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Sobolev spaces over the Wiener space based on an Ornstein-Uhlenbeck operator

By

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1. Introduction

Sobolev spaces over the Wiener space, or more generally over an abstract Wiener space, play a fundamental role in the Malliavin calculus. They are based on an *Ornstein-Uhlenbeck operator*, denoted by L, on the Wiener space. In this paper, we consider a different kind of Ornstein-Uhlenbeck operator L_A.

To be precise, let (B, H, μ) be an abstract Wiener space, i.e. B is a real separable Banach space and μ is a Gaussian measure with a reproducing kernel Hilbert space H. Let A be a strictly positive definite self-adjoint operator in H. We consider the following semigroup (called an Ornstein-Uhlenbeck semigroup)

$$T_t f(x)=\int_B f(e^{-tA}x+\sqrt{1-e^{-2tA}}y)\mu(dy). \tag{1.1}$$

The generator of $\{T_t\}$ is an operator in our consideration. We denote it by L_A. Moreover the associated Dirichlet form is given by

$$\mathcal{E}(f, g)=\int_B(\sqrt{A^*}Df(x), \sqrt{A^*}Dg(x))_{H^*}\mu(dx) \tag{1.2}$$

where D is the H-derivative. If $A=1$, then L_A is the usual Ornstein-Uhlenbeck operator L.

Its origin is in the quantum field theory. In physical literature, the Ornstein-Uhlenbeck operator in the Malliavin calculus is called the *Number operator* and our Ornstein-Uhlenbeck operator is the *free Hamiltonian*. So this operator is important in physics.

To construct Sobolev spaces in the Malliavin calculus, the Meyer equivalence which insists the equivalence of two kinds of norms defined by L and D, is essential. Such a problem was first discussed by P. A. Meyer [12] and then M. Krée, P. Krée [9] and H. Sugita [22] proved it in the higher degree case. In this paper, we will obtain an analogous equivalence. This problem was proposed by J. Potthoff [16]. The equivalence in our case is a little bit different from that in the Malliavin calculus because A is not bounded in general.

The organization of this paper is as follows. In the section 2, we give a precise

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definition of an Ornstein-Uhlenbeck operator and discuss the commutation relation of L_A and $\sqrt{A^*}D$ which is used later to show the equivalence of norms. We also prepare the Littlewood-Paley-Stein inequality. In the section 3, we give a proof of the equivalence of norms. In Meyer's proof, the Littlewood-Paley-Stein inequality was crucial. We also use the Littlewood-Paley-Stein inequality which is extended to Hilbert space valued functions in [19].

2. Ornstein-Uhlenbeck operator

Let (B, H, μ) be an abstract Wiener space: B is a separable real Banach space, H is a separable real Hilbert space which is embedded densely and continuously in B and μ is a Gaussian measure with

$$\hat{\mu}(l):=\int_B \exp\{\sqrt{-1}\,{}_{B^*}\langle l, x\rangle_B\}\mu(dx)=\exp\left\{-\frac{1}{2}|l|_{H^*}^2\right\}, \qquad l\in B^*\subset H^*.$$

Let us define an Ornstein-Uhlenbeck process. To do this, we give a Dirichlet form. Let A be a strictly positive definite self-adjoint operator in H and define $\mathcal{A}$ to be a set of all functions of the form

$$f(x)=p({}_{B^*}\langle l_1, x\rangle_B, \cdots, {}_{B^*}\langle l_n, x\rangle_B), \tag{2.1}$$

where $n\in\boldsymbol{N}$, p is a polynomial on $\boldsymbol{R}^n$ and $l_1, \cdots, l_n\in C^\infty(A^*)\cap B^*$, $A^*: H^*\to H^*$ being the dual operator of A and $C^\infty(A^*)=\cap_{k=1}^\infty \operatorname{Dom}(A^{*k})$. The Dirichlet form in our consideration is given by

$$\mathcal{E}(f, g)=\int_B(\sqrt{A^*}Df(x), \sqrt{A^*}Dg(x))_{H^*}\mu(dx) \qquad f, g\in\mathcal{A}. \tag{2.2}$$

Here $Df(x)$ is an H-derivative of f at x;

$$Df(x)[h]=\lim_{t\to 0}\frac{f(x+th)-f(x)}{t} \qquad \text{for } h\in H. \tag{2.3}$$

We assume that $C^\infty(A^*)\cap B^*$ is *dense in* $\operatorname{Dom}(A^{*k})$ *under the graph norm of* A^{*k} *for any* $k\in\boldsymbol{Z}_+$ and moreover there exists a symmetric diffusion process associated with the Dirichlet form (2.2) (see, e.g. [10, 1, 18] for the construction).

The Ornstein-Uhlenbeck semigroup is given as follows;

$$T_t f(x)=\int_B f(e^{-tA}x+\sqrt{1-e^{-2tA}}y)\mu(dy) \qquad \text{for } f\in\mathcal{A}. \tag{2.4}$$

The above expression is well-defined if the semigroup $\{e^{-tA}\}$ generated by A can be extended to a strongly continuous contraction semigroup in B such that, for $t>0$

$$\|e^{-tA}\|_{\mathcal{L}(B)}<1.$$

where $\|\cdot\|_{\mathcal{L}(B)}$ denotes the operator norm. In this case, $\{P_t\}$ is a Feller semigroup. But in our situation, (2.4) is well-defined ror $f\in\mathcal{A}$. We denote $\sqrt{A^*}D$ by D_A and the generator by L_A to specify A. We call L_A an *Ornstein-Uhlenbeck operator*. The generator L_A is given as follows, for $f(x)=p({}_{B^*}\langle l_1, x\rangle_B, \cdots, {}_{B^*}\langle l_n, x\rangle_B)$

$$L_A f(x)=\sum_{i,j=1}^{n}(A^*l_i,\ l_j)_{H^*}\frac{\partial^2 p}{\partial\xi^i\partial\xi^j}({}_{B^*}\langle l_1,\ x\rangle_B,\ \cdots,\ {}_{B^*}\langle l_n,\ x\rangle_B)$$

$$-\sum_{i=1}^{n}\langle A^*l_i,\ x\rangle\frac{\partial p}{\partial\xi^i}({}_{B^*}\langle l_1,\ x\rangle_B,\ \cdots,\ {}_{B^*}\langle l_n,\ x\rangle_B). \tag{2.5}$$

Here $\langle A^*l_i, x\rangle$ is the Wiener integral of $A^*l_i\in H^*$, so it is defined almost surely.

By H-differentiating both hands in (2.4), we have,

$$D(T_tf)(x)=\int_B e^{-tA^*}Df(e^{-tA}x+\sqrt{1-e^{-2tA}}y)\mu(dy)$$

$$=e^{-tA^*}T_tDf(x).$$

Hence we have the following commutation relation on $\mathcal{A}$;

$$D_AT_t=e^{-tA^*}T_tD_A. \tag{2.6}$$

By differentiating in t, we have

$$D_AL_A=(L_A-A^*)D_A. \tag{2.7}$$

This *commutation relation* plays an important role in this paper. It is convenient to use $1-L_A$ to define Sobolev spaces. So we consider $1-L_A$ and let $\{P_t\}$ be a semigroup generated by L_A-1: $P_t=e^{-t}T_t$.

We need a bigger Hilbert space. Let $H^{*\otimes n}$ be a Hilbert space of an n-fold tensor product of H^*: $H^{*\otimes n}=\underbrace{H^*\otimes\cdots\otimes H^*}_{n}$. Then A^* can be extended to $H^{*\otimes n}$ with the derivation property. In fact, let $d\Gamma(A^*)_n$ be an operator in $H^{*\otimes n}$ defined by;

$$d\Gamma(A^*)_nh_1^*\otimes\cdots\otimes h_n^*=\sum_{i=1}^{n}h_1^*\otimes\cdots\otimes A^*h_i^*\otimes\cdots\otimes h_n^*,\qquad h_1^*,\ \cdots,\ h_n^*\in H^*. \tag{2.8}$$

We consider the operator $1-L_A+d\Gamma(A^*)_n$ in $L^2(\mu)\otimes H^{*\otimes n}=L^2(\mu;H^{*\otimes n})$. To be more precise, we should write $1-L_A\otimes 1_{H^{*\otimes n}}+1_{L^2(\mu)}\otimes d\Gamma(A^*)_n$ in place of $1-L_A+d\Gamma(A^*)_n$. But there is no fear of confusion, we simply write $1-L_A+d\Gamma(A^*)_n$. Let $\{P_t^{(n)}\}_{t\geq 0}$ be a semigroup in $L^2(\mu;H^{*\otimes n})$ generated by $L_A-1-d\Gamma(A^*)_n$. Of course, for $n=0$, $P_t^{(0)}=P_t$. Then, $P_t^{(n)}$ can be represented as

$$P_t^{(n)}u(x)=E_x[e^{-t(1+d\Gamma(A^*)_n)}u(X_t)] \tag{2.9}$$

where E_x stands for the integration under P_x. Further it is easy to see that $\{P_t^{(n)}\}_{t\geq 0}$ defines a strongly continuous contraction semigroup in $L^p(\mu;H^{*\otimes n})$, for $p\geqq 1$. Now, we can extend the commutation relation (2.6) as follows. By noting $e^{-td\Gamma(A^*)_n}=\underbrace{e^{-tA^*}\otimes\cdots\otimes e^{-tA^*}}_{n}$,

$$D_AP_t^{(n)}=D_AP_t\underbrace{e^{-tA^*}\otimes\cdots\otimes e^{-tA^*}}_{n}$$

$$=P_t\underbrace{e^{-tA^*}\otimes\cdots\otimes e^{-tA^*}}_{n+1}D_A$$

$$=P_t^{(n+1)}D_A. \tag{2.10}$$

By differentiating in t we have

$$D_A(1-L_A+d\Gamma(A^*)_n)=(1-L_A+d\Gamma(A^*)_{n+1})D_A \qquad \text{on } \mathcal{A}(H^{*\otimes n}) \tag{2.11}$$

where $\mathcal{A}(H^{*\otimes n})=\mathcal{A}\otimes C^\infty(A^*)\otimes\cdots\otimes C^\infty(A^*)$ (algebraic tensor product).

We can introduce the L^p-Sobolev space by using L_A and D_A. We discuss the relation between them. It is well-known that they define equivalent norms in Malliavin calculus, i.e. in the case of $A=1$.

We need a Littlewood-Paley-Stein inequality. Let us review it quickly. Let μ_t, $t\geqq 0$ be a probability measure on $[0, \infty)$ such that

$$\int_0^\infty e^{-\alpha s}\mu_t(ds)=e^{-\sqrt{\alpha}t} \qquad \text{for } \alpha>0.$$

Let $\{Q_t^{(n)}\}$ be a Cauchy semigroup defined by

$$Q_t^{(n)}=\int_0^\infty P_s^{(n)}\mu_t(ds). \tag{2.12}$$

We simply denote $Q_t^{(0)}$ by Q_t. Further, we denote the generator of $\{Q_t^{(n)}\}$ by $-\sqrt{1-L_A+d\Gamma(A^*)_n}$ as usual. Now we define Littlewood-Paley's G-functions by

$$G_{(n)}^{\rightarrow}u(x)=\left\{\int_0^\infty t\left|\frac{\partial}{\partial t}Q_t^{(n)}u(x)\right|_{H^{*\otimes n}}^2 dt\right\}^{1/2}, \tag{2.13}$$

$$G_{(n)}^{\uparrow}u(x)=\left\{\int_0^\infty t|D_AQ_t^{(n)}u(x)|_{H^{*\otimes(n+1)}}^2 dt\right\}^{1/2}. \tag{2.14}$$

Then the following Littlewood-Paley-Stein inequality is fundamental (see [19] for the proof). For any $p\in(1, \infty)$, it holds that

$$\|u\|_p\lesssim\|G_{(n)}^{\rightarrow}u\|_p\lesssim\|u\|_p, \qquad u\in\mathcal{A}(H^{*\otimes n}), \tag{2.15}$$

$$\|G_{(n)}^{\uparrow}u\|_p\lesssim\|u\|_p, \qquad u\in\mathcal{A}(H^{*\otimes n}). \tag{2.16}$$

Here $\|\cdot\|_p$ stands for the L^p-norm and we denote $\|u\|_p\lesssim\|v\|_p$ if there exists a positive constant c_p depending only on p such that $\|u\|_p\leqq c_p\|v\|_p$. We use this convention throughout this paper.

3. Equivalence of norms

In this section we discuss the equivalence of norms defined by D_A and L_A. Let p be any number in $(1, \infty)$ and q be a conjugate exponent of p : $1/p+1/q=1$. We denote the inner product in L^2-space by $(\cdot, \cdot)_2$ and we use it as a natural pairing of L^p and L^q. First we have the following:

Theorem 3.1. *It holds that*

$$\|D_Au\|_p\lesssim\|\sqrt{1-L_A}u\|_p\lesssim\|D_Au\|_p+\|u\|_p \qquad \textit{for } u\in\mathcal{A}. \tag{3.1}$$

Proof. By using the commutation relation (2.10), we easily have

$$D_AQ_t=Q_t^{(1)}D_A.$$

Hence

$$D_A\sqrt{1-L_A}=\sqrt{1-L_A+A^*}D_A .$$

Then, for $u\in\mathcal{A}$,

$$\frac{\partial}{\partial t}Q_t^{(1)}D_Au=\sqrt{1-L_A+A^*}Q_t^{(1)}D_Au$$

$$=D_AQ_t\sqrt{1-L_A}u .$$

Therefore

$$G_{(1)}^{\rightarrow}D_Au=G_{(0)}^{\uparrow}\sqrt{1-L_A}u .$$

Now by using the Littlewood-Paley-Stein inequalities (2.15) and (2.16), we have

$$\|D_Au\|_p\lesssim\|G_{(1)}^{\rightarrow}D_Au\|_p=\|G_{(0)}^{\uparrow}\sqrt{1-L_A}u\|_p\lesssim\|\sqrt{1-L_A}u\|_p .$$

The reversed inequality is obtained by the duality. In fact, we note the equality

$$(\sqrt{1-L_A}u,\ \sqrt{1-L_A}v)_2=(D_Au,\ D_Av)_2+(u,\ v)_2 .$$

Then for $u,\ v\in\mathcal{A}$,

$$\begin{aligned}|(\sqrt{1-L_A}u,\ v)_2|&\leqq|(D_Au,\ D_A\sqrt{1-L_A}^{-1}v)_2|+|(u,\ \sqrt{1-L_A}^{-1}v)_2|\\&\leqq\|D_Au\|_q\|D_A\sqrt{1-L_A}^{-1}v\|_p+\|u\|_q\|\sqrt{1-L_A}^{-1}v\|_p\\&\lesssim\|D_Au\|_q\|v\|_p+\|u\|_q\|v\|_p .\end{aligned}$$

Thus we have

$$\|\sqrt{1-L_A}u\|_q\lesssim\|D_Au\|_q+\|u\|_q$$

as desired. □

Remark 3.1. By replacing $1-L_A$ with $-L_A$ in the above proof, we easily show

$$\|D_Au\|_p\lesssim\|\sqrt{-L_A}u\|_p\lesssim\|D_Au\|_p \qquad \text{for } u\in\mathcal{A} .$$

It is expected that similar results hold in higher degree cases. In fact, it is true in the case of the Malliavin calculus. On the contrary, the similar results do not hold in our case since A is not bounded in general. We need a slight modification.

Proposition 3.2. *For $u\in\mathcal{A}(H^{*\otimes n})$, it holds that*

$$\|D_A^ku\|_p\lesssim\|(1-L_A+d\Gamma(A^*)_n)^{k/2}u\|_p , \qquad k=1,\ 2,\ \cdots . \tag{3.2}$$

Proof. We prove this by induction on k. For $k=1$, by the same argument as in the proof of Theorem 3.1,

$$\begin{aligned}\|D_Au\|_p&\lesssim\|G_{(n+1)}^{\rightarrow}D_Au\|_p\\&=\|G_{(n)}^{\uparrow}\sqrt{1-L_A+d\Gamma(A^*)_n}u\|_p\\&\lesssim\|\sqrt{1-L_A+d\Gamma(A^*)_n}u\|_p .\end{aligned}$$

Next, assume (3.2) for k. Then by using (2.11)

$$
\begin{aligned}
\|D_A^{k+1}u\|_p &= \|D_A D_A^k u\|_p \\
&\lesssim \|\sqrt{1-L_A+d\Gamma(A^*)_{n+k}}\, D_A^k u\|_p \\
&= \|D_A^k \sqrt{1-L_A+d\Gamma(A^*)_n}\, u\|_p \\
&\lesssim \|(1-L_A+d\Gamma(A^*)_n)^{k/2}\sqrt{1-L_A+d\Gamma(A^*)_n}\, u\|_p \\
&= \|(1-L_A+d\Gamma(A^*)_n)^{(k+1)/2} u\|_p
\end{aligned}
$$

which completes the proof. □

The reversed inequality does not hold in general. But the following slightly modified inequality holds.

Proposition 3.3. *For $u \in \mathcal{A}(H^{*\otimes n})$, it holds that*

$$
\begin{aligned}
&\|(1-L_A+d\Gamma(A^*)_n)^{k/2}u\|_p \\
&\lesssim 2^k[\|D_A^k u\|_p + \|\sqrt{1-L_A+d\Gamma(A^*)_{n+k-1}}^{-1}(1+d\Gamma(A^*)_{n+k-1})D_A^{k-1}u\|_p \\
&\quad + \|(1+d\Gamma(A^*)_{n+k-2})D_A^{k-2}u\|_p \\
&\quad + \|\sqrt{1-L_A+d\Gamma(A^*)_{n+k-3}}^{-1}(1+d\Gamma(A^*)_{n+k-3})^2 D_A^{k-3}u\|_p \\
&\quad + \|(1+d\Gamma(A^*)_{n+k-4})^2 D_A^{k-4}u\|_p + \cdots \\
&\quad + \begin{cases} \|(1+d\Gamma(A^*)_n)^{k/2}u\|_p] & \textit{for } k\textit{: even} \\ \|\sqrt{1-L_A+d\Gamma(A^*)_n}^{-1}(1+d\Gamma(A^*)_n)^{(k+1)/2}u\|_p] & \textit{for } k\textit{: odd} \end{cases}
\end{aligned}
\tag{3.3}
$$

Proof. We prove this by induction on k. For $k=1$,

$$
\begin{aligned}
&|(\sqrt{1-L_A+d\Gamma(A^*)_n}\,u, v)_2| \\
&= |((1-L_A+d\Gamma(A^*)_n)u, \sqrt{1-L_A+d\Gamma(A^*)_n}^{-1}v)_2| \\
&\leqq |(D_A u, D_A\sqrt{1-L_A+d\Gamma(A^*)_n}^{-1}v)_2| \\
&\quad + |(\sqrt{1-L_A+d\Gamma(A^*)_n}^{-1}(1+d\Gamma(A^*)_n)u, v)_2| \\
&\leqq \|D_A u\|_p \|D_A\sqrt{1-L_A+d\Gamma(A^*)_n}^{-1}v\|_q \\
&\quad + \|\sqrt{1-L_A+d\Gamma(A^*)_n}^{-1}(1+d\Gamma(A^*)_n)u\|_p\|v\|_q \\
&\lesssim \|D_A u\|_p \|\sqrt{1-L_A+d\Gamma(A^*)_n}\sqrt{1-L_A+d\Gamma(A^*)_n}^{-1}v\|_q \\
&\quad + \|\sqrt{1-L_A+d\Gamma(A^*)_n}^{-1}(1+d\Gamma(A^*)_n)u\|_p\|v\|_q \\
&= \{\|D_A u\|_p + \|\sqrt{1-L_A+d\Gamma(A^*)_n}^{-1}(1+d\Gamma(A^*)_n)u\|_p\}\|v\|_q
\end{aligned}
$$

which proves (3.3) for $k=1$.

Assume that (3.3) holds up to k. We first prove in the case that k is odd.

$$
\begin{aligned}
&\|(1-L_A+d\Gamma(A^*)_n)^{(k+1)/2}u\|_p \\
&\quad=\|(1-L_A+d\Gamma(A^*)_n)^{k/2}\sqrt{1-L_A+d\Gamma(A^*)_n}u\|_p \\
&\quad\lesssim 2^k[\|D_A^k\sqrt{1-L_A+d\Gamma(A^*)_n}u\|_p \\
&\qquad+\|\sqrt{1-L_A+d\Gamma(A^*)_{n+k-1}}^{-1}(1+d\Gamma(A^*)_{n+k-1})D_A^{k-1} \\
&\qquad\quad\times\sqrt{1-L_A+d\Gamma(A^*)_n}u\|_p \\
&\qquad+\|(1+d\Gamma(A^*)_{n+k-2})D_A^{k-2}\sqrt{1-L_A+d\Gamma(A^*)_n}u\|_p \\
&\qquad+\|\sqrt{1-L_A+d\Gamma(A^*)_{n+k-3}}^{-1}(1+d\Gamma(A^*)_{n+k-3})^2D_A^{k-3} \\
&\qquad\quad\times\sqrt{1-L_A+d\Gamma(A^*)_n}u\|_p \\
&\qquad+\cdots+\|\sqrt{1-L_A+d\Gamma(A^*)_n}^{-1}(1+d\Gamma(A^*)_n)^{(k+1)/2} \\
&\qquad\quad\times\sqrt{1-L_A+d\Gamma(A^*)_n}u\|_p] \\
&\quad=2^k[\|\sqrt{1-L_A+d\Gamma(A^*)_{n+k}}D_A^ku\|_p+\|(1+d\Gamma(A^*)_{n+k-1})D_A^{k-1}u\|_p \\
&\qquad+\|\sqrt{1-L_A+d\Gamma(A^*)_{n+k-2}}(1+d\Gamma(A^*)_{n+k-2})D_A^{k-2}u\|_p \\
&\qquad+\|(1+d\Gamma(A^*)_{n+k-3})^2D_A^{k-3}u\|_p+\cdots+\|(1+d\Gamma(A^*)_n)^{(k+1)/2}u\|_p] \\
&\quad\lesssim 2^k[\|D_AD_A^ku\|_p+\|\sqrt{1-L_A+d\Gamma(A^*)_{n+k}}^{-1}(1+d\Gamma(A^*)_{n+k})D_A^ku\|_p \\
&\qquad+\|(1+d\Gamma(A^*)_{n+k-1})D_A^{k-1}u\|_p+\|D_A(1+d\Gamma(A^*)_{n+k-2})D_A^{k-2}u\|_p \\
&\qquad+\|\sqrt{1-L_A+d\Gamma(A^*)_{n+k-2}}^{-1}(1+d\Gamma(A^*)_{n+k-2})^2D_A^{k-2}u\|_p \\
&\qquad+\|(1+d\Gamma(A^*)_{n+k-3})^2D_A^{k-3}u\|_p+\cdots+\|(1+d\Gamma(A^*)_n)^{(k+1)/2}u\|_p] \\
&\quad\leqq 2^k[\|D_A^{k+1}u\|_p+\|\sqrt{1-L_A+d\Gamma(A^*)_{n+k}}^{-1}(1+d\Gamma(A^*)_{n+k})D_A^ku\|_p \\
&\qquad+\|(1+d\Gamma(A^*)_{n+k-1})D_A^{k-1}u\|_p+\|(1+d\Gamma(A^*)_{n+k-1})D_A^{k-1}u\|_p \\
&\qquad+\|\sqrt{1-L_A+d\Gamma(A^*)_{n+k-2}}^{-1}(1+d\Gamma(A^*)_{n+k-2})^2D_A^{k-2}u\|_p \\
&\qquad+\|(1+d\Gamma(A^*)_{n+k-3})^2D_A^{k-3}u\|_p+\cdots+\|(1+d\Gamma(A^*)_n)^{(k+1)/2}u\|_p] \\
&\quad\leqq 2^{k+1}[\|D_A^{k+1}u\|_p+\|\sqrt{1-L_A+d\Gamma(A^*)_{n+k}}^{-1}(1+d\Gamma(A^*)_{n+k})D_A^ku\|_p \\
&\qquad+\|(1+d\Gamma(A^*)_{n+k-1})D_A^{k-1}u\|_p \\
&\qquad+\|\sqrt{1-L_A+d\Gamma(A^*)_{n+k-2}}^{-1}(1+d\Gamma(A^*)_{n+k-2})^2D_A^{k-2}u\|_p \\
&\qquad+\|(1+d\Gamma(A^*)_{n+k-3})^2D_A^{k-3}u\|_p+\cdots+\|(1+d\Gamma(A^*)_n)^{(k+1)/2}u\|_p].
\end{aligned}
$$

Here we used that for $v\in\mathcal{A}(H^{*\otimes m})$, $l\in N$

$$
\begin{aligned}
|D_A(1+d\Gamma(A^*)_m)^lv(x)|_{H^{*\otimes(m+1)}}&=|(1+1_{H_*}\otimes d\Gamma(A^*)_m)^lD_Av(x)|_{H^{*\otimes(m+1)}} \\
&\leqq|(1+d\Gamma(A^*)_{m+1})^lD_Av(x)|_{H^{*\otimes(m+1)}}, \qquad (3.4)
\end{aligned}
$$

To see this, we note that

$$
\begin{aligned}
1_{H^{*\otimes(m+1)}}+1_{H^*}\otimes d\Gamma(A^*)_m &= 1_{H^{*\otimes(m+1)}}+\sum_{j=2}^{m+1} 1_{H^*}\otimes\cdots\otimes\overset{j}{\check{A}^*}\otimes\cdots\otimes 1_{H^*} \\
&\leqq 1_{H^{*\otimes(m+1)}}+\sum_{j=1}^{m+1} 1_{H^*}\otimes\cdots\otimes\overset{j}{\check{A}^*}\otimes\cdots\otimes 1_{H^*} \\
&= 1_{H^{*\otimes(m+1)}}+d\Gamma(A^*)_{m+1}
\end{aligned}
$$

since A^* is positive definite. Now (3.4) easily follows.

Similarly, we can prove in the case that k is even. This completes the proof. □

By noting that $\sqrt{1-L_A+d\Gamma(A^*)_n}^{-1}$ is bounded, we have

$$
\|(1-L_A+d\Gamma(A^*)_n)^{k/2}u\|_p \lesssim 2^k \sum_{l=0}^{k} \|(1+d\Gamma(A^*)_{n+k-l})^{[(l+1)/2]} D_A^{k-l} u\|_p \tag{3.5}
$$

where $[\cdot]$ stands for Gauss' symbol.

Now we proceed to estimate $(1+d\Gamma(A^*)_{n+m})^l D_A^m$. We need the following lemma.

Lemma 3.4. *For* $\alpha>\varepsilon>0$, $(1+d\Gamma(A^*)_n)^{\alpha-\varepsilon}(1-L_A+d\Gamma(A^*)_n)^{-\alpha}$ *is a bounded operator in* $L^p(\mu)\otimes H^{*\otimes n}$ *with*

$$
\begin{aligned}
&\|(1+d\Gamma(A^*)_n)^{\alpha-\varepsilon}(1-L_A+d\Gamma(A^*)_n)^{-\alpha}\|_{\mathcal{L}(L^p(\mu)\otimes H^{*\otimes n})} \\
&\leqq 2^{\alpha-\varepsilon-(1/2)}+2^{-1/2}\Gamma(\alpha)^{-1}\Gamma(2\alpha-2\varepsilon+1)^{1/2}\Gamma(\varepsilon)
\end{aligned}
$$

where $\|\cdot\|_{\mathcal{L}(L^p(\mu)\otimes H^{*\otimes n})}$ *denotes the operator norm in* $L^p(\mu)\otimes H^{*\otimes n}$.

Proof. We use the spectral decomposition of $1+d\Gamma(A^*)_n$;

$$
1+d\Gamma(A^*)_n=\int_1^\infty \lambda dE_\lambda .
$$

Further, $(1-L_A+d\Gamma(A^*)_n)^{-\alpha}$ is expressed as follows;

$$
(1-L_A+d\Gamma(A^*)_n)^{-\alpha}u=\Gamma(\alpha)^{-1}\int_0^\infty dt\int_1^\infty e^{-\lambda t}t^{\alpha-1}dE_\lambda T_t u .
$$

Hence we have

$$
\begin{aligned}
&\|(1+d\Gamma(A^*)_n)^{\alpha-\varepsilon}(1-L_A+d\Gamma(A^*)_n)^{-\alpha}u\|_p \\
&=\left\|\left|\int_0^\infty dt\int_1^\infty \lambda^{\alpha-\varepsilon}\Gamma(\alpha)^{-1}e^{-\lambda t}t^{\alpha-1}dE_\lambda T_t u\right|_{H^{*\otimes n}}\right\|_p \\
&=\left\|\int_0^\infty dt\Gamma(\alpha)^{-1}t^{\alpha-1}\left|\int_1^\infty \lambda^{\alpha-\varepsilon}e^{-\lambda t}dE_\lambda T_t u\right|_{H^{*\otimes n}}\right\|_p.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\left|\int_1^\infty \lambda^{\alpha-\varepsilon}e^{-\lambda t}dE_\lambda T_t u\right|^2_{H^{*\otimes n}} \\
&=\int_1^\infty \lambda^{2\alpha-2\varepsilon}e^{-2\lambda t}d(E_\lambda T_t u, T_t u)
\end{aligned}
$$

$$=\lambda^{2\alpha-2\varepsilon}e^{-2\lambda t}|_1^\infty-\int_1^\infty\{(2\alpha-2\varepsilon)\lambda^{2\alpha-2\varepsilon-1}-2\lambda^{2\alpha-2\varepsilon}t\}e^{-2\lambda t}(E_\lambda T_t u,\ T_t u)d\lambda$$

$$\leqq\int_1^\infty\lambda^{2\alpha-2\varepsilon-1}(2t\lambda-(2\alpha-2\varepsilon))e^{-2\lambda t}(E_\lambda T_t u,\ T_t u)d\lambda$$

$$\leqq\int_1^\infty 2\lambda^{2\alpha-2\varepsilon}te^{-2\lambda t}|T_t u|_{H^{*\otimes n}}^2 d\lambda$$

$$\leqq t|T_t u|_{H^{*\otimes n}}^2\int_0^\infty 2(\lambda+1)^{2\alpha-2\varepsilon}e^{-2(\lambda+1)t}d\lambda$$

$$\leqq e^{-2t}t|T_t u|_{H^{*\otimes n}}^2\int_0^\infty 2\cdot 2^{2\alpha-2\varepsilon-1}(\lambda^{2\alpha-2\varepsilon}+1^{2\alpha-2\varepsilon})e^{-2\lambda t}d\lambda$$

$$\leqq e^{-2t}t|T_t u|_{H^{*\otimes n}}^2\int_0^\infty\{(2\lambda)^{2\alpha-2\varepsilon}+2^{2\alpha-2\varepsilon}\}e^{-2\lambda t}d\lambda$$

$$\leqq e^{-2t}|T_t u|_{H^{*\otimes n}}^2\left\{\int_0^\infty\frac{1}{2}\left(\frac{\mu}{t}\right)^{2\alpha-2\varepsilon}e^{-\mu}d\mu+2^{2\alpha-2\varepsilon-1}\right\}\qquad(\mu=2t\lambda)$$

$$\leqq e^{-2t}|T_t u|_{H^{*\otimes n}}^2\{2^{-1}\Gamma(2\alpha-2\varepsilon+1)t^{2\varepsilon-2\alpha}+2^{2\alpha-2\varepsilon-1}\}$$

$$\leqq e^{-2t}|T_t u|_{H^{*\otimes n}}^2\{2^{-1/2}\Gamma(2\alpha-2\varepsilon+1)^{1/2}t^{\varepsilon-\alpha}+2^{\alpha-\varepsilon-(1/2)}\}^2 .$$

Thus we have,

$$\|(1+d\Gamma(A^*)_n)^{\alpha-\varepsilon}(1-L_A+d\Gamma(A^*)_n)^{-\alpha}u\|_p$$

$$\leqq\left\|\int_0^\infty dt\Gamma(\alpha)^{-1}t^{\alpha-1}e^{-t}|T_t u|_{H^{*\otimes n}}\{t^{\varepsilon-\alpha}2^{-1/2}\Gamma(2\alpha-2\varepsilon+1)^{1/2}+2^{\alpha-\varepsilon-(1/2)}\}\right\|_p$$

$$=\|u\|_p\Gamma(\alpha)^{-1}\left\{2^{-1/2}\Gamma(2\alpha-2\varepsilon+1)^{1/2}\int_0^\infty e^{-t}t^{\varepsilon-1}dt+2^{\alpha-\varepsilon-(1)/2)}\int_0^\infty e^{-t}t^{\alpha-1}dt\right\}$$

$$=\|u\|_p\{2^{-1/2}\Gamma(\alpha)^{-1}\Gamma(2\alpha-2\varepsilon+1)^{1/2}\Gamma(\varepsilon)+2^{\alpha-\varepsilon-(1/2)}\}$$

which completes the proof. □

Now we have the following proposition.

Proposition 3.5. *For any $l,\ m\in\boldsymbol{N}$, it holds that for $u\in\mathcal{A}(H^{*\otimes n})$,*

$$\|(1+d\Gamma(A^*)_{n+m})^l D_A^m u\|_p\lesssim 2^l\|(1-L_A+d\Gamma(A^*)_n)^{(m+2l+1)/2}u\|_p .\tag{3.6}$$

Proof. By Lemma 3.4, we have

$$\|(1+d\Gamma(A^*)_n)^l(1-L_A+d\Gamma(A^*)_n)^{-l-(1/2)}u\|_p\lesssim 2^l\|u\|_p .$$

Hence by using Proposition 3.2,

$$\|(1+d\Gamma(A^*)_n)^l D_A^m u\|_p$$

$$=\|(1+d\Gamma(A^*)_n)^l(1-L_A+d\Gamma(A^*)_n)^{-l-(1/2)}(1-L_A+d\Gamma(A^*)_{n+m})^{l+(1/2)}D_A^m u\|_p$$

$$\lesssim 2^l\|D_A^m(1-L_A+d\Gamma(A^*)_n)^{l+(1/2)}u\|_p$$

$$\lesssim 2^l\|(1-L_A+d\Gamma(A^*)_n)^{(m+2l+1)/2}u\|_p$$

which completes the proof. □

We summarize in a theorem.

Theorem 3.6. *For* $p\in(1,\infty)$, $\|(1-L_A+d\Gamma(A_*)_n)^{k/2}u\|_p$, $k=0, 1, 2, \cdots$, *and* $\|(1+d\Gamma(A^*)_{n+m})^l D_A^m u\|_p$, $l, m=0, 1, 2, \cdots$, *are equivalent systems of seminorms.*

If $p=2$, we can give an equivalent norms. In fact note that in this case, we can take $\varepsilon=0$ in Lemma 3.4, i.e. $\sqrt{1+d\Gamma(A^*)_n}\sqrt{1-L_A+d\Gamma(A^*)_n}^{-1}$ is a bounded operator with operator norm 1. So we have the following;

Theorem 3.7. *Suppose that* $p=2$. *Then for any* $l, m\in\boldsymbol{N}$ *such that* $l+m=k$, *it holds that for* $u\in\mathcal{A}(H^{*\otimes n})$,

$$\|(1+d\Gamma(A^*)_{n+m})^{l/2}D_A^m u\|_2\lesssim\|(1-L_A+d\Gamma(A^*)_n)^{k/2}u\|_2 . \tag{3.7}$$

Conversely, it holds that for $u\in\mathcal{A}(H^{*\otimes n})$,

$$\|(1-L_A+d\Gamma(A^*)_n)^{k/2}u\|_2\lesssim 2^k\sum_{l+m=k}\|(1+d\Gamma(A^*)_{n+m})^{l/2}D_A^m u\|_2 . \tag{3.8}$$

4. Sobolev spaces over an abstract Wiener space

We are now in position to define Sobolev space over an abstract Wiener space. Let notations be as before. For $s\in\boldsymbol{R}$, $p>1$, $n\in\boldsymbol{Z}_+$, we define a norm $\|\cdot\|_{s,p}$ by

$$\|u\|_{s,p}:=\|(1-L_A+d\Gamma(A^*)_n)^{s/2}u\|_p \qquad \text{for } u\in\mathcal{A}(H^{*\otimes n}). \tag{4.1}$$

We remark that the above definition is well-defined. First, let us recall that for $\tau\geqq0$, $(1-L_A+d\Gamma(A^*)_n)^{-\tau}$ is a contraction operator in $L^p(\mu; H^{*\otimes n})$. In fact, we can express $(1-L_A+d\Gamma(A^*)_n)^{-\tau}$ by using the semigroup $\{P_t^{(n)}\}$ as follows;

$$(1-L_A+d\Gamma(A^*)_n)^{-\tau}=\frac{1}{\Gamma(\tau)}\int_0^\infty t^{\tau-1}P_t^{(n)}dt .$$

By noting $\|P_t^{(n)}\|_{\mathcal{L}(L^p(\mu;H^{*\otimes n}))}\leqq e^{-t}$, we can see that $(1-L_A+d\Gamma(A^*)_n)^{-\tau}$ is a contraction operator in $L^p(\mu; H^{*\otimes n})$. Hence, for $s\leqq0$, (4.1) is well-defined for not only $u\in\mathcal{A}(H^{*\otimes n})$ but also all $u\in L^p(\mu; H^{*\otimes n})$.

For $s=2k$, $k\in\boldsymbol{Z}_+$, $(1-L_A+d\Gamma(A^*)_n)^k u$ is well-defined by noting the explicit form (2.5). For general $s\geqq0$ by choosing a $k\in\boldsymbol{Z}_+$, so that $2k\geqq s$, and define

$$(1-L_A+d\Gamma(A^*)_n)^{s/2}u=(1-L_A+d\Gamma(A^*)_n)^{-(2k-s)/2}(1-L_A+d\Gamma(A^*)_n)^k u .$$

In another way, by using the spectral decomposition of $1-L_A+d\Gamma(A^*)_n$, we can define $(1-L_A+d\Gamma(A^*)_n)^{s/2}$ as a self-adjoint operator in $L^2(\mu; H^{*\otimes n})$. In this case, noticing that

$$\mathcal{A}(H^{*\otimes n})\subseteqq\bigcap_{k=1}^\infty \operatorname{Dom}((1-L_A+d\Gamma(A^*)_n)^k),$$

(4.1) is well-defined for $u\in\mathcal{A}(H^{*\otimes n})$.

Now the Sobolev spaces can be defined as follows;

Definition 4.1. For each $s\in\boldsymbol{R}$, $1<p<\infty$, the Sobolev space $W^{s,p}(H^{*\otimes n})$ is a completion of $\mathcal{A}(H^{*\otimes n})$ with respect to the norm $\|\cdot\|_{s,p}$.

The following proposition is fundamental.

Proposition 4.1. (i) *For $p'\geqq p$ and $s'\geqq s$, it holds that*

$$\|u\|_{s,p}\leqq\|u\|_{s',p'} \qquad \textit{for } u\in\mathcal{A}(H^{*\otimes n}).$$

(ii) $\|\cdot\|_{s,p}$, $1<p<\infty$, $s\in\boldsymbol{R}$ *satisfy the consistency condition, i.e., for any* (s, p), (s', p') *and* $u_k\in\mathcal{A}(H^{*\otimes n})$, $k\in\boldsymbol{N}$, *such that* $\|u_k\|_{s,p}\to 0$ *and* $\|u_k-u_l\|_{s',p'}\to 0$ *as* $k, l\to\infty$, *it holds that* $\|u_k\|_{s',p'}\to 0$ *as* $k\to\infty$.

Proof. To prove (i), it suffices to show that

$$\|u\|_{s,p}\leqq\|u\|_{s',p} \tag{4.2}$$

since $\|u\|_{s,p}\leqq\|u\|_{s,p'}$, for $p\leqq p'$. But, by noticing that $(1-L_A+d\Gamma(A^*)_n)^{-(s'-s)}$ is contractive, (4.2) follows easily.

We shall show (ii). To do this, we may assume that $p\leqq p'$ and $s\leqq s'$ by (i). Set $v_k=(1-L_A+d\Gamma(A^*)_n)^{s'/2}u_k$. Then $\|v_k-v_l\|_{p'}\to 0$ as $k, l\to\infty$. Hence, there exists a $v\in L^{p'}(\mu; H^{*\otimes n})$ such that $\|v_k-v\|_{p'}\to 0$. It is enough to show $v=0$. Now we notice

$$\|(1-L_A+d\Gamma(A^*)_n)^{(s-s')/2}v_k\|_p=\|u_k\|_{p,s}\longrightarrow 0 \qquad \text{as } k\to\infty$$

and

$$(1-L_A+d\Gamma(A^*)_n)^{(s'-s)/2}w\in\bigcap_{p\geq 1}L^p(\mu; H^{*\otimes n}) \qquad \text{for } w\in\mathcal{A}(H^{*\otimes n}).$$

Thus, for $w\in\mathcal{A}(H^{*\otimes n})$,

$$\begin{aligned}
&\int_E(v(x), w(x))_{H^{*\otimes n}}\mu(dx)\\
&=\lim_{k\to\infty}\int_E(v_k(x), w(x))_{H^{*\otimes n}}\mu(dx)\\
&=\lim_{k\to\infty}\int_E(1-L_A+d\Gamma(A^*)_n)^{(s-s')/2}\\
&\qquad\times v_k(x), (1-L_A+d\Gamma(A^*)_n)^{(s'-s)/2}w(x))_{H^{*\otimes n}}\mu(dx)\\
&=0
\end{aligned}$$

which implies $v=0$. □

By the above proposition, we may regard $W^{s',p'}(H^{*\otimes n})\subseteq W^{s,p}(H^{*\otimes n})$ for $p\leqq p'$ and $s\leqq s'$. In particular, by noticing $W^{0,p}(H^{*\otimes n})=L^p(\mu; H^{*\otimes n})$, $W^{s,p}(H^{*\otimes n})$ is realized as a subspace of $L^p(\mu; H^{*\otimes n})$. In the sequel, we use this realization.

Proposition 4.2. (i) *For $1<p<\infty$ and $s\geqq 0$*

$$W^{s,p}(H^{*\otimes n})=(1-L_A+d\Gamma(A^*)_n)^{-s/2}(L^p(\mu; H^{*\otimes n})).$$

In other words, $\mathcal{A}(H^{*\otimes n})$ *is dense in* $(1-L_A+d\Gamma(A^*)_n)^{-s/2}(L^p(\mu;H^{*\otimes n}))$ *under the norm* $\|\cdot\|_{s,p}$.

(ii) *For* $s\geqq 0$, *the dual space of* $W^{s,p}(H^{*\otimes n})$ *is isomorphic to* $W^{-s,q}(H^{*\otimes n})$ *where* $1/p+1/q=1$. *Further for* $F\in W^{s,p}(H^{*\otimes n})$ *and* $G\in L^p(\mu;H^{*\otimes n})\subseteqq W^{-s,q}(H^{*\otimes n})$, *it holds that*

$$ {}_{W^{s,p}(H^{*\otimes n})}(F,G)_{W^{-s,p}(H^{*\otimes n})}=\int_E(F(x),G(x))_{H^{*\otimes n}}\mu(dx). $$

Proof. We first show (i). Set

$$ E=(1-L_A+d\Gamma(A^*)_n)^{-s/2}(L^p(\mu;H^{*\otimes n})). $$

Then E is a Banach space with the norm $\|\cdot\|_{s,p}$. In fact,

$$ (1-L_A+d\Gamma(A^*)_n)^{-s/2}:(L^p(\mu;H^{*\otimes n}),\|\cdot\|_p)\longrightarrow(E,\|\cdot\|_{s,p}) $$

gives rise to an isometric isomorphism. By this isomorphism, it is enough to prove that $(1-L_A+d\Gamma(A^*)_n)^{s/2}(\mathcal{A}(H^{*\otimes n}))$ is dense in $(L^p(\mu;H^{*\otimes n}),\|\cdot\|_p)$.

We first show this in the case that $s/2$ is a non-negative integer, say k. We use Nelson's technique. To do so, we introduce another class $\tilde{\mathcal{A}}(H^{*\otimes n})$. Recall that $\mathcal{A}$ is the set of all functions of the form (2.1). In the expression (2.1), we assumed that $l_1,\cdots,l_n\in C^\infty(A^*)\cap B^*$. By relaxing this condition as $l_1,\cdots,l_n\in C^\infty(A^*)$ we can define a wider class $\tilde{\mathcal{A}}$. Similarly, we define $\tilde{\mathcal{A}}(H^{*\otimes n})=\tilde{\mathcal{A}}\otimes C^\infty(A^*)\otimes\cdots\otimes C^\infty(A^*)$ (algebraic tensor product). The advantage to introduce this class $\tilde{\mathcal{A}}(H^{*\otimes n})$ is that it is stable under actions of $1-L_A+d\Gamma(A^*)_n$ and $P_t^{(n)}$, which can be seen from the explicit forms (2.4) and (2.5).

Take $v\in L^q(\mu;H^{*\otimes n})$ so that $\langle u,v\rangle=0$ for $u\in(1-L_A+d\Gamma(A^*)_n)^k(\tilde{\mathcal{A}}(H^{*\otimes n}))$. Here $\langle u,v\rangle=\int_B(u(x),v(x))_{H^{*\otimes n}}\mu(dx)$. For $u\in\tilde{\mathcal{A}}(H^{*\otimes n})$, set

$$ \varphi(t)=\langle v,P_t^{(n)}u\rangle \qquad \text{for } t\geqq 0. $$

$\varphi(t)\to 0$ as $t\to\infty$ since $\|P_t^{(n)}\|_{\mathcal{L}(L^p(\mu;H^{*\otimes n}))}\leqq e^{-t}$. By the assumption, we have

$$ \begin{aligned} \frac{d^k}{dt^k}\varphi(t)&=\langle v,(1-L_A+d\Gamma(A^*)_n)^k P_t^{(n)}u\rangle \\ &=0. \end{aligned} $$

Here we used the fact that $P_t^{(n)}u\in\tilde{\mathcal{A}}(H^{*\otimes n})$ since $\tilde{\mathcal{A}}(H^{*\otimes n})$ is stable under the action of $P_t^{(n)}$. Thus φ is a polynomial of degree less than k. Now it is easy to see that $\varphi(t)\equiv 0$ and in particular,

$$ \langle v,u\rangle=0 \qquad \text{for } u\in\tilde{\mathcal{A}}(H^{*\otimes n}). $$

Therefore we have $v=0$ since $\tilde{\mathcal{A}}(H^{*\otimes n})$ is dense in $L^p(\mu;H^{*\otimes n})$. By the Hahn-Banach theorem we have that $(1-L_A+d\Gamma(A^*)_n)^k(\tilde{\mathcal{A}}(H^{*\otimes n}))$ is dense in $L^p(\mu;H^{*\otimes n})$. From this, it is easy to see that $(1-L_A+d\Gamma(A^*)_n)^k(\mathcal{A}(H^{*\otimes n}))$ is dense. In fact, an element of $(1-L_A+d\Gamma(A^*)_n)^k(\mathcal{A}(H^{*\otimes n}))$ can be approximated by elements of $(1-L_A+d\Gamma(A^*)_n)^k(\mathcal{A}(H^{*\otimes n}))$ in $L^p(\mu;H^{*\otimes n})$ by noticing the explicit form of $1-L_A+d\Gamma(A^*)_n$ and the assumption that $C^\infty(A^*)\cap B^*$ is dense in $\mathrm{Dom}(A^{*l})$ under the graph norm of

A^{*l} for any $l\in \boldsymbol{Z}_+$.

Next we show that for general $s\geqq 0$, $(1-L_A+d\Gamma(A^*)_n)^{s/2}\mathcal{A}(H^{*\otimes n})$ is dense in $(L^p(\mu; H^{*\otimes n}), \|\cdot\|_p)$. Take an integer k so that $0\leqq 2k-s<2$. By the above result, we have

$$\begin{aligned}
&\overline{(1-L_A+d\Gamma(A^*)_n)^{s/2}(\mathcal{A}(H^{*\otimes n}))}^{\|\cdot\|_p}\\
&=\overline{(1-L_A+d\Gamma(A^*)_n)^{-(2k-s)/2}(1-L_A+d\Gamma(A^*)_n)^k(\mathcal{A}(H^{*\otimes n}))}^{\|\cdot\|_p}\\
&\supseteqq(1-L_A+d\Gamma(A^*)_n)^{-(2k-s)/2}\left(\overline{(1-L_A+d\Gamma(A^*)_n)^k(\mathcal{A}(H^{*\otimes n}))}^{\|\cdot\|_p}\right)\\
&=(1-L_A+d\Gamma(A^*)_n)^{-(2k-s)/2}(L^p(\mu; H^{*\otimes n}))\\
&\supseteqq(1-L_A+d\Gamma(A^*)_n)^{-1}(L^p(\mu; H^{*\otimes n}))\\
&=\mathrm{Dom}\,(1-L_A+d\Gamma(A^*)_n)\\
&\supseteqq\mathcal{A}(H^{*\otimes n}).
\end{aligned}$$

Here $\overline{}^{\|\cdot\|_p}$ denotes the closure in $L^p(\mu; H^{*\otimes n})$. Since $\mathcal{A}(H^{*\otimes n})$ is dense in $L^p(\mu; H^{*\otimes n})$, we have

$$\overline{(1-L_A+d\Gamma(A^*)_n)^{s/2}(\mathcal{A}(H^{*\otimes n}))}^{\|\cdot\|_p}\supseteqq\overline{\mathcal{A}(H^{*\otimes n})}^{\|\cdot\|_p}=L^p(\mu; H^{*\otimes n})$$

as desired.

Secondly, we show (ii). By (i), we obtained the isometric isomorphism A and C as follows;

$$\begin{aligned}
A&=(1-L_A+d\Gamma(A^*)_n)^{s/2}: W^{s,p}(H^{*\otimes n})\longrightarrow L^p(\mu; H^{*\otimes n}),\\
C&=(1-L_A+d\Gamma(A^*)_n)^{-s/2}: W^{-s,q}(H^{*\otimes n})\longrightarrow L^q(\mu; H^{*\otimes n}).
\end{aligned}$$

As is well-known, $L^p(\mu; H^{*\otimes n})^*$ and $L^q(\mu; H^{*\otimes n})$ are isometrically isomorphic and hence $W^{-s,q}(H^{*\otimes n})$ and $W^{s,p}(H^{*\otimes n})^*$ are isometrically isomorphic under the isomorphism A^*C. Moreover for $F\in W^{s,p}(H^{*\otimes n})$ and $G\in L^q(\mu; H^{*\otimes n})\subseteqq W^{-s,q}(H^{*\otimes n})$, we have

$$\begin{aligned}
&{}_{W^{s,p}(H^{*\otimes n})}\langle F, G\rangle_{W^{-s,q}(H^{*\otimes n})}\\
&={}_{W^{s,p}(H^{*\otimes n})}\langle F, A^*CG\rangle_{W^{-s,q}(H^{*\otimes n})}\\
&={}_{L^p(\mu;H^{*\otimes n})}\langle AF, CG\rangle_{L^q(\mu;H^{*\otimes n})}\\
&=\int_B(AF(x), CG(x))_{H^{*\otimes n}}\mu(dx)\\
&=\int_B((1-L_A+d\Gamma(A^*)_n)^{s/2}F(x), (1-L_A+d\Gamma(A^*)_n)^{-s/2}G(x))_{H^{*\otimes n}}\mu(dx)\\
&=\int_B(F(x), G(x))_{H^{*\otimes n}}\mu(dx)
\end{aligned}$$

which completes the proof. □

We set

$$W^{-\infty,1+}(H^{*\otimes n}):=\bigcup_{\substack{s\in R\\ p>1}}W^{s,p}(H^{*\otimes n}) \tag{4.3}$$

$$W^{\infty,\infty-}(H^{*\otimes n}):=\bigcap_{\substack{s\in R\\ p>1}}W^{s,p}(H^{*\otimes n}). \tag{4.4}$$

$W^{\infty,\infty-}(H^{*\otimes n})$ is a Fréchet space and $W^{-\infty,1+}(H^{*\otimes n})$ is its dual space.

Theorem 4.3. *For any $s\in\boldsymbol{R}$, $p>1$, $n\in\boldsymbol{Z}_+$ and $t>0$ the following operators are continuous.*

$$D_A: W^{s,p}(H^{*\otimes n})\longrightarrow W^{s-1,p}(H^{*\otimes(n+1)}) \tag{4.5}$$

$$D_A^*: W^{s,p}(H^{*\otimes(n+1)})\longrightarrow W^{s-1,p}(H^{*\otimes n}) \tag{4.6}$$

$$L_A: W^{s,p}(H^{*\otimes n})\longrightarrow W^{s-2,p}(H^{*\otimes n}) \tag{4.7}$$

$$P_t^{(n)}: W^{s,p}(H^{*\otimes n})\longrightarrow W^{\infty,p}(H^{*\otimes n}) \tag{4.8}$$

where $W^{\infty,p}(H^{*\otimes n})$ is a Fréchet space defined by

$$W^{\infty,p}(H^{*\otimes n}):=\bigcap_{s\in R}W^{s,p}(H^{*\otimes n}).$$

Moreover $W^{\infty,\infty-}=W^{\infty,\infty-}(\boldsymbol{R})$ forms an algebra.

Proof. By using the commutation relation (2.12) and Proposition 3.2, we have

$$\begin{aligned}\|D_Au\|_{s-1,p}&=\|(1-L_A+d\Gamma(A^*)_{n+1})^{(s-1)/2}D_Au\|_p\\&=\|D_A(1-L_A+d\Gamma(A^*)_n)^{(s-1)/2}u\|_p\\&\lesssim\|\sqrt{1-L_A+d\Gamma(A^*)_n}(1-L_A+d\Gamma(A^*)_n)^{(s-1)/2}u\|_p\\&=\|u\|_{s,p}\end{aligned}$$

which proves D_A in (4.5) is continuous. By the duality, we can prove that D_A^* in (4.6) is continuous. So L_A in (4.7) is continuous since $L_A=-D_A^*D_A$.

To prove that $P_t^{(n)}$ is continuous, we follow Sugita [23]. By noting (2.11), we have for $u\in\mathcal{A}(H^{*\otimes n})$,

$$D(P_t^{(n)}u)(x)[h]=\int_B e^{-t(1+d\Gamma(A^*)_n)}Df(e^{-tA}x+\sqrt{1-e^{-2tA}}y)[e^{-tA}h]\mu(dy).$$

Hence

$$\begin{aligned}&D_A(P_t^{(n)}u)(x)[h]\\&=\int_B e^{-t(1+d\Gamma(A^*)_n)}Df(e^{-tA}x+\sqrt{1-e^{-2tA}}y)[\sqrt{A}e^{-tA}h]\mu(dy)\\&=\int_B e^{-t(1+d\Gamma(A^*)_n)}D_y\{f(e^{-tA}x+\sqrt{1-e^{-2tA}}y)\}[(\sqrt{A}e^{-t}/\sqrt{1-e^{-tA}})h]\mu(dy)\\&=\int_B e^{-t(1+d\Gamma(A^*)_n)}f(e^{-tA}x+\sqrt{1-e^{-2tA}}y)D^*((\sqrt{A}e^{-tA}/\sqrt{1-e^{-tA}})h)\mu(dy)\\&=\int_B e^{-t(1+d\Gamma(A^*)_n)}f(e^{-tA}x+\sqrt{1-e^{-2tA}}y)\langle(\sqrt{A}e^{-tA}/\sqrt{1-e^{-tA}})h,\ y\rangle\mu(dy)\end{aligned}$$

where $\langle(\sqrt{A}e^{-tA}/\sqrt{1-e^{-tA}})h,\ y\rangle$ is a Wiener integral (so it is defined almost surely).

On the other hand, for any $g\in L^p(\mu; H^{*\otimes n})$, a linear mapping

$$h \longmapsto \int_B g(y)\langle h, y\rangle \mu(dy) \tag{4.9}$$

is of Hilbert-Schmidt class operator from H to $H^{*\otimes n}$. To see this, set

$$\Phi_g(h)=\int_B g(y)\langle h, y\rangle \mu(dy).$$

For a complete orthonormal system $\{h_i\}_{i=1}^{\infty}$ in H, $\{\langle h_i, \cdot\rangle\}_{i=1}^{\infty}$ forms a complete orthonormal system in the Wiener homogeneous chaos of order 1. Hence we have

$$|\Phi_g(h)|_{\mathcal{L}_{(2)}(H;H^{*\otimes n})}=\sum_{i=1}^{\infty}|\Phi_g(h_i)|_{H^{*\otimes n}}^2=\sum_{i=1}^{\infty}\left|\int_B g(y)\langle h, y\rangle\mu(dy)\right|_{H^{*\otimes n}}^2=\|J_1 g\|_2$$

where $\mathcal{L}_{(2)}(H; H^{*\otimes n})$ denotes a set of all Hilbert-Schmidt class operators from H to $H^{*\otimes n}$, $|\cdot|_{\mathcal{L}_{(2)}(H;H^{*\otimes n})}$ is its Hilbert-Schmidt norm and J_1 is a projection operator to the space of $H^{*\otimes n}$-valued Wiener homogeneous chaos of order 1, which we denote $Z_1(H^{*\otimes n})$. Since L^p-norm and L^2-norm are equivalent on the subspace $Z_1(H^{*\otimes n})$ and J_1 is a bounded operator also on $L^p(\mu; H^{*\otimes n})$ (see Sugita [22] Lemma 1.1 and Theorem 2.3), we have

$$\|J_1 g\|_2 \lesssim \|g\|_p .$$

Moreover, for a bounded operator $K: H\to H$, the composite mapping $\Phi_g\circ K$ is also of Hilbert-Schmidt class and

$$|\Phi_g\circ K|_{\mathcal{L}_{(2)}(H;H^{*\otimes n})}\leqq|\Phi_g|_{\mathcal{L}_{(2)}(H;H^{*\otimes n})}\|K\|_{\mathcal{L}(H;H)}\lesssim\|g\|_p\|K\|_{\mathcal{L}(H;H)} .$$

We can easily see that $\sqrt{A}e^{-A}/\sqrt{1-e^{-tA}}$ is a bounded operator by using the spectral decomposition. In fact, by noting that $\sqrt{\lambda}e^{-t\lambda}/\sqrt{1-e^{-t\lambda}}\leqq 1/\sqrt{2t}$ for $\lambda\geqq 0$, we have $\|(\sqrt{A}e^{-tA}/\sqrt{1-e^{-tA}})\|_{\mathcal{L}(H)}\leqq 1/\sqrt{2t}$. Thus we have

$$\begin{aligned}&|D_A(P_t^{(n)}u)(x)|_{H^{*\otimes(n+1)}}\\ &\lesssim\left\{\int_B|e^{-t(1+d\Gamma(A^*)_n)}u(e^{-tA}x+\sqrt{1-e^{-2tA}}y)|_{H^{*\otimes n}}^p\right\}^{1/p}\|(\sqrt{A}e^{-tA}/\sqrt{1-e^{-tA}})\|_{\mathcal{L}(H)}\\ &\leqq\left\{\int_B|u(e^{-tA}x+\sqrt{1-e^{-2tA}}y)|_{H^{*\otimes n}}^p\right\}^{1/p}\|(\sqrt{A}e^{-tA}/\sqrt{1-e^{-tA}})\|_{\mathcal{L}(H)} .\end{aligned}$$

Thus we obtain

$$\begin{aligned}\|D_A P_t^{(n)}u\|_p&\lesssim\left\{\int_B\int_B|u(e^{-tA}x+\sqrt{1-e^{-2tA}}y)|_{H^{*\otimes n}}^p\mu(dy)\mu(dx)\right\}^{1/p}/\sqrt{2t}\\ &\leqq\|u\|_p/\sqrt{2t} .\end{aligned}$$

On the other hand,

$$\|(1+d\Gamma(A^*)_n)P_t^{(n)}u\|_p$$
$$=\left\{\int_B\left|\int_B(1+d\Gamma(A^*)_n)e^{-t(1+d\Gamma(A^*)_n)}u(e^{-tA}x+\sqrt{1-e^{-2tA}}y)\mu(dy)\right|_{H^{*\otimes n}}\mu(dx)\right\}^{1/p}$$
$$\leqq\|(1+d\Gamma(A^*)_n)e^{-t(1+d\Gamma(A^*)_n)}\|_{\mathcal{L}(H)}\left\{\int_B\int_B|u(e^{-tA}x+\sqrt{1-e^{-2tA}}y)|_{H^{*\otimes n}}\mu(dy)\mu(dx)\right\}^{1/p}$$
$$\leqq\|u\|_p/et$$

since $\lambda e^{-t\lambda}\leqq 1/et$ for $\lambda\geqq 0$. Set $s=t/(l+m)$. Then by the semigroup property of $\{P_t^{(n)}\}$ and (2.9) we have

$$\|(1+d\Gamma(A^*)_{n+m})^l D_A^m P_t^{(n)}u\|_p$$
$$=\|(1+d\Gamma(A^*)_{n+m})P_s^{(n+m)}\cdots(1+d\Gamma(A^*)_{n+m})P_s^{(n+m)}$$
$$\times D_A P_s^{(n+m-1)}\cdots D_A P_s^{(n)}u\|_p$$
$$\lesssim(1/es)^l(1/\sqrt{2s})^m\|u\|_p\,.$$

Now by using (3.5) we can show that $P_t^{(n)}$ in (4.8) is continuous.

Lastly, we show that $W^{\infty,\infty-}$ forms an algebra. Take any $f, g\in W^{\infty,\infty-}$. It is enough to estimate the L^p norm of $(1+d\Gamma(A^*)_{n+k})^l D_A^k(fg)$. By the Leibniz rule, we have

$$d\Gamma(A^*)_k^l D_A^k(fg)[h_1,\cdots,h_k]$$
$$=\sum_{i=0}^k\sum_{j=0}^l\sum_\sigma {}_lC_j\frac{1}{j!(k-j)!}d\Gamma(A^*)_i^j D_A^i f\otimes d\Gamma(A^*)_{k-i}^{l-j}D_A^{k-i}g[h_{\sigma_1},\cdots,h_{\sigma_k}]$$

where σ runs over all permutations of $\{1,\cdots,k\}$. Then, by using the Hölder inequality we can get a desired estimate. □

Next we show the Rellich type theorem. If $A=1$ and B is an infinite dimensional space, then the inclusion $W^{s',p}(H^{*\otimes n})\hookrightarrow W^{s,p}(H^{*\otimes n})$ for $s'<s$ is not compact. So we impose the condition that A^{-1} is compact. Then we have;

Theorem 4.4. *Assume that A^{-1} is compact. Then for $s'<s$ the inclusion $W^{s',p}(H^{*\otimes n})\hookrightarrow W^{s,p}(H^{*\otimes n})$ is compact.*

Proof. By the definition of Sobolev spaces, it is enough to prove that for $\tau>0$,

$$(1-L_A+d\Gamma(A^*)_n)^{-\tau}: L^p(\mu; H^{*\otimes n})\longrightarrow L^p(\mu; H^{*\otimes n})$$

is compact. We set $S=(1-L_A+d\Gamma(A^*)_n)^{-\tau}$.

First we prove in case of $p=2$. Since A^{-1} is compact, A has only point spectra which tend to ∞, say $0<\lambda_1\leqq\lambda_2\leqq\lambda_3\cdots\to\infty$. Then, $d\Gamma(A^*)_n$ has only point spectra. In fact,

$$\sigma(d\Gamma(A^*)_n)=\{\lambda_{i_1}+\lambda_{i_2}+\cdots+\lambda_{i_n}; i_1, i_2,\cdots,i_n=1,2,\cdots\}$$

where $\sigma(d\Gamma(A^*)_n)$ denotes the spectrum of $d\Gamma(A^*)_n$. Moreover, by noting that $\sigma((d^2/dx^2)-x(d/dx))=\{0,-1,-2,\cdots\}$ we can easily see that

$$\sigma(S)=\left\{\left(1+\sum_{i=1}^{\infty} n_i\lambda_i+\lambda_{i_1}+\lambda_{i_2}+\cdots+\lambda_{i_n}\right)^{-\tau}\right.$$

$$\left.(n_1, n_2, \cdots)\in \boldsymbol{Z}_+^{\boldsymbol{N}}, \sum_{i=1}^{\infty} n_i<\infty, i_1, i_2, \cdots, i_n=1, 2, \cdots\right\}.$$

Hence S has only point spectra tending to 0. Now the compactness of S easily follows.

For general $p>1$, we borrow the argument of [6] Theorem 1.6.1. We first notice that S is a contraction operator. We can take a sequence of projection operators $\{\pi_n\}$ of finite rank such that $\{\pi_n\}$ converges strongly to the identity operator on $L^p(\mu; H^{*\otimes n})$ for each $p>1$ (e.g., take a conditional eqpectation under a finite σ-field). Since S is compact in $L^2(\mu; H^{*\otimes n})$

$$\lim_{n\to\infty} \|S-\pi_n S\|_{\mathcal{L}(L^2(\mu; H^{*\otimes n}))}=0 .$$

But, by the contraction property

$$\|S-\pi_n S\|_{\mathcal{L}(L^p(\mu; H^{*\otimes n}))}\leqq 2$$

for any $p>1$. For fixed $p\neq 2$, take p_1 so that $2<p<p_1$ or $p_1<p<2$. Then there exists $\theta\in(0, 1)$ so that $1/p=\theta/2+(1-\theta)/p_1$. By the Riesz-Thorin interpolation theorem, we have

$$\|S-\pi_n S\|_{\mathcal{L}(L^p(\mu; H^{*\otimes n}))}\leqq \|S-\pi_n S\|^{\theta}_{\mathcal{L}(L^2(\mu; H^{*\otimes n}))}\|S-\pi_n S\|^{\theta-1}_{\mathcal{L}(L^{p_1}(\mu; H^{*\otimes n}))}$$

$$\longrightarrow 0 \quad (\text{as } n\to\infty).$$

Since $\pi_n S$ is of finite rank, S is compact in $L^p(\mu; H^{*\otimes n})$. This completes the proof. □

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