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Hypoelliptic operators in $\mathbf{R}^3$ of the form $X_1^2 + X_2^2$

By

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Introduction and main results

It is well-known that a differential operator $D_1^2 + (x_1^k D_2 + x_1^l x_2^m D_3)^2$ is hypoelliptic in $\mathbf{R}^3$ if k, l ($k \neq l$) and m are non-negative integers. This is a direct consequence of the famous Hörmander Theorem (see [3]). If x_1^k, x_1^l and x_2^m are replaced by functions infinitely vanishing then the hypoellipticity of the operator is not obvious. In the present paper, we shall first study such a problem. Secondly we shall generalize one result about the above problem by using the symplectic geometry and give some sufficient conditions of the hypoellipticity for differential operators of the form $X_1^2 + X_2^2$, where X_j ($j = 1, 2$) are real vector fields in $\mathbf{R}^3$.

Let L_0 be a differential operator that has one of two forms

(1) $$L_0 = \mathrm{D}_1^2 + \alpha(x_1)^2(D_2 + f(x_1)g(x_2)D_3)^2$$

(2) $$L_0 = D_1^2 + \alpha(x_1)^2(f(x_1)D_2 + g(x_2)D_3)^2 ,$$

where $\alpha(t)$, $f(t)$ and $g(t)$ are real-valued, C^∞-functions with $\alpha(t)$, $f'(t)$, $g(t) \neq 0$ except for $t = 0$. In what follows, we admit that α, f' and g vanish infinitely at $t = 0$. Two forms (1) and (2) correspond to two cases $k < l$ and $l < k$, respectively, of the operator mentioned in the beginning.

Theorem 1. *Let L_0 be a differential operator of the form* (1) *or* (2). *Assume that $\alpha(t)$ is monotone in half lines $(-\infty, 0]$ and $[0, \infty)$, respectively. If α, f and g satisfy*

(3) $$\lim_{t \to 0} t \log |g(t)| = 0$$

(4) $$\lim_{t \to 0} t\alpha(t) \log |f'(t)| = 0$$

then L_0 is hypoelliptic in $\mathbf{R}^3$, furthermore,

(5) $$\mathrm{WF}\ L_0 v = \mathrm{WF}\ v \qquad \textit{for any } v \in \mathscr{D}'(\mathbf{R}^3) .$$

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The condition of the type (3) was first introduced by Kusuoka-Strook [6], who showed that the condition (3) is sufficient for the operator $D_1^2 + D_2^2 + g(x_2)^2 D_3^2$ to be hypoelliptic in $\mathbf{R}^3$ and also necessary if g is monotone in half lines $(-\infty, 0]$ and $[0, \infty)$ (c.f., Theorem 3 of [7]). We can also see the condition of the type (4) in Hoshiro [4], where the hypoellipticity of the operator $\alpha(x_2)^2 D_1^2 + D_2^2 + f'(x_2)^2 D_3^2$ was discussed. As in [6], it seems that the assumptions (3) and (4) are close to necessary condition for the hypoellipticity of L_0 (see the last remark in Section 7 of [8]).

We shall generalize Theorem 1 for L_0 of the form (1) under the restriction $\alpha \neq 0$. Let L be a differential operator of the form

$$L = -(X_1^2 + X_2^2), \tag{6}$$

where X_j $(j = 1, 2)$ are real vector fields in $\mathbf{R}^3$. Let $p_j(x, \xi)$ denote the symbol of $\sqrt{-1}X_j$ and set

$$\Sigma = \{(x, \xi) \in T^*\mathbf{R}^3\backslash 0; p_1(x, \xi) = p_2(x, \xi) = 0\} \tag{7}$$

$$\Gamma = \{(x, \xi) \in \Sigma; \{p_1, p_2\}(x, \xi) = 0\}, \tag{8}$$

where $\{p_1, p_2\} = H_1 p_2(x, \xi)$ and H_j $(j = 1, 2)$ denotes the Hamilton vector field of $p_j(x, \xi)$, that is, $H_j = \nabla_\xi p_j \cdot \nabla_x - \nabla_x p_j \cdot \nabla_\xi$. We assume that

$$d_\xi p_1 \text{ and } d_\xi p_2 \text{ are linearly independent on } \Sigma. \tag{9}$$

It follows from (9) that $\Sigma \cap \{|\xi| = 1\}$ consists of two connected components that are submanifolds of codimension 3 in $T^*\mathbf{R}^3$ parametrized by $x \in \mathbf{R}^3$. Hence we denote by $F(x)$ the restriction of $\{p_1, p_2\}$ on $\Sigma \cap \{|\xi| = 1\}$ in what follows.

The first result we shall state for the above L corresponds to Theorem 1 for L_0 of the form (1) in the case that $f'(0) = 0$ but α and g do not vanish. Assume that Γ is C^∞-hypersurface in Σ passing through $\rho_0 = (x_0, \xi_0) \in T^*\mathbf{R}^3\backslash 0$ and that

$$T\Gamma + (T\Sigma \cap T\Sigma^\perp) = T\Sigma \quad \text{at every point of } \Gamma. \tag{10}$$

Here $T\Sigma^\perp$ is the orthogonal space of $T\Sigma$ with respect to the symplectic form. Under the assumption (10) $T\Gamma \cap T\Sigma^\perp$ is of dimension 1 at every point. If V is a sufficiently small conic neighborhood of $\rho_0 \in \Gamma$ then we may assume without loss of generality that

$$H_1 \text{ is transversal to } \Gamma \cap \bar{V} \tag{11}$$

because, for each $\rho \in \Sigma$, $T_\rho \Sigma^\perp$ is equal to a linear subspace generated by $H_1(\rho)$ and $H_2(\rho)$. If $\rho \in \Gamma \cap \bar{V}$ and if γ_ρ is an integral curve of H_1 such that $\gamma_\rho = \gamma_\rho(s)$; $s \to \exp sH_1$, $\gamma_\rho(0) = \rho$ then we assume that the following formula holds uniformly with respect to $\rho \in \Gamma \cap \bar{V}$;

$$\lim_{s \to 0} s \log |F(\pi_x \gamma_\rho(s))| = 0. \tag{12}$$

Here π_x is the natural projection from $T^*\mathbf{R}_x^3$ to $\mathbf{R}_x^3$.

Theorem 2. *Let L be a differential operator of the form* (6) *satisfying* (9). *Assume that Γ is a C^∞-hypersurface in Σ containing $\rho_0 = (x_0, \xi_0)$ and that* (10)–(12) *hold. If $v \in \mathscr{D}'(\mathbf{R}^3)$ and $\rho_0 \notin \mathrm{WF}\, Lv$ then $\rho_0 \notin \mathrm{WF}\, v$.*

Next we shall state the result corresponding to the case that both f' and g of L_0 vanish at the origin (but $\alpha(0) \neq 0$). Assume that $\Gamma = \Gamma_1 \cup \Gamma_2$, $\Gamma_j = \{(x, \xi) \in \Sigma; f_j(x) = 0\}$ for $f_j(x) \in C^\infty$ $(j = 1, 2)$ satisfying

$$(13) \qquad \begin{cases} df_1 \wedge df_2 \quad \text{is non-degenerate on a linear subspace} \\ \text{of } T(T^*\mathbf{R}^3) \text{ generated by } H_1 \text{ and } H_2 \,. \end{cases}$$

It follows from (13) that Γ_j are C^∞-hypersurfaces in Σ. Let $\rho_0 = (x_0, \xi_0) \in \Gamma_1 \cap \Gamma_2$ and let $V \subset\subset W$ be conic neighborhoods of ρ_0. By means of (13), $\Sigma \cap \overline{W} \backslash \Gamma$ consists of four connected components Σ_j $(j = 1, \dots, 4)$. There exist a $\delta_0 > 0$ and a vector $(c_1^j, c_2^j) \in \mathbf{R}^2$ for each $j = 1, \dots, 4$ such that for any $\rho \in \Gamma \cap \overline{\Sigma}_j \cap \overline{V}$ an integral curve $\gamma_{\rho,j}(s)$; $s \to \exp(s\{c_1^j H_1 + c_2^j H_2\})$, $\gamma_{\rho,j}(0) = \rho$ satisfies

$$\gamma_{\rho,j}(s) \subset \Sigma_j \qquad \text{for } 0 < s \leq \delta_0 \,.$$

Furthermore, we assume that for each $j = 1, \dots, 4$ the following formula holds uniformly with respect to $\rho \in \Gamma \cap \overline{\Sigma}_j \cap \overline{V}$;

$$(14) \qquad \lim_{s \downarrow 0} s \log |F(\pi_x \gamma_{\rho,j}(s))| = 0 \,.$$

Theorem 3. *Let L be a differential operator of the form* (6) *satisfying* (9). *Assume that $\Gamma = \Gamma_1 \cup \Gamma_2$ as above and that* (13) *holds. If $\rho_0 \in \Gamma_1 \cap \Gamma_2$ and if* (14) *holds then $\rho_0 \notin \mathrm{WF}\, Lv$ implies $\rho_0 \notin \mathrm{WF}\, v$ for any $v \in \mathscr{D}'(\mathbf{R}^3)$.*

The last result we shall state is in a different situation from the above three theorems that required some growth order conditions such as (3), (4), (12) and (14). We assume that Γ is a C^∞-submanifold of codimension 2 in Σ and symplectic, that is,

$$(15) \qquad T\Gamma \cap T\Gamma^\perp = 0 \,.$$

Under (15), both H_1 and H_2 are transversal to Γ because $H_1, H_2 \in T\Sigma^\perp \subset T\Gamma^\perp$. If $\rho_0 = (x_0, \xi_0) \in \Gamma$ and if V is a conic neighborhood of ρ_0 we assume that

$$(16) \qquad \begin{cases} \text{there exist a } \delta_0 > 0 \text{ and a } C^\infty \text{ function } E(x) > 0 \text{ defined in a} \\ \text{neighborhood of } x_0 \text{ such that, for any } \rho \in V, (EF)(\pi_x \gamma_\rho(s)) \text{ has} \\ \text{a unique extremum in } (-\delta_0, \delta_0), \text{ which is } C^\infty \text{ with respect to } \rho. \end{cases}$$

Here $\gamma_\rho(s)$; $s \to \exp sH_1$, $\gamma_\rho(0) = \rho$.

Theorem 4. *Let L be a differential operator of the form* (6) *satisfying* (9). *Assume that Γ is a C^∞-symplectic submanifold and of codimension 2 in Σ. If $\rho_0 \in \Gamma$ and $v \in \mathscr{D}'(\mathbf{R}^3)$ then $\rho_0 \notin \mathrm{WF}\, Lv$ implies $\rho_0 \notin \mathrm{WF}\, v$, provided that the assumption* (16) *holds.*

Typical examples of L in Theorem 2 and 4 are, respectively, as follows:

$$D_1^2 + (D_2 + \exp(-|x_1|^{-\delta})D_3)^2 \qquad \text{with } 0 < \delta < 1\,,$$

$$D_1^2 + \left\{D_2 + \int_0^{x_1} \exp -(t^2 + x_2^2)^{-\delta/2} dt D_3\right\}^2 \qquad \text{with } \delta > 0\,.$$

Those examples are inspired by the works of Sjöstrand [9] and Grigis-Sjöstrand [2] who studied the analytic hypoellipticity by using the F.B.I. operator. More precisely, Theorem 2 and 4 are motivated by Theorem 4.2 of [9] and Theorem 4.1 of [2], respectively. In relation to the second example we remark that an operator $D_1^2 + \exp(-|x_1|^{-\delta})D_2^2$ is hypoelliptic in $\mathbf{R}^2$ for any $\delta > 0$ (see Fediĭ [1]).

Before talking about the plan of this paper, we recall a criterion of the hypoellipticity given in [7]. Let Ω be an open set in $\mathbf{R}^n$ and let $P = p(x, D_x)$ be a second order differential operators with $C^\infty(\Omega)$-coefficients, that is,

$$p(x, D_x) = \sum_{j,k=1}^{n} a_{jk} D_j D_k + \sum_{j=1}^{n} i b_j D_j + c\,, \tag{17}$$

where $a_{jk}(x)$, $b_j(x)$ and $c(x)$ belong to $C^\infty(\Omega)$. We assume that $a_{jk}(x)$, $b_j(x)$ are real valued and $a_{jk}(x)$ satisfy any x in Ω

$$\sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k \geq 0 \qquad \text{for all } \xi \in \mathbf{R}^n\,. \tag{18}$$

Let $\log \Lambda$ denote a pseudodifferential operator with symbol $\log \langle\xi\rangle$, where $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$. As for pseudodifferential operators we refer the reader to [5].

Theorem 5. *Let $\rho_0 = (x_0, \xi_0) \in T^*(\Omega)\backslash 0$ and let V be a conic neighborhood of γ. Let $0 \leq \varphi(x, \xi) \leq 1$ belong to $S^0_{1,0}$ and satisfy $\varphi = 1$ in $V \cap \{|\xi| \geq 1\}$. If for any $\varepsilon > 0$ the estimate*

$$\|(\log \Lambda)^2 \varphi(x, D)u\| \leq \varepsilon\|Pu\| + C_\varepsilon\|u\|\,, \qquad u \in \mathscr{S}\,, \tag{19}$$

holds with a constant C_ε then $\rho_0 \notin \mathrm{WF}\, Pv$ implies $\rho_0 \notin \mathrm{WF}\, v$ for any $v \in \mathscr{D}'(\Omega)$.

This is a microlocal version of Theorem 1 of [7] and similarly follows from the argument in Section 1 of [7]. In fact, the estimate (1.5) of Lemma 1.1 in [7] is derived from (19) instead of (3) of [7] because we have the estimate after (1.13) in [7]. We have the following corollary to Theorem 5 (cf., Corollary 2 of [7]).

Corollary 6. *Let $\rho_0 \in T^*(\Omega)\backslash 0$ and let $\varphi(x, \xi)$ be the same as in Theorem 5. If for any $\varepsilon > 0$ the estimate*

$$\|(\log \Lambda)\varphi(x, D)u\|^2 \leq \varepsilon \operatorname{Re}(Pu, u) + C_\varepsilon\|u\|^2\,, \qquad u \in \mathscr{S}\,, \tag{20}$$

holds with a constant C_ε then $\rho_0 \notin \mathrm{WF}\, Pv$ implies $\rho_0 \notin \mathrm{WF}\, v$ for any $v \in \mathscr{D}'(\Omega)$.

The estimate (19) is derived from (20). Indeed, let $\varphi_0(x, \xi) \in S^0_{1,0}$ satisfy $\operatorname{supp} \varphi_0 \subset\subset V$ and $0 \leq \varphi_0 \leq 1$. Replace u in (20) by $(\log \Lambda)\varphi_0(x, D)u$. Then, in view of Schwartz's inequality, we obtain (19) with φ replaced by φ_0 because the principal symbol of $[P, \log \Lambda]$ is purely imaginary and we have

$$\|(\log \Lambda)\varphi_0 u\|^2 \le \varepsilon \|(\log \Lambda)^2 \varphi_0 u\|^2 + C_\varepsilon \|u\|^2 .$$

The plan of this paper is as follows: In Section 1 we prove one part of Theorem 1, more precisely, Theorem 1 for L_0 of the form (2). Indeed, another part of Theorem 1 has been already proved in the previous paper [8]. Similarly as in [8], the criterion of hypoellipticity mentioned in the above can not be applied to the proof of Theorem 1 for L_0 of the form (2) because the estimate type of (20) no longer holds in general. In Section 1 we prepare a degenerate version of (20) (see (1.27)') by using arguments about the inequality of Poincaré type developed in Sections 1, 2, 4 and 7 of [8]. In the help of this estimate we prove the hypoellipticity of L_0 following the method in Section 5 of [8]. In Section 2 we prove Theorem 2 and 3 by means of Corollary 6. In order to derive (20) from the hypotheses we also employ the inequality of Poincaré type in [8]. Theorem 4 is proved in Section 3. By taking suitable coordinates, we search for inequalities between coefficients of L (see (3.1), (3.10) and (3.11)). Those inequalities enable us to estimate the commutator of L and cut functions in $T^*\mathbf{R}^3$. The proof of Theorem 4 is essentially confined in the classical method as in Fediĭ [1], differing from proofs of Theorem 1–3.

1. Proof of Theorem 1

As stated in Introduction, we shall prove Theorem 1 only for L_0 of the form (2) because the proof for L_0 of the form (1) was already given in the previous paper [8] under an additional assumption $g \ge 0$. This hypothesis $g \ge 0$ can be removed by comparing (7.1) of [8] with (1.1) in the below. We may assume that $f(0) = 0$. In fact, the form (2) with $f(0) \ne 0$ is reduced to the form (1) by replacing α by αf. Since $f'(t)$ is of the definite sign in half lines $(-\infty, 0]$ and $[0, \infty)$, $f(t)$ is monotone in each half lines. We may also assume that α, f, g and their derivatives of any order are all bounded because our consideration is local.

For a real η set $Y_\eta = f(x_1)D_2 + g(x_2)\eta$ and set

$$P_\eta = D_1 \pm iG(x)Y_\eta ,$$

where $G(x) = (\alpha^2 ff')(x_1)g(x_2)$. Then we have

$$P_\eta^* P_\eta = D_1^2 + Y_\eta G^2 Y_\eta \pm iG[D_1, Y_\eta] \pm i\{[D_1, G]Y_\eta - [Y_\eta, G]D_1\} . \tag{1.1}$$

Since $iG[D_1, Y_\eta] = \alpha^2 f'^2 g Y_\eta - \alpha^2 (f'g)^2 \eta$, for any compact $K \subset \mathbf{R}^2$ there exist constants c_K, $C_K > 0$ such that

$$\begin{aligned} (1.2)\quad 0 &\le \|P_\eta v\|^2 \\ &\le -(\{\pm(\alpha f'g)^2\eta\}v, v) + C_K\{\|D_1 v\|^2 + \|\alpha(x_1)Y_\eta v\|^2 + \|v\|^2\}, \qquad v \in C_0^\infty(K) . \end{aligned}$$

If we choose a suitable sign according to $\eta > 0$ or $\eta < 0$ then it follows from (1.2) that

$$(1.3)\qquad (\{(\alpha f'g)^2|\eta|\}v, v) \le C'_K\{\|D_1 v\|^2 + \|\alpha(x_1)Y_\eta v\|^2 + \|v\|^2\}$$
$$\le C''_K\{\|D_1 v\|^2 + \|\alpha(x_1)Y_\eta v\|^2\}, \qquad v \in C_0^\infty(K).$$

Here the last estimate follows from the usual Poincaré inequality,

$$\|v\| \le C_K\|D_1 v\|, \qquad v \in C_0^\infty(K).$$

If we replace $G(x)$ in (1.1) by $\alpha^2 f'$ then in place of (1.2) we have

$$(1.4)\qquad 0 \le \pm(\{(\alpha f')^2 D_2\}v, v) + C_K\{\|D_1 v\|^2 + \|\alpha(x_1)Y_\eta v\|^2\}, \qquad v \in C_0^\infty(K).$$

If we set $\beta(t) = g(t)^2$ and $\gamma(t) = (\alpha(t)f'(t))^2$ then from (3) and (4) we have

$$(1.5)\qquad \lim_{t\to 0} t \log \beta(t) = 0,$$

$$(1.6)\qquad \lim_{t\to 0} t\alpha(t) \log \gamma(t) = 0.$$

In view of (1.3), we prepare the following:

Lemma 1.1 (cf., Lemma 7.2 of [8]). *Let $\alpha(t)$, $\beta(t)$ and $\gamma(t) \in C(\mathbf{R}^1)$ satisfy α, β, $\gamma > 0$ except for $t \ne 0$. Assume that (1.5) and (1.6) hold. For $\zeta > 0$ set $V(x; \xi) = \gamma(x_1)\beta(x_2)\zeta^4$. Furthermore, set $Y_\eta = f(x_1)D_2 + g(x_2)\eta$ for $f(t)$, $g(t) \in C(\mathbf{R}^1)$ and $\eta \in \mathbf{R}$. Assume that α and f are monotone in half lines $(-\infty, 0]$ and $[0, \infty)$, respectively. Then for any $s > 0$ there exists a $\zeta_s > 0$ independent of η such that if $\zeta \ge \zeta_s$ the estimate*

$$(1.7)\qquad (\{D_1^2 + \alpha(x_1)^2 Y_\eta^2 + V(x; \zeta)\}u, u) \ge s(\alpha(x_1)^2 f(x_1)^2 (\log \zeta)^2 u, u)$$

holds for any $u \in C_0^\infty(I_0)$, where $I_0 = \{(x_1, x_2); |x_j| \le 1\}$.

Proof. It follows from (1.6) that for any $s > 0$ there exists a $\delta(s) > 0$ such that

$$(1.8)\qquad 0 \le -|x_1|\alpha(x_1)\log \gamma(x_1) < 1/s \qquad \text{if } |x_1| < \delta(s).$$

For the brevity we assume that α is even function because the proof in the general case will be obvious after proving this special case. Since α is monotone in $[0, \infty)$, for any $\zeta > 0$ there exists a unique positive root x_ζ such that

$$(1.9)\qquad s\alpha(x_\zeta)\log \zeta = x_\zeta^{-1}.$$

We may assume that x_ζ is smaller than $\delta(s)$ if ζ is sufficiently large. It follows from (1.8) that if $x_\zeta \le |x_1| < \delta(s)$ then

$$\gamma(x_1)\zeta = \exp\{\log \zeta + \log \gamma(x_1)\}$$
$$\ge \exp\{\log \zeta - (s|x_1|\alpha(x_1))^{-1}\} \ge 1.$$

Since $\gamma(x_1) \geq c_s > 0$ on $\{\delta(s) \leq |x_1| \leq 1\}$, we see that

$$\gamma(x_1)\zeta \geq 1 \qquad \text{on } \{x_1 \in \mathbf{R}^1; x_\zeta \leq |x_1| \leq 1\}, \tag{1.10}$$

if $\zeta \geq \zeta_s$ for a sufficiently large ζ_s. By means of (1.5) we see for any $s > 0$ that

$$\beta(x_2)\zeta \geq 1 \qquad \text{on } \{(s \log \zeta)^{-1} \leq |x_2| \leq 1\} \tag{1.11}$$

if $\zeta \geq \zeta_s$, by taking another sufficiently large ζ_s. Set $y_\zeta = (s \log \zeta)^{-1}$ and set

$$\omega_1 = \{x \in I_0; |x_1| < x_\zeta\},$$
$$\omega_2 = \{x \in I_0; |x_2| < y_\zeta\}.$$

Then $I_0 \backslash (\omega_1 \cup \omega_2)$ is composed of four congruent rectangles. We divide each rectangle into four smller congruent rectangles. We repeat this cutting procedure. Let $I_\nu = Q_1^\nu \times Q_2^\nu$ $(\subset \mathbf{R}_{x_1} \times \mathbf{R}_{x_2})$ denote one of congruent rectangles on some step, $\left(\text{that is, } I_0 \backslash (\omega_1 \cup \omega_2) = \bigcup_\nu I_\nu\right)$. We repeat the cutting and stop it if I_ν satisfies

$$\zeta^{1/2} \leq (\operatorname{diam} I_\nu)^{-2}. \tag{1.12}$$

Then we have $\zeta^{1/2} \geq (2 \operatorname{diam} I_\nu)^{-2}$. Note that $\operatorname{diam} I_\nu$ is equivalent to $\operatorname{diam} Q_j^\nu$ with $j = 1, 2$. By means of (1.10) and (1.11) we have

$$V(x; \zeta) \geq \zeta^2 \qquad \text{on } I_\nu \text{ if } \zeta \text{ is sufficiently large}. \tag{1.13}$$

We also divide $\bar{\omega}_1 \backslash \omega_2$ (and $\bar{\omega}_2 \backslash \omega_1$) into congruent smaller rectangles as follows:

$$\bar{\omega}_1 \backslash \omega_2 = \bigcup_{\nu'} J_{1\nu'}, \qquad J_{1\nu'} = [-x_\zeta, x_\zeta] \times Q_2^{\nu'}$$
$$\bar{\omega}_2 \backslash \omega_1 = \bigcup_{\nu''} J_{2\nu''}, \qquad J_{2\nu''} = Q_1^{\nu''} \times [-y_\zeta, y_\zeta],$$

where the diameter of $Q_2^{\nu'}$ (resp. $Q_1^{\nu''}$) is equal to that of Q_2^ν (resp. Q_1^ν). Set $\omega_1 \cap \omega_2 = \mathrm{K}_0$ $(= Q_0^1 \times Q_0^2 \subset \mathbf{R}_{x_1} \times \mathbf{R}_{x_2})$ and let K_0^* denote four times dilation of K_0. If $u \in C_0^\infty(I_0)$ then we have

$$\begin{aligned} &4(\{D_1^2 + \alpha(x_1)^2 Y_\eta^2 + V(x)\}u, u) \\ &\geq \int_{K_0^*} \{|D_1 u|^2 + |\alpha(x_1) Y_\eta u|^2 + V(x)|u|^2\} dx + \sum_\nu \int_{I_\nu} \{\cdot\} dx + \sum_{\nu'} \int_{J_{1\nu'}^\dagger} \{\cdot\} dx \\ &\quad + \sum_{\nu''} \int_{J_{2\nu''}^\dagger} \{\cdot\} dx \\ &\equiv \Omega_0 + \sum_\nu \Omega_\nu + \sum_{\nu'} \Omega_{\nu'} + \sum_{\nu''} \Omega_{\nu''}, \end{aligned} \tag{1.14}$$

where $J_{1\nu'}^\dagger = [-2x_\zeta, 2x_\zeta] \times Q_2^{\nu'}$ and $J_{2\nu''}^\dagger = Q_1^{\nu''} \times [-2y_\zeta, 2y_\zeta]$. Let $G(x_2)$ be a primitive function of $g(x_2)$ and set

$$\tilde{u}(x) = u(x) \exp\{iG(x_2)\eta / f(x_1)\} \qquad \text{for } x_1 \neq 0.$$

Then it follows from Lemma 1.1 of [8] (cf., (2.17) of [8]) that

$$(1.15)\quad \int_{K_0^*} |\alpha(x_1)Y_\eta u|^2 dx \geq \int_{Q_0^{1*}\backslash\omega_1} dx_1 \left\{ \int_{Q_0^{2*}} |\alpha(x_1)f(x_1)D_2\tilde{u}|^2 dx_2 \right\}$$

$$\geq c\int_{Q_0^1} dy_1/|Q_0^1| \left\{ \int_{Q_0^{1*}\backslash\omega_1} \left[\int_{Q_0^{2*}\times Q_0^2} \alpha(x_1)^2 f(x_1)^2 y_\zeta^{-2} \right.\right.$$

$$\left.\left. \times |\tilde{u}(x_1, x_2) - \tilde{u}(x_1, y_2)|^2 dy_2 dx_2 \right] \Big/ |Q_0^2| dx_1 \right\}$$

$$= c\int_{K_0} dx \left\{ \int_{K_0^*\backslash\omega_1} [\alpha(y_1)^2 f(y_1)^2 y_\zeta^{-2} \right.$$

$$\left. \times |\tilde{u}(y_1, x_2) - \tilde{u}(y_1, y_2)|^2 dy]/|K_0| \right\}.$$

In view of the monotoness of α and f, it follows from (2.17) of [8] and (1.15) that

$$(1.16)\quad \Omega_0 \geq c\int_{K_0} \left[\int_{K_0^*\backslash(\omega_1\cup\omega_2)} \{x_\zeta^{-2}|u(x) - u(y_1, x_2)|^2 \right.$$

$$\left. + \alpha(y_1)^2 f(y_1)^2 y_\zeta^{-2} |\tilde{u}(y_1, x_2) - \tilde{u}(y)|^2 + V(y)|u(y)|^2\} dy \right] \Big/ |K_0| dx$$

$$\geq c' s\alpha(x_\zeta)^2 f(x_\zeta)^2 (\log \zeta)^2 \int_{K_0} |u(x)|^2 dx$$

because of (1.9) and (1.13) with I_ν replaced by $K_0^*\backslash(\omega_1 \cup \omega_2)$. Similarly as in (1.15) we have

$$\int_{J_{1\nu'}^\dagger} |\alpha(x_1)Y_\eta u|^2 dx \geq c\int_{J_{1\nu'}} dx \left\{ \int_{J_{1\nu'}^\dagger\backslash\omega_1} [\alpha(y_1)^2 f(y_1)^2 y_\zeta^{-2} \right.$$

$$\left. \times |\tilde{u}(y_1, x_2) - \tilde{u}(y_1, y_2)|^2 dy]/|J_{1\nu'}| \right\}$$

Hence we obtain

$$(1.17)\quad \Omega_{\nu'} \geq c' s\alpha(x_\zeta)^2 f(x_\zeta)^2 (\log \zeta)^2 \int_{J_{1\nu'}} |u(x)|^2 dx.$$

More easily we have

$$(1.18)\quad \Omega_\nu \geq c''\zeta^{1/2} \int_{I_\nu} |u(x)|^2 dx.$$

Exchanging the order of D_1^2 and $\alpha^2 D_2^2$ and noting that $(\mathrm{diam}\, Q_1^{\nu''})^{-2} \sim \zeta^{1/2}$ we also have

$$(1.19)\qquad \Omega_{\nu''} \geq c \int_{J_{2\nu''}} \Big[\int_{J^{\dagger}_{2\nu''}\setminus \omega_2} \{\alpha(x_1)^2 f(x_1)^2 y_\zeta^{-2} |\tilde{u}(x) - \tilde{u}(x_1, y_2)|^2 + \zeta^{1/2} |u(x_1, y_2) - u(y)|^2 + V(y)|u(y)|^2\} dy \Big] \Big/ |J_{2\nu''}| dx \geq c' s (\log \zeta)^2 \int_{J_{2\nu''}} |\alpha(x_1) f(x_1) u(x)|^2 dx .$$

Summing up (1.16)–(1.19), in view of (1.14) we obtain the desired estimate (1.7). Q.E.D.

Let $h(t)$ be a $C_0^\infty(\mathbf{R}^1)$ function such that $0 \leq h \leq 1$, $h = 1$ in $|t| \leq 1$ and $\operatorname{supp} h \subset \{|t| \leq 3/2\}$. If we set $\chi_0(\xi; M) = 1 - h(|\xi|/M)$ for a parameter $M > 1$ then χ_0 belongs to a bounded set of the symbol class $S^0_{1,0}$ uniformly with respect to M.

Lemma 1.2. *Let δ_0 be any but a fixed positive. Let $\chi(\xi) \in S^0_{1,0}$ satisfy $0 \leq \chi \leq 1$, $\chi = 1$ in $\{|\xi'| \geq 2\delta_0 |\xi_3|\} \cap \{|\xi'| \geq 3\}$ and $\operatorname{supp} \chi \subset \{|\xi'| \geq \delta_0 |\xi_3|\}$, where $\xi' = (\xi_1, \xi_2)$. For any $s > 0$ and any compact set $K \subset \mathbf{R}^3$ there exist constants $M_{s,K}$ and $C_{s,K}$ such that if $M \geq M_{s,K}$*

$$(1.20)\qquad \|\alpha(x_1)(\log \Lambda^s)\chi(D)\chi_0(D; M)u\|^2 \leq (L_0 u, u) + C_{s,K}\|u\|^2_{-s}, \qquad u \in C_0^\infty(K),$$

where $\chi_0(\xi; M)$ is the same as in the above.

Proof. Let $\check{u}(x_1, x_2, \xi_3)$ denote the Fourier transform of $u(x)$ with respect to x_3. Substituting $\check{u}$ into (1.4) with $\eta = \xi_3$ we have with a $c_K > 0$

$$(1.21)\qquad \pm c_K(\alpha(x_1)^2 f'(x_1)^2 D_2 u, u) \leq \|D_1 u\|^2 + \|\alpha(f D_2 + g D_3)u\|^2 = (L_0 u, u), \qquad u \in C_0^\infty(K).$$

Take φ, $\psi \in C_0^\infty(\mathbf{R}^3_x)$ such that $\varphi = 1$ on K and $\varphi \subset\subset \psi$, (that is, $\psi = 1$ in a neighborhood of $\operatorname{supp} \varphi$). Let $\chi_\pm(\xi) \in S^0_{1,0}$ such that

$$\chi_\pm = 1 \qquad \text{on } \{\pm\xi_2 \geq \delta_0(\xi_1^2 + \xi_3^2)^{1/2}\} \cap \{|\xi_2| \geq 1/2\},$$

$$\chi_\pm = 0 \qquad \text{on } \{\pm\xi_2 \leq 2\delta_0(\xi_1^2 + \xi_3^2)^{1/2}\} \cup \{|\xi_2| \leq 1/3\},$$

where double sign takes its order. Substitute $\psi(x)\chi_+(D)u \in C_0^\infty$ into (1.21) with plus sign. Note that $[D_1, \psi\chi_+]$, $[fD_2 + gD_3, \psi\chi_+]$ and $[(\alpha f')^2 D_2, \chi_+]$ belong to $S^0_{1,0}$ and that $(1 - \psi)\chi_+\varphi$, $(1 - \psi)(\alpha f')^2 D_2 \chi_+ \varphi \in S^{-\infty}$. Then by using the usual Poincaré inequality we have

$$(1.22)\qquad (\{\alpha(x_1) f'(x_1)\}^2 |D_2| \chi_+^2(D)u, u) \leq C_K(L_0 u, u), \qquad u \in C_0^\infty(K).$$

Since it follows from (1.21) with minus sign that the similar formula as (1.22) holds for χ_- we have

$$(1.23)\quad (\{\alpha(x_1)f'(x_1)\}^2|D_2|(\chi_+^2+\chi_-^2)(D)u, u) \le C_K(L_0u, u), \qquad u \in C_0^\infty(K).$$

Set $\tilde{\chi}^2(\xi) = \chi_+^2(\xi) + \chi_-^2(\xi)$. Note that $\tilde{\chi}(\xi)$ can be written in the form

$$\tilde{\chi}(\xi) = \tilde{\chi}(0, \xi_2, \xi_3) + \xi_1 r(\xi)$$

for $r(\xi)$ such that $r(D)|D_2|$ is L^2 bounded operator. Since $|D_2|\tilde{\chi}^2(D) \in S_{1,0}^1$ it follows from (1.23) that

$$(1.24)\quad (\{\alpha(x_1)f'(x_1)\}^2|D_2|(\tilde{\chi}^2(0, D_2, D_3)u, u) \le C_K(L_0u, u), \qquad u \in C_0^\infty(K),$$

Here and in what follows we denote by the same C_K different constants depending on K. Since $\gamma(t) = \{\alpha(t)f'(t)\}^2$ satisfies (1.6) for any $s > 0$ there exists a $\zeta_s > 0$ such that for any $w \in C_0^\infty(\{|x_1| \le c\})$ with a $c > 0$ we have

$$(1.25)\qquad (\{D_1^2 + \{\alpha f'\}^2\zeta^2\}w, w) \ge s^2\|\alpha(x_1)(\log\zeta)w\|^2 \qquad \text{if } \zeta \ge \zeta_s.$$

In fact, this is nothing but Lemma 7.1 of [8]. Let $\tilde{u}(x_1, \xi_2, \xi_3)$ denote the Fourier transform of $u(x)$ with respect to (x_2, x_3). Substitute $\tilde{\chi}(0, \xi_2, \xi_3)(1 - h(|\xi_2|/M))\tilde{u}(x_1, \xi_2, \xi_3)$ into (1.25) with $|\zeta| = |\xi_2|^{1/2}$. If M satisfies $M \ge 2\zeta_s^2$ for $s > 0$ then in view of (1.24) we obtain

$$(1.26)\quad \|\alpha(x_1)(\log|D_2|^s)\tilde{\chi}(0, D_2, D_3)(1 - h(2|D_2|/M))u\|^2 \le C_K(L_0u, u), \quad u \in C_0^\infty(K).$$

Note that $\|\log(|D_1|)^s(1 - h(2|D_1|/M)\alpha(x_1)u\|$ is estimated above from $\|D_1\alpha u\| \le C_K(L_0u, u)$ if $M \ge M_s$ for a sufficiently large M_s. Since $\tilde{\chi}(0, \xi_2, \xi_3)(1 - h(2|\xi_2|/M) + 1 - h(2|\xi_1|/M)$ is non-zero on supp $\chi\chi_0$ we have

$$\|(\log\Lambda^s)\chi(D)\chi_0(D; M)\alpha(x_1)u\|^2 \le C_K(L_0u, u), \qquad u \in C_0^\infty(K).$$

It follows from the expansion fromula of $[(\log\Lambda)\chi(D)\chi_0(D; M), \alpha(x_1)]$ that the estimate

$$\begin{aligned}\|[(\log\Lambda^s)\chi(D)\chi_0(D; M), \alpha(x_1)]u\|^2 &\le sC_s(\|(1 - h(2|D|/M))u\|_{-1} + \|u\|_{-s})\\ &\le 2s(C_s/M)\|u\| + sC_s\|u\|_{-s}\end{aligned}$$

holds with a constant C_s. If M satisfies $M \ge sC_s$, furthermore, we obtain the desired estimate (1.20). Q.E.D.

If we apply the similar arguments as in the proof of Lemma 1.2 to estimates (1.3) and (1.7) with $\eta = \xi_3$ and $\zeta = |\xi_3|^{1/4}$ then for any $s > 0$ and any compact $K \subset \mathbf{R}^3$ there exists a $M_{s,K} > 0$ such that if $M \ge M_{s,K}$

$$\|\alpha(x_1)f(x_1)(\log(|D_3|^s)(1 - h(2|D_3|/M))u\|^2 \le (L_0u, u), \qquad u \in C_0^\infty(K).$$

The combination of this and (1.20) shows that for any $s > 0$ and any compact $K \subset \mathbf{R}^3$ there exist constants $M_{s,K}$ and $C_{s,K}$ such that if $M \ge M_{s,K}$

$$(1.27)\quad \|\alpha(x_1)f(x_1)(\log\Lambda^s)\chi_0(D; M)u\|^2 \le (L_0u, u) + C_{s,K}\|u\|_{-s}^2, \qquad u \in C_0^\infty(K).$$

From this we see that for any $\varepsilon > 0$ and for some $C_{\varepsilon,K}$ the estimate

$$(1.27)' \qquad \|(\log \Lambda)\alpha(x_1)f(x_1)u\|^2 \le \varepsilon(L_0 u, u) + C_{\varepsilon,K}\|u\|^2\,, \qquad u \in C_0^\infty(K)\,,$$

holds. By Corollary 6 in Introduction, (1.27)′ yields the formula (5) in the region $\{x_1 \neq 0\}$.

It follows from (1.27) that for any $s > 0$ and any compact K we have

$$(1.28) \quad \|\alpha(x_1)g(x_2)(\log \Lambda^s)\chi_0(D; M)u\|^2 \le (L_0 u, u) + C_{s,K}\|u\|_{-s}^2\,, \qquad u \in C_0^\infty(K)\,,$$

provided that $M \ge M_{s,K}$ for a sufficiently large $M_{s,K}$. In fact, we get

$$\|\alpha g(\log \Lambda^s)\chi_0 u\| \le \|\alpha(\log \Lambda^s)\chi_0\chi u\| + \|\alpha g(\log \Lambda^s)\chi_0(1-\chi)u\|\,.$$

The first term of the right hand side can be estimated by using (1.20). Note that the second term is estimated above from $\|\alpha(fD_2 + gD_3)\{D_3^{-1}(1-\chi)\chi_0 \log \Lambda^s\}u\| + \|\alpha f(\log \Lambda^s)\chi_0\{D_2 D_3^{-1}(1-\chi)\}u\|$. Since $D_3^{-1}(1-\chi)\chi_0(\log \Lambda^s + D_2)$ is a L^2 bounded operator with a fixed bound we obtain (1.28) in the help of (1.27).

We shall prove that if $\rho_0 = (x_0, \xi_0) = (0, (0, 0, \pm 1))$ and if $v \in \mathscr{E}'$ then

$$(1.29) \qquad \rho_0 \notin \mathrm{WF}\, L_0 v \qquad \text{implies } \rho_0 \notin \mathrm{WF}\, v\,.$$

We prepare some special cut functions as in Section 5 of [8]. For a $\delta > 0$ let $\psi_\delta(\xi) \in S_{1,0}^0$ be real valued and satisfy $\psi_\delta = 1$ in $\{\pm\delta\xi_3 \ge |\xi'|\} \cap \{|\xi_3| \ge 3/2\delta\}$ and $\psi_\delta = 0$ in $\{\pm 3\delta\xi_3 \le 2|\xi'|\} \cup \{|\xi_3| \le \delta^{-1}\}$. Here we choose one of $\pm$ signs according to $\xi_0 = (0, 0, 1)$ or $(0, 0, -1)$. We assume that ψ_δ can be written as $\psi_\delta(\xi) = \tilde{\psi}_\delta(\xi_3, \xi_1)\tilde{\psi}_\delta(\xi_3, \xi_2)$ for some $\tilde{\psi}_\delta(t, t') \in C^\infty(\mathbf{R}^2)$ such that $\tilde{\psi}_\delta = 1$ in $\{\pm\delta t \ge |t'|\} \cap \{|t| \ge 3/2\delta\}$ and $\psi_\delta = 0$ in $\{\pm 3\delta t \le 2\sqrt{2}|t'|\} \cup \{|t| \le \delta^{-1}\}$. Here we also take one of $\pm$ signs following the above convention. Set $\varphi(x) = \prod_{k=1}^{3} h(x_k)$ and set $\varphi_\delta(x) = \varphi(x/\delta)$. If we set $\Psi_\delta(\xi) = \Psi_\delta(\xi; M) = h((M^{-1}|\xi_3| - 3)/\delta)\psi_\delta(\xi)$ for a parameter $M \ge 1$, then for any multi-index β there exists a C_β such that

$$(1.30) \qquad |D_\xi^\beta \Psi_\delta| \le C_\beta M^{-s}\langle\xi\rangle^{-|\beta|+s}$$

with any real $0 \le s \le |\beta|$ because with a $C > 0$ we have $C^{-1} \le M/\langle\xi\rangle \le C$ on $\operatorname{supp} D_\xi^\beta \Psi_\delta$.

Fix an integer $N > 0$. Take a sequence $\{\Psi_j(\xi)\}_{j=0}^N \subset S_{1,0}^0$ such that

$$\Psi_\delta = \Psi_0 \subset\subset \Psi_1 \subset\subset \Psi_2 \subset\subset \cdots \subset\subset \Psi_{N-1} \subset\subset \Psi_N = \Psi_{2\delta}$$

and for any multi-index β the estimate

$$(1.31) \qquad |D_\xi^\beta \Psi_j| \le C_\beta N^{|\beta|}M^{-s}\langle\xi\rangle^{-|\beta|+s}\,, \qquad 0 \le s \le |\beta|\,,$$

holds with a constant C_β independent of N and j. It should be noted that Ψ_j can be taken of the form $\Psi_j = h_j(\xi_3; M)\psi_j(\xi) = h_j(\xi_3; M)\tilde{\psi}_j(\xi_3, \xi_1)\tilde{\psi}_j(\xi_3, \xi_2)$ with $\tilde{\psi}_j = 1$ in $\{\pm\delta\xi_3 \ge |\xi'|\} \cap \{|\xi| \ge 3/2\delta\}$. Here one of $\pm$ signs is chosen under the above convention. Similarly, take a sequence $\{\varphi_j(x)\}_{j=0}^N \subset C_0^\infty(\mathbf{R}^3)$ such that

$$\varphi_\delta = \varphi_0 \subset\subset \varphi_1 \subset\subset \varphi_2 \subset\subset \cdots \subset\subset \varphi_{N-1} \subset\subset \varphi_N = \varphi_{2\delta}$$

and for any β the estimate

$$|D_x^\beta \varphi_j| \le C'_\beta N^{|\beta|} \tag{1.32}$$

holds with a constant C'_β independent of N and j. We may also assume that φ_j can be written as in $\varphi_j(x) = \prod_{k=1}^{3} h_j(x_k)$.

For the proof of (1.29) we need the following lemma corresponding to Lemma 5.4 of [8]. We fix a sufficiently small $\delta > 0$ such that $\psi_{2\delta}(D)\varphi_{2\delta}(x)Lv \in \mathscr{S}$.

Lemma 1.3. *Let $K = \{x \in \mathbf{R}^3; |x_j| \le 4\delta\}$. There exist a constant C_0 independent of M and N such that for any $s > 0$ and some $C_s > 0$ we have*

$$(\log M^s)^2 \operatorname{Re}([L_0, \varphi_j(x)\Psi_j(D)]u, \varphi_j(x)\Psi_j(D)u) \tag{1.33}$$
$$\le (C_0 N)^2[(L_0 u, u) + C_s\{\|u\|_{-s}^2 + N^{2s+8}M^{-s}\|u\|^2\}], \qquad u \in C_0^\infty(K),$$

provided that $\log M^s \ge C_0 N$ and $M \ge M_s$ for a sufficiently large $M_s > 0$.

Proof. Note that

$$[L_0, \varphi_j(x)\Psi_j(D)] = [L_0, \varphi_j(x)]\Psi_j(D) + \varphi_j(x)[L_0, \Psi_j(D)]. \tag{1.34}$$

We see that

$$\operatorname{Re}([\alpha^2(fD_2 + gD_3)^2, \varphi_j(x)]u, \varphi_j(x)u) = \|[\alpha(fD_2 + gD_3), \varphi_j]u\|^2$$
$$\le (CN)^2\{\|\alpha f u\|^2 + \|\alpha g u\|^2\} \quad \text{for } u \in \mathscr{S}.$$

For a moment, we denote different constants independent of N, M and s by the same notation C. From this we have

$$(\log M^s)^2 \operatorname{Re}([\alpha^2(fD_2 + gD_3)^2, \varphi_j(x)]\Psi_j(D)u, \varphi_j(x)\Psi_j(D)u) \tag{1.35}$$
$$\le (CN)^2\{\|(\log M^s)\Psi_j(D)\alpha f u\|^2 + \|(\log M^s)\Psi_j(D)\alpha g u\|^2$$
$$+ (\log M^s)^2\|[\alpha f, \Psi_j(D)]u\|^2 + (\log M^s)^2\|[\alpha g, \Psi_j(D)]u\|^2\}.$$

It follows from (1.27) that for any $s > 0$ we have

$$\|(\log \Lambda^s)\chi_0(D; M)\alpha(x_1)f(x_1)u\|^2 \le (L_0 u, u) + C_s\|u\|_{-s}^2, \qquad u \in C_0^\infty(K),$$

if $M \ge M_s$ for a sufficiently large M_s. Hence

$$\|(\log M^s)\Psi_j(D)\alpha f u\|^2 \le C\|(\log \Lambda^s)\Psi_{2\delta}(D)\chi_0(D; M)\alpha f u\|^2 \tag{1.36}$$
$$\le C(L_0 u, u) + C_s\|u\|_{-s}^2 \qquad \text{for } u \in C_0^\infty(K).$$

if $M \ge M_s$ for a large $M_s > 0$. Here and in what follows we denote by C_s defferent constants depending on s but independent of N and M. By means of (1.28) we see that $\|(\log M^s)\Psi_j(D)\alpha g u\|^2$ is also estimated above from the right hand side of (1.36). It follows from Lemma 5.3-i) of [8] that

$$\begin{aligned}&(\log M^s)^2\{\|[\alpha f, \Psi_j(D)]u\|^2 + \|[\alpha g, \Psi_j(D)]u\|^2\}\\ &\quad\le (\log M^s)^2(CN)^2M^{-1}\{(\|u\|^2 + C_sN^{2s+8}M^{-s}\|u\|^2\}\\ &\quad\le (\log M^s)^4M^{-1}\{C_K(L_0u, u) + C_sN^{2s+8}M^{-s}\|u\|^2\}, \qquad u\in C_0^\infty(K),\end{aligned}$$

if $\log M^s \ge CN$. Therefore, if $\log M^s \ge CN$ and M is sufficiently large such that $(\log M^s)^4M^{-1} \le 1$ then we have

$$\begin{aligned}(1.37)\quad &(\log M^s)^2 \operatorname{Re}([\alpha^2(fD_2 + gD_3)^2, \varphi_j]\Psi_j u, \varphi_j\Psi_j u)\\ &\quad\le (CN)^2[(L_0u, u) + C_s\{\|u\|_{-s}^2 + N^{2s+8}M^{-s}\|u\|^2\}] \equiv \Omega, \qquad u\in C_0^\infty(K).\end{aligned}$$

Note that

$$\begin{aligned}(1.38)\qquad &(\log M^s)^2 \operatorname{Re}([D_1^2, \varphi_j(x)]\Psi_j(D)u, \varphi_j(x)\Psi_j(D)u)\\ &\quad\le (CN)^2(\log M^s)^2\|\tilde h(x_1)\Psi_j(D)u\|^2\\ &\quad\le (CN)^2\{\|(\log \Lambda^s)\Psi_{2\delta}(D)\tilde h(x_1)u\|^2\\ &\qquad + (\log M^s)^2\|[\tilde h(x_1), \Psi_j(D)]u\|^2\}, \qquad u\in C_0^\infty(K),\end{aligned}$$

where $\tilde h(t)$ is C_0^∞ function such that $0 \le \tilde h \le 1$, $\operatorname{supp}\tilde h \subset [\delta, 4\delta]$. If M is large enough then we have

$$\|(\log \Lambda^s)\Psi_{2\delta}(D)\tilde h(x_1)u\| \le \|\tilde h(x_1)(\log \Lambda^s)\chi_0(D; M)u\| + \|u\| + C_s\|u\|_{-s}, \quad u\in C_0^\infty(K).$$

It follows from (1.27) that the first term of the right hand side of (1.38) is estimated above from Ω. Applying (5.13) of [8] to the second term of the right hand side of (1.38), we obtain

$$(\log M^s)^2 \operatorname{Re}([D_1^2, \varphi_j(x)]\Psi_j(D)u, \varphi_j(x)\Psi_j(D)u) \le \Omega, \qquad u\in C_0^\infty(K),$$

if M satisfies the same condition as in (1.37). From this and (1.37) we obtain

$$(1.39)\qquad \operatorname{Re}([L_0, \varphi_j(x)]\Psi_j(D)u, \varphi_j(x)\Psi_j(D)u) \le \Omega, \qquad u\in C_0^\infty(K),$$

if $\log M^s \ge CN$ and $M \ge M_s$ for a sufficiently large M_s. In view of (1.34), the proof of the lemma will be completed if we show

$$(1.40)\quad (\log M^s)^2 \operatorname{Re}(\varphi_j(x)[X^2, \Psi_j(D)]u, \varphi_j(x)\Psi_j(D)u) \le \Omega, \qquad u\in C_0^\infty(K),$$

where $X = \alpha(x_1)(f(x_1)D_2 + g(x_2)D_3)$. Note that

$$\begin{aligned}(1.41)\quad \operatorname{Re}(\varphi_j[X^2, \Psi_j]u, \varphi_j\Psi_j u) &= \operatorname{Re}(Xu, \{[\Psi_j\varphi_j^2, [X, \Psi_j]] + [X, \Psi_j][\Psi_j, \varphi_j^2]\}u)\\ &\quad + \operatorname{Re}([X, \varphi_j^2\Psi_j]u, [X, \Psi_j]u).\end{aligned}$$

The first term of the right hand side is estimated above from

$$\begin{aligned}&C\|Xu\|\{N^3M^{-1} + C_sN^{2s+10}M^{-(s+1)}\}\|u\|\\ &\quad\le CN^3/M\{\|Xu\|^2 + \|u\|^2 + C_sN^{2s+8}M^{-s}\|u\|^2\}\\ &\quad\le (\log M^s)^{-2}\Omega, \qquad u\in C_0^\infty(K).\end{aligned}$$

if $\log M^s \geq CN$ and M is sufficiently large such that $(\log M^s)^5 \leq M$. Note that the principal symbols of $[X, \Psi_j]$ and $[\alpha, \Psi_j]$ are contained in $\{|\xi'| \geq \delta|\xi_3|\}$ and $\{|\xi_1| \geq \delta|\xi_3|\}$, respectively, because of the form of Ψ_j. Hence the second term of the right hand side of (1.41) is estimated above from

$$CN^2\{(\log M^s)^{-2}\|(\log \Lambda^s)\chi_0(D; M)\chi(D)\alpha u\|^2 + M^{-1}\|D_1 u\|^2 + C_s N^{2s+8} M^{-s}\|u\|^2\},$$

where χ is the same as in Lemma 1.3 with $\delta_0 < \delta/10$. By means of (1.20), those terms multiplied by $(\log M^s)^2$ are also estimated above from $(CN)^2\Omega$. In view of (1.41) we obtain (1.40). Q.E.D.

The implication (1.28) follows immediately from Lemma 1.3 because the arguments on and after Lemma 5.5 of [8] can be carried out quite similarly. In fact, the difference between Lemma 5.4 of [8] and Lemma 1.3 is the presence of $\|u\|_{-s}^2$ in (1.33). This term is harmless because we employ (1.33) with u replaced by $\varphi_j \Psi_j u$ and hence we estimate $\|\varphi_j \Psi_j u\|_{-s}$ by $M^{-s}\|u\|$ (see the proof of Lemma 5.5 of [8]).

The implication (1.28) also holds even if we replace ρ_0 by $((0, x_{02}, x_{03}), (0, 0, \pm 1))$ with $(x_{02}, x_{03}) \neq (0, 0)$. In fact, Lemma 1.3 still holds for $\varphi_j(x)$ corresponding to $\tilde{\varphi}_\delta(x) = h(x_1/\delta) \prod_{j=2}^{3} h((x_j - x_{0j})/\delta)$. In view of Lemma 1.2, the preceding argument also yields (1.28) for $\rho_0 = (x_0, \xi_0)$ with $\xi_0 \neq (0, 0, \pm 1)$ if we modify $\Psi_\delta(\xi)$ to correspond to the direction ξ_0. Thus the proof of Theorem 1 is completed.

2. Proofs of Theorem 2 and 3

We shall first prove Theorem 3. It follows from (13) that $d_x f_1$ and $d_x f_2$ are linearly independent. By taking a suitable coordinates, we may assume $f_j(x) = x_j$, $j = 1, 2$. Write

$$p_j(x, \xi) = a_{j1}(x)\xi_1 + a_{j2}(x)\xi_2 + a_{j3}(x)\xi_3, \qquad j = 1, 2. \tag{2.1}$$

It follows from (13) that

$$D(x) \equiv a_{11}a_{22} - a_{12}a_{21} \neq 0. \tag{2.2}$$

If $(b_{ij}(x))$ is the inverse matrix of $(a_{ij}(x))$ then we have

$$\begin{cases} \xi_1 - c_1(x)\xi_3 = b_{11}p_1 + b_{12}p_2 \\ \xi_2 - c_2(x)\xi_3 = b_{21}p_1 + b_{22}p_2 \end{cases} \tag{2.3}$$

for some $c_j(x) \in C^\infty$. From this we have

$$\begin{aligned} G(x)\xi_3 &\equiv \{\xi_1 - c_1(x)\xi_3, \xi_2 - c_2(x)\xi_3\} \\ &= D(x)^{-1}\{p_1, p_2\} + \sum_{j=1}^{2} \alpha_j(x) p_j(x, \xi) \end{aligned}$$

for some $\alpha_j(x) \in C^\infty$. Under the above choise of the coordinates we see that

$$\Sigma = \{(x, \xi) \in T^*\mathbf{R}^3 \backslash 0; \xi_j - c_j(x)\xi_3 = 0, j = 1, 2\}, \tag{2.4}$$

$$\Gamma_j = \{(x, \xi) \in \Sigma; x_j = 0\}, \quad j = 1, 2. \tag{2.5}$$

If $\rho_0 \in \Gamma_1 \cap \Gamma_2$ then we may write $\rho_0 = (0, \xi_0)$ with

$$\xi_0 \in \{\xi = (\xi', \xi_3); |\xi'| \leq C_0 |\xi_3|\} \tag{2.6}$$

for a sufficiently large $C_0 > 0$. Furthermore, the function $F(x)$ defined in Introduction can be written as in the form

$$F(x) = D(x)G(x)\xi_3 \quad \text{with } \xi_3 = \pm 1/\sqrt{1 + c_1(x)^2 + c_2(x)^2}. \tag{2.7}$$

Let $z_j(x)$ $(j = 1, 2)$ be a solution to

$$(\partial z_j/\partial x_1)(x) + c_j(x', z_j(x)) = 0, \qquad z_j(x)|_{x_1=0} = x_3,$$

where $x' = (x_1, x_2)$. It is clear that $z_j(x)$ exists in a small neighborhood of the origin. Let $u \in C_0^\infty(\mathbf{R}^3)$ satisfy $\operatorname{supp} u \subset \{|x| \leq 2\delta\}$ for a sufficiently small $\delta > 0$. Then there exists a $C_1 > 0$ independent of x' such that

$$\begin{aligned} C_1^{-1} \|u(x', \cdot)\|_{L^2(\mathbf{R})}^2 &\leq \|u(x', z_j(x', \cdot))\|_{L^2(\mathbf{R})}^2 \\ &\leq C_1 \|u(x', \cdot)\|_{L^2(\mathbf{R})}^2 . \end{aligned} \tag{2.8}$$

Since Lemma 2.1 of [8] holds with the absolute value $|\cdot|$ replaced by the norm $\|\cdot\|$ we have

$$\begin{aligned} &\int_I \|D_1 u(x', \cdot)\|_{L^2(\mathbf{R})}^2 dx' \\ &\geq c \frac{(\operatorname{diam} Q_1)^{-2}}{|I|} \int_{I \times I} \|u(x_1, x_2, \cdot) - u(y_1, x_2, \cdot)\|_{L^2(\mathbf{R})}^2 dx' dy' \end{aligned} \tag{2.9}$$

for any rectangle $I = Q_1 \times Q_2 \subset \mathbf{R}_{x'}^2$. Note that $D_1 u(x', z_1(x)) = \{(D_1 - c_1 D_3)u\}(x', z_1(x))$. In view of (2.8), it follows from (2.9) that

$$\begin{aligned} &\int_I \|(D_1 - c_1 D_3)u(x', \cdot)\|_{L^2(\mathbf{R})}^2 dx' \\ &\geq c' \frac{(\operatorname{diam} Q_1)^{-2}}{|I|} \int_{I \times I} \|u(x_1, x_2, \cdot) - u(y_1, x_2, \cdot)\|_{L^2(\mathbf{R})}^2 dx' dy', \end{aligned} \tag{2.10}$$

if $\operatorname{supp} u \subset \{|x| \leq 2\delta\}$. Similarly we have

$$\begin{aligned} &\int_I \|(D_2 - c_2 D_3)u(x', \cdot)\|_{L^2(\mathbf{R})}^2 dx' \\ &\geq c' \frac{(\operatorname{diam} Q_2)^{-2}}{|I|} \int_{I \times I} \|u(y_1, x_2, \cdot) - u(y_1, y_2, \cdot)\|_{L^2(\mathbf{R})}^2 dx' dy'. \end{aligned} \tag{2.11}$$

Set $Y_j = D_j - c_j(x)D_3$, $j = 1, 2$. Then, by means of (2.3) we have for any compact $K \subset \mathbf{R}^3$

$$\|Y_1u\|^2 + \|Y_2u\|^2 \le C_K\{\|X_1u\|^2 + \|X_2u\|^2\} \le C'_K\{\mathrm{Re}\,(Lu, u) + \|u\|^2\}, \qquad u \in C_0^\infty(K). \tag{2.12}$$

If $P = Y_1 \pm iG(x)Y_2$ then

$$P^*P = Y_1^*Y_1 + Y_2^*G^2Y_2 \pm i\{[Y_1^*, G]Y_2 - [Y_2^*, G]Y_1\} \pm iG\{(Y_1^* - Y_1)Y_2 + (Y_2^* - Y_2)Y_1\} \pm iG[Y_1, Y_2].$$

In view of $i[Y_1, Y_2] = G(x)D_3$ we have

$$\pm(G(x)^2D_3u, u) \le C_K\{\|Y_1u\|^2 + \|Y_2u\|^2 + \|u\|^2\}, \qquad u \in C_0^\infty(K). \tag{2.13}$$

Let $h(t) \in C_0^\infty(\mathbf{R}^1)$ be the same as in Section 1. For a large parameter $M > 0$ and a small $\delta > 0$ set

$$\chi_\pm(\xi_3; M) = h((\pm M^{-1}\xi_3 - 3)/\delta).$$

It follows from (2.13) that

$$\|G(x)|D_3|^{1/2}h(x_3/\delta)\chi_\pm(D_3; M)u\|^2 \le C_K\{\|Y_1u\|^2 + \|Y_2u\|^2 + \|u\|^2\}, \qquad u \in C_0^\infty(K). \tag{2.14}$$

Set $\chi(\xi_3; M) = \chi_+(\xi_3; M) + \chi_-(\xi_3; M)$ $(= h((M^{-1}|\xi_3| - 3)/\delta))$. Since $2M \le |\xi_3| \le 4M$ on $\mathrm{supp}\,\chi$ it follows from (2.14) that

$$\|G(x)M^{1/2}h(x_3/\delta)\chi(D_3; M)u\|^2 \le C_K\{\|Y_1u\|^2 + \|Y_2u\|^2 + \|u\|^2\}, \qquad u \in C_0^\infty(K). \tag{2.15}$$

Assume that x belongs to a sufficiently small neighborhood V_0 of the origin such that $V_0 \subset\subset \pi_x V$. Here V is the conic neighborhood of ρ_0 given between (13) and (14) in Introduction. In view of (2.7), it follows from (13) and (14) that for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that

$$|G(x)| \ge \exp\{-\varepsilon/\min(|x_1|, |x_2|\} \qquad \text{if } 0 < \min(|x_1|, |x_2|) \le \delta(\varepsilon). \tag{2.16}$$

Set $x_M = 4\varepsilon/\log M$. We may assume that $x_M < \delta(\varepsilon)$ if M is sufficiently large. It follows from (2.16) that

$$|G(x)|M^{1/2} \ge M^{1/4} \qquad \text{on } \{x \in V_0;\, x_M \le \min(|x_1|, |x_2|) \le \delta(\varepsilon)\}.$$

Since $|G(x)| \ge c_\varepsilon > 0$ on $\{x \in V_0;\, \min(|x_1|, |x_2|) \ge \delta(\varepsilon)\}$ we see that

$$|G(x)|M^{1/2} \ge M^{1/4} \qquad \text{on } \{x \in V_0;\, x_M \le \min(|x_1|, |x_2|)\} \tag{2.17}$$

if $M \ge M_\varepsilon$ for a sufficiently large $M_\varepsilon > 0$.

Let $\delta > 0$ be sufficiently small such that

$$I_0 \equiv \{|x'| \le 2\delta\} \subset\subset \pi_{x'}V_0.$$

Here $\pi_{x'}$ is a natural projection from $\mathbf{R}_x^3$ to $\mathbf{R}_{x'}^2$. Set $\omega_j = \{x \in I_0; |x_j| < x_M\}$, $j = 1, 2$. Similarly as in the proof of Lemma 1.1, divide $I_0 \backslash (\omega_1 \cup \omega_2)$ into congruent squares $I_\nu = Q_1^\nu \times Q_2^\nu$ such that $I_0 \backslash (\omega_1 \cup \omega_2) = \bigcup_\nu I_\nu$ and

$$M^{1/2} \le (\operatorname{diam} Q_j^\nu)^{-2} \le 4M^{1/2} . \tag{2.18}$$

We also divide $\bar{\omega}_1 \backslash \omega_2$ (and $\bar{\omega}_2 \backslash \omega_1$) into congruent smaller rectangles as follows:

$$\bar{\omega}_1 \backslash \omega_2 = \bigcup_{\nu'} J_{1\nu'} , \qquad J_{1\nu'} = [-x_M, x_M] \times Q_2^{\nu'}$$

$$\bar{\omega}_2 \backslash \omega_1 = \bigcup_{\nu''} J_{2\nu''} , \qquad J_{2\nu''} = Q_1^{\nu''} \times [-x_M, x_M] ,$$

where the diameter of $Q_2^{\nu'}$ (resp. $Q_1^{\nu''}$) is equal to that of Q_2^ν (resp. Q_1^ν). Set $\omega_1 \cap \omega_2 = K_0$ $(= Q_0^1 \times Q_0^2 \subset \mathbf{R}_{x_1} \times \mathbf{R}_{x_2})$ and let K_0^* denote four times dilation of K_0. If $u \in C_0^\infty(\{|x| \le \delta\})$ and if $h_\delta \chi = h(x_3/\delta)\chi(D_3; M)$ then we have

$$\begin{aligned}
(2.19)\quad & 4\{\|Y_1 h_\delta \chi u\|^2 + \|Y_2 h_\delta \chi u\|^2 + \|G(x) M^{1/2} h_\delta \chi u\|^2\} \\
& \ge \int_{K_0^*} \{\|Y_1 h_\delta \chi u(x', \cdot)\|_{L^2}^2 + \|Y_2 h_\delta \chi u(x', \cdot)\|_{L^2}^2 + \|G M^{1/2} h_\delta \chi u(x', \cdot)\|_{L^2}^2\} dx' \\
& \quad + \sum_\nu \int_{I_\nu} \{\cdot\} dx' + \sum_{\nu'} \int_{J_{1\nu'}^\dagger} \{\cdot\} dx' + \sum_{\nu''} \int_{J_{2\nu''}^\dagger} \{\cdot\} dx' \\
& \equiv \Omega_0 + \sum_\nu \Omega_\nu + \sum_{\nu'} \Omega_{\nu'} + \sum_{\nu''} \Omega_{\nu''} ,
\end{aligned}$$

where $J_{1\nu'}^\dagger = [-2x_M, 2x_M] \times Q_2^{\nu'}$ and $J_{2\nu''}^\dagger = Q_1^{\nu''} \times [-2x_M, 2x_M]$. Here $\| \ \| = \| \ \|_{L^2(\mathbf{R}^3)}$ and $\| \ \|_{L^2} = \| \ \|_{L^2(\mathbf{R})}$. It follows from (2.10) and (2.11) with I and u replaced by K_0^* and $\tilde{u} \equiv h_\delta \chi u$, respectively, that

$$\begin{aligned}
(2.20)\quad \Omega_0 \ge c \int_{K_0} \Big[\int_{K_0^* \backslash (\omega_1 \cup \omega_2)} & \{x_M^{-2} \|\tilde{u}(x', \cdot) - \tilde{u}(y_1, x_2, \cdot)\|_{L^2}^2 \\
& + x_M^{-2} \|\tilde{u}(y_1, x_2, \cdot) - \tilde{u}(y', \cdot)\|_{L^2}^2 + \|G M^{1/2} \tilde{u}(y', \cdot)\|_{L^2}^2\} dy' \Big] \Big/ |K_0| dx' ,
\end{aligned}$$

because

$$\int_{K_0^*} \|G M^{1/2} \tilde{u}(x', \cdot)\|_{L^2}^2 dx' = \int_{K_0} \Big[\int_{K_0^*} \|G M^{1/2} \tilde{u}(y', \cdot)\|_{L^2}^2 dy' \Big] \Big/ |K_0| dx' .$$

By means of (2.17) and (2.20) we obtain

$$\begin{aligned}
(2.21)\quad \Omega_0 & \ge c' \varepsilon^{-2} (\log M)^2 \int_{K_0} \Big[\int_{K_0^* \backslash (\omega_1 \cup \omega_2)} \|\tilde{u}(x', \cdot)\|_{L^2}^2 dy' \Big] \Big/ |K_0| dx' \\
& \ge c'' \varepsilon^{-2} (\log M)^2 \int_{K_0} \|\tilde{u}(x', \cdot)\|_{L^2}^2 dx' .
\end{aligned}$$

It follows from (2.17) and (2.18) that

$$(2.22)\quad \Omega_{v'} \geq c \int_{J_{1v'}} \left[\int_{J_{1v'}^{\dagger}\setminus\omega_1} \{x_M^{-2} \|\tilde{u}(x', \cdot) - \tilde{u}(x_1, y_2, \cdot)\|_{L^2}^2 + M^{1/2}\|\tilde{u}(x_1, y_2, \cdot) - \tilde{u}(y', \cdot)\|_{L^2}^2 + \|GM^{1/2}\tilde{u}(y', \cdot)\|_{L^2}^2\} dy' \right] \Big/ |J_{1v'}| dx'$$

$$\geq c'\varepsilon^{-2}(\log M)^2 \int_{J_{1v'}} \|\tilde{u}(x', \cdot)\|_{L^2}^2 dx' .$$

Similarly we have

$$(2.23)\qquad \Omega_{v''} \geq c'\varepsilon^{-2}(\log M)^2 \int_{J_{2v''}} \|\tilde{u}(x', \cdot)\|_{L^2}^2 dx' ,$$

$$(2.24)\qquad \Omega_{v} \geq c' M^{1/2} \int_{I_v} \|\tilde{u}(x', \cdot)\|_{L^2}^2 dx' .$$

Summing up (2.21–24), in view of (2.19) we obtain

$$(2.25)\qquad \|Y_1 h_\delta \chi u\|^2 + \|Y_2 h_\delta \chi u\|^2 + \|G(x)M^{1/2} h_\delta \chi u\|^2 \geq c'\varepsilon^{-2}(\log M)^2 \|h_\delta \chi u\|^2 , \qquad u \in C_0^\infty(\{|x| \leq \delta\})$$

if M is large enough. Note that $[Y_j, \chi(D_3; M)]$ $(j = 1, 2)$ are L^2 bounded operators uniformly with respect to M. It follows from (2.12), (2.15) and (2.25) that for any $\varepsilon > 0$ we have

$$(\log M)^2 \|h(x_3/\delta)\chi(D_3; M)u\|^2 \leq \varepsilon^2 \left\{ \sum_{j=1}^{2} \|X_j u\|^2 + \|u\|^2 \right\} , \qquad u \in C_0^\infty(\{|x| \leq \delta\}) ,$$

provided that $M \geq M_\varepsilon$ for a sufficiently large $M_\varepsilon > 0$. From this we see that for any $M \in [1, \infty)$ the estimate

$$(\log M)^2 \|h(x_3/\delta)\chi(D_3; M)u\|^2 \leq \varepsilon \sum_{j=1}^{2} \|X_j u\|^2 + C_\varepsilon \|u\|^2 , \qquad u \in C_0^\infty(\{|x| \leq \delta\}) ,$$

holds with any $\varepsilon > 0$ and some constant C_ε. Note that $h(x_3/\delta) = 1$ on $\operatorname{supp} u$ and $2M \leq |\xi_3| \leq 4M$ on $\operatorname{supp} \chi$. Since $M[h(x_3/\delta), \chi(D_3; M)]$ is L^2 bounded uniformly with respect to M we have

$$(2.26)\quad \|(\log |D_3|)\chi(D_3; M)u\|^2 \leq 4\|(\log M)^2 \chi(D_3; M)u\|^2 \leq \varepsilon \sum_{j=1}^{2} \|X_j u\|^2 + C_\varepsilon \|u\|^2 , \qquad u \in C_0^\infty(\{|x| \leq \delta\}) .$$

Let $\psi(\xi) \in S_{1,0}^0$ be real valued and let ψ satisfy $\psi = 1$ in $\{|\xi'| \leq C_0|\xi_3|\} \cap \{|\xi_3| \geq 1\}$ and $\operatorname{supp} \psi \subset \{|\xi'| \leq 2C_0|\xi_3|\}$. Here C_0 is the same constant as in (3.6). Set $\varphi(x) = \prod_{k=1}^{3} h(2x_k/\delta)$ and $\chi_{2\delta}(\xi_3; M) = h((M^{-1}|\xi_3| - 3)/2\delta)$. (Note that $\chi(\xi_3; M) = \chi_\delta(\xi_3; M)$). Let $u \in \mathscr{S}$ and substitute $\varphi(x)\chi_{2\delta}(D_3; M)\psi(D)u$ into (3.26).

Then we have

$$\text{(2.27)} \qquad \|\chi_\delta(D_3; M)\varphi(x)\psi(D)(\log \Lambda)u\|^2$$
$$\leq \varepsilon \sum_{j=1}^{2} \|\chi_{2\delta}(D_3; M)X_j u\|^2 + C_\varepsilon\{\|\chi_{4\delta}(D_3; M)u\|^2 + M^{-2}\|u\|^2\}$$

by noting the expansion formula of pseudodifferential operators. Integrate with respect to $M \in [1, \infty)$ after dividing both sides of (2.27) by M. By Lemma 5.6 of [8] we have

$$\text{(2.28)} \qquad \|(\log \Lambda)\varphi(x)\psi(D)u\|^2 \leq \varepsilon \sum_{j=1}^{2} \|X_j u\|^2 + C_\varepsilon\|u\|^2$$
$$\leq \varepsilon \operatorname{Re}(Lu, u) + C'_\varepsilon\|u\|^2, \qquad u \in \mathscr{S}.$$

By means of Corollary 6 in Introduction, (2.28) shows that $\rho_0 = (0, \xi_0) \notin \mathrm{WF}\, Lu$ implies $\rho_0 \notin \mathrm{WF}\, u$ for any $u \in \mathscr{D}'(\mathbf{R}^3)$. We have completed the proof of Theorem 3.

Now the proof of Theorem 2 is an easy exercise. Taking a suitable coordinates, by means of (9) we may write

$$\text{(2.29)} \qquad p_1 = \xi_1, \qquad p_2 = a_1(x)\xi_1 + a_2(x)\xi_2 + a_3(x)\xi_3$$

with $a_2(x) \neq 0$. Then

$$\text{(2.30)} \qquad \Sigma = \{(x, \xi) \in T^*\mathbf{R}^3\backslash 0; \xi_1 = \xi_2 + b(x)\xi_3 = 0\},$$

where $b(x) = a_3(x)/a_2(x)$. It follows from (11) that we may assume

$$\text{(2.31)} \qquad \Gamma = \{(x, \xi) \in \Sigma; x_1 = 0\}.$$

If $\rho_0 \in \Gamma$ then we may write $\rho_0 = (0, \xi_0)$ with ξ_0 satisfying (2.6). Setting $G(x)\xi_3 = \{\xi_1, \xi_2 + b(x)\xi_3\}$ $(= \partial_{x_1} b(x)\xi_3)$, instead of (2.7) we have

$$\text{(2.32)} \qquad F(x) = a_2(x)G(x)\xi_3 \qquad \text{with } \xi_3 = \pm 1/\sqrt{1 + b(x)^2}.$$

Set $Y_1 = D_1$ and $Y_2 = D_2 + b(x)D_3$. Then we also have (2.12) and (2.15). Let V_0 be a sufficiently small neighborhood of the origin such that $V_0 \subset\subset \pi_x V$, where V is the conic neighborhood of ρ_0 given between (10) and (11) in Introduction. If for any $\varepsilon > 0$ we set $x_M = 4\varepsilon/\log M$ then it follows from (12) and (2.32) that

$$\text{(2.33)} \qquad |G(x)|M^{1/2} \geq M^{1/4} \qquad \text{on } \{x \in V_0; x_M \leq |x_1|\}$$

if $M \geq M_\varepsilon$ for a sufficiently large $M_\varepsilon > 0$. Using (2.33) we obtain, in place of (2.25),

$$\text{(2.34)} \qquad \|Y_1 h_\delta \chi u\|^2 + \|G(x)M^{1/2}h_\delta \chi u\|^2$$
$$\geq c'\varepsilon^{-2}(\log M)^2\|h_\delta \chi u\|^2, \qquad u \in C_0^\infty(\{|x| \leq \delta\}).$$

Since (2.12) and (2.15) still holds we obtain (2.26) and hence (2.28), which leads us to the conclusion of Theorem 2.

3. Proof of Theorem 4

Similarly as in the proof of Theorem 3, it follows from (9) that we may write without loss of generality

$$p_1 = \xi_1 , \qquad p_2 = a_1(x)\xi_1 + a_2(x)\xi_2 + a_3(x)\xi_3 \tag{3.1}$$

with $a_2(x) \neq 0$. Then

$$\Sigma = \{(x, \xi) \in T^*\mathbf{R}^3\backslash 0; \xi_1 = \xi_2 + b(x)\xi_3 = 0\} , \tag{3.2}$$

$$\Gamma = \{(x, \xi) \in \Sigma; \partial_{x_1} b(x) = 0\} , \tag{3.3}$$

where $b(x) = a_3(x)/a_2(x)$. If $\rho_0 \in \Gamma \cap \{|\xi| = 1\}$ then we may write $\rho_0 = (0, \xi_0)$. By taking the change of variables $x_j = y_j$ $(j = 1, 2)$, $x_3 = b(0)y_2 + y_3$, if necessary, we may assume that $b(0) = 0$. In view of (3.2) we see $\rho_0 = (0, (0, 0, \pm 1))$. Since $H_1, H_2 \in T\Sigma^\perp \subset T\Gamma^\perp$ it follows from (15) that we can find a $c_0 > 0$ satisfying the following; for any $0 < \delta \leq c_0$

$$\partial_{x_1} b(x) \neq 0 \qquad \text{on } \{|x_3| \leq c_0\delta\} \cap \{|x_j| \geq \delta, j = 1, 2\} . \tag{3.4}$$

Since $\pi_x \Gamma$ is a submanifold in $\mathbf{R}^3$ of codimension 2, $\partial_{x_1} b(x)$ has a definite sign. Note that

$$|\partial_{x_1} b(x)| = \sqrt{1 + b(x)^2}\,|a_2(x)F(x)| \qquad \text{(cf., (2.32))}.$$

It follows from (16) that there exists a C^∞ function $\tilde{E}(x) > 0$ defined in a neighborhood of the origin such that $(\tilde{E}\partial_{x_1} b)(s, x_2, x_3)$ has a unique extremum in $(-\delta_0, \delta_0)$ if $|x_j|$ are small enough. For each $x'' = (x_2, x_3)$ let $s(x'') = s(x_2, x_3)$ denote the extremal point. If we set $\tilde{b}(x) = \int_{s(x'')}^{x_1} \partial_{x_1} b(\tau, x'')d\tau$ then in a small neighborhood of the origin we have

$$\begin{aligned} |\tilde{b}(x)| &\leq C\left|\int_{s(x'')}^{x_1} |(\tilde{E}\partial_{x_1} b)(\tau, x'')|\,d\tau\right| \\ &\leq C'|(\tilde{E}\partial_{x_1} b)(x)| \leq C''|\partial_{x_1} b(x)| . \end{aligned} \tag{3.5}$$

Let $z(y'') = z(y_2, y_3)$ be a solution to

$$\partial z/\partial y_2 = b(s(y_2, z), y_2, z) , \qquad z(0, y_3) = y_3 .$$

It is clear that $z(y'')$ exists in a small neighborhood of the origin in $\mathbf{R}^2$. Take the change of variables

$$x_j = y_j \ (j = 1, 2) , \qquad x_3 = z(y_2, y_3) . \tag{3.6}$$

Since $b(x) = \tilde{b}(x) + b(s(x''), x'')$ we see that D_1 and $D_2 + b(x)D_3$ are transformed to D_1 and $D_2 + B(y)D_3$, respectively, where

$$B(y) = \tilde{b}(y_1, y_2, z(y''))/(\partial z/\partial y_3)(y'') . \tag{3.7}$$

Note that $\partial z/\partial y_3$ is close to 1 near $y''=0$. Since $\partial_{y_1}B(y)=(\partial_{x_1}b)(y_1, y_2, z(y''))/(\partial z/\partial y_3)(y'')$ it follows from (3.5) that

$$|B(y)| \le C|\partial_{y_1}B(y)| \qquad \text{for } |y| \text{ small enough}. \tag{3.8}$$

The direct calculation gives

$$|\partial_{y_3}B(y)|$$

$$\le C_1|\partial_{x_1}b(s(y_2, z(y'')), y_2, z(y''))| + C_2\left|\int_{s(y_2,z(y''))}^{y_1} |(\partial_{x_1}\partial_{x_3}b)(\tau, y_2, z(y''))|d\tau\right|.$$

The first term of the right hand side is estimated above from $C|\partial_{y_1}B(y)|$ because $|(\tilde{E}\partial_{x_1}b)(s(x''), x'')| \le |(\tilde{E}\partial_{x_1}b)(x)|$. Since $\partial_{x_1}b$ has a definite sign we have $|\partial_{x_1}\partial_{x_3}b(x)| \le C|\partial_{x_1}b(x)|^{1/2}$ in a neighborhood of the origin. The second term is estimated above from

$$C\left|\int_{s(x'')}^{x_1} |(\tilde{E}\partial_{x_1}b)(\tau, x'')|^{1/2}d\tau\right| \le C'|\partial_{x_1}b(x)|^{1/2}$$

with $x=(y_1, y_2, z(y''))$. The last estimate follows from the similar argument as in (3.5). Hence we have

$$|\partial_{y_3}B(y)| \le C|\partial_{y_1}B(y)|^{1/2} \qquad \text{for } |y| \text{ small enough}. \tag{3.9}$$

From now on we denote new variables y in (3.6) and $B(y)$ by x and $b(x)$, respectively. Furthermore we assume that $a_j(x)$ in (3.1) are written by new variables. Since $a_3=a_2b$ it follows from (3.8) and (3.9) that

$$|a_3(x)| \le C|\partial_{x_1}b(x)|, \tag{3.10}$$

$$|\partial_{x_3}a_3(x)| \le C|\partial_{x_1}b(x)|^{1/2} \qquad \text{for } |x| \text{ small enough}. \tag{3.11}$$

We may assume that (3.4) holds by taking another small $c_0>0$, if necessary. If $P=D_1 \pm i(D_2+bD_3)$ then $P^*P=D_1^2+(D_2+D_3b)(D_2+bD_3) \pm ((\partial_{x_3}b)D_1+(\partial_{x_1}b)D_3)$. Since $\partial_{x_1}b$ has a definite sign we have

$$\begin{aligned}\pm(|\partial_{x_1}b|D_3u, u) &\le C\{\|D_1u\|^2+\|D_2+bD_3u\|^2+\|u\|^2\} \\ &\le C'\{\mathrm{Re}\,(Lu, u)+\|u\|^2\}\end{aligned} \tag{3.12}$$

if $u\in C_0^\infty(\{|x|\le 100\delta\})$ for a sufficiently small $\delta>0$.

Since $\Gamma \ni \rho_0=(0, \xi_0)$ with $\xi_0=(0, 0, \pm 1)$ we prepare similar cut functions as in Section 1. For a $\delta>0$ let $\psi_\delta(\xi)$ and $\Psi_\delta(\xi; M)$ be the same as in Section 1. Considering (3.4), we modify the definition of $\varphi_\delta(x)$ as follows; $\varphi_\delta(x)=h(10x_3/c_0\delta)\prod_{k=1}^{2} h(x_k/\delta)$. For any integer $N>0$ we take the same sequences $\{\Psi_j(\xi)\}_{j=0}^N$ and $\{\varphi_j(x)\}_{j=0}^N$ as in Section 1. In what follows we shall only use estimates

$$|D_\xi^\beta \Psi_j| \le C_\beta M^{-s}\langle\xi\rangle^{-|\beta|+s}, \qquad 0 \le s \le |\beta|, \tag{3.13}$$

$$|D_x^\beta \varphi_j| \le C'_\beta, \tag{3.14}$$

in place of the precise estimates (1.31) and (1.32). We still require that φ_j can be written as in $\varphi_j(x) = \prod_{k=1}^{3} h_j(x_k)$.

Note that $|\partial_{x_1} b(x)|\varphi_{2\delta}(x)^2(|\xi_3| - M)\Psi_{2\delta}(\xi)^2 \ge 0$ belongs to $S^1_{1,0}$. By the sharp Gårding inequality (see Theorem 4.4 of [5]), it follows from (3.12) that

$$M\| |\partial_{x_1} b|^{1/2} \varphi_{2\delta}(x)\Psi_{2\delta}(D)u\|^2 \le C\{\mathrm{Re}\,(Lu, u) + \|u\|^2\}, \tag{3.15}$$

if $u \in C_0^\infty(\{|x| \le 100\delta\})$ for a sufficiently small $\delta > 0$. For the proof of Theorem 4 we need the following lemma that corresponds to Lemma 1.3 in Section 1.

Lemma 3.1. *Let $\kappa = 1/4$ and let $K = \{x \in \mathbf{R}^3; |x| \le 10\delta\}$. There exist a constant C_0 independent of M such that*

$$\begin{aligned} M^{2\kappa}\,\mathrm{Re}\,([L, \varphi_j(x)\Psi_j(D)]u, \varphi_j(x)\Psi_j(D)u) \\ \le C_0\{(Lu, u) + \|u\|^2\}, \qquad u \in C_0^\infty(K). \end{aligned} \tag{3.16}$$

Proof. Note that

$$[L, \varphi_j(x)\Psi_j(D)] = [L, \varphi_j(x)]\Psi_j(D) + \varphi_j(x)[L, \Psi_j(D)]. \tag{3.17}$$

If $X = a_1D_1 + a_2D_2 + a_3D_3$ we see that

$$\mathrm{Re}\,([X^2, \varphi_j(x)]u, \varphi_j(x)u) = \mathrm{Re}\,([X^*X, \varphi_j]u, \varphi_j u) + \mathrm{Re}\,([(X - X^*)X, \varphi_j]u, \varphi_j u).$$

Since the first term of the right hand side is equal to $\|[X, \varphi_j]u\|^2$ and the second term is not bigger than $C\|[X, \varphi_j]u\|\,\|u\|$ we have

$$\mathrm{Re}\,([X^2, \varphi_j(x)]u, \varphi_j(x)u) \le C\{M^{2\kappa}\|[X, \varphi_j]u\|^2 + M^{-2\kappa}\|u\|^2\} \qquad \text{for } u \in \mathcal{S}.$$

From this and the similar formula with X replaced by D_1 we have

$$\begin{aligned} &M^{2\kappa}\,\mathrm{Re}([L, \varphi_j(x)]\,\Psi_j(D)u, \varphi_j(x)\Psi_j(D)u) \\ &\le C\Big\{\sum_{k=1}^{2} M\|\tilde{h}(x_k)\varphi_{2\delta}(x)\Psi_{2\delta}(D)u\|^2 \\ &\quad + M\|a_3(x)^{1/2}\varphi_{2\delta}(x)\Psi_{2\delta}(D)u\|^2 + \|u\|^2\Big\}, \qquad u \in \mathcal{S}, \end{aligned}$$

where $\tilde{h}(t)$ is the same as in (1.38). In view of (3.4) and (3.10), it follows from (3.15) that

$$\begin{aligned} M^{2\kappa}\,\mathrm{Re}([L, \varphi_j(x)]\,\Psi_j(D)u, \varphi_j(x)\Psi_j(D)u) \\ \le C\{\mathrm{Re}\,(Lu, u) + \|u\|^2\}, \qquad u \in C_0^\infty(K). \end{aligned} \tag{3.18}$$

Note that

$$(3.19)\quad \mathrm{Re}\,(\varphi_j[X^2, \Psi_j]u, \varphi_j\Psi_j u)$$
$$= \mathrm{Re}\,(\varphi_j[(X - X^*)X, \Psi_j]u, \varphi_j\Psi_j u) + \mathrm{Re}\,([X, \varphi_j^2\Psi_j]u, [X, \Psi_j]u)$$
$$+ \mathrm{Re}\,(Xu, \{[\Psi_j\varphi_j^2, [X, \Psi_j]] + [X, \Psi_j][\Psi_j, \varphi_j^2]\}u).$$

Since $\Psi_j(\xi)$ has the form $\Psi_j = h_j(\xi_3; M)\psi_j(\xi)$ we see that

$$[a_3(x)D_3, \Psi_j(D)] = [a_3, \psi_j]h_j D_3 + \psi_j[a_3, h_j]D_3\,.$$

Note that the principal symbol of $[a_3, \psi_j]$ is contained in $\{|\xi'| \geq \delta|\xi_3|\}$. The first term of the right hand side of (3.19) is estimated above from

$$C[\{M^{-1}\|D_1u\| + M^{-1}\|\varphi_{2\delta}\Psi_{2\delta}D_2u\| + \|a_3\varphi_{2\delta}\Psi_{2\delta}u\|$$
$$+ \|(\partial_{x_3}a_3)\varphi_{2\delta}\Psi_{2\delta}u\|\}\|u\| + M^{-1}\|u\|^2]$$
$$\leq C'M^{-1}\{\|D_1u\|^2 + \|Xu\|^2 + \|u\|^2 + M\|a_3\varphi_{2\delta}\Psi_{2\delta}u\|^2$$
$$+ M\|(\partial_{x_3}a_3)\varphi_{2\delta}\Psi_{2\delta}u\|^2\}$$

On account of (3.10) and (3.11), it follows from (3.15) that the first term of the right hand side of (3.19) is estimated above from $C''M^{-1}\{\mathrm{Re}\,(Lu, u) + \|u\|^2\}$. Similarly we can estimate the second term of the right hand side of (3.19). Because the third term is not bigger than $C\|Xu\|\,\|u\|/M$, we have

$$(3.20)\quad M\,\mathrm{Re}\,(\varphi_j[X^2, \Psi_j]u, \varphi_j\Psi_j u) \leq C\{\mathrm{Re}\,(Lu, u) + \|u\|^2\}\,, \qquad u \in C_0^\infty(K)\,.$$

In view of (3.17) we obtain the desired estimate (3.16) from (3.18) and (3.20).
Q.E.D.

If $\delta > 0$ is sufficiently small then we have

$$(3.21)\qquad \|u\|^2 \leq C\,\mathrm{Re}\,(Lu, u)\,, \qquad u \in C_0^\infty(\{|x| \leq 10\delta\})\,.$$

In fact, if W is a small neighborhood of the origin then there exists a $C(W) > 0$ depending only on W such that

$$\|D_1u\|^2 \leq C(W)\{\mathrm{Re}\,(Lu, u) + \|u\|^2\}\,, \qquad u \in C_0^\infty(W)\,.$$

From this we have (3.21) because the Poincaré inequality

$$\|u\| \leq c_1\delta^2\|D_1u\|^2\,, \qquad u \in C_0^\infty(\{|x| \leq 10\delta\})\,,$$

holds with an absolute constant c_1. By (3.21) it follows from (3.16) that

$$(3.16)'\qquad M^{2\kappa}\,\mathrm{Re}\,([L, \varphi_j(x)\Psi_j(D)]u, \varphi_j(x)\Psi_j(D)u)$$
$$\leq C(Lu, u)\,, \qquad u \in C_0^\infty(\{|x| \leq 10\delta\})\,.$$

Using (3.21) and (3.16)′, by the same method as in the proof of Lemma 5.5 of

[8] we obtain for any $M \geq 1$

$$M^{2N\kappa}\|\varphi_\delta \Psi_\delta u\|^2 \leq C\{M^{2N\kappa}\|\Psi_{2\delta}\varphi_{2\delta}Lu\|\,\|u\| + M^2\|u\|^2\} \tag{3.22}$$
$$\leq C'\{M^{4N\kappa}\|\Psi_{2\delta}\varphi_{2\delta}Lu\|^2 + M^2\|u\|^2\}, \qquad u \in \mathscr{S}.$$

Here constants C and C' depend on N, of course. Recall that $N > 0$ is arbitrary integer. Then the argument after (5.33) of [8] can be carried out by using (3.22) in place of (5.32) of [8]. Thus the proof of Theorem 4 is accomplished.

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