the projection of X_{U_i} to U_i. Note that $f_{i*}(\tilde{E}_i(m)|_{X_{U_i}})$ is a locally free $\mathcal{O}_{U_i}$-module of rank $H_i(m)$. Hence, the quotient map $f_{i*}(\phi_i)$ defines a morphism of $\gamma_i\colon U_i\to G_i$ where $G_i=\mathrm{Grass}(V_m\otimes_\Xi\mathcal{O}_S, H_i(m))$.

In §3 of [13], we have constructed a closed subscheme Γ of $Q^0\times{}_S\prod_{i=1}^{l}U_i$ and a flat family of parabolic sheaves $(\tilde{E}(m)_{X_\Gamma}, \tilde{F}_*\,\alpha_*)$ of length l and a surjection $\phi\otimes 1\colon V_m\otimes_\Xi\mathcal{O}_{X_\Gamma}\to\tilde{E}(m)_{X_\Gamma}$ where $\tilde{F}_*$ is a filtration of $\tilde{E}(m)_{X_\Gamma}$ such that

$\tilde{E}(m)_{X_{\Gamma}}/\tilde{F}_{i+1}(\tilde{E}(m)_{X_{\Gamma}})$ is isomorphic to $\tilde{E}_i(m)_{X_{\Gamma}}$ as quotients of $V_m \otimes_{\Xi} \mathcal{O}_{X_{\Gamma}}$. These have the following universal property.

(2.4.1) Let T be an object of (Sch/S). Assume that a flat family of parabolic sheaves (E, F_*, α_*) of length l on X_T and a surjection $\phi'\colon V_m \otimes_{\Xi} \mathcal{O}_{X_T} \to E$ have the following properties.

1. For all geometric points t of T and all i, the Hilbert polynomial of E (or, $E/F_{i+1}(E)$) on X_t is $H[m]$ (or, $H_i[m]$, resp.) and $H^j((E_t/F_{i+1}(E)_t)) = 0$ for all $j \geq 1$.
2. For all i, the natural homomorphism of $V_m \otimes_{\Xi} \mathcal{O}_T$ to $(f_T)_*(E/F_{i+1}(E))$ is surjective.

Then there exists a unique morphism of T to Γ such that (E, F_*, α_*) and ϕ' are given by the pull back of $(\tilde{E}_{X_{\Gamma}}, \tilde{F}_*, \alpha_*)$ and $\phi_{\Gamma}\colon V_m \otimes_{\Xi} \mathcal{O}_{X_{\Gamma}} \to \tilde{E}(m)_{X_{\Gamma}}$.

As in [9], let P be a finite union of connected components of $\mathrm{Pic}_{X/S}$ which have a non-empty intersection with $\nu(Q)$ where ν is the morphism of Q to $\mathrm{Pic}_{X/S}$ determined by $\det(\tilde{E}(m))$. Let Z be a Gieseker space such that Z is a $\mathbf{P}^N$-bundle over P in étale topology and for each K-valued geometric point x of P, the fibre Z_x is isomorphic to

$$\mathbf{P}(\mathrm{Hom}_K(\wedge^r(V_m \otimes_{\Xi} K), H^0(L_x))^{\vee})$$

where L_x is the invertible sheaf corresponding to x and r is the rank of $\tilde{E}$ on fibres. Then we have a morphism τ of Q to Z defined in §4 of [9], roughly speaking, it maps a point of Q which corresponds to a quotient $v\colon V_m \otimes_{\Xi} \mathcal{O}_X \to E$ to a point

$$\wedge^r(V_m \otimes_{\Xi} \mathcal{O}_X) \xrightarrow{\wedge^r(v)} \wedge^r E \longrightarrow \det E$$

of $\mathbf{P}(\mathrm{Hom}_K(\wedge^r(V_m \otimes_{\Xi} K), H^0(\det E))^{\vee})$. Note that, by Proposition 4.9 of [9], $\tau|_{Q^0}$ is an immersion. Let Ψ be the restriction of $\tau \times \prod \gamma_i\colon Q^0 \times \prod U_i \to Z \times \prod G_i$ to Γ.

$$\begin{array}{ccc} \Gamma & \xrightarrow{\quad\Psi\quad} & Z \times \prod_{i=1}^{l} G_i \\ {\scriptstyle \text{closed immersion}} \searrow & & \nearrow {\scriptstyle \tau \times \prod \gamma_i} \\ & Q^0 \times \prod_{i=1}^{l} U_i & \end{array}$$

By Proposition 2.2, there exists a coherent $\mathcal{O}_{\Gamma}$-module $H(\tilde{E}(m)_{X_{\Gamma}}, (\tilde{E}(m) \otimes_X \Omega)_{X_{\Gamma}})$ such that the scheme $\mathbf{V}(H(\tilde{E}(m)_{X_{\Gamma}}, (\tilde{E}(m) \otimes_X \Omega)_{X_{\Gamma}}))$ represents a functor

$$(Sch/\Gamma) \ni T \longmapsto \mathrm{Hom}_{X_T}(\tilde{E}(m)_{X_T}, (\tilde{E}(m) \otimes_X \Omega)_{X_T}).$$

By Corollary 2.3, it is easy to see that there exists a closed subscheme R of $\mathbf{V}(H(\tilde{E}(m)_{X_{\Gamma}}, (\tilde{E}(m) \otimes_X \Omega)_{X_{\Gamma}}))$ which represents a sub-functor of the above

$$(Sch/\Gamma) \ni T \longmapsto$$
$$\{\varphi \mid \varphi(\tilde{F}_i(\tilde{E}(m)_{X_T})) \subseteq (\tilde{F}_i(\tilde{E}(m)) \otimes_X \Omega)_{X_T} \text{ for all } i \text{ and } \varphi \wedge \varphi = 0\}$$

where $\varphi \wedge \varphi$ is the homomorphism defined as in Definition 1.1. From now on, let us denote $\tilde{E}(m)_{X_R}$ (or, $\phi \otimes 1: V_m \otimes_{\Xi} \mathcal{O}_{X_R} \to \tilde{E}(m)_{X_R}$) by $\tilde{E}(m)$ (or, $\phi: V_m \otimes_{\Xi} \mathcal{O}_{X_R} \to \tilde{E}(m)$, resp.). Thus, on X_R, we have a universal family of parabolic pairs $((\tilde{E}(m), \tilde{F}_*, \alpha_*), \tilde{\varphi})$ and a surjection ϕ where $\tilde{\varphi}$ is a "parabolic" homomorphism

$$\tilde{\varphi}: \tilde{E}(m) \longrightarrow \tilde{E}(m) \otimes_X \Omega.$$

Since $\tilde{\varphi} \wedge \tilde{\varphi} = 0$, we have a homomorphism

$$\tilde{\varphi}^a: \tilde{E}(m) \otimes_X S^*(\Omega^\vee) \longrightarrow \tilde{E}(m)$$

which is naturally defined by $\tilde{\varphi}$.

Now, by the condition (2.0.4), Remark 2.1 and Proposition 4.7 of [26], there exists a P-scheme $\tilde{Z}$ such that $\tilde{Z}$ is $\mathbf{P}^N$-bundle in étale topology and for a K-valued geometric point x of P, the fiber $\tilde{Z}_x$ is isomorphic to

$$\mathbf{P}(\mathrm{Hom}_{\mathcal{O}_{X_K}}(\wedge^r(V_m \otimes_{\Xi} S_r^*(\Omega^\vee)_{X_K}), L_x)^\vee).$$

By the argument in §4 in [26], we obtain a morphism

$$\tilde{\tau}: R \longrightarrow \tilde{Z}$$

which is determined by the following homomorphism

$$\wedge^r(V_m \otimes_{\Xi} S_r^*(\Omega^\vee)_{X_R}) \xrightarrow{\wedge^r(\phi \otimes 1)} \wedge^r(\tilde{E}(m) \otimes_X S_r^*(\Omega^\vee))$$
$$\xrightarrow{\wedge^r \tilde{\varphi}^a} \quad \wedge^r(\tilde{E}(m)) \quad \longrightarrow \quad \det(\tilde{E}(m)).$$

Therefore, by $\tilde{\tau}$ and the morphism $R \to \Gamma \xrightarrow{\Psi} Z \times \prod_{i=1}^l G_i \to \prod_{i=1}^l G_i$, we obtain a morphism

$$\tilde{\Psi}: R \longrightarrow \tilde{Z} \times \prod_{i=1}^{l} G_i.$$

By virtue of Proposition 1.11, there exists an open subscheme R^{ss} (or, R^s) of R such that a geometric point x of R is contained in R^{ss} (or, R^s, resp.) if and only if the corresponding parabolic Ω-pair $((\tilde{E}(m), \tilde{F}_*, \alpha_*), \tilde{\varphi})|_{X_x}$ is strictly parabolic e-semi-stable (or, parabolic e-stable, resp.) and the homomorphism

$$H^0(\phi|_{X_x}): V_m \otimes_{\Xi} k(x) \longrightarrow H^0(\tilde{E}(m)|_{X_x}) \tag{2.4.2}$$

is an isomorphism.

Proposition 2.5. *The morphism* $\tilde{\Psi}: R^{ss} \to \tilde{Z} \times \prod_{i=1}^l G_i$ *is an immersion.*

Proof. Let $\tilde{Q}$ be the Quot-scheme $\mathrm{Quot}(V_m \otimes_{\Xi} S_r^*(\Omega^\vee), H[m])$ and let $\tilde{\phi}: V_m \otimes_{\Xi} S_r^*(\Omega^\vee)_{X_{\tilde{Q}}} \to \tilde{\tilde{E}}(m)$ be the universal quotient. By $\tilde{Q}^0$, we denote an open subscheme of $\tilde{Q}$ consisting of all points x such that $\tilde{\tilde{E}}(m)|_{X_x}$ is torsion free and the restriction of $\tilde{\phi} \otimes k(x)$ to $V_m \otimes_{\Xi} \mathcal{O}_{X_x} = V_m \otimes_{\Xi} S^0(\Omega^\vee)_{X_x} \subseteq V_m \otimes_{\Xi} S_r^*(\Omega^\vee)_{X_x}$ is surjective. $\tilde{\phi} \otimes k(x)|_{V_m \otimes \mathcal{O}_{X_x}}$ defines a morphism of $\tilde{Q}^0$ to Q^0. The surjection on X_R

$$V_m \otimes S_r^*(\Omega^\vee)_{X_R} \xrightarrow{\phi \otimes 1} \tilde{E}(m) \otimes_X S_r^*(\Omega^\vee) \xrightarrow{\tilde{\phi}^a} \tilde{E}(m)$$

defines a morphism of R to $\tilde{Q}^0$. Moreover, the homomorphism

$$\wedge^r(V_m \otimes_{\Xi} S_r^*(\Omega^\vee)_{X_{\tilde{Q}}}) \xrightarrow{\wedge^r(\tilde{\phi} \otimes 1)} \wedge^r(\tilde{\tilde{E}}(m)) \longrightarrow \det(\tilde{\tilde{E}}(m))$$

defines a morphism of $\tilde{Q}$ to $\tilde{Z}$ whose restriction to $\tilde{Q}^0$ is an immersion by Proposition 4.7 of [26]. The composite of these two morphism is clearly $\tilde{\tau}: R \to \tilde{Z}$. Thus, we obtain the following commutative diagram.

$$\begin{array}{ccccccc} R & \longrightarrow & \tilde{Q}^0 \times \prod U_i & \longrightarrow & Q^0 \times \prod U_i & \hookleftarrow & \Gamma \\ \downarrow{\scriptstyle \tilde{\Psi}} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \Psi} \\ \tilde{Z} \times \prod G_i & \hookleftarrow & \tilde{Q}^0 \times \prod G_i & \longrightarrow & Q^0 \times \prod G_i & \hookrightarrow & Z \times \prod G_i \end{array}$$

Note that $\tilde{Q}^0$ is isomorphic to $\mathbf{V}(H(V_m \otimes_{\Xi} (\bigoplus_{i=1}^{(r-1)\mathrm{rk}(\Omega)} S^i(\Omega^\vee))_{X_Q}, \tilde{E}(m)))$ as a Q-scheme. Then using Corollary 2.3 repeatedly, we can easily show that the morphism of R to $\tilde{Q}^0 \times \prod U_i$ is a closed immersion.

In the proof of Proposition 3.1 of [13], we have constructed a subscheme Δ_i^0 of $Q \times Q_i \times G_i$ which is characterized by the following property. For an S-morphism $g: T \to Q \times Q_i \times G_i$, let $g_Q: V_m \otimes_{\Xi} \mathcal{O}_{X_T} \twoheadrightarrow E$ ($g_{Q_i}: V_m \otimes_{\Xi} \mathcal{O}_{X_T} \twoheadrightarrow E_i$ or $g_{G_i}: V_m \otimes_{\Xi} \mathcal{O}_T \twoheadrightarrow \mathscr{H}_i$) be the quotient corresponding to the T-valued point of Q (Q_i or G_i, resp.) which is determined by g. Then g factors through Δ_i^0 if and only if (i) E_i is a quotient of E and $\mathscr{H}_i \otimes_T \mathcal{O}_{X_T}$ as the quotient of $V_m \otimes_{\Xi} \mathcal{O}_{X_T}$ and (ii) in the exact commutative diagram obtained by (i)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J_i \otimes_T \mathcal{O}_{X_T} & \longrightarrow & V_m \otimes_{\Xi} \mathcal{O}_{X_T} & \longrightarrow & \mathscr{H}_i \otimes_T \mathcal{O}_{X_T} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F_{i+1} & \longrightarrow & E & \longrightarrow & E_i & \longrightarrow & 0, \end{array}$$

$J_i \otimes_T \mathcal{O}_{X_T} \to F_{i+1}$ is surjective.

Set $\Delta = \Delta_1^0 \times_Q \cdots \times_Q \Delta_l^0$. It is a subscheme of $Q \times \prod Q_i \times \prod G_i$. We have proved in [13] that the projection η of Δ to $Q \times \prod G_i$ is an immersion. By the Q-morphism of R to $Q^0 \times \prod G_i$ and to $Q^0 \times \prod U_i$, we obtain a morphism of R to $Q^0 \times \prod U_i \times \prod G_i$. By the conditions (2.0.2), (2.0.3) and the above property of Δ_i^0, the morphism of R^{ss} to $Q^0 \times \prod U_i \times \prod G_i$ is factored by a subscheme

$$\Delta \cap (Q^0 \times \prod U_i \times \prod G_i).$$

It follows that the morphism of R^{ss} to $\tilde{Q}^0 \times \prod U_i \times \prod G_i$ is factored by

$$\Delta' = (\Delta \cap (Q^0 \times \prod U_i \times \prod G_i)) \times_{Q^0} \tilde{Q}^0$$

The morphism $R^{ss} \to \Delta'$ is an immersion because $R^{ss} \to \Delta' \to \tilde{Q}^0 \times \prod U_i$ is the immersion. Moreover, the projection $\eta \times 1_{\tilde{Q}^0}$ of Δ' to $\tilde{Q}^0 \times \prod G_i$ is also an immersion. Thus, $\tilde{\Psi}|_{R^{ss}}$ is a composite of the following three immersions

$$R^{ss} \hookrightarrow \Delta' \hookrightarrow \tilde{Q}^0 \times \prod G_i \hookrightarrow \tilde{Z} \times \prod G_i. \qquad \square$$

Set $G = \mathrm{SL}(V_m)$ which acts on R, $\tilde{Z}$ and G_i. It is easy to see that $\tilde{\Psi}$ is a G-morphism and R^{ss} (or, R^s, resp.) is a G-invariant open set of R. We shall say that a $\mathbf{Q}$-invertible sheaf L has a G-linearization when there exists an integer m such that $L^{\otimes m}$ is an invertible sheaf and has a G-linearization. As in the case of parabolic sheaves, we choose a G-linearized $\mathbf{Q}$-invertible sheaf

$$L = \mathcal{O}_{\tilde{Z}}(\text{par-}\, P_{E_*}(m)) \otimes \bigotimes_{i=1}^{l} \mathcal{O}_{G_i}(\varepsilon_i).$$

on $\tilde{Z} \times \prod_{i=1}^{l} G_i$, where E_* is an underlying parabolic sheaf for some member of $\mathscr{F}^e_\Omega(H, H_*, \alpha_*)$, $\varepsilon_i = \alpha_{i+1} - \alpha_i$ for $i = 1, \ldots, l$ ($\alpha_{l+1} = 1$) and $\mathcal{O}_{\tilde{Z}}(1)$ (or, $\mathcal{O}_{G_i}(1)$) is the tautological $\mathbf{Q}$-invertible sheaf on $\tilde{Z}$ (or, G_i, resp.) which has the canonical G-linearization. ($\mathcal{O}_{\tilde{Z}}(n+1)$ is an invertible sheaf ($n = \dim(X/S)$) but in general $\mathcal{O}_{\tilde{Z}}(1)$ is not invertible.) The open set consisting of all semi-stable (or, stable) points with respect to this G-linearization is denoted by $(\tilde{Z} \times \prod_{i=1}^{l} G_i)^{ss}$ (or, $(\tilde{Z} \times \prod_{i=1}^{l} G_i)^s$, resp.). Those are G-invariant open subsets of $\tilde{Z} \times \prod_{i=1}^{l} G_i$.

Recall some facts on stable points of $\tilde{Z}$ or G_i. Let x be a K-valued geometric point of $\tilde{Z} \times \prod_{i=1}^{l} G_i$. We denote the point of $\tilde{Z}(K)$ (or, $G_i(K)$) determined by x by T_x (or, $g_{i,x}$, resp.). We use the same symbol $g_{i,x}$ for the surjection $g_{i,x}\colon V_m \otimes K \to J_{i,x}$ which corresponds to x and moreover, we denote its kernel by $W_{i,x}$. T_x is regarded as a K-valued point of a Gieseker space $P_{S^*_r(\Omega^\vee)_{X_x}}(V_m \otimes_{\Xi} K, r, L_x)$ (cf. §3) where L_x is the invertible sheaf corresponding to $\rho(x) \in P(K)$.

For the convenience of readers, we shall recall some notations and definitions on Gieseker spaces (cf. [26]). Let $P_\Omega(V, r, L) = \mathbf{P}(\mathrm{Hom}_X(\wedge^r(V \otimes_k \Omega), L)^\vee)$ be a Gieseker space where X is a scheme over a field k, V is a k-vector space, Ω (or, L) is a locally free $\mathcal{O}_X$-module (or, an invertible sheaf, resp.) and r is a positive integer. For vector subspaces $V_1, \ldots, V_t$ of V_K and non-negative integers $r_1, \ldots, r_t$, we denote by $[V_1 ; r_1, \ldots, V_t; r_t]$ an image of the following natural homomorphism:

$$\wedge^{r_1}(V_1 \otimes_k \Omega) \otimes_{X_K} \cdots \otimes_{X_K} \wedge^{r_t}(V_t \otimes_k \Omega) \longrightarrow \wedge^r(V_K \otimes_k \Omega).$$

If $r_i = 1$, a symbol $[\cdots, V_i, \cdots]$ is simply used instead of $[\cdots, V_i; 1, \cdots]$. Moreover, if V_i is generated by one element e, then $[\cdots, e, \cdots]$ is used. Let T be a K-valued point of $P_\Omega(V, r, L)$ which is identified as a non-zero homomorphism $T\colon \wedge^r(V_K \otimes_k \Omega) \to L_K$. Vectors $e_1, \ldots, e_i$ of V_K are said to be T-independent if $T|_{[e_1, \ldots, e_i, V_K; r-i]} \neq 0$. Otherwise, those are said to be T-dependent. Note that those vectors may contain same vectors. Let W be a subspace of V_K. Vectors $e_1, \ldots, e_i$ in W is called a T-basis of W if $e_1, \ldots, e_i$ are T-independent and for all vectors e in W, $e_1, \ldots, e_i$, e are T-dependent. The maximal (or, minimal) length of T-basis of W is denoted by $\overline{\dim}_T W$ (or, $\underline{\dim}_T W$, resp.) and called the maximal (or, minimal, resp.) T-dimension of W. In general, $\underline{\dim}_T W \leq \overline{\dim}_T W$ and if equality holds, then it is denoted by $\dim_T W$.

We need the following criterion of semi-stability of points of $\tilde{Z} \times \prod_{i=1}^{l} G_i$.

Lemma 2.6. *Let x be a K-valued geometric point of $\tilde{Z} \times \prod_{i=1}^{l} G_i$. Assume that the point T_x in $\tilde{Z}(K)$ has the following property:*

(2.6.1) *For all subspaces W of $V_m \otimes_{\Xi} K$, $\underline{\dim}_{T_x} W = \overline{\dim}_{T_x} W$.*

Then the point x is semi-stable (or, stable) with respect to the G-linearized invertible sheaf L if and only if for all non-trivial vector subspaces W of $V_m \otimes_{\Xi} K$, the following inequality holds

$$\text{par-}P_{E_*}(m)(\dim_{T_x} W \cdot \dim_K (V_m \otimes_{\Xi} K) - r \dim_K W) + \sum_{i=1}^{l} \varepsilon_i (\dim_K W \cdot \dim_K W_{i,x} - \dim_K (V_m \otimes_{\Xi} K) \cdot \dim_K (W_{i,x} \cap W)) \geq 0$$

(or, > 0, resp.).

Proof. Set $T = T_x$, $V = V_m \otimes_{\Xi} K$ and $N = H(m) = \dim_K V$. Let λ be a non-trivial one parameter subgroup of G and let $e_1, \dots, e_N$ be a basis of V such that $e_i^{\lambda(\alpha)} = \alpha^{r_i} e_i$ where $r_1 \leq \dots \leq r_N$ and $\sum r_i = 0$. Then by Proposition 2.3 of [16], we see easily that

$$\mu^{\mathcal{O}_{\tilde{Z}}(1)}(T, \lambda) = - \min_{1 \leq d_1, \dots, d_r \leq N} \{ r_{d_1} + \dots + r_{d_r} \mid T|_{[e_{d_1}, \dots, e_{d_r}]} \neq 0 \}.$$

Let W^i be the vector subspace generated by $e_1, \dots, e_i$. Let k_1 be the minimum integer such that e_{k_1} is T-independent. If a sequence of integers $k_1, \dots, k_p$ are defined, then let k_{p+1} be the minimum integer such that $e_{k_1}, \dots, e_{k_{p+1}}$ are T-independent. Thus we have a sequence of integers $k_1 \leq \dots \leq k_r$. By the proof of the claim (3.3.2) of [13], we have that

$$\mu^{\mathcal{O}_{\tilde{Z}}(1)}(T, \lambda) = - \sum_{i=1}^{r} r_{k_i}.$$

If i appears a_i-times in the sequence $k_1, \dots, k_r$, then $\dim_T W^i = \dim_T W^{i-1} + a_i$. Thus we have that

$$\mu^{\mathcal{O}_{\tilde{Z}}(1)}(T, \lambda) = - \sum_{i=1}^{N} (\dim_T W^i - \dim_T W^{i-1}) r_i.$$

The rest of the proof is completely same as that of Lemma 3.3 of [13]. □

Let $\sigma(W, x)$ be the left-hand side of the above inequality. Since $\dim_K (V_m \otimes_{\Xi} K) = H(m)$ and $\dim_K W_{i,x} = H(m) - H_i(m)$, we have the following description of $\sigma(W, x)$:

$$\sigma(W, x) = H(m)\cdot(\text{par-}P_{E_*}(m)\cdot \dim_{T_x} W - \sum_{i=1}^{l} \varepsilon_i \dim_K (W_{i,x}\cap W) - \alpha_1 \dim_K W).$$

Proposition 2.7. 1) $\tilde{\Psi}(R^{ss}) \subseteq (\tilde{Z} \times \prod_{i=1}^{l} G_i)^{ss}$.

2) $\tilde{\Psi}(R^s) \subseteq (\tilde{Z} \times \prod_{i=1}^{l} G_i)^s$.

3) *If a point* x *is in* R^{ss} *but not in* R^s, *then* $\tilde{\Psi}(x)$ *is not in* $(\tilde{Z} \times \prod_{i=1}^{l} G_i)^s$.

Proof. Let x be a K-valued geometric point of R^{ss} and W be a non-trivial vector subspace of $V_m \otimes_{\Xi} K$. Then we have a parabolic pair $(E_*(m), \varphi)$ on X_x and a surjection $\phi_x\colon V_m \otimes_{\Xi} \mathcal{O}_{X_x} \to E(m)$ which correspond to $x \in R^{ss}(K)$. Let φ' be the following surjection

$$\varphi'\colon V_m \otimes_{\Xi} S^*(\Omega^{\vee})_{X_x} \xrightarrow{\phi_x \otimes 1} E(m) \otimes_X S^*(\Omega^{\vee}) \xrightarrow{\varphi^a} E(m)$$

and let φ'_r be its restriction to $V_m \otimes_{\Xi} S^*_r(\Omega^{\vee})_{X_x}$. The K-valued point $T_{\tilde{\Psi}(x)}$ of $\tilde{Z}$ corresponds to the homomorphism

$$\wedge^r(V_m \otimes_{\Xi} S^*_r(\Omega^{\vee})_{X_x}) \xrightarrow{\wedge^r \varphi'_r} \wedge^r(E(m)) \longrightarrow \det(E(m)).$$

By Lemma 3.7 in the next §3, we have that

$$\underline{\dim}_{T_{\tilde{\Psi}(x)}} W = \overline{\dim}_{T_{\tilde{\Psi}(x)}} W = \operatorname{rk}(\varphi'_r(W \otimes_K S^*_r(\Omega^{\vee})_{X_x})). \tag{2.7.1}$$

By Lemma 1.2 of [26] and Remark 2.1.2, $\varphi'_r(W \otimes_K S^*_r(\Omega^{\vee})_{X_x})$ is generically isomorphic to $\varphi'(W \otimes_K S^*(\Omega^{\vee})_{X_x})$. Let $F(m)$ be a subsheaf of $E(m)$ containing $\varphi'(W \otimes_K S^*(\Omega^{\vee})_{X_x})$ such that $E(m)/F(m)$ is torsion free and $F(m)/\varphi'(W \otimes_K S^*(\Omega^{\vee})_{X_x})$ is a torsion sheaf. Then by (2.7.1), $\dim_{T_{\tilde{\Psi}(x)}} W = \operatorname{rk}(F)$. Since $H^0(\phi_x)\colon V_m \otimes_{\Xi} K \to H^0(E(m))$ is an isomorphism, we know that (cf. [13] (3.4.2) and (3.4.3)) $\dim_K W \le h^0(F(m))$, $\dim_K (W \cap W_{i,\tilde{\Psi}(x)}) \le h^0(F(m) \cap F_{i+1}(E(m)))$ and therefore

$$\sigma(W, \tilde{\Psi}(x)) \ge H(m)\cdot\left(\text{par-}P_{E_*}(m)\cdot \operatorname{rk}(F) - \int_0^1 h^0(F_\alpha(m))\,d\alpha\right)$$

where $F_*(m)$ has the induced structure. Since $\varphi'(W \otimes_K S^*(\Omega^{\vee})_{X_x})$ is φ-invariant, so is $F(m)$. Hence, the assertions 1), 2) follows from Propositions 1.9, (2.7.1) and Lemma 2.6. To prove 3), let E'_* be the φ-invariant parabolic subsheaf of E_* given in 2) of Proposition 1.9. Note that the parabolic structure of E'_* is the induced structure. Set $W = H^0(E'(m))$. Since $E'(m)$ is generated by its global sections and φ-invariant, $\varphi'_r(W \otimes_K S^*_r(\Omega^{\vee})_{X_x}) = E'(m)$. Hence, by the above argument, $\dim_{T_{\tilde{\Psi}(x)}} W = \operatorname{rk}(E')$ and

$$\sigma(W, \tilde{\Psi}(x)) = H(m)\cdot\left(\text{par-}P_{E_*}(m)\cdot \operatorname{rk}(E') - \int_0^1 h^0(E'_\alpha(m))\,d\alpha\right) = 0.$$

By Lemma 2.6, $\tilde{\Psi}(x)$ is not in $(\tilde{Z} \times \prod G_i)^s$. □

Let $\tilde{R}^{ss}$ be the scheme theoretic image of R^{ss} in $(\tilde{Z} \times \prod_{i=1}^{l} G_i)^{ss}$. Then by virtue of Theorem 4 of [20], there exists a good quotient $\xi: R^{ss} \to Y$ and Y is projective over S. Set $\bar{M}^{H_*, \alpha_*, e}_{\Omega/D/X/S} = Y - \xi(\tilde{R}^{ss} - R^{ss})$. Then $\bar{M}^{H_*, \alpha_*, e}_{\Omega/D/X/S}$ is quasi-projective over S. Moreover $\bar{M}^{H_*, \alpha_*, e}_{\Omega/D/X/S}$ contains $M^{H_*, \alpha_*, e}_{\Omega/D/X/S} = \xi(R^s)$ as an open subscheme.

Theorem 2.8. *$M^{H_*, \alpha_*, e}_{\Omega/D/X/S}$ is a coarse moduli scheme of* par-$\Sigma^{H_*, \alpha_*, e}_{\Omega/D/X/S}$, *that is, $M^{H_*, \alpha_*, e}_{\Omega/D/X/S}$ has the following properties.*

(2.8.1) *For each geometric point s of S, there exist a natural bijection:*

$$\theta_s: \text{par-}\Sigma^{H_*, \alpha_*, e}_{\Omega/D/X/S}(k(s)) \longrightarrow M^{H_*, \alpha_*, e}_{\Omega/D/X/S}(k(s)).$$

(2.8.2) *For $T \in (Sch/S)$ and $[(E_*, \varphi)] \in$ par-$\Sigma^{H_*, \alpha_*, e}_{\Omega/D/X/S}(T)$, there exists a morphism*

$$f^e_{[(E_*, \varphi)]}: T \longrightarrow M^{H_*, \alpha_*, e}_{\Omega/D/X/S}$$

such that for all points t in $T(k(s))$, $f^e_{[(E_*, \varphi)]}(t) = \theta_s((E_*, \varphi)|_{X_t})$. Moreover, for a morphism $g: T' \to T$ in (Sch/S),

$$f^e_{[(E_*, \varphi)]} \circ g = f^e_{[(1_X \times g)^*(E_*, \varphi)]}.$$

(2.8.3) *If $M' \in (Sch/S)$ and maps*

$$\theta'_s: \text{par-}\Sigma^{H_*, \alpha_*, e}_{\Omega/D/X/S}(k(s)) \longrightarrow M'(k(s))$$

$$f'_{[(E_*, \varphi)]}: T \longrightarrow M'$$

have the properties (2.8.1) *and* (2.8.2), *then there exists a unique S-morphism Υ of $M^{H_*, \alpha_*, e}_{\Omega/D/X/S}$ to M' such that $\Upsilon(k(s)) \circ \theta_s = \theta'_s$ and $\Upsilon \circ f^e_{[(E_*, \varphi)]} = f'_{[(E_*, \varphi)]}$ for all geometric points s of S and for all $[(E_*, \varphi)] \in$ par-$\Sigma^{H_*, \alpha_*, e}_{\Omega/D/X/S}(T)$.*

(2.8.4) *$M^{H_*, \alpha_*, e}_{\Omega/D/X/S}$ is quasi-projective over S.*

Proof. Though the proof is essentially the same as in the case of moduli of stable sheaves, we give the proof for completeness. (2.8.4) is already proved. Set $\Sigma =$ par-$\Sigma^{H_*, \alpha_*, e}_{\Omega/D/X/S}$ and $M = M^{H_*, \alpha_*, e}_{\Omega/D/X/S}$. Let s be a geometric point of S. We have a natural map

$$\pi_s: R^s(k(s))/G(k(s)) \longrightarrow \Sigma(k(s)).$$

For each pair (E_*, φ) in $\Sigma(k(s))$, by (2.0.3), $h^0(E(m)) = H(m)$. Taking an isomorphism $V_m \otimes_{\Xi} k(s) \simeq H^0(E(m))$, by (2.0.2), we obtain a surjective homomorphism $\phi': V_m \otimes_{\Xi} \mathcal{O}_{X_s} \to E(m)$. By the universal property of R^s, $(\phi', (E_*, \varphi))$ defines a $k(s)$-valued point of R^s. Hence, π_s is surjective. Moreover, by the property (2.4.1) for points of R^s, π_s is injective. Since $\xi(k(s)): R^s(k(s)) \to M(k(s))$ induces a bijection $R^s(k(s))/G(k(s)) \simeq M(k(s))$, we obtain a bijection $\theta_s: \Sigma(k(s)) \simeq M(k(s))$.

To prove (2.8.2), assume that $T \in (Sch/S)$ and $[(E_*, \varphi)] \in \Sigma(T)$ are given. Then by virtue of the properties (2.0.2) and (2.0.3), $E' = (f_T)_*(E(m))$ is locally free of rank $H(m)$ and the canonical map $(f_T)^*(E') \to E(m)$ is surjective where f_T is the projection of X_T to T. Let $T = \bigcup_\lambda T_\lambda$ be an open covering of T such that $E'|_{T_\lambda}$ is free for each λ. Take an isomorphism $\beta_\lambda: V_m \otimes_\Xi \mathcal{O}_{T_\lambda} \simeq E'|_{T_\lambda}$. By the universal property of R^s, a surjection

$$V_m \otimes_\Xi \mathcal{O}_{X_{T_\lambda}} \xrightarrow{\beta_\lambda} (f_T)^*(E'|_{T_\lambda}) \longrightarrow E(m)|_{X_{T_\lambda}}.$$

and $(E_*, \varphi)|_{X_{T_\lambda}}$ defines a morphism h_λ of T_λ to R^s. On $T_{\lambda\mu} = T_\lambda \cap T_\mu$, a $T_{\lambda\mu}$-valued point $\beta_\mu \circ \beta_\lambda$ of G transforms h_λ to h_μ. Since M is a geometric quotient of R^s, $\xi \circ h_\lambda = \xi \circ h_\mu$ on $T_{\lambda\mu}$. Thus we have a morphism $f^e_{(E_*,\varphi)}$ of T to M. Note that by the same argument, $f^e_{(E_*,\varphi)} = f^e_{(E_*,\varphi)\otimes L}$ for each invertible sheaf on T. Clearly, $f^e_{(E_*,\varphi)}$ is the desired one.

Finally, let us prove (2.8.3). The universal parabolic pair $(\tilde{E}(m)_*, \tilde{\varphi})|_{X_{R^s}}$ (simply denote by $(\tilde{E}(m)_*, \tilde{\varphi})$) on X_{R^s} determines a morphism

$$\omega: R^s \longrightarrow \text{par-}\Sigma^{H_*,\alpha_*,e}_{\Omega/D/X/S}.$$

Since $\sigma^*((\tilde{E}(m)_*, \tilde{\varphi})) \simeq p_2^*((\tilde{E}(m)_*, \tilde{\varphi}))$, we have a commutative diagram

$$\begin{array}{ccc} G \times R^s & \xrightarrow{\sigma} & R^s \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle \omega} \\ R^s & \xrightarrow{\omega} & \text{par-}\Sigma^{H_*,\alpha_*,e}_{\Omega/D/X/S}. \end{array}$$

where σ is the action of G on R^s and p_2 is the projection. The property (2.8.2) implies that there exists a morphism δ of par-$\Sigma^{H_*,\alpha_*,e}_{\Omega/D/X/S}$ to M'. Moreover, we have that

$$\delta \circ \omega = f'_{[(\tilde{E}(m)_*,\tilde{\varphi})]}.$$

Hence, we see that $f'_{[(\tilde{E}(m)_*,\tilde{\varphi})]} \circ \sigma = f'_{[(\tilde{E}(m)_*,\tilde{\varphi})]} \circ p_2$. Since M is a geometric quotient of R^s by G, there exists a unique morphism $\Upsilon: M \to M'$ with $\Upsilon \circ \xi = f'_{[(\tilde{E}(m)_*,\tilde{\varphi})]}$. Then by the universality of R^s, we know easily that Υ has the property in (2.8.3).

By the similar arguments as in §5 of [9], we know that there exists a unique morphism $\nu_{e,e'}$ of $M^{H_*,\alpha_*,e}_{\Omega/D/X/S}$ to $M^{H_*,\alpha_*,e'}_{\Omega/D/X/S}$ if $e \le e'$. Moreover, $M^{H_*,\alpha_*,e}_{\Omega/D/X/S}$ can be regarded as an open subscheme of $M^{H_*,\alpha_*,e'}_{\Omega/D/X/S}$ through $\nu_{e,e'}$. Taking inductive limit of $\{M^{H_*,\alpha_*,e}_{\Omega/D/X/S}\}$, an S-scheme $M^{H_*,\alpha_*}_{\Omega/D/X/S}$ is obtained.

Theorem 2.9. *$M^{H_*,\alpha_*}_{\Omega/D/X/S}$ is a coarse moduli scheme of* par-$\Sigma^{H_*,\alpha_*}_{\Omega/D/X/S}$. *Moreover, $M^{H_*,\alpha_*}_{\Omega/D/X/S}$ is separated and locally of finite type over S.*

Proof. Since $M = M^{H_*,\alpha_*}_{\Omega/D/X/S}$ is the union of open subschemes $M^e = M^{H_*,\alpha_*,e}_{\Omega/D/X/S}$ which are quasi-projective over S, it is locally of finite type over S. Moreover, $M \times_S M$ is covered by open subschemes $M^e \times_S M^e$. Let Δ (or, Δ^e) be the diagonal morphism $M \to M \times_S M$ (or, $M^e \to M^e \times_S M^e$, resp.). Then $\Delta \cap M^e \times_S M^e$ is closed in $M^e \times_S M^e$. Hence, Δ is closed in $M \times_S M$, i.e. M is separated over S. For all K-valued geometric points s of S, $M(k(s)) = \bigcup_e M^e(k(s))$ and $\Sigma(k(s)) = \bigcup_e \Sigma^e(k(s))$ where $\Sigma = \text{par-}\Sigma^{H_*,\alpha_*}_{\Omega/D/X/S}$ and $\Sigma^e = \text{par-}\Sigma^{H_*,\alpha_*,e}_{\Omega/D/X/S}$. Hence, clearly $M(k(s)) = \Sigma(k(s))$ in a natural way. It is easy to see that $\Sigma = \varinjlim_e \Sigma^e$ and there exists natural commutative diagrams for $e \leq e'$

$$\begin{array}{ccc} \Sigma^e & \longrightarrow & \Sigma^{e'} \\ \downarrow{\scriptstyle g^e} & & \downarrow{\scriptstyle g^{e'}} \\ M^e & \xrightarrow{\nu_{e,e'}} & M^{e'} \end{array}$$

where g^e is a morphism given by the property (2.8.2). Hence, there exists a natural morphism g of Σ to M. Finally for each morphism h of Σ to $M' \in (Sch/S)$, there is a morphism v^e of M^e to M' such that $v^e \circ g^e = h|_{\Sigma^e}$. By the property (2.8.3), we know that $v^{e'} \circ \nu_{e,e'} = v^e$ for $e \leq e'$. Hence, we get a unique morphism v of M to M' whose restriction to M^e is v^e. Since the restriction of $v \circ g$ to Σ^e is the same as that of h, $v \circ g = h$. Clearly, such v is unique. □

3. GL(V)-orbits of $\tilde{Z} \times \prod G_i$

In this section, we shall analyze orbit spaces of $(\tilde{Z} \times \prod G_i)^{ss}$ with respect to GL(V).

Let X be a smooth, projective variety over a field k and $\mathcal{O}_X(1)$ a very ample invertible sheaf. Fix a locally free $\mathcal{O}_X$-module Ω of finite rank. As in §3 of [26], for a k-vector space V of dimension N, a non-negative integer r and an invertible sheaf L on X, we denote by $P_\Omega(V, r, L)$ a Gieseker space $\mathbf{P}(\mathrm{Hom}_{\mathcal{O}_X}(\wedge^r(V \otimes_k \Omega), L)^\vee)$ on which the algebraic group $G = \mathrm{GL}(V)$ acts and there is the G-linearized invertible sheaf $\mathcal{O}(1)$. Let $\alpha_* = \{\alpha_1, \ldots, \alpha_l\}$ (or, $N_* = \{N_1, \ldots, N_l\}$) be a set of rational numbers (or, positive integers, resp.) such that $0 \leq \alpha_1 < \cdots < \alpha_l < 1$ (or, $0 < N_1 < \cdots < N_l < N$, resp.). Set $\varepsilon_i = \alpha_{i+1} - \alpha_i$ ($\alpha_{l+1} = 1$). We denote by $G(V, N_i)$ the Grassmann variety Grass(V, N_i). On $G(V, N_i)$, we have a natural G-linearized invertible sheaf $\mathcal{O}_{G(V,N_i)}(1)$. Moreover, we denote by $\Theta_\Omega(V, r, L, N_*, \alpha_*)$ the scheme

$$P_\Omega(V, r, L) \times F(V, N_*)$$

with a G-linearized $\mathbf{Q}$-invertible sheaf

$$\mathcal{O}_\Theta(1) = \mathcal{O}\left(\frac{N - \sum_i \varepsilon_i N_i}{r}\right) \otimes \bigotimes_{i=1}^{l} \mathcal{O}_{G(V,N_i)}(\varepsilon_i),$$

where $F(V, N_*)$ is a flag variety consisting of all flags $V \supset W_1 \cdots \supset W_l$ with $\dim_k W_i = N - N_i$ and where $\bigotimes_{i=1}^l \mathcal{O}_{G(V,N_i)}(\varepsilon_i)$ is regarded as a $\mathbf{Q}$-invertible sheaf on $F(V, N_*)$ by a canonical inclusion $F(V, N_*) \subsetneq \prod_{i=1}^l G(V, N_i)$.

In this section, we shall fix Ω, hence we denote $\Theta_\Omega(V, r, L, N_*, \alpha_*)$ (or, $P_\Omega(V, r, L)$) simply by $\Theta(V, r, L, N_*, \alpha_*)$ (or, $P(V, r, L)$, resp.). Moreover for $\Theta = \Theta(V, r, L, N_*, \alpha_*)$, the above l (or, ε_i) is sometimes denoted by $l(\Theta)$ (or, $\varepsilon(\Theta)_i$, resp.) and $l(\Theta)$ is called the length of Θ. For a K-valued point x of $\Theta(V, r, L, N_*, \alpha_*)$, we denote by T_x (or, $g_{i,x}$) the point of $P(V, r, L)(K)$ (or,

$G(V, N_i)(K)$, resp.) determined by x. We use the same symbol $g_{i,x}$ for the surjection $g_{i,x}\colon V \otimes_k K \to J_{i,x}$ which corresponds to x and its kernel is denoted by $F_{i,x}(V)$. Moreover, for each $0 \le \alpha \le 1$, we set

$$V_x^\alpha = F_{i-1,x}(V) \qquad \text{if } \alpha_{i-1} < \alpha \le \alpha_i,$$

where $\alpha_0 = \alpha_l - 1$, $\alpha_{l+1} = 1$ and $F_{0,x}(V) = V \otimes_k K$. We denote by $F(V, N_*, \alpha_*)$ the scheme $F(V, N_*)$ when this additional structure "$\alpha \mapsto V_x^\alpha$" is given for each flag $F_{*,x}(V)$ which corresponds to each point x on $F(V, N_*)$. Note that

$$\int_0^1 \dim_K V_x^\alpha \, d\alpha = N - \sum_{i=1}^{l} \varepsilon_i N_i > 0. \tag{3.0.1}$$

From now on, set $\Theta = \Theta(V, r, L, N_*, \alpha_*)$, $\Theta' = \Theta(V', r', L', N'_*, \alpha'_*)$ and $\Theta'' = \Theta(V'', r'', L'', N''_*, \alpha''_*)$. Let us recall that the notion of extension of points in Gieseker spaces (cf. [10] and [26]). It is generalized for our case.

Definition 3.1. Let T, T' and T'' be K-valued geometric points of $P(V, r, L)$, $P(V', r', L')$ and $P(V'', r'', L'')$, respectively and let $\phi\colon L' \otimes_X L'' \to L$ be an injective homomorphism. The point T is said to be a ϕ-extension or, simply, an extension of T'' by T' if the following conditions are satisfied;

(3.1.1) $r = r' + r''$,

(3.1.2) there exists an exact sequence

$$0 \longrightarrow V' \otimes_k K \xrightarrow{f} V \otimes_k K \xrightarrow{g} V'' \otimes_k K \longrightarrow 0$$

such that

(3.1.2.1) the following diagram is commutative (mod $K^\times$).

$$\begin{array}{ccc} \wedge^{r'}(V'_K \otimes_K \Omega_K) \otimes_{X_K} \wedge^{r''}(V_K \otimes_K \Omega_K) & \longrightarrow & \wedge^{r}(V_K \otimes_K \Omega_K) \\ \downarrow {\scriptstyle T' \otimes (T'' \circ \wedge^{r''}(g \otimes id_\Omega))} & & \downarrow {\scriptstyle T} \\ L'_K \otimes_{X_K} L''_K & \xrightarrow{\phi_K} & L_K \end{array}$$

In this case, T' (or, T'') is said to be a subpoint (or, quotient point, resp.) of T.

Let x, x' and x'' be K-valued geometric points of Θ, Θ' and Θ'', respectively. The point x is said to be a ϕ-quasi-extension or, simply, a quasi-extension of x'' by x' if T_x is a ϕ-extension of $T_{x''}$ by $T_{x'}$ i.e. the above conditions (3.1.1) and (3.1.2) are satisfied and moreover, in (3.1.2), the following holds.

(3.1.2.2) For all $0 \le \alpha \le 1$, $f(V'^\alpha_{x'}) \subseteq V^\alpha_x$ and $g(V^\alpha_x) \subseteq V''^\alpha_{x''}$.

Moreover, the point x is said to be ϕ-extension (or, extension) if, in addition, the following induced sequence is exact for all $0 \le \alpha \le 1$.

(3.1.3) $$0 \longrightarrow V'^{\alpha}_{x'} \xrightarrow{f} V^{\alpha}_{x} \xrightarrow{g} V''^{\alpha}_{x''} \longrightarrow 0$$

Remark 3.2. If x is a ϕ-extension of x'' by x' as above, then by virtue of (3.0.1) and (3.1.3), we have that

(3.2.1) $$N - \sum_{i=1}^{l} \varepsilon_i N_i = (N' - \sum_{i=1}^{l'} \varepsilon'_i N'_i) + (N'' - \sum_{i=1}^{l''} \varepsilon''_i N''_i).$$

Definition 3.3. Let x, x' and x'' be K-valued geometric points of Θ, Θ' and Θ'', respectively and let $\phi\colon L' \otimes_X L'' \to L$ be an injective homomorphism. Assume that x is a ϕ-extension of x'' by x' and let

$$0 \longrightarrow V' \otimes_k K \xrightarrow{f} V \otimes_k K \xrightarrow{g} V'' \otimes_k K \longrightarrow 0$$

be the underlying exact sequence of the extension. Then x is said to be a ϕ-direct sum of x' and x'' if there exists a K-linear map $i\colon V'' \otimes_k K \to V \otimes_k K$ such that $g \circ i = id_{V'' \otimes_k K}$, $i(V''^{\alpha}_{x''}) \subseteq V^{\alpha}_{x}$ for all $0 \leq \alpha \leq 1$ and that $T_x|_{[i(V''_K);s, V_K;r-s]} = 0$ whenever $s > r''$.

The notion of isomorphisms of K-valued geometric points is naturally defined, for two K-valued geometric points x and y in Θ, $x \simeq y$ if and only if these are in the same $\mathrm{GL}(V)$-orbit. If x_1 and x_2 are two ϕ-direct sums of x' and x'', then we know that $x_1 \simeq x_2$ (see Lemma 2.16 of [10]). Thus a direct sum of x' and x'' can be denoted by $x' \oplus x''$. Moreover, let x'_i be a K-valued geometric point of $\Theta(V'(i), l_i, L'(i), N'(i)_*, \alpha'(i)_*)$ $(1 \leq i \leq t)$ and put $r_i = l_1 + \cdots + l_i$ and $V(i) = V'(1) \oplus \cdots \oplus V'(i)$. Let $\phi_i\colon L(i-1) \otimes L'(i) \to L(i)$ be a sequence of injective homomorphisms $(1 \leq i \leq t,\ L(0) = \mathcal{O}_X)$. We can define ϕ_i-direct sum of x_{i-1} and x'_i inductively. Each x_i is a K-valued geometric point of $\Theta(V(i), r_i, L(i), N(i)_*, \alpha(i)_*)$ and it is denoted by $(\cdots((x'_1 \oplus x'_2) \oplus x'_3) \oplus \cdots) \oplus x'_i)$. By a similar argument as in Lemma 2.19 and Corollary 2.19.1 of [10], we can denote x_i by $x'_1 \oplus \cdots \oplus x'_i$.

Lemma 3.4. *Let Z' (or, Z'') be a $\mathrm{GL}(V')$ (or, $\mathrm{GL}(V'')$, resp.)-invariant closed subset of Θ' (or, Θ'', resp.) and let $\phi\colon L' \otimes_X L'' \to L$ be an injective homomorphism. Then there exists a $\mathrm{GL}(V)$-invariant closed subset Z of Θ such that for all algebraically closed fields K containing k,*

$$Z(K) = \{x \in \Theta(K) \mid x \textit{ has one of the following properties } (3.4.1), (3.4.2)\}.$$

(3.4.1) *x is a ϕ-quasi-extension of x'' in $Z''(K)$ by x' in $Z'(K)$.*

(3.4.2) *There exist points x' in $Z'(K)$, x'' in $Z''(K)$ and an exact sequence $0 \to V'_K \xrightarrow{f} V_K \xrightarrow{g} V''_K \to 0$ such that $T_x|_{[f(V'_K);r', V_K;r'']} = 0$ and for all $0 \leq \alpha \leq 1$, $f(V'^{\alpha}_{x'}) \subseteq V^{\alpha}_{x}$ and $g(V^{\alpha}_{x}) \subseteq V''^{\alpha}_{x''}$.*

Proof. We can find that there exists a subscheme U_0 of $\mathrm{Hom}_k(V', V) \times_k \mathrm{Hom}_k(V, V'')$ such that for all fields K containing k, $U_0(K) = \{(f, g) \mid 0 \to V'_K \xrightarrow{f} V_K \xrightarrow{g} V''_K \to 0$ is exact$\}$ (cf. Lemma 2.6 of [10]) and on U_0, we have a universal exact sequence

$$0 \longrightarrow V'_{U_0} \xrightarrow{\tilde{f}} V_{U_0} \xrightarrow{\tilde{g}} V''_{U_0} \longrightarrow 0.$$

Let

$$\gamma: \wedge^{r'}(V'_{U_0} \otimes_k \Omega) \otimes \wedge^{r''}(V_{U_0} \otimes_k \Omega) \longrightarrow \wedge^{r}(V_{U_0} \otimes_k \Omega)$$

be a homomorphism defined by

$$\begin{aligned} &\gamma((v_1 \wedge \cdots \wedge v_{r'}) \otimes (w_1 \wedge \cdots \wedge w_{r''})) \\ &= (\tilde{f} \otimes 1)(v_1) \wedge \cdots \wedge (\tilde{f} \otimes 1)(v_{r'}) \wedge w_1 \wedge \cdots \wedge w_{r''} \end{aligned}$$

where v_i (or, w_i) are local sections of $V'_{U_0} \otimes_k \Omega$ (or, $V_{U_0} \otimes_k \Omega$, resp.). Taking $\mathrm{Hom}_{X_{U_0}}(-, L_{U_0})^\vee$, we obtain a homomorphism

$$\begin{array}{ccc} \mathrm{Hom}_{X_{U_0}}(\wedge^{r'}(V'_{U_0} \otimes_k \Omega) \otimes_{X_{U_0}} \wedge^{r''}(V_{U_0} \otimes_k \Omega), L_{U_0})^\vee & \xrightarrow{\simeq} & M_{U_0} \\ \downarrow & & \downarrow{\scriptstyle \gamma'} \\ \mathrm{Hom}_{X_{U_0}}(\wedge^{r}(V_{U_0} \otimes_k \Omega), L_{U_0})^\vee & \xrightarrow{\simeq} & N_{U_0} \end{array}$$

where $M = \mathrm{Hom}_X(\wedge^{r'}(V' \otimes_k \Omega) \otimes_X \wedge^{r''}(V \otimes_k \Omega), L)^\vee$ and $N = \mathrm{Hom}_X(\wedge^r(V \otimes_k \Omega), L)^\vee$. Let $\mathcal{O}_{M_{U_0}}(1)$ be the tautological invertible sheaf on $\mathbf{P}(M_{U_0}) \simeq \mathbf{P}(M) \times_k U_0$ and let M' be the kernel of the natural quotient map

$$q^*(M_{U_0}) \longrightarrow \mathcal{O}_M(1)$$

where q is the projection of $\mathbf{P}(M_{U_0})$ to U_0. Set

$$Z_0 = \mathbf{P}(q^*(N_{U_0})/q^*(\gamma')(M')) \subset \mathbf{P}(q^*(N_{U_0})) \simeq \mathbf{P}(M) \times_k \mathbf{P}(N) \times_k U_0.$$

Then we obtain the following commutative diagram

$$(*) \qquad \begin{array}{ccccc} & & Z_0 & & \\ & {\scriptstyle q'}\swarrow & & \searrow{\scriptstyle p'} & \\ \mathbf{P}(N) \times_k U_0 & & \cdots\longrightarrow & & \mathbf{P}(M) \times_k U_0 \\ & {\scriptstyle p}\searrow & & \swarrow{\scriptstyle q} & \\ & & U_0 & & \end{array}$$

Let $x = (T_x, (f_x, g_x))$ be a K-valued geometric point of $\mathbf{P}(M) \times_k U_0$ where T_x is a homomorphism (mod $K^\times$)

$$T_x: \wedge^{r'}(V'_K \otimes_k \Omega) \otimes \wedge^{r''}(V_K \otimes_k \Omega) \longrightarrow L_K$$

and (f_x, g_x) determines an exact sequence

$$0 \longrightarrow V'_K \xrightarrow{f_x} V_K \xrightarrow{g_x} V''_K \longrightarrow 0.$$

Then the fibre $p'^{-1}(x)$ is a closed subscheme of $\mathbf{P}(N)_K \simeq P(V, r, L)_K$ and we can find that

(3.4.3) $T \in P(V, r, L)_K(K)$ is in $p'^{-1}(x)(K)$ if and only if $T \circ (\wedge^{r'}(f \otimes 1) \wedge (\wedge^{r''} id)) = T_x$ as a point of $P(V, r, L)_K(K)$, or $T \circ (\wedge^{r'}(f \otimes 1) \wedge (\wedge^{r''} id)) = 0$.

On the other hand, from a surjection

$$\wedge^{r'}(V'_{U_0} \otimes_k \Omega) \otimes_{X_{U_0}} \wedge^{r''}(V_{U_0} \otimes_k \Omega) \twoheadrightarrow \wedge^{r'}(V'_{U_0} \otimes_k \Omega) \otimes_{X_{U_0}} \wedge^{r''}(V''_{U_0} \otimes_k \Omega)$$

defined by $\tilde{g}$ (i.e. $(\wedge^{r'} id) \wedge (\wedge^{r''}(\tilde{g} \otimes 1)))$ and the injection ϕ: $L' \otimes_X L'' \to L$, we obtain a surjection

$$\begin{aligned} &\mathrm{Hom}_{X_{U_0}}(\wedge^{r'}(V'_{U_0} \otimes_k \Omega), L'_{U_0})^\vee \otimes_{U_0} \mathrm{Hom}_{X_{U_0}}(\wedge^{r''}(V''_{U_0} \otimes_k \Omega), L''_{U_0})^\vee \\ &\simeq \mathrm{Hom}_{X_{U_0}}(\wedge^{r'}(V'_{U_0} \otimes_k \Omega) \otimes_{X_{U_0}} \wedge^{r''}(V''_{U_0} \otimes_k \Omega), (L' \otimes_X L'')_{U_0})^\vee \\ &\twoheadrightarrow M_{U_0} \end{aligned}$$

Hence, by the Veronese embedding, we have a closed immersion:

$$\iota\colon P(V', r', L') \times_k P(V'', r'', L'') \times_k U_0 \hookrightarrow \mathbf{P}(M) \times_k U_0.$$

Now set $F = F(V, N_*, \alpha_*)$, $F' = F(V', N'_*, \alpha'_*)$, $F'' = F(V'', N''_*, \alpha''_*)$ and $\tilde{F} = F \times_k F' \times_k F''$. Then there exists a closed subscheme U_1 of $U_0 \times_k \tilde{F}$ such that for all algebraically closed fields K containing k,

$$U_1(K) = \left\{ ((f, g), V^*, V'^*, V''^*) \,\middle|\, \begin{array}{l} f(V'^\alpha) \subset V^\alpha \text{ and } g(V^\alpha) \subset V''^\alpha \\ \text{for all } 0 \le \alpha \le 1 \end{array} \right\}.$$

Taking the product of (*) and $\tilde{F}$ and combining natural isomorphisms

$$\Theta \times U_0 \times F' \times F'' \simeq \mathbf{P}(N) \times U_0 \times \tilde{F},$$

$$\Theta' \times \Theta'' \times U_0 \times F \simeq P(V', r', L') \times P(V'', r'', L'') \times U_0 \times \tilde{F},$$

we obtain the following commutative diagram.

$$\begin{array}{ccccc} Z_0 \times \tilde{F} & \xrightarrow{\tilde{p}'} & \mathbf{P}(M) \times U_0 \times \tilde{F} & \xleftarrow{\iota} & Z' \times Z'' \times U_0 \times F \\ \downarrow \tilde{q}' & & \downarrow \tilde{q} & & \\ \Theta \times U_0 \times F' \times F'' & \xrightarrow{\tilde{p}} & U_0 \times \tilde{F} & \hookleftarrow & U_1 \end{array}$$

Set

$$Z_1 = \tilde{p}'^{-1}(\tilde{q}^{-1}(U_1) \cap \iota(Z' \times Z'' \times U_0 \times F)).$$

By virtue of (3.4.3) it is easy to see that $Z = \pi(\tilde{q}'(Z_1))$ is the desired set where π is the projection of $\Theta \times U_0 \times F' \times F''$ to Θ. We must prove that Z is closed and $\mathrm{GL}(V)$-invariant. It is not difficult to verify that the properties (3.4.1) and (3.4.2) are $\mathrm{GL}(V)$-invariant, hence Z is $\mathrm{GL}(V)$-invariant.

Set $H = \mathrm{GL}(V') \times_k \mathrm{GL}(V'')$. It is easy to see that all morphisms in (*) are H-morphisms and hence Z_1 is H-invariant. Moreover, U_0 is a principal H-bundle over the Grassmann scheme $G(V, \dim V')$ and Z_1 is proper over U_0, therefore by Proposition 7.1 of [16] and its proof, Z_1 is a principal H-bundle over a proper $G(V, \dim V')$-scheme Z_2. Since H acts on Θ trivially, the projection $\pi: Z_1 \to \Theta$ factors through Z_2. Therefore Z is closed because it is the image of the complete k-scheme Z_2. □

For a K-valued geometric point x of Θ, $o(x)$ denote its G-orbit.

Lemma 3.5. *Let x, x' and x'' be K-valued geometric points of Θ, Θ' and Θ'', respectively and let $\phi: L' \otimes_X L'' \to L$ be an injective homomorphism. Assume that x is a ϕ-extension of x'' by x'. Then the closure of* GL(V)*-orbit $\overline{o(x)}$ contains the orbit $o(x' \oplus x'')$.*

Proof. Let R be a discrete valuation ring over K with residue field K. Let i be a section of the underlying exact sequence

$$0 \longrightarrow V'_K \xrightarrow{u} V_K \xrightarrow{v} V''_K \longrightarrow 0$$

of the extension x. Then $V_K = U_1 \oplus U_2$ with $U_1 = u(V'_K)$ and $U_2 = i(V''_K)$. We may assume that $i(V''^{\alpha}_{x''}) \supset V^{\alpha}_x$. Take an automorphism $g = id_{U_1} \oplus \pi \cdot id_{U_2}$ of $V_{Q(R)}$ where π is a uniformizing parameter of R. Set $x_1 = \sigma(g, x)$. Using a natural decomposition

$$\begin{aligned}&\mathrm{Hom}_{X_K}(\wedge^r(V_K \otimes_K \Omega_K), L_K)\\ &= \textstyle\bigoplus_{s=0}^r \mathrm{Hom}_{X_K}(\wedge^s(U_1 \otimes_K \Omega_K) \otimes \wedge^{r-s}(U_2 \otimes_K \Omega_K), L_K),\end{aligned}$$

T_x can be denoted by $\sum_{s=0}^r T_x^s$ with

$$T_x^s \in \mathrm{Hom}_{X_K}(\wedge^s(U_1 \otimes_K \Omega_K) \otimes \wedge^{r-s}(U_2 \otimes_K \Omega_K), L_K).$$

By the condition (3.1.2), for all s with $r - s > r''$, $T_x^s = 0$ and $T_x^{r'} \neq 0$. Hence, we have

$$T_{x_1} = \sum_{s=r'}^{r} \pi^{s-r} T_x^s.$$

The point $\tilde{T} = \sum_{s=r'}^r \pi^{s-r'} T_x^s$ can be regarded as an R-valued point of $P(V, r, L)$. Since $T_{x_1} = \tilde{T}$ as point of $P(V, r, L)(Q(R))$, $\tilde{T}(\mathrm{mod}\ \pi) = T_x^{r'}$ and $T_x^{r'}|_{[U_2; s, V_K; r-s]} = 0$ whenever $s > r''$, we know that $\tilde{T}(\mathrm{mod}\ \pi)$ and the flag structure for x determines a point $x' \oplus x''$. □

Recall the definition of excellent points of Gieseker spaces (cf. [10], [26]).

Definition 3.6. A K-valued geometric point T of $P(V, r, L)$ is said to be excellent, if T has the following properties.

(3.6.1) For all vector subspaces W of V_K,

$$\underline{\dim}_T W = \overline{\dim}_T W.$$

(3.6.2) For all subpoints $T' \in P(V', r', L')(K)$ of T and for all vector subspaces W of V'_K, if $e_1, \dots, e_t$ is a T'-basis of W, then $f(e_1), \dots, f(e_t)$ is a T-basis of W where f: $V' \otimes_k K \to V \otimes_k K$ is an injection which makes T' the subpoint of T.

A K-valued geometric point of Θ is said to be excellent if T_x is excellent.

Note that if T is excellent, then for all subpoints $T' \in P(V', r', L')(K)$ of T and for all vector subspaces W of V'_K,

$$\underline{\dim}_{T'} W = \overline{\dim}_{T'} W = \dim_T W.$$

The following lemma is (5.3.1) of [26] and is a natural generalization of a part of Lemma 4.4 of [10]. We give a proof since it is omitted in [26].

Lemma 3.7. *Let T be a K-valued geometric point of $P(V, r, L)$. Assume that there exists a surjective homomorphism φ: $V_K \otimes_K \Omega_K \to E$ with E a coherent $\mathcal{O}_{X_K}$-module of rank $r > 0$ such that $\det E \simeq L_K$ and T is given by the following homomorphism.*

$$\wedge^r (V_K \otimes_K \Omega_K) \xrightarrow{\wedge^r \varphi} \wedge^r E \longrightarrow \det E \simeq L_K.$$

Then T is excellent. Moreover, for all K-vector subspaces W of V_K,

$$\dim_T W = \mathrm{rk}\,(\varphi(W \otimes_K \Omega)).$$

Proof. We may assume that $k = K$ and $\det E = L_K$. Since $\det E$ is torsion free, for every submodule M of $\wedge^r(V \otimes_k \Omega)$, $T|_M = 0$ if and only if $T_\xi|_{M_\xi} = 0$ where ξ is the generic point of X. Let $e_1, \dots, e_i$ be a T-basis of W. Then $T|_{[e_1, \dots, e_i, V; r-i]} \neq 0$. Therefore there exist elements $w_1, \dots, w_i$ of Ω_ξ and $v_1, \dots, v_{r-i}$ of $(V \otimes_k \Omega)_\xi$ such that

$$T_\xi((e_1 \otimes w_1) \wedge \cdots \wedge (e_i \otimes w_i) \wedge v_1 \wedge \cdots \wedge v_{r-i}) \neq 0.$$

Hence, $\varphi(e_1 \otimes w_1), \dots, \varphi(e_i \otimes w_i)$ is linearly independent over $k(\xi)$. Since any element e of W is T-dependent on $e_1, \dots, e_i$, we know that for any element w of Ω_ξ, $\varphi(e \otimes w)$ is linearly dependent on $\varphi(e_1 \otimes w_1), \dots, \varphi(e_i \otimes w_i)$, and so is $\varphi(v)$ for any element v of $(W \otimes_k \Omega)_\xi$. Therefore, the length i must be equal to $\mathrm{rk}\,(\varphi(W \otimes_k \Omega))$. Thus the condition (3.6.1) and the last assertion of our lemma were proved.

Next, let us prove the condition (3.6.2). Assume that T is a ϕ-extension of $T'' \in P(V'', r'', L'')(K)$ by $T' \in P(V', r', L'')(K)$. Let W be a K-vector subspace of V'_K. For vectors $e_1, \dots, e_i$ in W, we have a following commutative diagram.

$$
\begin{array}{ccc}
[e_1,\dots,e_i, V'_K; r'-i] \otimes \wedge^{r''} V_K & \searrow & \\
\Big\downarrow & & [e_1,\dots,e_i, V'_K; r'-i, V_K; r''] \\
 & & \Big\downarrow \\
[e_1,\dots,e_i, V'_K; r'-i] \otimes \wedge^{r''} V''_K & & [e_1,\dots,e_i, V_K; r-i] \\
\Big\downarrow {\scriptstyle T_{x'}\otimes T_{x''}} & & \Big\downarrow {\scriptstyle T_x} \\
L' \otimes L'' & \xrightarrow{\quad\phi\quad} & L
\end{array}
$$

By this commutative diagram, if $e_1,\dots,e_i$ are T'-independent, then so are T-independent. Conversely, assume that $e_1,\dots,e_i$ are T-independent. It is easy to see that $\underline{\dim}_T V'_K = r'$ and there exist vectors $e_{i+1},\dots,e_{r'}$ in V'_K such that $e_1,\dots,e_{r'}$ is a T-basis of V'_K. Then $T|_{[e_1,\dots,e_{r'},V_K;r'']} \neq 0$. Hence, $T'|_{[e_1,\dots,e_{r'}]} \neq 0$. In particular, $e_1,\dots,e_i$ are T'-independent.

Lemma 2.3 is rewritten for Θ as follows.

Lemma 3.8. *Let x be a K-valued geometric point of Θ. Assume that T_x has the property* (3.6.1). *Then the point x is semi-stable* (*or, stable*) *with respect to $\mathcal{O}_\Theta(1)$ if and only if for all non-trivial vector subspaces W of V_K, the following inequality holds*

$$
(3.8.1) \qquad (N - \sum_i \varepsilon_i N_i) \dim_{T_x} W - r \int_0^1 \dim_K (W \cap V_x^\alpha) d\alpha \geq 0
$$

(*or*, > 0, *resp.*).

Lemma 3.9. *Let x, x' and x'' be K-valued geometric points of Θ, Θ' and Θ'' respectively and let $\phi\colon L' \otimes_X L'' \to L$ be an injective homomorphism. Assume that x is excellent and is a ϕ-extension of x'' by x' with underlying exact sequence*

$$
0 \longrightarrow V'_K \stackrel{f}{\longrightarrow} V_K \stackrel{g}{\longrightarrow} V''_K \longrightarrow 0.
$$

Then x' and x'' satisfy the condition (3.6.1) *and for all vector subspaces W of V'_K, the following inequality holds*

$$
(3.9.1) \qquad \dim_{T_x} W \geq \dim_{T_{x'}} f^{-1}(W) + \dim_{T_{x''}} g(W).
$$

Moreover, if W contains $f(V'_K)$, then

$$
(3.9.2) \qquad \dim_{T_x} W = r' + \dim_{T_{x''}} g(W).
$$

Proof. We regard V'_K as a subspace of V_K by f. Let $v_1,\dots,v_d$ be a $T_{x'}$-basis of $W \cap V'_K$ and let $w_1,\dots,w_e$ be elements of W such that $g(w_1),\dots,g(w_e)$ is a $T_{x''}$-basis of $g(W)$. Then, by virtue of (3.1.2.1), we obtain a following natural commutative diagram

$$\begin{array}{ccc}
[v_1,\dots,v_d,\ V'_K;\ r'-d]\otimes[w_1,\dots,w_e,\ V_K;\ r''-e] & \searrow & \\
\downarrow & & [v_1,\dots,v_d,\ V'_K;\ r'-d,\ w_1,\dots,w_e,\ V_K;\ r''-e] \\
[v_1,\dots,v_d,\ V'_K;\ r'-d]\otimes[g(w_1),\dots,g(w_e),\ V''_K;\ r''-e] & & \downarrow i \\
 & & [v_1,\dots,v_d,\ w_1,\dots,w_e,\ V_K;\ r-d-e] \\
\downarrow T_{x'}\otimes T_{x''} & & \downarrow T_x \\
L'\otimes L'' & \xrightarrow{\ \ \phi\ \ } & L
\end{array}$$

Since $T_{x'}|_{[v_1,\dots,v_d,V'_K;r'-d]}\neq 0$ and $T_{x''}|_{[g(w_1),\dots,g(w_e),V''_K;r''-e]}\neq 0$, we have

$$T_{x'}\otimes T_{x''}|_{[v_1,\dots,v_d,V'_K;r'-d]\otimes[g(w_1),\dots,g(w_e),V''_K;r''-e]}\neq 0.$$

Hence, by the above diagram, $T_x|_{[v_1,\dots,v_d,w_1,\dots,w_e,V_K;r-d-e]}\neq 0$. It follows that

$$\dim_{T_x} W\geq d+e. \tag{3.9.3}$$

Since x is excellent, x' satisfies (3.6.1) and hence $d=\dim_{T_{x'}}f^{-1}(W)$. It is sufficient to prove that x'' satisfies the condition (3.6.1) and that for $W\supseteq V'_K$, (3.9.2) holds.

If $W\supseteq V'_K$, then $d=\dim_{T_{x'}}V'_K=r'$. Fix a $T_{x'}$-basis $v_1,\dots,v_{r'}$. For vectors $w_1,\dots,w_e$ of W, the injection i in the above diagram is an isomorphism. Hence, we know that $T_x|_{[v_1,\dots,v_d,w_1,\dots,w_e,V_K;r-r'-e]}\neq 0$ if and only if $T_x|_{[g(w_1),\dots,g(w_e),V''_K;r''-e]}$ $\neq 0$. This fact implies that $g(w_1),\dots,g(w_e)$ is a $T_{x''}$-basis of $g(W)$ if and only if $v_1,\dots,v_{r'},\ w_1,\dots,w_e$ is a T_x-basis of W. Hence each $T_{x''}$-basis of $g(W)$ has the same length $\dim_{T_x}W-r'$. Thus x'' satisfies the condition (3.6.1) and we obtain the equality (3.9.2). □

Lemma 3.10. *Under the same situation as in Lemma* 3.9, *assume, moreover, that*

$$\frac{1}{r}\Big(N-\sum_i\varepsilon_iN_i\Big)=\frac{1}{r'}\Big(N'-\sum_i\varepsilon'_iN'_i\Big)=\frac{1}{r''}\Big(N''-\sum_i\varepsilon''_iN''_i\Big). \tag{3.10.1}$$

Then

(1) *x' and x'' are semi-stable when x is semi-stable.*

(2) *If x' and x'' are excellent and semi-stable, then x is semi-stable.*

Proof. (1) Assume that x is semi-stable. For each vector subspace $W\neq 0$ of V'_K, by (3.6.2), (3.8.1) and (3.10.1), we have

$$\begin{aligned}
0&\leq\Big(N-\sum_i\varepsilon_iN_i\Big)\dim_{T_x}W-r\int_0^1\dim_K(W\cap V_x^{\alpha})\,d\alpha\\
&=\frac{r}{r'}\left(\Big(N'-\sum_i\varepsilon'_iN'_i\Big)\dim_{T_{x'}}W-r'\int_0^1\dim_K(W\cap V_{x'}'^{\alpha})\,d\alpha\right).
\end{aligned}$$

Hence, by Lemma 3.8, x' is semi-stable.

To prove that x'' is semi-stable, take $0 \neq W \subseteq V''_K$. Since the sequence (3.1.3) is exact, we have an exact sequence

$$0 \longrightarrow V'^{\alpha}_{x'} \xrightarrow{f} g^{-1}(W) \cap V^{\alpha}_{x} \xrightarrow{g} W \cap V''^{\alpha}_{x''} \longrightarrow 0.$$

Then, by (3.8.1), (3.9.2) and (3.10.1),

$$0 \leq (N - \sum_i \varepsilon_i N_i) \dim_{T_x} g^{-1}(W) - r \int_0^1 \dim_K (g^{-1}(W) \cap V^{\alpha}_x) \, d\alpha$$

$$= (N - \sum_i \varepsilon_i N_i)(\dim_{T_{x''}} W + r')$$

$$- r\left(\int_0^1 \dim_K (W \cap V''^{\alpha}_{x''}) \, d\alpha + \int_0^1 \dim_K V'^{\alpha}_{x'} \, d\alpha \right)$$

$$= \frac{r}{r''}\left((N'' - \sum_i \varepsilon''_i N''_i) \dim_{T_{x''}} W - r'' \int_0^1 \dim_K (W \cap V''^{\alpha}_{x''}) \, d\alpha \right).$$

Hence, by Lemma 3.8, x'' is semi-stable.

(2) Assume that x' and x'' are excellent and semi-stable. Let $W \neq 0$ be a vector subspace of V_K. Then, by the exact sequence (3.1.3), we obtain the following exact sequence

$$0 \longrightarrow V'^{\alpha}_{x'} \cap W \longrightarrow V^{\alpha}_{x} \cap W \longrightarrow V''^{\alpha}_{x''} \cap g(W).$$

Hence, by (3.8.1), (3.9.1) and (3.10.1), we have that

$$(N - \sum_i \varepsilon_i N_i) \dim_{T_x} W - r \int_0^1 \dim_K (W \cap V^{\alpha}_x) \, d\alpha$$

$$\geq (N - \sum_i \varepsilon_i N_i)(\dim_{T_{x'}} (W \cap V'_{x'}) + \dim_{T_{x''}} g(W))$$

$$- r \int_0^1 (\dim_K (V'^{\alpha}_{x'} \cap W) + \dim_K (V''^{\alpha}_{x''} \cap g(W))) \, d\alpha$$

$$= \frac{r}{r'}\left((N' - \sum_i \varepsilon'_i N'_i) \dim_{T_{x'}} (W \cap V'_{x'}) - r' \int_0^1 \dim_K (V'^{\alpha}_{x'} \cap W) \, d\alpha \right)$$

$$+ \frac{r}{r''}\left((N'' - \sum_i \varepsilon''_i N''_i) \dim_{T_{x''}} g(W) - r'' \int_0^1 \dim_K (V''^{\alpha}_{x''} \cap g(W)) \, d\alpha \right)$$

$$\geq 0.$$

Therefore, by Lemma 3.8, x is semi-stable. □

Proposition 3.11. *Let $\phi_i: L_{i-1} \otimes L'_i \to L_i$ be injective homomorphisms ($1 \leq i \leq t$, $L_0 = \mathcal{O}_X$), $0 < r_1 < \cdots < r_t = r$ be a sequence of integers and let D_i be a $\mathrm{GL}(V_i)$-invariant closed set of $\Theta_i = \Theta(V_i, r_i, L_i, N^i_*, \alpha^i_*)$ ($1 \leq i \leq t$). Assume that* *for every algebraically closed field K containing k, all the points of $D_i(K)$ are excellent and that*

$$(3.11.1)\qquad \frac{1}{r_1}(N^1 - \sum_{j=1}^{l(\Theta_1)} \varepsilon(\Theta_1)_j N_j^1) = \cdots = \frac{1}{r_t}(N^t - \sum_{j=1}^{l(\Theta_t)} \varepsilon(\Theta_t)_j N_j^t).$$

Let S_i be a k-rational, stable, excellent point in $\Theta_i' = \Theta(V_i', r_i', L_i', N_*'^i, \alpha_*'^i)(\bar{k})$ where $r_i' = r_i - r_{i-1}$ and $\bar{k}$ is the algebraic closure of k. Then there exists a $\mathrm{GL}(V_t)$-invariant closed set $Z_t = Z(S_1, \ldots, S_t)$ of $D_t^{ss} = D_t^{ss}(\mathcal{O}_{\Theta_t}(1) \otimes \mathcal{O}_{D_t})$ such that for every algebraically closed field K containing k,

$$Z_t(K) = \{x \in D_t(K) \mid x \text{ has the following property } (*)_t\}.$$

$(*)_t$: *There exists a K-valued geometric point x_i in each $D_i^{ss} = D_i^{ss}(\mathcal{O}_{\Theta_i}(1) \otimes \mathcal{O}_{D_i})$ such that $x_1 \simeq S_1$, x_i is a ϕ_i-extension of S_i by x_{i-1} $(2 \le i \le t)$ and $x = x_t$.*

Moreover if $Z(S_1, \ldots, S_t)$ is not empty, then $\mathrm{GL}(V_t)$-orbit $o(S_1, \ldots, S_t)$ of $S_1 \oplus \cdots \oplus S_t$ is a unique closed orbit in $Z(S_1, \ldots, S_t)$.

Proof. If $(*)_t$ holds, then by (3.2.1), we have that

$$(3.11.2)\qquad N'^i - \sum_j \varepsilon_j'^i N_j'^i$$
$$= (N^i - \sum_j \varepsilon_j^i N_j^i) - (N^{i-1} - \sum_j \varepsilon_j^{i-1} N_j^{i-1})$$

where $\varepsilon_j^i = \varepsilon(\Theta_i)_j$ and $\varepsilon_j'^i = \varepsilon(\Theta_i')_j$. Hence, we may assume that (3.11.2) holds because otherwise $Z_t = \emptyset$ is desired one.

We prove the first assertion by induction on t. When $t = 1$, set $Z_1 = o(S_1)$. Since S_1 is stable, Z_1 is closed in D_1^{ss}. Obviously, Z_1 is desired one. Assume the assertion holds for $t-1$. Then there exists a $\mathrm{GL}(V_{t-1})$-invariant closed set Z_{t-1} of D_{t-1} such that Z_{t-1} satisfies the property $(*)_{t-1}$. Let $\overline{Z_{t-1}}$ (or, $\overline{o(S_t)}$) be the closure of Z_{t-1} (or, $o(S_t)$, resp.) in D_{t-1} (or, Θ_t', resp.). Then by Lemma 3.4, we obtain a $\mathrm{GL}(V_t)$-invariant closed subset Z of Θ_t such that a K-valued geometric point x of Θ_t is contained in $Z(K)$ if and only if x has one of the following properties.

(3.11.3) x is a ϕ_t-quasi-extension of a x'' in $\overline{o(S_t)}(K)$ by a x' in $\overline{Z_{t-1}}(K)$.

(3.11.4) There exist points x' in $\overline{Z_{t-1}}(K)$, x'' in $\overline{o(S_t)}(K)$ and an exact sequence

$$0 \longrightarrow V_{t-1} \otimes_k K \xrightarrow{f} V_t \otimes_k K \xrightarrow{g} V_t' \otimes_k K \longrightarrow 0$$

such that $T_x|_{[f(V_{t-1} \otimes_k K); r_{t-1}, V_t \otimes_k K; r_t]} = 0$ and for all $0 \le \alpha \le 1$, $f^{-1}((V_t)_x^\alpha) \supseteq (V_{t-1})_{x'}^\alpha$ and $g((V_t)_x^\alpha) \subseteq (V_t')_{x''}^\alpha$.

We claim that $Z_t = Z \cap D_t^{ss}$ is desired one. Let x be a K-valued geometric point of Z_t. If x has the property (3.11.4), then $\dim_{T_x} f(V_{t-1} \otimes_k K) < r_{t-1}$. By virtue of (3.11.1), we have that

$$(N_t - \sum_j \varepsilon_j^t N_j^t) \dim_{T_x} f(V_{t-1} \otimes_k K) - r_t \int_0^1 \dim_K (f(V_{t-1} \otimes_k K) \cap (V_t)_x^\alpha) \mathrm{d}\alpha$$
$$< (N_t - \sum_j \varepsilon_j^t N_j^t) r_{t-1} - r_t \int_0^1 \dim_K (V_{t-1})_{x'}^\alpha \mathrm{d}\alpha = 0.$$

By Lemma 3.8, this inequality contradicts to semi-stability of x. Hence, x satisfies the condition (3.11.3).

We claim that x is not only a ϕ_t-quasi-extension but also a ϕ_t-extension. Let

$$0 \longrightarrow V_{t-1} \otimes_k K \xrightarrow{f} V_t \otimes_k K \xrightarrow{g} V_t' \otimes_k K \longrightarrow 0$$

be the underlying exact sequence of the quasi-extension. We must prove that for all $0 \le \alpha \le 1$, $f^{-1}((V_t)_x^\alpha) = (V_{t-1})_{x'}^\alpha$ and $g((V_t)_x^\alpha) = (V_t')_{x''}^\alpha$. These are trivial for $\alpha = 0$. By the condition (3.1.2.2), $f^{-1}((V_t)_x^\alpha) \supseteq (V_{t-1})_{x'}^\alpha$ and $g((V_t)_x^\alpha) \subseteq (V_t')_{x''}^\alpha$. Hence, it is enough to prove that

$$\int_0^1 \dim_K f^{-1}((V_t)_x^\alpha)/(V_{t-1})_{x'}^\alpha \mathrm{d}\alpha = 0 \quad \text{and}$$
$$\int_0^1 \dim_K (V_t')_{x''}^\alpha / g((V_t)_x^\alpha) \mathrm{d}\alpha = 0.$$

Since x is excellent and semi-stable, by Lemma 3.8, Lemma 3.9, (3.0.1) and (3.11.1),

$$\begin{aligned} 0 &\le \int_0^1 \dim_K f^{-1}((V_t)_x^\alpha)/(V_{t-1})_{x'}^\alpha \mathrm{d}\alpha \\ &\le \frac{1}{r_t}(N^t - \sum_i \varepsilon_i^t N_i^t) \dim_{T_x} f(V_{t-1} \otimes_k K) - (N^{t-1} - \sum_i \varepsilon_i^{t-1} N_i^{t-1}) \\ &= \frac{r_{t-1}}{r_t}(N^t - \sum_i \varepsilon_i^t N_i^t) - (N^{t-1} - \sum_i \varepsilon_i^{t-1} N_i^{t-1}) \\ &= 0. \end{aligned}$$

Hence, for all α with $0 \le \alpha \le 1$, $f^{-1}((V_t)_x^\alpha) = (V_{t-1})_{x'}^\alpha$. Therefore we obtain that $g((V_t)_x^\alpha) \simeq (V_t)_x^\alpha / f((V_{t-1})_{x'}^\alpha)$. Hence by (3.0.1) and (3.11.2), we have that

$$\begin{aligned} &\int_0^1 \dim_K g((V_t)_x^\alpha) \mathrm{d}\alpha \\ &= \int_0^1 \dim_K (V_t)_x^\alpha / f((V_{t-1})_{x'}^\alpha) \mathrm{d}\alpha \\ &= \int_0^1 \dim_K (V_t')_{x''}^\alpha \mathrm{d}\alpha. \end{aligned}$$

Hence, for all $0 \le \alpha \le 1$, we have that $g((V_t)_x^\alpha) = (V_t')_{x''}^\alpha$.

By equalities (3.11.1), (3.11.2) and $r_t - r_{t-1} = r_t'$, we get easily equalities

$$\frac{1}{r_t}(N^t - \sum_i \varepsilon_i^t N_i^t) = \frac{1}{r_{t-1}}(N^{t-1} - \sum_i \varepsilon_i^{t-1} N_i^{t-1}) = \frac{1}{r_t'}(N'' - \sum_i \varepsilon_i'' N_i'').$$

Hence, by virtue of Lemma 3.10, x' and x'' are semi-stable. Therefore x' (or, x'') is in $Z_{t-1}(K) = \overline{Z_{t-1}} \cap D_{t-1}^{ss}(K)$ (or, $o(S_t)(K) = \overline{o(S_t)} \cap \Theta_t'^{ss}(K)$, resp.). Thus x satisfies the condition $(*)_t$.

Conversely, if a K-valued geomeric point x in $D_t(K)$ satisfies the condition $(*)_t$, then x is a ϕ-extension of a x'' in $o(S_t)$ by x' in Z_{t-1} and by Lemma 3.10, x is semi-stable. Hence, x is in $Z_t(K) = Z \cap D_t^{ss}(K)$.

Let us prove the last assertion by induction on t. If $t = 1$, then $Z(S_1) = o(S_1)$ is closed in D_1^{ss}. Assume that our assertion holds for $t - 1$. Then $o(S_1, \dots, S_{t-1})$ is a unique closed orbit of $Z(S_1, \dots, S_{t-1})$. Let x be a K-valued geometric point of $Z(S_1, \dots, S_t)$ such that $o(x)$ is closed in $Z(S_1, \dots, S_t)$. Then by $(*)_{t-1}$, there exists x' in $Z(S_1, \dots, S_{t-1})(K)$ such that x is a ϕ_t-extension of S_t by x'. By virtue of Lemma 3.5, $\overline{o(x' \oplus S_t)} \supseteq o(S_1, \dots, S_t)$. Hence,

$$o(x) = \overline{o(x)} \supseteq \overline{o(x' \oplus S_t)} \supseteq o(S_1, \dots, S_t).$$

Hence, $o(x) = o(S_1, \dots, S_t)$. □

4. Moduli of parabolic semi-stable pairs

In this section, under the situation (2.0.1), we shall show that the functor par-$\bar{\Sigma}_{\Omega/D/X/S}^{H_*, \alpha_*}$ has a moduli scheme. We may assume that S is connected and $\mathscr{F} = \mathscr{F}_\Omega(H, H_*, \alpha_*)$ is not empty. r or $r_{\mathscr{F}}$ denotes the rank of members of $\mathscr{F}$. Set

$$H_{\mathscr{F}} = H - \sum_i \varepsilon_i H_i$$
$$P_{\mathscr{F}} = H_{\mathscr{F}} / r_{\mathscr{F}}.$$

$H_{\mathscr{F}}$ is the parabolic Hilbert polynomial of members of $\mathscr{F}$. Let $\mathscr{G}$ be the family of parabolic Ω-pairs such that (E_*', φ') is contained in $\mathscr{G}$ if and only if there is a strictly parabolic e-semi-stable Ω-pair $(E_*, \varphi) \in \mathscr{F}$ and a Jordan-Hölder filtration $E = E^0 \supset E^1 \supset \cdots \supset E^m = 0$ of (E_*, φ) such that (E_*', φ') is isomorphic to some $((E^i/E^j)_*, \varphi_{i,j})$ where $(E^i/E^j)_*$ has the induced structure defined by the parabolic structure of E_* and $\varphi_{i,j}$ is the parabolic homomorphism of $(E^i/E^j)_*$ to $(E^i/E^j)_* \otimes \Omega$ induced from φ. For such (E_*', φ'), we have par-$P_{E_*'} = P_{\mathscr{F}}$. Therefore there exists an integer M depending only on $\mathscr{F}$ such that $\deg E' \geq M$. By virtue of Corollary 1.2.1 of [12], it is easy to see that $\mathscr{G}$ is bounded. Hence, there is a finite set of families

$$\mathscr{F}_1 = \mathscr{F}_\Omega(H^1, H_*^1, \alpha_*^1), \dots, \mathscr{F}_N = \mathscr{F}_\Omega(H^N, H_*^N \alpha_*^N)$$

such that $\mathscr{G} \subset \bigcup_{i=1}^N \mathscr{F}_i$ and $\mathscr{G} \cap \mathscr{F}_i \neq 0$ for all i. Note that for all i, $P_{\mathscr{F}} = P_{\mathscr{F}_i}$. We may assume that $\mathscr{F} = \mathscr{F}_1$.

By Proposition 1.9 and the proof of Proposition 2.5 of [13], we have

Lemma 4.1. *For each non-negative integer e, there exists an integer m_e such that if $m \geq m_e$, then for all geometric points s of S and for all strictly parabolic e-semi-stable Ω-pairs (E_*, φ) on X_s which is contained in some $\mathscr{F}_i$, the conditions* (2.0.2), (2.0.3), (2.0.4) *and the following condition are satisfied;*

(4.1.1) *for all φ-invariant parabolic subsheaves E'_* of E_* with $E' \neq 0$,*

$$\int_0^1 h^0(E'_\alpha(m))\,\mathrm{d}\alpha \leq \mathrm{rk}\,(E') \cdot \text{par-}P_{E_*}(m)$$

and moreover, the equality holds if and only if

$$\text{par-}P_{E'_*}(m) = \text{par-}P_{E_*}(m) = P_{\mathscr{F}}(m).$$

We may assume that $m_e \geq m_{e'}$ if $e \geq e'$. Set $\mathscr{F}_i^e = \mathscr{F}_\Omega^e(H^i, H^i_*, \alpha^i_*)$. Let $V_{i,e}$ be a free Ξ-module of rank $H^i(m_e)$ and let R_i and P_i be the schemes constructed in §2 for $\mathscr{F}_i^e$ and $V_{i,e}$ instead of $\mathscr{F}_\Omega^e(H, H_*, \alpha_*)$ and V_m. On X_{R_i}, we have a flat family of parabolic sheaves $(\tilde{E}^i(m_e), \tilde{F}^i_*, \alpha^i_*)$, a universal parabolic homomorphism $\tilde{\varphi}^i\colon \tilde{E}^i(m_e)_* \to \tilde{E}^i(m_e)_* \otimes_X \Omega$ and surjections;

$$V_{i,e} \otimes_\Xi \mathcal{O}_{X_{R_i}} \xrightarrow{\phi^i} \tilde{E}^i(m_e) \xrightarrow{\phi^i_{li}} \tilde{E}^i_{l^i}(m_e) \xrightarrow{\phi^i_{li-1}} \cdots \xrightarrow{\phi^i_1} \tilde{E}^i_1(m_e),$$

where $\tilde{E}^i_j(m_e)$ is $\tilde{E}^i(m_e)/\tilde{F}^i_{j+1}(\tilde{E}^i(m_e))$.

Moreover, let $\tilde{Z}_i$ be a P_i-scheme such that $\tilde{Z}_i$ is a $\mathbf{P}^M$-bundle in étale topology and for a K-valued geometric point x of P_i, the fiber $(\tilde{Z}_i)_x$ over x is a Gieseker space

$$P_{S^*_r(\Omega^\vee)_{X_K}}(V_{i,e} \otimes K, r_{\mathscr{F}_i}, L_x)\ ^1$$

where L_x is an invertible sheaf corresponding to x. Then as in §2, we have a $\mathrm{GL}\,(V_{i,e})$-morphism $\tilde{\tau}_i\colon R_i \to \tilde{Z}_i$ and

$$\begin{array}{ccc} R_i & \xrightarrow{\Psi} & \tilde{Z}_i \times \prod_j G_{i,j} \\ & {\scriptstyle \nu_i} \searrow \quad \swarrow {\scriptstyle \pi_i} & \\ & P_i & \end{array}$$

where $G_{i,j}$ is a Grassmann scheme $\mathrm{Grass}\,(V_{i,e}, H^i_j(m_e))$.

By virtue of Proposition 1.11, for each integer e' with $0 \leq e' \leq e$, there exists an open subscheme $R_i^{ss}(e, e')$ (or, $R_i^s(e, e')$) of R_i such that a geometric point x of R_i is contained in $R_i^{ss}(e, e')$ (or, $R_i^s(e, e')$, resp.) if and only if the corresponding parabolic Ω-pair $((\tilde{E}^i(m_e), \tilde{F}^i_*, \alpha^i_*), \tilde{\varphi}^i) \otimes k(x)$ is strictly parabolic e'-semi-stable (or, parabolic e'-stable, resp.) and the homomorphism

[1] $\tilde{Z}_i$ is slightly different from $\tilde{Z}$ which is defined in §2, that is, if we define $\tilde{Z}_i$ as in §2, then the fibre $(\tilde{Z}_i)_x$ must be $P_{S^*_{r_{\mathscr{F}_i}}(\Omega^\vee)_{X_K}}(V_{i,e} \otimes_\Xi K, r_{\mathscr{F}_i}, L_x)$. But all arguments in §2 hold good for this modification because we have a relation $r \geq r_{\mathscr{F}_i}$.

(4.1.2) $$H^0(\phi^i \otimes k(x))\colon V_{i,e} \otimes k(x) \longrightarrow H^0(\tilde{E}^i(m_e) \otimes k(x))$$

is an isomorphism.

By virtue of Proposition 2.5 and Proposition 2.7, the $\mathrm{GL}(V_{i,e})$-morphism

$$\tilde{\Psi}_i\colon R_i^{ss}(e, e') \longrightarrow (\tilde{Z}_i \times \prod_j G_{i,j})^{ss}$$

is an immersion. Let $\tilde{R}_i^{ss}(e, e')$ be the scheme theoretic closed image of $R_i^{ss}(e, e')$ in $(\tilde{Z}_i \times \prod_{j=1}^{l^i} G_{i,j})^{ss}$.

Lemma 4.2. *For all k-valued geometric points y of P_i, every geometric point of $\tilde{R}_i^{ss}(e, e')_y$ is excellent in $(\tilde{Z}_i)_y = P_{S_r^*(\Omega^\vee)_{X_k}}(V_{i,e} \otimes_{\Xi} k, r_{\mathscr{F}_i}, L_y)$.*

Proof. Let $\tilde{Q}_i$ be a Quot-scheme $\mathrm{Quot}(V_{i,e} \otimes_{\Xi} S_r^*(\Omega^\vee), H^i[m_e])$ and let

$$\tilde{\phi}^i\colon V_{i,e} \otimes_{\Xi} S_r^*(\Omega^\vee)_{X_{\tilde{Q}_i}} \longrightarrow \tilde{\tilde{E}}^i(m_e)$$

be the universal quotient. The homomorphism

$$\wedge^{r_{\mathscr{F}_i}}(V_{i,e} \otimes_{\Xi} S_r^*(\Omega^\vee)_{X_{\tilde{Q}_i}}) \xrightarrow{\wedge^{r_{\mathscr{F}_i}}(\tilde{\phi}^i \otimes 1)} \wedge^{r_{\mathscr{F}_i}}(\tilde{\tilde{E}}^i(m_e)) \longrightarrow \det(\tilde{\tilde{E}}^i(m_e))$$

defines a morphism of $\tilde{Q}_i$ to $\tilde{Z}_i$. Let us denote its scheme theoretic image by $\bar{Q}_i$. By virtue of Lemma 3.7, every geometric point of $(\bar{Q}_i)_y$ is excellent in $(\tilde{Z}_i)_y$. Since the morphism $\tilde{\tau}_i\colon R_i \to \tilde{Z}_i$ is factored by $\tilde{Q}_i$, $\tilde{R}_i^{ss}(e, e')$ is a subscheme of $\bar{Q}_i \times \prod_j G_{i,j}$. Therefore, every geometric point of $\tilde{R}_i^{ss}(e, e')_y$ is excellent. □

Let s be a k-valued geometric point of S and let (E_*, φ), (E'_*, φ') and (E''_*, φ'') be parabolic Ω-pairs on X_s satisfying the conditions (2.0.2) and (2.0.3) with $N_0 = 0$. Assume that we have an exact sequence of parabolic pairs

(4.3.0) $$0 \longrightarrow (E'_*, \varphi') \xrightarrow{f} (E_*, \varphi) \xrightarrow{g} (E''_*, \varphi'') \longrightarrow 0.$$

Set $V = H^0(E)$, $r = \mathrm{rk}(E)$, $L = \det E$, $N_i = \dim_k H^0(E/F_{i+1}(E))$ and let $\alpha_1, \dots, \alpha_l$ be weights of E_*. For E'_* (or, E''_*), let us denote similarly those for E'_* (or, E''_*) by attaching $'$ (or, $''$, resp.), for example $V' = H^0(E')$. We have a natural surjections $\eta\colon V \otimes_k \mathcal{O}_{X_s} \to E$ and also have η' or η'' for E' or E'' respectively. Then η defines a k-valued point of $P_{S_r^*(\Omega^\vee)}(V, r, L)$ by

$$\wedge^r(V \otimes_k S_r^*(\Omega^\vee)) \xrightarrow{\wedge^r \tilde{\eta}} \wedge^r E \longrightarrow \det E = L.$$

where $\tilde{\eta} = \varphi^a \circ (\eta \otimes 1)$. Moreover, for each i, a natural surjection $\phi_i\colon V = H^0(E) \to H^0(E/F_{i+1}(E))$ defines a k-valued point of $G(V, N_i)$. Thus, η and $\phi_1, \dots, \phi_l$ defines a k-valued point x of $\Theta = \Theta_{S_r^*(\Omega^\vee)}(V, r, L, N_*, \alpha_*)$. Similarly, we get a k-valued point x' (or, x'') of $\Theta' = \Theta_{S_r^*(\Omega^\vee)}(V', r', L', N'_*, \alpha'_*)$ (or, $\Theta'' = \Theta_{S_r^*(\Omega^\vee)}(V'', r'', L'', N''_*, \alpha''_*)$, resp.).

Lemma 4.3. *Under the above situation, x is an extension of x'' by x' with the underlying exact sequence*

$$(4.3.1) \qquad 0 \longrightarrow V' \xrightarrow{H^0(f)} V \xrightarrow{H^0(g)} V'' \longrightarrow 0.$$

Moreover, x is a direct sum of x' and x'' under (4.3.1) *if and only if the sequence* (4.3.0) *splits as an exact sequence of parabolic pairs.*

Proof. Set $\tilde{\Omega} = S_r^*(\Omega^\vee)$. We have an exact commutative diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & V' \otimes_k \tilde{\Omega} & \xrightarrow{H^0(f)\otimes 1} & V \otimes_k \tilde{\Omega} & \xrightarrow{H^0(g)\otimes 1} & V'' \otimes_k \tilde{\Omega} & \longrightarrow 0 \\ & \downarrow \tilde{\eta}' & & \downarrow \tilde{\eta} & & \downarrow \tilde{\eta}'' & \\ 0 \longrightarrow & E' & \xrightarrow{f} & E & \xrightarrow{g} & E'' & \longrightarrow 0. \end{array}$$

The condition (3.1.1) is clear. To prove the commutativity of the diagram in (3.1.2.1), we may assume that $\tilde{\Omega}$, E, E' and E'' are free and $E \simeq E' \oplus E''$ because it is enough to prove that on a open set U of X_s with $\operatorname{codim}(X_s, U) \leq 2$ and moreover, the question is local on X_s. For $a_1, \ldots, a_{r'} \in V' \otimes_k \tilde{\Omega}$ and $b_1, \ldots, b_{r''} \in V \otimes_k \tilde{\Omega}$, we have that

$$\begin{aligned} &\phi(T_{x'}(a_1, \ldots, a_{r'}) \otimes T_{x''}(g'(b_1), \ldots, g'(b_r))) \\ &= \phi((\tilde{\eta}'(a_1) \wedge \cdots \wedge \tilde{\eta}'(a_{r'})) \otimes (\tilde{\eta}''(g'(b_1)) \wedge \cdots \wedge \tilde{\eta}''(g'(b_{r''})))) \\ &= \phi((\tilde{\eta}'(a_1) \wedge \cdots \wedge \tilde{\eta}'(a_{r'})) \otimes (g(\tilde{\eta}(b_1)) \wedge \cdots \wedge g(\tilde{\eta}(b_{r''})))) \\ &= f(\tilde{\eta}'(a_1)) \wedge \cdots \wedge f(\tilde{\eta}'(a_{r'})) \wedge \tilde{\eta}(b_1) \wedge \cdots \wedge \tilde{\eta}(b_{r''}) \\ &= \tilde{\eta}(f'(a_1)) \wedge \cdots \wedge \tilde{\eta}(f'(a_{r'})) \wedge \tilde{\eta}(b_1) \wedge \cdots \wedge \tilde{\eta}(b_{r''}) \\ &= T_x(f'(a_1), \ldots, f'(a_{r'}), b_1, \ldots, b_{r''}) \end{aligned}$$

where $f' = H^0(f)$, $g' = H^0(g) \otimes 1$ and ϕ is a natural isomorphism of $L' \otimes L''$ to L. Thus, the condition (3.1.2) is proved. For each $0 \leq \alpha \leq 1$, $V_x^\alpha = \ker(\phi^{i-1})$ and $E_\alpha = F_i(E)$ when $\alpha_{i-1} < \alpha \leq \alpha_i$. By virtue of (2.0.3), $\ker(\phi^{i-1}) = H^0(F_i(E))$. Hence, we have that $H^0(E_\alpha) = V_x^\alpha$ for each $0 \leq \alpha \leq 1$. Therefore, the condition (3.1.2.2) and (3.1.3) hold by the exactness of (4.3.0).

To prove the second assertion, assume that there exists a k-linear map $i: V'' \to V$ satisfying the conditions in Definition 3.3. Let E''' be the image of $\tilde{\eta} \circ (i \otimes 1): V'' \otimes_k S^*(\Omega^\vee) \to E$. Then E''' is φ-invariant. Since i is a section of g', $g(E''') = E''$. Hence, $E' + E''' = E$. If $E' \cap E'''$ is not zero, then $r''' = \operatorname{rk}(E''') > \operatorname{rk}(E'') = r''$. Take local sections $a_1, \ldots, a_{r'''}$ of E''' and $a_{r'''+1}, \ldots, a_r$ of E' so that $a_1, \ldots, a_r$ are linearly independent over $k(\xi)$ where ξ is the generic point of X_s. Moreover, take local sections $b_1, \ldots, b_{r'''}$ of $i(V'') \otimes_k S^*(\Omega^\vee)$ and $b_{r'''+1}, \ldots, b_r$ of $f(V') \otimes_k S^*(\Omega^\vee)$ such that $\tilde{\eta}(b_j) = a_j$ for all j. Then $T_x(b_1, \ldots, b_r) \neq 0$ which contradicts the condition $T_x|_{[i(V''); r''', V; r - r''']} = 0$ in Definition 3.3. Hence, $E''' \cap E'$ is zero and so $g|_{E'''}: E''' \to E''$ is the isomorphism. Applying this argument to E_α, we know that the image of $\tilde{\eta} \circ (i \otimes 1): V_x''^\alpha \otimes_k S^*(\Omega^\vee) \to E$ is a φ-invariant submodule E_α''' of E''' such that $g|_{E_\alpha'''}: E_\alpha''' \to E_\alpha''$ is an isomorphism. Thus $g|_{E'''}^{-1}$ is the desired section. Conversely, if (4.3.0) splits as an exact sequence of parabolic pairs. If i is the given section, then $H^0(i)$ is clearly the section of $H^0(g)$ and

$H^0(i)(V_{x''}^{\prime\prime\alpha}) \subset V_x^{\alpha}$. Since $\eta''(H^0(i)(V'')\otimes_k \mathcal{O}_{X_s}) \simeq E''$ and its rank is r'', $T_x|_{[i(V'');s,V;r-s]} = 0$ for $s > r''$. □

By virtue of Proposition 3.11, we obtain the following results on $\mathrm{GL}(V_{1,e})$-orbits of $R_1^{ss}(e, e')$.

Proposition 4.4. *Let s be a k-valued geometric point of S and let $((\bar{E}_1)_*, \varphi_1), \dots, ((\bar{E}_t)_*, \varphi_t)$ be parabolic e'-stable Ω-pairs on X_s such that $\bigoplus_i((\bar{E}_i)_*, \varphi_i)$ is in $\mathscr{F} = \mathscr{F}_1$. Let y be a k-valued point of P_1 corresponding to an invertible sheaf $\bigotimes_{i=1}^t \det(\bar{E}_i(m_e))$. Then there exists a $\mathrm{GL}(V_{1,e})$-invariant closed subset $Z(((\bar{E}_1)_*, \varphi_1), \dots, ((\bar{E}_t)_*, \varphi_t))$ of $(R_1^{ss}(e, e'))_y = (\nu_1)^{-1}(y) \cap R_1^{ss}(e, e')$ such that*

(4.4.1) $\tilde{\Psi}_1(Z(((\bar{E}_1)_*, \varphi_1), \dots, ((\bar{E}_t)_*, \varphi_t)))$ *is closed in* $(\tilde{Z}_1 \times \prod_i G_{1,i})_y^{ss}$,

(4.4.2) *for every algebraically closed field K containing k,*

$Z(((\bar{E}_1)_*, \varphi_1), \dots, ((\bar{E}_t)_*, \varphi_t))(K) =$

$\{x \in (R_1^{ss}(e, e'))_y(K) \mid \mathrm{gr}((\tilde{E}^1(m_e)_*, \tilde{\varphi}^1)_x) \simeq \bigoplus_i(\bar{E}_i(m_e)_*, \varphi_i(m_e))_K\}$,

(4.4.3) *the* $\mathrm{GL}(V_{1,e})$*-orbit of x_0 corresponding to $\bigoplus_i(\bar{E}_i(m_e)_*, \varphi_i(m_e))$ is the unique closed orbit in $Z(((\bar{E}_1)_*, \varphi_1), \dots, ((\bar{E}_t)_*, \varphi_t))$.*

Proof. Assume that $((\bar{E}_i)_*, \varphi_i)$ (or, $\bigoplus_{j=1}^i((\bar{E}_j)_*, \varphi_j)$) is in $\mathscr{F}_{\zeta_i}$ (or, $\mathscr{F}_{\iota_i}$, resp.) and let y_i' (or, y_i, resp.) be the k-valued geometric point of P_{ζ_i} (or, P_{ι_i}, resp.) corresponding to $L_i' = \det(\bar{E}_i(m_e))$ (or, $L_i = \bigotimes_{j=1}^i \det(\bar{E}_j(m_e))$, resp.). Then we have a natural isomorphism $\phi_i\colon L_{i-1} \otimes L_i' \to L_i$. Note that $\mathscr{F}_1 = \mathscr{F} = \mathscr{F}_{\iota_t}$, hence, $\iota_t = 1$. By virtue of Lemma 4.1, $\bar{E}_i(m_e)$ is generated by its global sections for each i. Hence, we have a surjection

$$\bar{\eta}_i\colon V_{\zeta_i,e} \otimes_{\Xi} \mathcal{O}_{X_s} \longrightarrow \bar{E}_i(m_e).$$

Let x_i' be a k-valued point of $R_{\zeta_i}^{ss}(e, e')_{y_i'}$ corresponding to $(\bar{\eta}_i, (\bar{E}_i(m_e)_*, \varphi_i(m_e)))$. Set

$$z_i' = \tilde{\Psi}_{\zeta_i}(x_i') \in (\tilde{Z}_{\zeta_i} \times \prod_j G_{\zeta_i,j})_{y_i'}.$$

By virtue of Lemma 4.2, applying Proposition 3.11 to the case where $D_i = \tilde{R}_{\iota_i}^{ss}(e, e')_{y_i}$ and $S_i = z_i'$, we obtain a $\mathrm{GL}(V_{1,e})$-invariant closed set $Z(z_1', \dots, z_t')$ of $R' = \tilde{R}_1^{ss}(e, e')_y$ which satisfies the condition $(*)_t$ in Proposition 3.11. For a permutation δ of $\{1, \dots, t\}$, a $\mathrm{GL}(V_{1,e})$-invariant closed set $Z(z_{\delta(1)}', \dots, z_{\delta(t)}')$ is similarly defined. Set $Z' = \bigcup_{\delta \in \mathscr{S}_t} Z(z_{\delta(1)}', \dots, z_{\delta(t)}')$ where $\mathscr{S}_t$ is the permutation group of $\{1, \dots, t\}$. Then Z' is the $\mathrm{GL}(V_{1,e})$-invariant closed set of R' (hence, of $(\tilde{Z}_1 \times \prod_i G_{1,i})_y^{ss}$). We claim that Z' is closed subset of $R'' = \tilde{\Psi}_1((R_1^{ss}(e, e'))_y)$. Since $C = R' - R''$ is a $\mathrm{GL}(V_{1,e})$-invariant closed set, if $Z' \cap C$ is not empty, it contains the unique closed orbit $o(z_1', \dots, z_t')$ of $z_1' \oplus \cdots \oplus z_t'$ in Z'. By virtue of Lemma 4.3, $o(z_1', \dots, z_t') = \tilde{\Psi}_1(o(\bar{\eta}, \bigoplus_i(\bar{E}_i(m_e)_*, \varphi(m_e)))$ where $\bar{\eta}$ is a natural surjection

$$V_{1,e}\otimes_{\Xi}\mathcal{O}_{X_x}\simeq\bigoplus_i V_{\zeta_i,e}\otimes_{\Xi}\mathcal{O}_{X_s}\longrightarrow\bigoplus_i\bar{E}_i(m_e).$$

Hence, $Z'\cap C$ is empty. Set $Z=Z(((\bar{E}_1)_*,\varphi_1),\dots,((\bar{E}_t)_*,\varphi_t))=\tilde{\Psi}_1^{-1}(Z')$. Let us prove that Z is the desired set. Since $\tilde{\Psi}_1(Z)=Z'$, (4.4.1) and (4.4.3) are already proved.

Finally, let us prove (4.4.2). Let x be a K-valued point of $(R_1^{ss}(e,e'))_y$ such that $\mathrm{gr}((\tilde{E}^1(m_e)_*,\tilde{\varphi}^1)_x)\simeq\bigoplus_i(\bar{E}_i(m_e)_*,\varphi_i(m_e))_K$. Then we can find a Jordan-Hölder filtration of $(E_*,\varphi)=(\tilde{E}^1(m_e)_*,\tilde{\varphi}^1)_x$

$$0=J_0(E)\subset J_1(E)\subset\cdots\subset J_t(E)=E$$

where $J_j(E)$ are φ-invariant subsheaves of E. Set $\bar{J}_j(E)=J_j(E)/J_{j-1}(E)$. Then $(J_j(E)_*,\varphi_j)$ and $(\bar{J}(E)_*,\bar{\varphi}_j)$ are strictly parabolic e'-semi-stable (see Lemma 3.5 of [10] which can be easily extended to our case) with respect to the induced structures where φ_j and $\bar{\varphi}_j$ are the canonical induced parabolic homomorphisms. By our assumption, there is a permutation δ of $\{1,\dots,t\}$ such that $(\bar{J}_j(E)_*,\bar{\varphi}_j)\simeq(\bar{E}_{\delta(j)}(m_e)_*,\varphi_{\delta(j)}(m_e))_K$. Now by virtue of Lemma 4.3, we conclude that $\tilde{\Psi}_1(x)$ is in $Z(z'_{\delta(1)},\dots,z'_{\delta(t)})(K)\subset Z'(K)$. Hence, x is in $Z(K)$.

Conversely, assume that x is in $Z(K)$. Take a Jordan-Hölder filtration as above. Then $(\bar{J}_i(E)(-m_e)_*,\bar{\varphi}_i(-m_e))$ is a member of some $\mathscr{F}_{\lambda_i}$. Hence, we obtain a K-valued point w'_i of $(\tilde{Z}_{\lambda_i}\times\prod_j G_{\lambda_i,j})_{u'_i}$ where u'_i is a K-valued point of P_{λ_i} corresponding to $\det(\bar{J}_i(E))$ as z'_i is obtained from $((\bar{E}_i)_*,\varphi_i)$. Moreover, we know that $\tilde{\Psi}_1(x)$ is in $Z(w'_1,\dots,w'_t)$. On the other hand, $\tilde{\Psi}_1(x)$ is in $Z'(K)$. Since $Z(w'_1,\dots,w'_t)$ and $Z'(K)$ are $\mathrm{GL}(V_{1,e})$-invariant closed subsets of $R'=\tilde{R}_1^{ss}(e,e')_y$, $Z(w'_1,\dots,w'_t)\cap Z'(K)$ contains a closed orbit. By the uniqueness of the closed orbit in $Z(w'_1,\dots,w'_t)$ or $Z'(K)$, we conclude that $o(z'_1,\dots,z'_t)=o(w'_1,\dots,w'_t)$. Therefore, $\bigoplus_i(\bar{E}_i(m_e)_*,\varphi(m_e))$ and $\bigoplus_i(\bar{J}_i(E)_*,\bar{\varphi}_i)$ are in the same orbit, equivalently $\bigoplus_i(\bar{E}_i(m_e)_*,\varphi(m_e))\simeq\bigoplus_i(\bar{J}_i(E)_*,\bar{\varphi}_i)$. □

By virtue of Theorem 4 of [20], there exists a good quotient

$$\xi\colon\tilde{R}_1^{ss}(e,e')\longrightarrow Y$$

and Y is projective over S. Set

$$\bar{M}^{H_*,\alpha_*,e,e'}_{\Omega/D/X/S}=Y-\xi(\tilde{R}_1^{ss}(e,e')-R_1^{ss}(e,e')).$$

Then $\bar{M}^{H_*,\alpha_*,e,e'}_{\Omega/D/X/S}$ is quasi-projective over S. Moreover, it contains $M^{H_*,\alpha_*,e,e'}_{\Omega/D/X/S}=\xi(R_1^s(e,e'))$ as an open subscheme.

Proposition 4.5. *$\bar{M}^{H_*,\alpha_*,e,e'}_{\Omega/D/X/S}$ has the following properties*:

(4.5.1) *For each geometric point s of S, there exists a natural bijection*

$$\bar{\theta}_s\colon\text{par-}\bar{\Sigma}^{H_*,\alpha_*,e'}_{\Omega/D/X/S}(\mathrm{Spec}(k(s)))\longrightarrow\bar{M}^{H_*,\alpha_*,e,e'}_{\Omega/D/X/S}(k(s)).$$

(4.5.2) *For $T\in(Sch/S)$ and a flat family of strictly parabolic e'-semi-stable Ω-pairs (E_*,φ) on X_T/T, there exists a morphism $\bar{f}^{e,e'}_{(E_*,\varphi)}$ of T to $\bar{M}^{H_*,\alpha_*,e,e'}_{\Omega/D/X/S}$ such that*

$$\bar{f}^{e,e'}_{(E_*,\varphi)}(t) = \bar{\theta}_s([(E_* \otimes_T k(t), \varphi \otimes_T k(t))])$$

for all points t in $T(k(s))$. *Moreover, for a morphism* $g: T' \to T$ *in* (Sch/S),

$$\bar{f}^{e,e'}_{(E_*,\varphi)} \circ g = \bar{f}^{e,e'}_{(1_X \times g)^*(E_*,\varphi)}.$$

(4.5.3) *If* $\bar{M}' \in (Sch/S)$ *and maps* $\bar{\theta}'_s$: par-$\bar{\Sigma}^{H_*,\alpha_*,e,e'}_{\Omega/D/X/S}(\mathrm{Spec}(k(s))) \to \bar{M}'(k(s))$ *have the above property* (4.5.2), *then there exists a unique S-morphism* $\bar{\Psi}$ *of* $\bar{M}^{H_*,\alpha_*,e,e'}_{\Omega/D/X/S}$ *to* $\bar{M}'$ *such that* $\bar{\Psi}(k(s)) \circ \bar{\theta}_s = \bar{\theta}'_s$ *and* $\bar{\Psi} \circ \bar{f}^{e,e'}_{(E_*,\varphi)} = \bar{f}'_{(E_*,\varphi)}$ *for all geometric points s of S and for all* (E_*, φ), *where* $\bar{f}'_{(E_*,\varphi)}$ *is the morphism given by the property* (4.5.2), *for* $\bar{M}'$ *and* $\bar{\theta}'_s$.

Proof. For two K-valued geometric points x_1 and x_2 of $\tilde{R}^{ss}_1(e, e')$, $\xi(x_1) = \xi(x_2)$ if and only if $\overline{o(x_1)} \cap \overline{o(x_2)}$ is not empty. Let K be an algebraically closed field. For K-valued point x of $(R^{ss}_1(e, e'))_y$, set $\mathrm{gr}(x) = \mathrm{gr}((\tilde{E}^1(m_e)_*, \tilde{\varphi}^1)_x)$ If $\mathrm{gr}(x) \simeq \bigoplus_i (\bar{E}_i(m_e)_*, \varphi_i(m_e))$, by Proposition 4.4, x is contained in $\mathrm{GL}(V_{1,e})$-invariant closed subset $Z(x) = Z(((\bar{E}_1)_*, \varphi_1), \dots, ((\bar{E}_t)_*, \varphi_t))$ of $(R^{ss}_1(e, e'))_y$ satisfying conditions (4.4.1), (4.4.2) and (4.4.3). By (4.4.2), x is in $Z(x)$. By (4.4.1) and (4.4.3), we conclude that for x and x' in $(R^{ss}_1(e, e'))_y(K)$, $\xi(x) = \xi(x')$ if and only if $\mathrm{gr}(x) \simeq \mathrm{gr}(x')$. Moreover, if $x \in (R^{ss}_1(e, e'))_y(K)$ and $x' \in \tilde{R}^{ss}_1(e, e') - R^{ss}_1(e, e')$, since $\tilde{R}^{ss}_1(e, e') - R^{ss}_1(e, e')$ and $Z(x)$ are closed in $\tilde{R}^{ss}_1(e, e')$, $\xi(x) \neq \xi(x')$. Thus (4.5.1) is proved. The construction of the morphisms in (4.5.2) is completely same as that of (2.8.2). Finally, the morphism of (4.5.3) is similarly constructed as in the proof of (2.8.3) by the isomorphism $\sigma^*((\tilde{E}(m)_*, \tilde{\varphi})) \simeq p_2^*((\tilde{E}(m)_*, \tilde{\varphi}))$ and the fact that $\bar{M}^{H_*,\alpha_*,e,e'}_{\Omega/D/X/S}$ is the geometric quotient. □

The construction of a moduli scheme of the functor par-$\bar{\Sigma}^{H_*,\alpha_*}_{\Omega/D/X/S}$ is completely same as in §4 of [10], that is, $\bar{M}^{H_*,\alpha_*}_{\Omega/D/X/S} = \varinjlim_e \bar{M}^{H_*,\alpha_*,e,e}_{\Omega/D/X/S}$.

Theorem 4.6. *In the situation of* (2.0.1), *there exists an S-scheme* $\bar{M}^{H_*,\alpha_*}_{\Omega/D/X/S}$ *with the following properties*:

(4.6.1) $\bar{M}^{H_*,\alpha_*}_{\Omega/D/X/S}$ *is locally of finite type and separated over S.*

(4.6.2) *There exists a coarse moduli scheme* $M^{H_*,\alpha_*}_{\Omega/D/X/S}$ *of* par-$\Sigma^{H_*,\alpha_*}_{\Omega/D/X/S}$ *and it is contained in* $\bar{M}^{H_*,\alpha_*}_{\Omega/D/X/S}$ *as an open subscheme.*

(4.6.3) *For each geometric point s of S, there exists a natural bijection*

$$\bar{\theta}_s: \text{par-}\bar{\Sigma}^{H_*,\alpha_*}_{\Omega/D/X/S}(\mathrm{Spec}(k(s))) \longrightarrow \bar{M}^{H_*,\alpha_*}_{\Omega/D/X/S}(k(s)).$$

(4.6.4) *For* $T \in (Sch/S)$ *and a flat family of parabolic semi-stable pairs* (E_*, φ) *on* X_T/T, *there exists a morphism* $\bar{f}_{(E_*,\varphi)}$ *of T to* $\bar{M}^{H_*,\alpha_*}_{\Omega/D/X/S}$ *such that for all points t in* $T(k(s))$, $\bar{f}_{(E_*,\varphi)}(t) = \bar{\theta}_s([(E_* \otimes_T k(t), \varphi \otimes_T k(t))])$. *Moreover, for a morphism* $g: T' \to T$ *in* (Sch/S),

$$\bar{f}_{(E_*,\varphi)} \circ g = \bar{f}_{(1_X \times g)^*_{(E_*,\varphi)}}.$$

(4.6.5) *If* $\bar{M}' \in (Sch/S)$ *and maps* $\bar{\theta}'_s$: par-$\bar{\Sigma}^{H_*,\alpha_*}_{\Omega/D/X/S}(\mathrm{Spec}(k(s))) \to \bar{M}'(k(s))$ *have* *the above property* (4.6.4), *then there exists a unique S-morphism* $\bar{\Psi}$ *of* $\bar{M}^{H_*,\alpha_*}_{\Omega/D/X/S}$ *to* $\bar{M}'$ *such that* $\bar{\Psi}(k(s)) \circ \bar{\theta}_s = \bar{\theta}'_s$ *and* $\bar{\Psi} \circ \bar{f}_{(E_*,\varphi)} = \bar{f}'_{(E_*,\varphi)}$ *for all geometric points s of S and for all* (E_*, φ), *where* $\bar{f}'_{(E_*,\varphi)}$ *is the morphism given by the property* (4.6.4) *for* $\bar{M}'$ *and* $\bar{\theta}'_s$.

Proof. The proof of (4.6.1) is completely same as Theorem 2.9. (4.6.2) is already proved. (4.6.3) is clear, because we have that

$$\text{par-}\bar{\Sigma}^{H_*,\alpha_*}_{\Omega/D/X/S}(\operatorname{Spec}(k(s))) = \bigcup_e \text{par-}\bar{\Sigma}^{H_*,\alpha_*,e}_{\Omega/D/X/S}(\operatorname{Spec}(k(s))).$$

Moreover, (4.6.4) and (4.6.5) are easy by Proposition 1.11 and the proof of Theorem 2.9. □

If $\mathcal{F}^{H_*,\alpha_*}_{\Omega/D/X/S}$ is bounded, then there exists an integer e such that $\mathcal{F}^{H_*,\alpha_*}_{\Omega/D/X/S} = \mathcal{F}^{H_*,\alpha_*,e}_{\Omega/D/X/S}$. Hence, we have

Corollary 4.7. *If* $\mathcal{F}^{H_*,\alpha_*}_{\Omega/D/X/S}$ *is bounded, then* $\bar{M}^{H_*,\alpha_*}_{\Omega/D/X/S}$ *is quasi-projective over S.*

5. Compactness of moduli spaces

In this sction, we shall prove some compactness theorems. As in the case of semi-stable Higgs bundles, we shall construct a morphism from the moduli scheme of parabolic semi-stable pairs to an affine space of characteristic polynomials and prove the properness of the map along the method of S. G. Langton [8].

First of all, let us generalize the notion of (μ-)semi-stability of parabolic pairs. Let k be a field. For a parabolic sheaf E_* on a fibre X_k, set

$$\text{par-}P_{E_*} = \frac{d \cdot m^n}{n!} + \sum_{i=1}^{n} \mu_i(E_*) m^{n-i}.$$

where d is the degree of X_k and n is the dimension of X_k. Let us introduce the lexicographic order into $\mathbf{R} \times \cdots \times \mathbf{R}$, i.e. $(\mu_1, \ldots, \mu_r) < (\mu'_1, \ldots, \mu'_r)$ if and only if $\mu_j < \mu'_j$ for $j = \min\{i | \mu_i \neq \mu'_i\}$. In § 1, ($\mu$-)semi-stability was defined only on parabolic pairs on schemes over algebraically closed fields, but we need them over arbitrary fields.

Definition 5.1. Let k be a field over S and let (E_*, φ) be a parabolic Ω-pair on a fibre X_k. (E_*, φ) is said to be semi-stable of level i, if for all $\varphi_{\bar{k}}$-invariant coherent subsheaves F of $E_{\bar{k}}$ with $0 \neq F \neq E_{\bar{k}}$ and with torsion free quotient $E_{\bar{k}}/F$, we have

$$(\mu_1(F_*), \ldots, \mu_i(F_*)) \leq (\mu_1(E_*), \ldots, \mu_i(E_*))$$

where $\bar{k}$ is the algebraic closure of k and F_* has the induced structure, i.e. $F_\alpha = F \cap (E_\alpha)_{\bar{k}}$.

Remark 5.2. Semi-stability of level 1 is equivalent to μ-semi-stability and semi-stability of level $n = \dim X_k$ is equivalent to semi-stability. Clearly, for each i, semi-stability of level i implies that of level $i-1$.

Definition 5.3. Let k be a field over S and let (E_*, φ) be a parabolic Ω-pair on a fibre X_k. A filtration of (E_*, φ)

$$0 \subset (E_*^1, \varphi^1) \subset \cdots \subset (E_*^l, \varphi^l) = (E_*, \varphi)$$

is said to be a Harder-Narasimhan filtration of level i, if for each j, the following conditions are satisfied;

(5.3.1) (E_*^j, φ^j) has the induced structure.

(5.3.2) $\bar{E}^j = E^j/E^{j-1}$ is torsion free.

(5.3.3) $((\bar{E}^j)_*, \bar{\varphi}^j)$ with the induced structure is semi-stable of level i.

(5.3.4) $(\mu_1(\bar{E}_*^j), \dots, \mu_i(\bar{E}_*^j)) > (\mu_1(\bar{E}_*^{j+1}), \dots, \mu_i(\bar{E}_*^{j+1}))$.

Harder-Narasimhan filtrations of level 1 (or, of level n) are sometimes called μ-Harder-Narasimhan filtrations (or, Harder-Narasimhan filtrations, resp.).

Proposition 5.4. *Let k be a field over S. Every parabolic Ω-pair (E_*, φ) on a fibre X_k has a unique Harder-Narasimhan filtration of level i.*

Proof. First of all, let us prove the proposition by induction on the rank of E under the situation that the base field k is algebraically closed. If $\mathrm{rk}(E) = 0$, there is nothing to prove. Assume the assertion holds for all parabolic pairs of rank $< \mathrm{rk}(E)$.

Let $\mathscr{F}$ be a set of all φ-invariant coherent subsheaves F of E such that E/F is torsion free and that

$$(\mu_1(F_*), \dots, \mu_i(F_*)) \geq (\mu_1(E_*), \dots, \mu_i(E_*)),$$

where F_* has the induced structure. Note that by Riemann-Roch theorem,

$$\mu_1(E_*) = \frac{1}{(n-1)!}\left\{\left(\mu(E) + \frac{c_1(X)}{2}\right) + \frac{\mathrm{wt}(E_*)}{\mathrm{rk}(E)}\right\}.$$

Moreover, we have inequalities $0 \leq \mathrm{wt}(E_*) \leq \mathrm{rk}(E) \deg D$. Hence, the set of degrees of members of $\mathscr{F}$ is bounded below. By virtue of Corollary 1.2.1 of [12], $\mathscr{F}$ is bounded and hence, the set of polynomials

$$\{\text{par-}P_{F_*}(m) \mid F \in \mathscr{F} \text{ and } F_* \text{ has the induced structure}\}$$

is a finite set. Thus, there exists a member F in $\mathscr{F}$ such that $(\mu_1(F_*), \dots, \mu_i(F_*))$ is maximal among all members of $\mathscr{F}$ (with respect to induced structures). Let us take such a member F in $\mathscr{F}$ whose rank is maximal among all such members. Then (F_*, φ) with induced structure is semi-stable of level i. By our induction hypothesis, $((E/F)_*, \bar{\varphi})$ with induced structure has a unique Harder-Narasimhan filtration of level i;

$$0 \subset ((E^1/F)_*, \bar{\varphi}) \subset \cdots \subset ((E^t/F)_*, \bar{\varphi}) = ((E/F)_*, \bar{\varphi}).$$

We claim that the filtration

$$0 \subset (F_*, \varphi) \subset (E^1_*, \varphi) \subset \cdots \subset (E^t_*, \varphi) = (E_*, \varphi)$$

gives a Harder-Narasimhan filtration of level i where each (E^j_*, φ) has the induced structure. We may assume that $E/F \neq 0$. It is enough to prove that

$$(\mu_1(F_*), \ldots, \mu_i(F_*)) > (\mu_1((E^1/F)_*), \ldots, \mu_i((E^1/F)_*)).$$

We have an equality

$$\text{par-}\chi(F_*(m)) + \text{par-}\chi((E^1/F)_*(m)) = \text{par-}\chi(E^1_*(m)).$$

Hence, we get

$$\text{rk}(F)(\text{par-}P_{F_*}(m) - \text{par-}P_{E^1_*}(m)) = \text{rk}(E^1/F)(\text{par-}P_{E^1_*}(m) - \text{par-}P_{E^1/F_*}(m)).$$

Therefore, by the choice of F,

$$(\mu_1(F_*), \ldots, \mu_i(F_*)) > (\mu_1(E^1_*), \ldots, \mu_i(E^1_*)) > (\mu_1((E^1/F)_*), \ldots, \mu_i((E^1/F)_*)).$$

Now we shall prove the uniqueness. The following lemma is easily proved, so we omit its proof.

Lemma 5.5. *Let (E_*, φ) and (E'_*, φ') be parabolic pairs which are semi-stable of level i. If there exists a non-zero homomorphism of parabolic pairs of (E_*, φ) to (E'_*, φ'), then the following inequality holds*

$$(\mu_1(E_*), \ldots, \mu_i(E_*)) \leq (\mu_1(E'_*), \ldots, \mu_i(E'_*)).$$

Let $0 \subset (E'^1_*, \varphi) \subset \cdots \subset (E'^t_*, \varphi) = (E_*, \varphi)$ be another Harder-Narasimhan filtration of level i. If $F \subseteq E'^j$ and $F \nsubseteq E'^{j-1}$ $(E'^0 = 0)$, then we have a natural non-zero homomorphism of parabolic pairs

$$(F_*, \varphi) \longrightarrow ((E'^j/E'^{j-1})_*, \bar{\varphi}).$$

Hence, by Lemma 5.5, we obtain an inequality

$$(\mu_1(F_*), \ldots, \mu_i(F_*)) \leq (\mu_1((E'^j/E'^{j-1})_*), \ldots, \mu_i((E'^j/E'^{j-1})_*)).$$

If $j \geq 2$, then the right-hand side is less than $(\mu_1(E'^1_*), \ldots, \mu_i(E'^1_*))$. It contradicts the choice of F. Hence, $j = 1$. Then by the above inequality and the choice of F, we conclude that $F = E'^1$. This and our induction hypothesis imply the assertion.

Now let us prove our proposition over arbitrary fields k. Let $\bar{k}$ be the algebraic closure of k. By induction on the rank of E, it is enough to prove that the first term of the Harder-Narasimhan filtration of level i of $(E_*, \varphi)_{\bar{k}}$ is defined over the base field k. Set $Q = \text{Quot}(E/X_k/k)$. The first term of the Harder-Narasimhan filtration of level i of $(E_*, \varphi)_{\bar{k}}$ corresponds to a $\bar{k}$-valued point of Q. Let x be the scheme point of Q defined by the $\bar{k}$-valued point. We claim that the residue field K of the local ring $\mathcal{O}_{Q,x}$ must be k. Note that the extension of fields K/k is a finite extension. If K/k is not purely inseparable, then there exist at least two embeddings of fields of K to $\bar{k}$ over k. It contradicts the uniqueness of the Harder-Narasimhan filtration of level i. Hence, K/k is a purely inseparable field extension. Set $B = K \otimes_k \bar{k}$. B is an artinian local ring with residue field $\bar{k}$. We have an exact sequence of $\mathcal{O}_{X_K}$-modules

$$0 \longrightarrow F \xrightarrow{f} E_K \xrightarrow{g} G \longrightarrow 0$$

such that $F_{\bar{k}}$ gives the first term of the Harder-Narasimhan filtration of level i of $(E_{\bar{k}*}, \varphi_{\bar{k}})$. Note that F and G are φ_K-invariant and torsion free because these hold over $\bar{k}$. Hence, setting $F_\alpha = (E_\alpha)_K \cap F$, (F_*, φ_K) becomes a parabolic pair. Let us consider the following homomorphism

$$\xi\colon F \otimes_K B \xrightarrow{f_B} E_B \xrightarrow{g_{\bar{k}} \otimes 1_B} G \otimes_K \bar{k} \otimes_{\bar{k}} B.$$

Let $\mathbf{m}$ be the maximal ideal of B. If ξ is not zero, then for some i, we have $\xi(F \otimes_K B) \subseteq G \otimes_K \bar{k} \otimes_{\bar{k}} \mathbf{m}^i$ and $\xi(F \otimes_K B) \nsubseteq G \otimes_K \bar{k} \otimes_{\bar{k}} \mathbf{m}^{i+1}$. Hence, we get a non-zero homomorphism

$$\bar{\xi}\colon F_{\bar{k}} \simeq F \otimes_K B \otimes_B B/\mathbf{m} \longrightarrow G \otimes_K \bar{k} \otimes_{\bar{k}} \mathbf{m}^i/\mathbf{m}^{i+1} \simeq G_{\bar{k}} \otimes_{\bar{k}} \mathbf{m}^i/\mathbf{m}^{i+1}.$$

There is an element δ of $\mathrm{Hom}_{\bar{k}}(\mathbf{m}^i/\mathbf{m}^{i+1}, \bar{k})$ such that $(1_{G_{\bar{k}}} \otimes \delta) \circ \bar{\xi}$ is not zero. Since $\xi(F_\alpha \otimes_K B) \subseteq G_\alpha \otimes_K \bar{k} \otimes_{\bar{k}} \mathbf{m}^i$, we obtain a non-zero homomorphism of parabolic pairs

$$(1_{G_{\bar{k}}} \otimes \delta) \circ \bar{\xi}\colon (F_*)_{\bar{k}} \longrightarrow (G_*)_{\bar{k}}.$$

By the same argument used in the above proof of the uniqueness of Harder-Narasimhan filtrations, such a non-zero homomorphism does not exist. Hence, we conclude that ξ must be zero. It implies that two quotients

$$E_B \xrightarrow{g_B} G_B \text{ and } E_B \xrightarrow{g_{\bar{k}} \otimes 1_B} G \otimes_K \bar{k} \otimes_{\bar{k}} B$$

are same quotients. Hence, corresponding B-valued homomorphisms of Q are also the same one, i.e.

$$(K \ni a \longrightarrow a \otimes 1 \in B) = (K \ni a \longrightarrow 1 \otimes a \in B).$$

Therefore, $K = k$ and F is defined over k. □

Corollary 5.6. *(E_*, φ) is semi-stable of level i if and only if for all φ-invariant coherent subsheaves F of E with $0 \neq F \neq E$, the following inequality holds*

$$(\mu_1(F_*), \ldots, \mu_i(F_*)) \leq (\mu_1(E_*), \ldots, \mu_i(E_*)),$$

where F_* has the induced structure.

From now on, let R be a discrete valuation ring over S and let K (k or π)

be the quotient field (residue field or uniformizing parameter, resp.) of R. For a coherent $\mathcal{O}_{X_R}$-module E, we denote $E/\pi E = E \otimes_R k$ by $\bar{E}$.

Theorem 5.7. *Let (E_*, φ) be a flat family of parabolic Ω-pairs on X_R/R. Assume that $(E_*, \varphi)_K$ is semi-stable of level i. Then there exists a φ-invariant $\mathcal{O}_X$-submodule E' of E such that $E'_K = E_K$ and (E'_*, φ') is a flat family of parabolic pairs on X_R/R and $(E'_*, \varphi')_k$ is semi-stable of level i where $E'_\alpha = E' \cap E_\alpha$ for $\alpha \geqq 0$ and φ' is a parabolic homomorphism induced from φ.*

Proof. Let $(\bar{F}_*, \bar{\varphi})$ be the first term of the Harder-Narasimhan filtration of level i of $(\bar{E}_*, \bar{\varphi})$. Set $E^{(1)} = \ker(E \to \bar{E} \to \bar{E}/\bar{F})$ and $E^{(1)}_\alpha = E_\alpha \cap E^{(1)}$ for all $\alpha \geq 0$. Then $E^{(1)}_\alpha$ is φ-invariant.

We claim that $(E^{(1)}_*, \varphi)$ is a flat family of parabolic Ω-pairs on X/R. It is sufficient to prove that for all $\alpha \geq 0$, $E^{(1)}/E^{(1)}_\alpha$ is R-flat, $E^{(1)}_{\alpha+1} = E^{(1)}_\alpha(-D)$ and that $\bar{E}^{(1)}$ is torsion free $\mathcal{O}_{\bar{X}}$-module. Since $\pi E/\pi E^{(1)} \simeq E/E^{(1)} \simeq \bar{E}/\bar{F}$ and $\bar{F} \simeq E^{(1)}/\pi E$, we have an exact sequence

$$0 \longrightarrow \bar{E}/\bar{F} \xrightarrow{\ f\ } \bar{E}^{(1)} \xrightarrow{\ g\ } \bar{F} \longrightarrow 0. \tag{5.7.1}$$

Hence, $\bar{E}^{(1)}$ is torsion free. For all $\alpha \geq 0$, since $E^{(1)}/E^{(1)}_\alpha$ is a subsheaf of a torsion free R-module E/E_α, $E^{(1)}/E^{(1)}_\alpha$ is R-torsion free, i.e. R-flat. Finally, since $E_\alpha/E^{(1)}_\alpha$ is a subsheaf of a torsion free $\mathcal{O}_{\bar{X}}$-module $E/E^{(1)}$, we get natural injections

$$E_\alpha(-D)/E^{(1)}_\alpha(-D) \simeq (E_\alpha/E^{(1)}_\alpha) \otimes \mathcal{O}_X(-D) \hookrightarrow E_\alpha/E^{(1)}_\alpha \hookrightarrow E/E^{(1)}.$$

Hence, $E^{(1)}_{\alpha+1} = E^{(1)} \cap E_\alpha(-D) = E^{(1)}_\alpha(-D)$.

Let $(\bar{F}^{(1)}_*, \bar{\varphi})$ be the first term of the Harder-Narasimhan filtration of level i of $(\bar{E}^{(1)}_*, \bar{\varphi})$. We claim that

$$(\mu_1(\bar{F}^{(1)}_*), \ldots, \mu_i(\bar{F}^{(1)}_*)) \leq (\mu_1(\bar{F}_*), \cdots, \mu_i(\bar{F}_*)), \tag{5.7.2}$$

where the equality holds only if g maps $\bar{F}^{(1)}$ to $\bar{F}$ injectively. Let α be a non-negative real number. Note that $\bar{F} \simeq E^{(1)}/\pi E$ and that $\bar{E}_\alpha \simeq E_\alpha + \pi E/\pi E$. We have natural isomorphisms

$$\bar{E}^{(1)}_\alpha = E^{(1)}_\alpha + \pi E^{(1)}/\pi E^{(1)},$$
$$(\bar{E}/\bar{F})_\alpha = \bar{E}_\alpha + \bar{F}/\bar{F} \simeq E_\alpha + E^{(1)}/E^{(1)} \simeq \pi E_\alpha + \pi E^{(1)}/\pi E^{(1)} \text{ and}$$
$$\bar{F}_\alpha = \bar{F} \cap \bar{E}_\alpha = (E^{(1)} \cap (E_\alpha + \pi E))/\pi E = E^{(1)}_\alpha + \pi E/\pi E,$$

where $(\bar{E}/\bar{F})_*$ has the induced structure from $\bar{E}_*$. Moreover, since

$$\pi E \cap (E^{(1)}_\alpha + \pi E^{(1)}) = (\pi E \cap E^{(1)} \cap E_\alpha) + \pi E^{(1)} = \pi E_\alpha + \pi E^{(1)},$$

we have an exact sequence

$$0 \longrightarrow \pi E_\alpha + \pi E^{(1)}/\pi E^{(1)} \xrightarrow{\ f\ } E^{(1)}_\alpha + \pi E^{(1)}/\pi E^{(1)} \xrightarrow{\ g\ } E^{(1)}_\alpha + \pi E/\pi E \longrightarrow 0.$$

Compatibility of f (or, g) and $\bar{\varphi}$ is easily verified. Thus, the sequence (5.7.2) induces an exact sequence of parabolic pairs

$$0 \longrightarrow ((\bar{E}/\bar{F})_*, \bar{\varphi}) \xrightarrow{f} (\bar{E}^{(1)}_*, \bar{\varphi}) \xrightarrow{g} (\bar{F}_*, \bar{\varphi}) \longrightarrow 0.$$

Now, let us provide $(f^{-1}(\bar{F}^{(1)})_*, \bar{\varphi})$ (or, $(g(\bar{F}^{(1)})_*, \bar{\varphi})$) with the induced substructure (or, quotient structure, resp.) from $(\bar{F}^{(1)}_*, \bar{\varphi})$. Then $(f^{-1}(\bar{F}^{(1)})_*, \bar{\varphi})$ (or, $(g(\bar{F}^{(1)})_*, \bar{\varphi})$) becomes a sub-pair of $((\bar{E}/\bar{F})_*, \bar{\varphi})$ (or, $(\bar{F}_*, \bar{\varphi})$, resp.) and we have an exact sequence of parabolic pairs

$$0 \longrightarrow (f^{-1}(\bar{F}^{(1)})_*, \bar{\varphi}) \longrightarrow (\bar{F}^{(1)}_*, \bar{\varphi}) \longrightarrow (g(\bar{F}^{(1)})_*, \bar{\varphi}) \longrightarrow 0.$$

If $g(\bar{F}^{(1)})$ is not zero, then we have

$$\begin{aligned}(\mu_1(\bar{F}^{(1)}_*), \ldots, \mu_i(\bar{F}^{(1)}_*)) &\leq (\mu_1(g(\bar{F}^{(1)})_*), \ldots, \mu_i(g(\bar{F}^{(1)})_*)) \\ &\leq (\mu_1(\bar{F}_*), \ldots, \mu_i(\bar{F}_*)).\end{aligned}$$

If $f^{-1}(\bar{F}^{(1)})$ is not zero, then

$$(\mu_1(f^{-1}(\bar{F}^{(1)})_*), \ldots, \mu_i(f^{-1}(\bar{F}^{(1)})_*)) < (\mu_1(\bar{F}_*), \ldots, \mu_i(\bar{F}_*)). \tag{5.7.3}$$

Hence, the inequality (5.7.2) always holds. Suppose that the equality holds, then $g(\bar{F}^{(1)})$ is not zero and we obtain the equality

$$(\mu_1(\bar{F}^{(1)}_*), \ldots, \mu_i(\bar{F}^{(1)}_*)) = (\mu_1(g(\bar{F}^{(1)})_*), \ldots, \mu_i(g(\bar{F}^{(1)})_*))$$

If $f^{-1}(\bar{F}^{(1)})$ is not zero, then by the equality

$$\begin{aligned}&\operatorname{rk}(f^{-1}(\bar{F}^{(1)})) \cdot (\text{par-}P_{f^{-1}(\bar{F}^{(1)}_*)}(m) - \text{par-}P_{\bar{F}^{(1)}_*}(m)) \\ &= \operatorname{rk}(g(\bar{F}^{(1)})) \cdot (\text{par-}P_{\bar{F}^{(1)}_*}(m) - \text{par-}P_{g(\bar{F}^{(1)})_*}(m)),\end{aligned}$$

we get

$$(\mu_1(f^{-1}(\bar{F}^{(1)})_*), \ldots, \mu_i(f^{-1}(\bar{F}^{(1)})_*)) = (\mu_1(\bar{F}^{(1)}_*), \ldots, \mu_i(\bar{F}^{(1)}_*)).$$

which contradicts (5.7.3). Thus we conclude our claim.

Let us construct $(E^{(m)}_*, \varphi)$ and $(\bar{F}^{(m)}_*, \bar{\varphi})$ inductively. Set $(E^{(0)}_*, \varphi) = (E_*, \varphi)$ and $(\bar{F}^{(0)}_*, \bar{\varphi}) = (\bar{F}_*, \bar{\varphi})$. Repeating the construction of $(E^{(1)}_*, \varphi)$ from (E_*, φ), we obtain a sequence of flat families of parabolic pairs $(E^{(m)}_*, \varphi)$ $(m = 1, 2, \ldots)$ which is called the sequence of elementary transformations of level i of (E_*, φ). Let $(\bar{F}^{(m)}_*, \bar{\varphi})$ be the first term of the Harder-Narasimhan filtration of level i of $(\bar{E}^{(m)}_*, \varphi)$. Then $E^{(m+1)} = \ker(E^{(m)} \to \bar{E}^{(m)}/\bar{F}^{(m)})$. By virtue of the above argument, we have that for all m,

$$(\mu_1(\bar{F}^{(m+1)}_*), \ldots, \mu_i(\bar{F}^{(m+1)}_*)) \leq (\mu_1(\bar{F}^{(m)}_*), \ldots, \mu_i(\bar{F}^{(m)}_*)),$$

where the equality holds only if the natural map ϕ_m of $\bar{F}^{(m+1)}$ to $\bar{F}^{(m)}$ is injective. We can easily prove the theorem using the following lemma successively.

Lemma 5.8. *Let (E_*, φ) be a flat family of parabolic pairs on X_R/R.*

Assume that $(E_*, \varphi)_K$ *is semi-stable of level i and* $(E_*, \varphi)_k$ *is semi-stable of level* $i-1$. *Let* $(E_*^{(m)}, \varphi)$ $(m = 1, 2, \ldots)$ *be the sequence of elementary transformations of level i. Then for sufficiently large integer m,* $(\bar{E}_*^{(m)}, \bar{\varphi})$ *is semi-stable of level i.*

Proof of Lemma 5.8. We have inequalities

$$\begin{aligned}(\mu_1(\bar{F}_*),\ldots,\mu_i(\bar{F}_*)) &\geq (\mu_1(\bar{F}_*^{(1)}),\ldots,\mu_i(\bar{F}_*^{(1)}))\\ &\geq (\mu_1(\bar{F}_*^{(2)}),\ldots,\mu_i(\bar{F}_*^{(2)}))\\ &\geq \cdots \geq (\mu_1(\bar{E}_*),\ldots,\mu_i(\bar{E}_*)).\end{aligned}$$

Since $\bar{E}_*$ is semi-stable of level $i-1$, we have

$$(\mu_1(\bar{F}_*),\ldots,\mu_{i-1}(\bar{F}_*)) \leq (\mu_1(\bar{E}_*),\ldots,\mu_{i-1}(\bar{E}_*)).$$

Hence, for all m, we have

$$(\mu_1(\bar{F}_*^{(m)}),\ldots,\mu_{i-1}(\bar{F}_*^{(m)})) = (\mu_1(\bar{E}_*),\ldots,\mu_{i-1}(\bar{E}_*)).$$

Therefore, we have a descending sequence of rational numbers

$$\mu_i(\bar{F}_*) \geq \mu_i(\bar{F}_*^{(1)}) \geq \mu_i(\bar{F}_*^{(2)}) \geq \cdots \geq \mu_i(\bar{E}_*).$$

Since the system of weights of $\bar{F}_*^{(m)}$ is a subset of that of $\bar{E}_*^{(m)}$ i.e. that of $\bar{E}_*$, there exists an integer M such that for all m, $\mu_i(\bar{F}_*^{(m)}) \in \frac{1}{M}\mathbf{Z}$. Hence, for sufficiently large integer N,

$$\mu_i(\bar{F}_*^{(N)}) = \mu_i(\bar{F}_*^{(N+1)}) = \cdots = \mu_i.$$

Then the natural homomorphism ϕ_m of $\bar{F}^{(m+1)}$ to $\bar{F}^{(m)}$ is injective for all $m \geq N$. We may assume that $\mathrm{rk}\,(\bar{F}^{(N)}) = \mathrm{rk}\,(\bar{F}^{(N+1)}) = \cdots = p$. For all $m \geq N$, we have

$$0 = \mu_1(\bar{F}_*^{(m)}) - \mu_1(\bar{F}_*^{(m+1)}) = \frac{1}{P(n-1)!} \cdot \int_0^1 \deg\,(\bar{F}_\alpha^{(m)}/\phi_m(\bar{F}_\alpha^{(m+1)}))\,d\alpha.$$

Hence, $\deg \bar{F}_\alpha^{(m)} = \deg \bar{F}_\alpha^{(m+1)}$ for all $\alpha \geq 0$ and for all $m \geq N$. We may assume without loss of generality that $N = 0$. We claim that $\mu_i = \mu_i(\bar{E}_*)$, then since $(\mu_1(\bar{F}_*^{(m)}),\ldots,\mu_i(\bar{F}_*^{(m)})) = (\mu_1(\bar{E}_*^{(m)}),\ldots,\mu_i(\bar{E}_*^{(m)}))$ for $m \geq N$, $(\bar{E}_*^{(m)}, \bar{\varphi})$ is semi-stable of level i. We may assume that R is complete.

By the argument in the proof of Lemma 2 in §5 of [8], we have the following.

Lemma 5.9 (S. G. Langton). *Assume that R is complete. Let E be a torsion free coherent* $\mathcal{O}_X$*-module of rank r and let* $E = E^{(0)} \supseteq E^{(1)} \supseteq \cdots \supseteq E^{(m)} \supseteq \cdots$ *be a sequence of* $\mathcal{O}_X$*-submodules such that* $E^{(m+1)} \supseteq \pi E^{(m)}$ *and* $E^{(m)}/E^{(m+1)}$ *is a torsion free* $\mathcal{O}_{\bar{X}}$*-module for all m. Let* $\bar{F}^{(m)}$ *be the image of the natural homomorphism* $\phi_m\colon \bar{E}^{(m+1)} \to \bar{E}^{(m)}$. *Assume that for all m,* ϕ_m *maps* $\bar{F}^{(m+1)}$ *to* $\bar{F}^{(m)}$ *injectively,* $\mathrm{rk}\,(\bar{F}^{(m+1)}) = \mathrm{rk}\,(\bar{F}^{(m)}) = p$ *and* $\deg(\bar{F}^{(m+1)}) = \deg(\bar{F}^{(m)})$. *Then there exists an integer N such that for all* $m \geq N$, $\phi_m\colon \bar{F}^{(m+1)} \to \bar{F}^{(m)}$ *is an isomorphism and there* *exists a coherent $\mathcal{O}_X$-submodule F of $E^{(N)}$ such that $F/\pi^m F \simeq E^{(m+N)}/\pi^m E^{(N)}$.*

Let us apply this lemma to our sequence

$$E_\alpha = E_\alpha^{(0)} \supseteq E_\alpha^{(1)} \supseteq \cdots \supseteq E_\alpha^{(m)} \supseteq \cdots$$

for $\alpha \geq 0$. Then we obtain an integer N_α such that for all $m \geq N_\alpha$, $\phi_m\colon \bar{F}_\alpha^{(m+1)} \to \bar{F}_\alpha^{(m)}$ is an isomorphism and there exists a coherent $\mathcal{O}_X$-submodule F_α of $E_\alpha^{(N)}$ such that

$$F_\alpha/\pi^m F_\alpha \simeq E_\alpha^{(m+N_\alpha)}/\pi^m E_\alpha^{(N)}. \tag{5.8.1}$$

Since $E_{\alpha+1}^{(m)} = E_\alpha^{(m)}(-D)$, we may assume that N_α is independent on α. Set $N = N_\alpha$. By virtue of (5.8.1), we have that $F_\alpha + \pi^m E_\alpha^{(N)} = E_\alpha^{(m+N)}$. Hence,

$$F_\alpha = \bigcap_{m \geq 0} (F_\alpha + \pi^m E_\alpha^{(N)}) = \bigcap_{m \geq 0} E_\alpha^{(m+N)}.$$

Therefore, for all $\alpha \geq 0$, F_α is φ-invariant and $F_\alpha = F \cap E_\alpha^{(N)} (F = F_0)$.

We claim that (F_*, φ) is a flat family of parabolic pairs. We must prove that F is flat over R, $F/\pi F$ is a torsion free $\mathcal{O}_{X_k}$-module and that for all $\alpha \geq 0$, F/F_α is flat over R and $F_{\alpha+1} = F_\alpha(-D)$. We have a natural injection

$$F/\pi F \simeq E^{(N+1)}/\pi E^{(N)} \longrightarrow E^{(N)}/\pi E^{(N)}.$$

Hence, $F/\pi F$ is a torsion free $\mathcal{O}_{X_k}$-module and $E^{(N)}/F$ is flat over R. Since for all $\alpha \geq 0$, $F/F_\alpha \subseteq E^{(N)}/E_\alpha^{(N)}$ and $E^{(N)}/E_\alpha^{(N)}$ is R-torsion free, F/F_α is R-torsion free i.e. R-flat. $E_\alpha^{(N)}/F_\alpha$ is relatively torsion free. In fact, $E_\alpha^{(N)}/F_\alpha$ is a subsheaf of $E^{(N)}/F$ which is flat over R. Hence, $E_\alpha^{(N)}/F_\alpha$ is flat over R. We have an isomorphism

$$(E_\alpha^{(N)}/F_\alpha)_k \simeq E_\alpha^{(N)}/E_\alpha^{(N+1)}.$$

Therefore, $(E_\alpha^{(N)}/F_\alpha)_k$ is a torsion free $\mathcal{O}_{X_k}$-module. Since torsion freeness is open property (cf. [12]), $(E_\alpha^{(N)}/F_\alpha)_K$ is also torsion free $\mathcal{O}_{X_K}$-module. Thus, by Remark 1.10, the natural homomorphism

$$E_\alpha^{(N)}(-D)/F_\alpha(-D) \simeq (E_\alpha^{(N)}/F_\alpha) \otimes_X \mathcal{O}_X(-D) \longrightarrow E_\alpha^{(N)}/F_\alpha \longrightarrow E^{(N)}/F$$

is injective. Therefore, $F_{\alpha+1} = E_\alpha^{(N)}(-D) \cap F = F_\alpha(-D)$.

Now, $(\bar{F}_*, \bar{\varphi})_k$ is isomorphic to $(\bar{F}_*^{(N)}, \bar{\varphi})$. In fact, by (5.8.1), for all $\alpha \geq 0$,

$$F_\alpha/\pi F_\alpha \simeq E_\alpha^{(N+1)} + \pi E^{(N)}/\pi E^{(N)} \simeq \bar{F}_\alpha^{(N)}.$$

$(F_*, \varphi)_K$ is a sub-pair of $(E_*^{(N)}, \varphi)_K$. Hence,

$$(\mu_1((F_*)_K), \ldots, \mu_i((F_*)_K)) \leq (\mu_1((E_*^{(N)})_K), \ldots, \mu_i((E_*^{(N)})_K)).$$

Since (F_*, φ) and $(E_*^{(N)}, \varphi)$ are flat families of parabolic pairs, by the above inequality, we obtain

$$(\mu_1(\bar{F}_*^{(N)}), \dots, \mu_i(\bar{F}_*^{(N)})) = (\mu_1((F_*)_k), \dots, \mu_i((F_*)_k)) \\ \le (\mu_1(\bar{E}_*^{(N)}), \dots, \mu_i(\bar{E}_*^{(N)})).$$

In this inequality, the equality holds because $(\bar{F}_*^{(N)}, \bar{\varphi})$ is the first term of the Harder-Narasimhan filtration of level i of $(\bar{E}_*^{(N)}, \bar{\varphi})$. Thus our claim holds. □

Now, let us define a morphism of the moduli scheme to a space of characteristic polynomials. For a parabolic Ω-pair (E_*, φ) on X_k, its characteristic polynomial is defined as follows. Let t be an indeterminate and let

$$t - \varphi : E \otimes_X S^*(\Omega)[t] \longrightarrow E \otimes_X S^*(\Omega)[t]$$

be an $S^*(\Omega)[t]$-homomorphism defined by

$$(t - \varphi)(e \otimes a) = e \otimes at - \varphi(e)a$$

where e (or, a) is a local section of E (or, $S^*(\Omega)[t]$, resp.) and $\varphi(e)$ is regarded as a local section of $E \otimes_X \Omega \subset E \otimes_X S^*(\Omega)[t]$. Let r be the rank of E. Taking r-th exterior product over $S^*(\Omega)[t]$, we get a homomorphism

$$\wedge^r(t - \varphi) : (\wedge^r E) \otimes_X S^*(\Omega)[t] \longrightarrow (\wedge^r E) \otimes_X S^*(\Omega)[t].$$

Let U be the maximal open set of X_k such that $E|_U$ is locally free and let η be the natural inclusion of U to X_k. Since E is torsion free, codim $(X_k - U, X_k) \ge 2$. Hence, $\det E \simeq \eta_*(\wedge^r E|_U)$. Thus we obtain a homomorphism

$$\eta_*(\wedge^r(t - \varphi)|_U) : (\det E) \otimes_X S^*(\Omega)[t] \longrightarrow (\det E) \otimes_X S^*(\Omega)[t].$$

Tensoring $(\det E)^\vee$ and taking the image of 1 of $S^*(\Omega)[t]$, we obtain an element $\phi_{(E_*, \varphi)}(t)$ of $H^0(X_k, S^*(\Omega)[t]_{X_k})$. Let us call it the characteristic polynomial of (E_*, φ). $\phi_{(E_*, \varphi)}(t)$ is determined by its restriction on U. Moreover, for each open set $U' \subset U$ such that $E|_{U'}$ is free, $\phi_{(E_*, \varphi)}(t)|_{U'}$ is in fact the characteristic polynomial of the $r \times r$ matrix with elements in $H^0(U', \Omega_{U'})$. Therefore, $\phi_{(E_*, \varphi)}(t)$ is in

$$\otimes_{i=0}^r H^0(X_k, S^i(\Omega)_{X_k}) t^{r-i} \subset H^0(X_k, S^*(\Omega)[t]_{X_k})$$

and the coefficient of t^r is 1. Set

$$\phi_{(E_*, \varphi)}(t) = t^r + a_1((E_*, \varphi)) t^{r-1} + \cdots + a_r((E_*, \varphi))$$

where $a_i((E_*, \varphi))$ is in $H^0(X_k, S^i(\Omega)_{X_k})$.

By Proposition 2.2, there exists a coherent $\mathcal{O}_S$-module $H(\mathcal{O}_X, \oplus_{i=0}^{r-1} S^i(\Omega))$ such that $A = \mathbf{V}(H(\mathcal{O}_X, \oplus_{i=0}^{r-1} S^i(\Omega)))$ represents a functor

$$(Sch/S) \ni T \longmapsto \mathrm{Hom}_{X_T}(\mathcal{O}_{X_T}, \oplus_{i=0}^{r-1} S^i(\Omega)_{X_T}).$$

In particular, for a field k over S, we have the natural identification

$$A(k) \simeq \bigoplus_{i=0}^{r-1} H^0(X_k, S^i(\Omega)_{X_k}).$$

The polynomial $\phi_{(E_*, \varphi)}(t)$ is regarded as an element of $A(k)$ which corresponds to $(a_1((E_*, \varphi)),\dots,a_r((E_*, \varphi)))$ under the above identification.

Characteristic polynomials determine an S-morphism Φ of the moduli scheme $\bar{M}^{H_*,\alpha_*}_{\Omega/D/X/S}$ to A. In fact, we have constructed $\bar{M}^{H_*,\alpha_*}_{\Omega/D/X/S}$ as an inductive limit $\varinjlim_e \bar{M}^{H_*,\alpha_*,e,e}_{\Omega/D/X/S}$ and $\bar{M}^{H_*,\alpha_*,e,e}_{\Omega/D/X/S}$ is a good quotient of $\tilde{R}^{ss}_1(e, e)$. On $X_{\tilde{R}^{ss}_1(e,e)}$, we have a universal (parabolic) homomorphism

$$\tilde{\varphi}: \tilde{E} \longrightarrow \tilde{E} \otimes_X \Omega.$$

Let U be the maximal open set of $X_{\tilde{R}^{ss}_1(e,e)}$ such that $\tilde{E}|_U$ is locally free. Then we know that

$$\eta_*(\det \tilde{E}|U) = \det \tilde{E}$$

where η is the natural inclusion map of U to $X_{\tilde{R}^{ss}_1(e,e)}$ (cf. the proof of Lemma 4.2 of [9]). Thus $\eta_*(\wedge^r(t - \tilde{\varphi}))$ determines a morphism $\tilde{\Phi}_e$ of $\tilde{R}^{ss}_1(e, e)$ to A. This is clearly a $GL(V_1, e)$-morphism with respect to the trivial action of $GL(V_1, e)$ on A. Since $\bar{M}^{H_*,\alpha_*,e,e}_{\Omega/D/X/S}$ is a categorical quotient of $\tilde{R}^{ss}_1(e, e)$, $\tilde{\Phi}_e$ induces a morphism Φ_e of $\bar{M}^{H_*,\alpha_*,e,e}_{\Omega/D/X/S}$ to A. It is easy to see that for $e' \geq e$, $\Phi_e = \Phi_{e'} \circ j_{e,e'}$ for the natural open immersion $j_{e,e'}: \bar{M}^{H_*,\alpha_*,e,e}_{\Omega/D/X/S} \to \bar{M}^{H_*,\alpha_*,e',e'}_{\Omega/D/X/S}$. Thus we obtain a morphism Φ of $\bar{M}^{H_*,\alpha_*}_{\Omega/D/X/S}$ to A. Clearly, for each parabolic pair (E_*, φ) on a geometric fibre X_k which corresponds to a k-valued point x of $\bar{M}^{H_*,\alpha_*,e,e}_{\Omega/D/X/S}$, $\Phi(x) = \phi_{(E_*,\varphi)}$ as a point of $A(k)$.

Theorem 5.10. *Let R be a discrete valuation ring over A. Then the natural map* v: $\mathrm{Hom}_A(\mathrm{Spec}(R), \bar{M}^{H_*,\alpha_*}_{\Omega/D/X/S}) \to \mathrm{Hom}_A(\mathrm{Spec}(K), \bar{M}^{H_*,\alpha_*}_{\Omega/D/X/S})$ *is bijective.*

Proof. By Theorem 4.6, $\bar{M}^{H_*,\alpha_*}_{\Omega/D/X/S}$ is separated and locally of finite type over S. Hence, Φ is separated and locally of finite type. Therefore, by the valuative criterion of separatedness, v is injective. To prove the surjectivity, let us take an A-morphism g of $\mathrm{Spec}(K)$ to $\bar{M}^{H_*,\alpha_*}_{\Omega/D/X/S}$. Then g is contained in $\bar{M}^{H_*,\alpha_*,e}_{\Omega/D/X/S}$ for some e. Hence, there exist a finite extension K' of K and a K'-valued point x of $\tilde{R}^{ss}_1(e, e)$ such that $\xi(x)$ is the K'-valued point

$$g': \mathrm{Spec}(K') \longrightarrow \mathrm{Spec}(K) \xrightarrow{g} \bar{M}^{H_*,\alpha_*}_{\Omega/D/X/S}.$$

Let R' be an extension of R whose quotient field is K'. Thus we have a commutative diagram:

$$\begin{array}{ccccc} \mathrm{Spec}(K') & \longrightarrow & \mathrm{Spec}(K) & \xrightarrow{g} & \bar{M}^{H_*,\alpha_*,e}_{\Omega/D/X/S} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(R') & \longrightarrow & \mathrm{Spec}(R) & \longrightarrow & A \end{array}$$

If g' is extended to an A-morphism of $\mathrm{Spec}(R')$ to $\bar{M}^{H_*,\alpha_*}_{\Omega/D/X/S}$ then since $R' \cap K = R$, we obtain a desired extension of g. Hence, we may assume that $R = R'$, $K = K'$ and $g = g'$. The K-valued point x corresponds to a strictly e-semi-stable parabolic pair (E^1_*, φ^1) on a fibre X_K. Let E' be a coherent $\mathcal{O}_{X_R}$-submodule of

$i_*(E^1)$ such that $i^*(E') = E^1$ where i is a canonical open immersion of X_K to X_R. Set

$$\varphi = i_*(\varphi^1): i_*(E^1) \longrightarrow i_*(E^1 \otimes_X \Omega) \simeq i_*(E^1) \otimes_X \Omega.$$

Since $\varphi \wedge \varphi = 0$, φ induces a homomorphism $\varphi^a: i_*(E^1) \otimes_X S^*(\Omega^\vee) \to i_*(E^1)$. Let E be the image of $E' \otimes_X S^*(\Omega^\vee)$ by the homomorphism φ^a. Then E is φ-invariant.

We claim that E is a coherent $\mathcal{O}_{X_R}$-module. Let U be an open subset of X_K such that $E^1|_U$ is locally free. Then, by virtue of the Cayley-Hamilton theorem, $\phi_{(E^1_*, \varphi^1)}(\varphi^1)|_U = 0$ as a homomorphism of $E^1|_U$ to $(E^1 \otimes_X S^r(\Omega))_U$. Since E^1 is torsion free, $\phi_{(E^1_*, \varphi^1)}(\varphi^1) = 0$. Let $\phi(t) \in \bigoplus_{i=0}^r H^0(X_R, S^i(\Omega)_{X_R}) t^{r-i}$ be the polynomial which corresponds to the given morphism Spec$(R) \to A$. Since $\phi(t)|_{X_K}$ is given by $\phi_{(E^1_*, \varphi^1)}(t)$, $\phi(t)$ is monic. Set

$$\phi(t) = t^r + a_1 t^{r-1} + \cdots + a_r$$

where a_i is in $H^0(X_R, S^i(\Omega)_{X_R})$ for each i. Since $\phi(\varphi)|_{X_K} = 0$ as a homomorphism of E^1 to $E^1 \otimes_X S^r(\Omega)$, $\phi(\varphi) = 0$ as a homomorphism of $i_*(E^1)$ to $i_*(E^1) \otimes_X S^r(\Omega)$. Hence, for each local section θ of $\Omega^\vee$, a local section of algebras $S^*(\Omega^\vee)_{X_R}$:

$$\theta^r + \theta(a_1)\theta^{r-1} + \cdots + \theta(a_r)$$

annihilates $i_*(E^1)$. Therefore, we know that the image $E = \varphi^a(E' \otimes_X S^*(\Omega^\vee))$ is same as $\varphi^a(E' \otimes_X S^*_r(\Omega^\vee))$ (cf. the proof of Lemma 1.2 of [26] and Remark 2.1.2). Thus E is coherent. By Proposition 6 of [8], there exists a coherent $\mathcal{O}_{X_R}$-module $E' \subset i_*(E|_{X_K})$ such that $i^*(E') = E|_{X_K}$ and E'_{X_k} is torsion free. We prove this fact by giving E' explicitly as follows.

Lemma 5.11. *Let E be a coherent $\mathcal{O}_{X_R}$-module flat over R. Assume that $E|_{X_K}$ is $\mathcal{O}_{X_K}$-torsion free. Then $E' = E^{\vee\vee} \cap i_*(E|_{X_K})$ is a relatively torsion free coherent $\mathcal{O}_{X_R}$-module and the restriction of a natural injection $E \hookrightarrow E'$ to the fibre X_K (or, X_k) is isomorphism (or, generically isomorphism, resp.) where $E^{\vee\vee}$ and $i_*(E|_{X_K})$ are regarded as submodules of $i_*(E^{\vee\vee}|_{X_K})$.*

Proof. Note that $i_*(E|_{X_K})$ is quasi-coherent. Hence, the module E' is coherent because it is a quasi-cherent submodule of a coherent module $E^{\vee\vee}$. Let U be an open subscheme of X_R such that $E|_U$ is locally free and $U \cap X_K$ is not empty. Then clearly $E|_U = E'|_U$. Hence, $E|_{X_K} = E'|_{X_K}$ and $E|_{X_k}$ is generically isomorphic to $E'|_{X_k}$. Since E' is R-torsion free, it is R-flat. By the assumption, $E'|_{X_K} = E|_{X_K}$ is torsion free. To prove the torsion freeness of $E'|_{X_k}$, we may assume that $X_R = \text{Spec}(B)$ and E is the sheaf associated with a torsion free finite B-module M. We must prove that $M^{\vee\vee} \cap M_\pi / \pi(M^{\vee\vee} \cap M_\pi)$ is a torsion free $B/\pi B$-module. Note that an element $m/\pi^n \in M_\pi$ is in $M^{\vee\vee}$ if and only if for all elements $f \in M^\vee$, $f(m)$ is in $\pi^n B$. Take an element $m/\pi^n \in M^{\vee\vee} \cap M_\pi$ and assume that there is an element $b \in B \setminus \pi B$ such that $b \cdot m/\pi^n \in \pi(M^{\vee\vee} \cap M_\pi)$. Then for all elements $f \in M^\vee$, $f(bm) = bf(m) \in \pi^{n+1} B$. Since $f(m) \in \pi^n B$ and $b \notin \pi B$, $f(m)$

is in $\pi^{n+1}B$, i.e. m/π^n is in $\pi(M^{\vee\vee} \cap M_\pi)$.

Now, set $E' = E^{\vee\vee} \cap i_*(E|_{X_K})$. Then E' is φ-invariant and relatively torsion free. By the properness of Quot-schemes, there exists a unique coherent subsheaf E'_α of E' such that $E'_\alpha|_{X_K} = E^1_\alpha$ and E'/E'_α is flat over R for each $0 \leq \alpha \leq 1$. Since $E' \otimes_X \mathcal{O}_X(-D)|_{X_K} \to E'|_{X_k}$ is injective, $E'/E' \otimes_X \mathcal{O}_X(-D)$ is flat over R. Hence, $E'_1 = E' \otimes_X \mathcal{O}_X(-D)$. Thus, we obtain the flat family of parabolic pairs (E'_*, φ). By virtue of Theorem 5.7, we have a φ-invariant coherent subsheaf E'' of E' such that (E''_*, φ) with the induced structure is a flat family of parabolic pairs, $(E''_*, \varphi)_K = (E_*, \varphi)_K$ and $(E''_*, \varphi)_k$ is semi-stable. Thus by virtue of Theorem 4.6, we can extend the given A-morphism of Spec(K) to $\bar{M}^{H_*,\alpha_*}_{\Omega/D/X/S}$ over Spec(R). □

If $\mathscr{F}_\Omega(H, H_*, \alpha_*)$ is bounded, then for some e, $\bar{M}^{H_*,\alpha_*}_{\Omega/D/X/S} = \bar{M}^{H_*,\alpha_*,e,e}_{\Omega/D/X/S}$ which is quasi-projective over S. Hence Φ is a quasi-projective morphism. By the valuative criterion of properness and Theorem 5.10, we have the following.

Corollary 5.12. *If the family $\mathscr{F}_\Omega(H, H_*, \alpha_*)$ is bounded, then the morphism Φ: $\bar{M}^{H_*,\alpha_*}_{\Omega/D/X/S} \to A$ is projective. In particular, if S is a noetherian scheme over a field of characteristic zero, then Φ: $\bar{M}^{H_*,\alpha_*}_{\Omega/D/X/S} \to A$ is projective.*

In the case that $\Omega = 0$, we have that $A = S$ and that $\bar{M}^{H_*,\alpha_*}_{\Omega/D/X/S} = \bar{M}^{H_*,\alpha_*}_{D/X/S}$. Therefore we have

Corollary 5.13. *If the family $\mathscr{F}(H, H_*, \alpha_*)$ is bounded, then the moduli scheme of semi-stable parabolic sheaves $\bar{M}^{H_*,\alpha_*}_{D/X/S}$ is projective over S. In particular, if S is a noetherian schame over a field of characteristic zero, then $\bar{M}^{H_*,\alpha_*}_{D/X/S}$ is projective over S.*

A. Compactification of moduli of parabolic sheaves in the case of characteristic zero

In this appendix, we shall deal with only parabolic sheaves and assume that Ξ contains a field of characteristic zero. Then we have the following "strong" boundedness results (see for the proof, [11]).

Proposition A.1. *For each positive integer r, there exists a non-negative integer e such that all μ-semi-stable sheaves with its rank $\leq r$ are of c-type e.*

By Proposition 3.4 1) of [13], we have a morphism Ψ: $\Gamma^{ss} \to (Z \times \prod G_i)^{ss}$. If, in §2, we set $\Omega = 0$, then $\Psi = \tilde{\Psi}$, $\Gamma^{ss} = R^{ss}$ and $(Z \times \prod G_i)^{ss} = (\tilde{Z} \times \prod G_i)^{ss}$. Hence, we use notations in §2 assuming $\Omega = 0$. The construction of Ψ: $\Gamma \to (Z \times \prod G_i)$ depends on m fixed at the first part of §2. Hence, we denote Ψ, Γ, Z and G_i by Ψ_m, Γ_m, Z_m and $G_{i,m}$ respectively. Our aim in this section is to prove the following.

Proposition A.2. *Assume that $\alpha_1 > 0$. Then there exists an integer m such that the morphism Ψ_m: $\Gamma^{ss}_m \to (Z_m \times \prod^l_{i=1} G_{i,m})^{ss}$ is proper. Hence, it is a closed immersion.*

In the case of curves, this is proved by U. N. Bhosle [2] Proposition 3. Since we deal with higher dimensional cases, we need some boundedness results.

For an integer L, we set

$$\mathscr{P}\mathscr{A}^L(H, H_*, \alpha_*) = \left\{ E_* \,\middle|\, \begin{array}{l} E_* \text{ is a parabolic sheaf on some fiber } X_s \\ \text{with the properties (A.2.1) and (A.2.2).} \end{array} \right\}.$$

(A.2.1) The Hilbert polynomial of E (or, $F_{i+1}(E)$) is H (or, H_i, resp.) and the system of weights of E_* is α_*.

(A.2.2) For some integer $m \geq L$, there exists a generically surjective homomorphism φ of $V_m \otimes \mathcal{O}_{X_s}$ to $E(m)$ such that

(A.2.2.1) for all i, there exists a vector subspace W_i of $V_m \otimes k(s)$ of dimension $H(m) - H_i(m)$ such that $H^0(\varphi)(W_i) \subseteq H^0(F_{i+1}(E)(m))$ and

(A.2.2.2) the point of $Z_m \times \prod G_{i,m}$ determined by $V_m \otimes k(s) \twoheadrightarrow V_m \otimes k(s)/W_i$ and $\wedge^r \varphi\colon \wedge^r (V_m \otimes k(s)) \to H^0(\det(E(m)))$ is contained in $(Z_m \times \prod G_{i,m})^{ss}(k(s))$.

Lemma A.3. *There exists an integer L_0 such that the family $\mathscr{P}\mathscr{A}^{L_0}(H, H_*, \alpha_*)$ is bounded.*

Proof. We need the following lemma which is equivalent to "Fundamental lemma" in [9].

Lemma A.4 (Lemma 2.6 in [13]). *Let S be a locally noetherian, connected scheme, $f\colon X \to S$ be a smooth, projective, geometrically integral morphism of relative dimension n and let $\mathcal{O}_X(1)$ be an f-very ample invertible sheaf on X. Let a be a rational number, r be a positive integer and $P(m)$ be a polynomial of degree n with the highest term $hm^n/n!$ where h is the degree of $\mathcal{O}_X(1)$ on fibers of f. Then there exist integers L and M such that if F is a torsion free coherent $\mathcal{O}_{X_s}$-module of rank $r' \leq r$ for some geometric point s of S and if F has the properties;*

1) *for general non-singular curves $C = D_1 \cdot D_2 \cdot \cdots \cdot D_{n-1}$, $D_i \in |\mathcal{O}_{X_s}(1)|$, every coherent subsheaf $E(\neq 0)$ of $F \otimes_{X_s} \mathcal{O}_C$ has a degree $\leq \operatorname{rk}(E)a$,*
2) *$\mu(F) \leq M$,*

then for all $m \geq L$, the following inequality holds;

$$h^0(F(m)) \leq r' P(m).$$

Apply Lemma A.4 to the case where $a = \mu_0 + e$ ($\mu_0 = \mu(E)$ and e is a non-negative integer as in Proposition A.1), $P(m)$ is a polynomial such that $P(m) \prec \text{par-}P_{E_*}(m)$ and $r = r$. Then there exist integers L_0 and M such that if a coherent sheaf F of rank $\leq r$ on a fiber X_s satisfies the above conditions 1) and 2), then for all integers $m \geq L_0$, we have

$$h^0(F(m))/\operatorname{rk}(F) \leq P(m).$$

We may assume that for all $m \geq L_0$, $P(m) < \text{par-}P_{E_*}(m)$. Hence, for all $m \geq L_0$, we have

$$h^0(F(m))/\mathrm{rk}\,(F) < \text{par-}P_{E_*}(m). \tag{A.4.1}$$

To prove the boundedness of $\mathscr{P}\mathscr{A}^L(H, H_*, \alpha_*)$, it is sufficient to prove that there exists an integer β such that for all members E_* of $\mathscr{P}\mathscr{A}^L(H, H_*, \alpha_*)$, E is of type β. Let E_* be a member of $\mathscr{P}\mathscr{A}^L(H, H_*, \alpha_*)$ and E' be the last term of the Harder-Narasimhan filtration of E. Set

$$W = \ker(V_m \otimes_{\Xi} k(s) \longrightarrow H^0(E(m)) \longrightarrow H^0(E'(m))).$$

If x is the point defined in (A.2.2.2), by Lemma 2.6, we have

$$\begin{aligned} 0 &\le \sigma(W, x) \\ &= H(m)(\text{par-}P_{E_*}(m)\dim_{T_x} W - \textstyle\sum \varepsilon_i \dim_{k(s)}(W_i \cap W) - \alpha_1 \dim_{k(s)} W). \end{aligned}$$

By the condition (A.2.2.1),

$$\dim_{k(s)}(W_i \cap W) \ge \dim_{k(s)} W_i - h^0(F_{i+1}(E'(m))),$$

where $F_{i+1}(E'(m)) = \mathrm{Image}\,(F_{i+1}(E(m)) \subsetneq E(m) \twoheadrightarrow E'(m))$. Hence we have

$$\begin{aligned} 0 &\le \text{par-}P_{E_*}(m)\dim_{T_x} W - \textstyle\sum \varepsilon_i \dim_{k(s)} W_i - \alpha_1 \dim_{k(s)} W \\ &\quad + \textstyle\sum \varepsilon_i h^0(F_{i+1}(E'(m))) \\ &= \text{par-}P_{E_*}(m)(\dim_{T_x} W - \mathrm{rk}\,(E)) + \alpha_1 \dim_{k(s)}(V_m \otimes k(s)/W) \\ &\quad + \textstyle\sum \varepsilon_i h^0(F_{i+1}(E'(m))) \\ &\le -\text{par-}P_{E_*}(m)\,\mathrm{rk}\,(E') + \int_0^1 h^0(E'_\alpha(m))\,d\alpha \end{aligned}$$

where $E'_*(m)$ has the induced struture. Therefore

$$\text{par-}P_{E_*}(m) \le \int_0^1 h^0(E'_\alpha(m))\,d\alpha/\mathrm{rk}\,(E'). \tag{A.4.2}$$

On the other hand, by virtue of Proposition A.1, for general non-singular curves $C = D_1 \cdot D_2 \cdot \cdots \cdot D_{n-1}$, $D_i \in |\mathcal{O}_{X_s}(1)|$ and for all non-trivial coherent subsheaves E'' of $E'|_C$,

$$\mu(E'') \le \mu(E'|_C) + e \le \mu_0 + e = a.$$

Clearly, all E'_α $(0 \le \alpha \le 1)$ satisfies the above condition. Hence, by (A.2.1), if $\mu(E') \le M$ (hence, $\mu(E'_\alpha) \le M$), then for all $m \ge L_0$, we have

$$\int_0^1 h^0(E'_\alpha(m))\,d\alpha/\mathrm{rk}\,(E') < \text{par-}P_{E_*}(m).$$

Therefore, if we take $L = L_0$, $\mu(E') > M$. It follows that there exists a integer β depending only on μ_0, M and r such that E is of type β. □

Therefore, there exists an integer $L_1 \geq L_0$ such that for all $m \geq L_1$ and $E_* \in \mathscr{P}\mathscr{A}^{L_0}(H, H_*, \alpha_*)$, we have

(A.4.3) $h^i(E(m)) = 0$, $h^i(F_j(E)(m)) = 0$ and $h^i((E/F_j(E))(m)) = 0$ for all j and $i > 0$.

(A.4.4) $E(m)$, $F_j(E)(m)$ and $(E/F_j(E))(m)$ are generated by its global sections.

Lemma A.5. *Let E_* be a member of $\mathscr{P}\mathscr{A}^{L_1}(H, H_*, \alpha_*)$ and let m be an integer $\geq L_1$. If a generically surjective homomorphism $\varphi: V_m \otimes \mathcal{O}_{X_s} \to E(m)$ and $W_i \subset V_m \otimes k(s)$ satisfy the conditions* (A.2.2.1) *and* (A.2.2.2), *then $H^0(\varphi): V_m \otimes k(s) \to H^0(E(m))$ is an isomorphism. Hence φ is surjective. Moreover, there exists an integer $L_2 \geq L_1$ such that all members E_* of $\mathscr{P}\mathscr{A}^{L_2}(H, H_*, \alpha_*)$ are parabolic semi-stable.*

Proof. Set $W = \ker(H^0(\varphi))$. Then $\dim_{T_x} W = 0$ and by the condition (A.2.2.2), we have

$$0 \leq \frac{\sigma(W, x)}{H(m)} = -\sum \varepsilon_i \dim(W_i \cap W) - \alpha_1 \dim W.$$

Since we assume $\alpha_1 > 0$, we have $W = 0$. By the condition (A.4.3), $H^0(\varphi)$ is an isomorphism.

Set

$$\mathscr{B}^L(H, H_*, \alpha_*) = \left\{ E'_* \,\middle|\, \begin{array}{l} E'_* \text{ is the last term of the Harder-Narasimhan} \\ \text{filtration of some member } E_* \text{ of } \mathscr{P}\mathscr{A}^L(H, H_*, \alpha_*) \end{array} \right\}.$$

Since the Harder-Narasimhan filtration is unique, the family $\mathscr{B}^{L_1}(H, H_*, \alpha_*)$ is bounded. Therefore, there exists an integer L_2 such that for all members E'_*, all $0 \leq \alpha \leq 1$ and all $j > 0$, $h^j(E'_\alpha) = 0$ and par-$P_{E'_*}(m) <$ par-$P_{E_*}(m)$ for all $m \geq L_2$ if $E' \neq E$. Note that the inequality (A.4.1) hold for E'_*. Hence for all members E'_* of $\mathscr{B}^{L_2}(H, H_*, \alpha_*)$, we see that there is an integer $m \geq L_2$ such that

$$\text{par-}P_{E_*}(m) \leq \text{par-}P_{E'_*}(m).$$

It follows that $E' = E$. □

Proof of Proposition A.2. Let m be an integer $\geq L_2$. We use the valuative criterion. Let R be a discrete valuation ring with the residue field k and the quotient field K. Set $C = \text{Spec}(R)$. Assume we have the following commutative diagram:,

$$\begin{array}{ccc} C - p & \longrightarrow & \Gamma^{ss} \\ \downarrow & & \downarrow \Psi \\ C & \xrightarrow{g} & (Z \times \prod G_i)^{ss} \end{array}$$

where p is the closed point of C. Γ^{ss} is the subscheme of $Q \times \prod Q_i$. Since $Q \times \prod Q_i$ is prober over S, the S-morphism of $C - p$ to $Q \times \prod Q_i$ is uniquely extended to a S-morphism θ of C to $Q \times \prod Q_i$. Hence on X_C, there exist surjections $V_m \otimes \mathcal{O}_{X_C} \to E(m)$ and $V_m \otimes \mathcal{O}_{X_C} \to E_i(m)$.

Consider the canonical homomorphism $E(m) \to E^{\vee\vee}(m)$. Since $E^{\vee\vee}(m)$ is a torsion free $\mathcal{O}_{X_C}$-module, it is flat over C. Moreover it is easy to see that $E^{\vee\vee}(m)$ is f_C-torsion free. If the Hilbert polynomial of $E^{\vee\vee}$ on fibers is H', then $E(m) \otimes_R K \subsetneq E^{\vee\vee}(m) \otimes_R K$ defines a morphism of Spec(K) to the Quot-scheme Quot$(E^{\vee\vee}(m), H' - H[m])$ and by properness of Quot$(E^{\vee\vee}(m), H' - H[m])$ over C, it is extended to a unique morphism δ of C to Quot$(E^{\vee\vee}(m), H' - H[m])$. The kernel $\bar{E}(m)$ of the quotient map defined by δ is f_C-torsion free and its Hilbert polynomial on fibers is $H[m]$. Then we have injections:

$$E(m) \overset{\xi}{\hookrightarrow} \bar{E}(m) \hookrightarrow E^{\vee\vee}(m),$$

where ξ has the following properties:

(A.5.1) $\xi \otimes_R K : E(m) \otimes_R K \to \bar{E}(m) \otimes_R K$ is an isomorphism.

(A.5.2) $\xi \otimes_R k : E(m) \otimes_R k \to \bar{E}(m) \otimes_R k$ is generically isomorphism and its kernel is the torsion sheaf of $E(m) \otimes_R k$.

Since we have surjections on X_K:

$$\bar{E}(m) \otimes_R K \simeq E(m) \otimes_R K \twoheadrightarrow E_l(m) \otimes_R K \twoheadrightarrow \cdots \twoheadrightarrow E_1(m) \otimes_R K.$$

There exist surjections of $\mathcal{O}_{X_C}$-modules flat over C;

$$\bar{E}(m) \twoheadrightarrow \bar{E}_l(m) \twoheadrightarrow \cdots \twoheadrightarrow \bar{E}_1(m)$$

which induce the above sequence of surjections on X_K. We obtain a flat family of parabolic sheaves on X_C. Hence, on X_k, we have

$$V_m \otimes \mathcal{O}_{X_k} \longrightarrow \bar{E}(m) \otimes_R k \twoheadrightarrow \bar{E}_l(m) \otimes_R k \twoheadrightarrow \cdots \twoheadrightarrow \bar{E}_1(m) \otimes_R k,$$

where $V_m \otimes \mathcal{O}_{X_k} \to \bar{E}(m) \otimes_R k$ is generically surjective. Let x be the k-valued point defined by the morphism g of C to $(Z \times \prod G_i)^{ss}$. Easily we see that the homomorphism $V_m \otimes k \to H^0(\bar{E}(m) \otimes \mathcal{O}_{X_k})$ maps $W_{i,x}$ to $H^0(F_{i+1}(\bar{E}(m)) \otimes \mathcal{O}_{X_k})$ where $F_{i+1}(\bar{E}(m)) = \ker(\bar{E}(m) \to \bar{E}_i)$. Therefore $\bar{E}(m)_* \otimes_R k$ is a member of $\mathcal{PA}^{L_2}(H, H_*, \alpha_*)$. By virtue of Lemma A.5, we see that $V_m \otimes \mathcal{O}_{X_k} \to \bar{E}(m) \otimes_R k$ is surjective and $\bar{E}(m) \otimes_R k$ is parabolic semi-stable. Hence, $\xi \otimes_R k : E(m) \otimes_R k \to \bar{E}(m) \otimes_R k$ is an isomorphism and therefore the image of p in $Q \times \prod Q_i$ is contained in Γ^{ss}.

Remark A.6. In the case of $\alpha_1 = 0$, if we change the system of weights by $\alpha_* + \varepsilon = \{\alpha_1 + \varepsilon, \ldots, \alpha_l + \varepsilon\}$ so that $0 < \alpha_1 + \varepsilon < \cdots < \alpha_l + \varepsilon < 1$, then we can apply Proposition A.2. Changing weights, (semi-)stability may be different to the original one. But by such a change as above, μ-(semi-)stability is not changed. Hence, in the case of curves, or, in higher dimensional cuses if μ-(semi-)stability is same as (semi-)stability, we can recover the case "$\alpha_1 = 0$".

By virtue of Theorem 4 of [21], there exists a good quotient $\bar{M}$ of Γ^{ss}. Let s be a k-valued geometric point. If we prove the following (1) and (2), then we can easily prove the next Corollary. The proof of (1) and (2) is easier than the analysis of orbits in $(Z_m \times \prod_i G_{i,m})^{ss}$ given in §3, so we omit its proof.

(1) For parabolic stable sheaves $(E_1)_*, \dots, (E_t)_*$ on X_s such that $\bigoplus_i (E_i)_*$ corresponds to a k-valued point of Γ^{ss}, the orbit of x_0 of $\Gamma^{ss}(k)$ corresponding to $\bigoplus_i (E_i)_*$ is closed.

(2) For each semi-stable parabolic sheaf E_* corresponding to a k-valued point x of Γ^{ss}, the closure of orbit of x contains the point corresponding to $\mathrm{gr}(E_*)$.

Corollary A.7. *$\bar{M}$ has the following properties.*

1) *For each geometric point s of S, there exists a natural bijection:*

$$\bar{\theta}_s\colon \text{par-}\bar{\Sigma}^{H_*,\alpha_*}_{D/X/S}(k(s)) \longrightarrow \bar{M}(k(s)).$$

2) *For $T \in (Sch/S)$ and a flat family of parabolic sheaves E_* on X_T/T such that E_* has the property* (1.14.1) *of* [13] *and for every geometric point t of T, $E_*|_{X_t}$ is parabolic semi-stable, then there exists a morphism*

$$f_{E_*}\colon T \longrightarrow \bar{M}$$

such that for all points t in $T(k(s))$, $f_{E_*}(t) = \bar{\theta}([E_*|_{X_t}])$ where $[\cdot]$ means the equivalence class defined by (1.14.2) *of* [13]. *Moreover, for a morphism $g\colon T' \to T$ in (Sch/S), we have*

$$f_{E_*} \circ g = f_{(1_X \times g)^*(E_*)}.$$

3) *If $M' \in (Sch/S)$ and maps*

$$\theta'_s\colon \text{par-}\bar{\Sigma}^{H_*,\alpha_*,e}_{D/X/S}(k(s)) \longrightarrow M'(k(s))$$

have the above property 2), *then there exists a unique S-morphism Υ of M_e to M' such that $\Upsilon(k(s)) \circ \theta_s = \theta'_s$ and $\Upsilon \circ f^e_{E_*} = f'_{E_*}$ for all geometric points s of S and for all E_*, where f'_{E_*} is a morphism given by the property* 2) *for M' and θ'_s.*

4) *$\bar{M}$ is projective over S.*

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