J. Math. Kyoto Univ. (JMKYAZ)
32-2 (1992) 387-399

Range characterization of Radon transforms on complex projective spaces

By

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§ 0. Introduction

The purpose of this paper is to characterize the ranges of Radon transforms on P^nC by invariant differential operators.

Range characterization of a Radon transform by a differential operator was first treated by F. John [8]. Consider the set M of all lines in R^3 of the form

$$l:\ x=\alpha_1 t+\beta_1,\qquad y=\alpha_2 t+\beta_2,\qquad z=t,\qquad (t\text{: parameter}).$$

We define a coordinate system on M by $l\mapsto(\alpha_1, \alpha_2, \beta_1, \beta_2)\in R^4$. Let $R: C_0^\infty(R^3)\to C_0^\infty(M)$ be the Radon transform defined by

$$Rf(l)=\int_{-\infty}^{\infty} f(\alpha_1 t+\beta_1, \alpha_2 t+\beta_2, t)dt \qquad \text{for } f\in C_0^\infty(R^3).$$

Then it is easily chacked that the range of R is included in the kernel of an ultra-hyperbolic differential operator P defined by

$$P=\frac{\partial^2}{\partial\alpha_1\partial\beta_2}-\frac{\partial^2}{\partial\alpha_2\partial\beta_1}, \tag{0.1}$$

and, in fact, F. John showed that Ker P=Im R, that is, the range of R is characterized by P. Gelfand, Graev, and Gindikin [1] later extended F. John's result to Radon transforms on R^n and C^n. They characterized the range of the d-plane Radon transform on R^n (resp. on C^n) $(d<n-1)$ by a system of second order differential operators on a corresponding real (resp. complex) affine Grassmann manifold.

For Radon transforms on compact symmetric spaces, there exists Grinberg's result [4]. He showed that the range of the projective d-plane Radon transform on a real or complex projective space is characterized by an invariant system of second order differential operators in a corresponding compact Grassman manifold. We notice that his construction of the system was led by representation theoretical consideration.

On the other hand, our approach is based on the idea of F. John, which yields characterization by a single invariant differential operator on a Grassmann manifold. In fact, the range-characterizing operator can be represented as an ultrahyperbolic differential operator like (0.1) on a vector bundle, but we here treat the one reduced to a single differential operator for the sake of simplicity.

Communicated by Prof. N. Iwasaki, May 15, 1990

Let M be the set of all projective l-planes in $\boldsymbol{P}^n\boldsymbol{C}$, which is a complex Grassmann manifold and is a compact symmetric space of rank $\min\{l+1, n-l\}$.

We define a Radon transform $R: C^\infty(\boldsymbol{P}^n\boldsymbol{C}) \to C^\infty(M)$ by

$$Rf(\xi)=\frac{1}{\mathrm{Vol}\,(\boldsymbol{P}^l\boldsymbol{C})}\int_{x\in\xi} f(x)dv_\xi(x), \qquad \xi\in M, \qquad f\in C^\infty(\boldsymbol{P}^n\boldsymbol{C}),$$

where $dv_\xi(x)$ denotes the measure on $\xi(\subset\boldsymbol{P}^n\boldsymbol{C})$ induced by the canonical measure on $\boldsymbol{P}^n\boldsymbol{C}$.

We assume that $\mathrm{rank}\, M\geqq 2$, that is, $1\leqq l\leqq n-2$. The main theorem of this paper is as follows.

Theorem. *There exists a fourth order invariant differential operator P on M such that the range of the Radon transform R is characterized by P, i.e.,* $\mathrm{Ker}\, P=\mathrm{Im}\, R$.

The explicit form of P will be given in the next section.

The author would like to thank Professor Chiaki Tsukamoto for suggesting the problem and for many valuable discussions.

§1. Invariant differential operator P

Let M be the set of all $(l+1)$-dimensional complex vector subspaces of $\boldsymbol{C}^{n+1}$, that is, the set of projective l-planes in $\boldsymbol{P}^n\boldsymbol{C}$. Then M is a compact symmetric space $SU(n+1)/S(U(l+1)\times U(n-l))$ of rank $\min\{l+1, n-l\}$. We assume that $r:=\mathrm{rank}\, M\geqq 2$, that is, $1\leqq l\leqq n-2$.

For a Lie group G and its closed subgroup H, we denote by $C^\infty(G, H)$ the set $\{f\in C^\infty(G)\,;\, f(gh)=f(g)\ \forall g\in G \text{ and } \forall h\in H\}$, and we identify $C^\infty(G, H)$ with $C^\infty(G/H)$. We define an action L_g of G on $C^\infty(G)$ by $(L_g f)(x)=f(g^{-1}x)$ for $x\in G$, and $f\in C^\infty(G)$. Similarly we define an action R_g of G on $C^\infty(G)$ by $(R_g f)(x)=f(xg)$. A differential operator D on G is called left-G-invariant if $L_g D=DL_g$ for all $g\in G$. Similarly, D is called right-H-invariant if $R_h D=DR_h$ for all $h\in H$. We identify a right-H-invariant differential operator on G with a differential operator on G/H.

Let G, K, and K' be the groups $SU(n+1)$, $S(U(l+1)\times U(n-l))$, and $S(U(1)\times U(n))$, respectively. Then we have $M=G/K$, $\boldsymbol{P}^n\boldsymbol{C}=G/K'$, and by the above identification, $C^\infty(G, K)=C^\infty(M)$, $C^\infty(G, K^n)=C^\infty(\boldsymbol{P}^n\boldsymbol{C})$. We choose a Killing form metrics on G, which induces metrics on K, K', M, and $\boldsymbol{P}^n\boldsymbol{C}$. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of G and K, respectively,

$$\mathfrak{g}=\{X\in M_{n+1}(\boldsymbol{C})\,;\, X+X^*=0,\ \mathrm{tr}\, X=0\},$$

$$\mathfrak{k}=\left\{\begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}\in\mathfrak{g}\,;\, X_1\in M_{l+1}(\boldsymbol{C}),\ X_2\in M_{n-l}(\boldsymbol{C})\right\}.$$

Let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{m}$ be the Cartan decomposition, where $\mathfrak{m}$ is all the matrices of the form

$$Z=\begin{pmatrix} 0 & \cdots & 0 & -\bar{z}_{l+2,1} & \cdots & -\bar{z}_{n+1,1} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & -\bar{z}_{l+2,l+1} & \cdots & -\bar{z}_{n+1,l+1} \\ z_{l+2,1} & \cdots & z_{l+2,l+1} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ z_{n+1,1} & \cdots & z_{n+1,l+1} & 0 & \cdots & 0 \end{pmatrix}.$$

We define second order differential operators $L_{ij,\alpha\beta}$ ($l+2\leqq i<j\leqq n+1$, $1\leqq\alpha<\beta\leqq l+1$) and a fourth order differential operator P on G as follows.

$$L_{ij,\alpha\beta}f(g)=\Big(\frac{\partial^2}{\partial z_{i\alpha}\partial z_{j\beta}}-\frac{\partial^2}{\partial z_{i\beta}\partial z_{j\alpha}}\Big)f(g\exp Z)|_{Z=0},\qquad f\in C^\infty(G),$$

$$P=\sum_{\substack{l+2\leq i<j\leq n+1\\ 1\leq\alpha<\beta\leq l+1}} L^*_{ij,\alpha\beta}L_{ij,\alpha\beta},$$

where $L^*_{ij,\alpha\beta}$ denotes the adjoint operator of $L_{ij,\alpha\beta}$ and is given by

$$L^*_{ij,\alpha\beta}f(g)=\Big(\frac{\partial^2}{\partial \bar{z}_{i\alpha}\partial \bar{z}_{j\beta}}-\frac{\partial^2}{\partial \bar{z}_{i\beta}\partial \bar{z}_{j\alpha}}\Big)f(g\exp Z)|_{Z=0}.$$

Lemma 1.1. *P is a right-K-invariant differential operator.*

Proof. We define Ad-K-invariant polynomials $F_j(Z)$ ($j=1, 2, \cdots$) on $\mathfrak{m}$ by

$$\det(\lambda I+Z)=\lambda^{n+1}+F_1(Z)\lambda^{n-1}+F_2(Z)\lambda^{n-3}+\cdots.$$

Then we have

$$F_2(Z)=\sum_{\substack{l+2\leq i<j\leq n+1\\ 1\leq\alpha<\leq\beta l+1}}(\bar{z}_{i\alpha}\bar{z}_{j\beta}-\bar{z}_{i\beta}\bar{z}_{j\alpha})(z_{i\alpha}z_{j\beta}-z_{i\beta}z_{j\alpha}). \tag{1.1}$$

On the other hand, we have

$$R_kPR_{k^{-1}}=\sum_{\substack{l+2\leq i<j\leq n+1\\ 1\leq\alpha<\beta\leq l+1}} R_kL^*_{ij,\alpha\beta}R_{k^{-1}}\circ R_kL_{ij,\alpha\beta}R_{k^{-1}},$$

where

$$R_kL_{ij,\alpha\beta}R_{k^{-1}}f(g)=\Big(\frac{\partial^2}{\partial z_{i\alpha}\partial z_{j\beta}}-\frac{\partial^2}{\partial z_{i\beta}\partial z_{j\alpha}}\Big)f(g\exp kZk^{-1})|_{Z=0},$$

$$R_kL^*_{ij,\alpha\beta}R_{k^{-1}}f(g)=\Big(\frac{\partial^2}{\partial \bar{z}_{i\alpha}\partial \bar{z}_{j\beta}}-\frac{\partial^2}{\partial \bar{z}_{i\alpha}\partial \bar{z}_{j\alpha}}\Big)f(g\exp kZk^{-1})|_{Z=0},$$

for $f\in C^\infty(G)$ and $k\in K$.

Thus, we have only to prove that P is invariant under the linear transform $Z\mapsto kZk^{-1}=Ad_kZ$, which follows easily from the fact that the polynomial $F_2(Z)$ is Ad-K-invariant. ∎

It is obvious that P is left-G-invariant. Therefore, P is well-defined as an invariant differential operator on M.

The purpose of this paper is to prove the following theorem.

Theorem 1.2. *The range of the Radon transform R is characterized by the invariant differential operator P, that is,* $\operatorname{Ker} P=\operatorname{Im} R$.

Remark 1.3. The above differential operators $L_{ij,\alpha\beta}$ and $L^{*}_{ij,\alpha\beta}$ are of the form similar to (0.1). In this sense, we can say that the range of the Radon transform on $\boldsymbol{P}^n\boldsymbol{C}$ is also characterized by an *ultrahyperbolic* differential operator.

§2. Proof that Im $R\subset$Ker P

We first prove that $\operatorname{Im} R\subset\operatorname{Ker} P$. By the identification $C^{\infty}(G, K(=C^{\infty}(M)$ and $C^{\infty}(G, K')=C^{\infty}(\boldsymbol{P}^n\boldsymbol{C})$, we consider the Redon transform R as a map from $C^{\infty}(G, K')$ to $C^{\infty}(G, K)$. Then R is given by

$$(2.1)\qquad (Rf)(g)=\frac{1}{\operatorname{Vol}(K)}\int_{k\in K}f(gk)dk,\qquad f\in C^{\infty}(G, K').$$

From this section, we use the representation of the form (2.1).

We define a bilinear form $\langle\cdot,\cdot\rangle$ on $\boldsymbol{C}^{n+1}\times\boldsymbol{C}^{n+1}$ by $\langle u, v\rangle=\sum_{j=1}^{n+1}u_jv_j$ for $u=(u_1, \cdots, u_{n+1})$, $v=(v_1, \cdots, v_{n+1})$, and a function $h^m_{a,b}\in C^{\infty}(G)$ by $h^m_{a,b}(g)=\langle a, g\boldsymbol{e}_1\rangle^m\langle b\overline{g\boldsymbol{e}_1}\rangle^m$, where $a, b\in\boldsymbol{C}^{n+1}$, $\boldsymbol{e}_1=(1, 0, \cdots, 0)\in\boldsymbol{C}^{n+1}$ and m is a nonnegative integer. It is easily checked that $h^m_{a\,b}\in C_{\infty}(G, K')$, that is, $h^m_{a,b}\in C^{\infty}(\boldsymbol{P}^n(\boldsymbol{C})$. Moreover the following lemma holds.

Lemma 2.1. *Let V_m denote the subspace of $C^{\infty}(\boldsymbol{P}^n\boldsymbol{C})$ generated by the set $\{h^m_{a,b}; \langle a, b\rangle=0\}$. Then V_m is the eigenspace of $\Delta_{\boldsymbol{P}^n\boldsymbol{C}}$, the Laplacian on $\boldsymbol{P}^n\boldsymbol{C}$, corresponding to the m-th eigenvalue and V_m is irreducible under the action of G.*

For the proof, see [10] §14.

Proposition 2.2. Im $R\subset$Ker P.

Proof. Since P and R are G-invariant operators and $L_{g^{-1}}h^m_{a,b}=h^m_{g*a,g*b}$, we have

$$P(R(h^m_{a,b}))(g)=P(R(h^m_{g*a,g*b}))(I),$$

where I denotes the $(n+1)\times(n+1)$ identity matrix.

Since the direct sum $\bigoplus_{m=0}^{\infty}R(V^m)$ is dense in Im R in C^{∞}-topology, we have only to prove $P(R(h^m_{a,b}))(I)=0$, or,

$$\begin{aligned}&L_{ij,\alpha\beta}(R(h^m_{a\,b}))(I)\\&=\frac{1}{\operatorname{Vol}(K)}\Big(\frac{\partial^2}{\partial z_{i\alpha}\partial z_{j\beta}}-\frac{\partial^2}{\partial z_{i\beta}\partial z_{j\alpha}}\Big)\int_{k\in K}h^m_{a,b}((\exp Z)k)dk\,|_{Z=0}\\&=0.\end{aligned}$$

Here we have

$$\left(\frac{\partial^2}{\partial z_{i\alpha}\partial z_{j\beta}}-\frac{\partial^2}{\partial z_{i\beta}\partial z_{j\alpha}}\right)\{\langle a, (\exp Z)ke_1\rangle^m\langle b, \overline{(\exp Z)ke_1\rangle^m}\}|_{Z=0}$$

$$=m(m-1)\langle a, ke_1\rangle^{m-2}\langle b, \overline{ke_1}\rangle^m$$

$$\times\left\{\frac{\partial}{\partial z_{i\alpha}}\langle a, Zke_1\rangle\frac{\partial}{\partial z_{j\beta}}\langle a, Zke_1\rangle-\frac{\partial}{\partial z_{j\alpha}}\langle a, Zke_1\rangle\frac{\partial}{\partial z_{i\beta}}\langle a, Zke_1\rangle\right\}$$

$$=m(m-1)(a_ik_{\alpha 1}a_jk_{\beta 1}-a_ik_{\beta 1}a_jk_{\alpha 1})\langle a, ke_1\rangle^{m-2}\langle b, \overline{ke_1}\rangle^m=0\,,$$

where $k\in K$, and k_{ij} denotes the (i, j) entry of k. In the above computation, we used the fact that the polynomial $\langle b, \overline{Zke_1}\rangle$ on $\mathfrak{m}$ is a linear combination of $\bar{z}_{pq}$'s and the fact that the polynomial $\langle a, Z^2ke_1\rangle$ and $\langle b, \overline{Z^2ke_1}\rangle$ on $\mathfrak{m}$ consist only of the terms of the form (constant)$\times z_{pq}\bar{z}_{p'q'}$.)

Therefore the assertion is verified. ■

§ 3. The inversion Formula

We construct a continuous linear map $S: C^\infty(M)\to C^\infty(\boldsymbol{P}^n\boldsymbol{C})$ such that $SR=Id$, where Id denotes the identity map.

Let Ξ denote the set of $(n-1)$ dimensional complex projective subspaces of $\boldsymbol{P}^n\boldsymbol{C}$. Then $\Xi=SU(n+1)/S(U(n)\times U(1))$, and we put $K''=S(U(n)\times U(1))$. We define a Radon transform $\mathfrak{F}: C^\infty(\boldsymbol{P}^n\boldsymbol{C})\to C^\infty(\Xi)$ and its dual Radon transform $\check{\mathfrak{F}}: C^\infty(\Xi)\to C^\infty(\boldsymbol{P}^n\boldsymbol{C})$ by

$$\mathfrak{F}f(g)=\frac{1}{\mathrm{Vol}\,(K'')}\int_{k''\in K''}f(gk'')dk''\,,\qquad f\in C^\infty(G, K'),$$

$$\check{\mathfrak{F}}\phi(g)=\frac{1}{\mathrm{Vol}\,(K')}\int_{k'\in K'}\phi(gk')dk'\,,\qquad \phi\in C^\infty(G, K'').$$

We define a polynomial $\Phi(t)$ by

$$\Phi(t)=\left(t+\frac{(n-1)1}{n+1}\right)\left(t+\frac{(n-2)2}{n+1}\right)\cdots\left(t+\frac{1(n-1)}{n+1}\right).$$

Theorem 3.1 (Helgason [6], Ch. 1, Theorem 4.11). *We have the inversion formula*

$$c_n\Phi(\Delta_{P^nC})\check{\mathfrak{F}}\mathfrak{F}=Id\,,$$

where c_n is a constant depending on n.

Proposition 3.2. *There exists an inversion map $S: C^\infty(M)\to C^\infty(\boldsymbol{P}^n\boldsymbol{C})$ such that $SR=Id$.*

Proof. We define a continuous linear map $\tilde{R}: C^\infty(M)\to C^\infty(\Xi)$ by

$$\tilde{R}f(g)=\frac{1}{\mathrm{Vol}\,(K'')}\int_{k''\in K''}f(gk'')dk''\,,\qquad f\in C^\infty(G, K).$$

Then, it is easily checked that $\tilde{R}R=\mathfrak{F}$. Therefore, if we put

$$S=c_n\Phi(\Delta_{P^nC})\check{\mathfrak{F}}\tilde{R}\,,\tag{3.1}$$

we get $SR=Id$ by Theorem 3.1.

§ 4. Representation of (G, K)

In this section, we describe the root, the weight, and the Weyl group of (G, K).

Let $\mathfrak{a}\subset\mathfrak{m}$ be the set of all matrices of the form

$$H(t)=H(t_1, \cdots, t_r)=\sqrt{-1}\begin{pmatrix} 0 & \cdots & 0 & t_1 & & & & \\ \vdots & & \vdots & & \ddots & & & \\ 0 & \cdots & 0 & & & t_r & & \\ t_1 & & & & & 0 & \cdots & 0 \\ & \ddots & & & & \vdots & & \vdots \\ & & t_r & & & & & \\ & & & & & 0 & \cdots & 0 \end{pmatrix}.$$

where we put $r=\operatorname{rank} M(=\operatorname{rank} G/K)$ in Section 1 and $t=(t_1, \cdots, t_r)\in \boldsymbol{R}^r$. Then, $\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{m}$. We identify $\mathfrak{a}$ with $\boldsymbol{R}^r$ by the mapping $H(t)\mapsto t$.

Let $(\cdot, \cdot)$ denote an invariant inner product on $\mathfrak{g}$ defined by

$$(X, Y)=-2(n+1)\operatorname{tr}(XY), \qquad X, Y\in\mathfrak{g},$$

which is a minus signed Killing form on $\mathfrak{g}$.

For $\alpha\in\mathfrak{a}$, we set $\mathfrak{g}_\alpha:=\{X\in\mathfrak{g}^C; [H, X]=\sqrt{-1}(\alpha, H)X$ for all $H\in\mathfrak{a}]$, and α is called a root of $(\mathfrak{g}, \mathfrak{a})$ when $\mathfrak{g}_\alpha\neq\{0\}$. We put $m_\alpha=\dim_C\mathfrak{g}_\alpha$, and call it a multiplicity of α.

We put $H_i=H(0, \cdots, \overset{(i)}{1}, \cdots, 0)$ $(1\leqq i\leqq r)$. Then the roots of $(\mathfrak{g}, \mathfrak{a})$ and their multiplicities are given by the table:

α	m_α	
$\pm\dfrac{1}{2(n+1)}H_j$	1	$(1\leqq j\leqq r)$,
$\pm\dfrac{1}{4(n+1)}H_j$	$2(n+1-2r)$	$(1\leqq j\leqq r)$,
$\pm\dfrac{1}{4(n+1)}(H_j\pm H_k)$	2	$(1\leqq j<k\leqq r)$,

We fix a lexicographical order $<$ on $\mathfrak{a}$ such that $H_1>\cdots>H_r>0$. Then the positive roots are $(1/2(n+1))H_j$, $(1/4(n+1))H_j$, $(1\leqq j\leqq r)$, $(1/4(n+1))(H_j\pm H_k)$, $(1\leqq j<k\leqq r)$. The simple roots are $(1/4(n+1))(H_1-H_2)$, $(1/4(n+1))(H_2-H_3)$, $\cdots$, $(1/4(n+1))(H_{r-1}-H_r)$, $(1/4(n+1))H_r$. We define the positive Weyl chamber $\mathcal{A}^+$ by $\{t\in\boldsymbol{R}^r; 0<t_j<\pi/2$ $(1\leqq j\leqq r)$, $t_1>\cdots>t_r\}$.

We set

$$\Omega((\exp H(t))K):=\Big|\prod_{\alpha:\text{ positive root}}(e^{\sqrt{-1}(\alpha, H(t))}-e^{-\sqrt{-1}(\alpha, H(t))})^{m_\alpha}\Big|.$$

We consider Ω as a density function on $\boldsymbol{R}^+$, and we have

$$\Omega=\sigma\omega^{2},$$

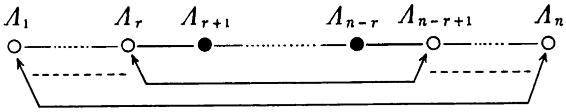
(4.1) where $\sigma=2^{r(2n-2r+3)}\left|\prod_{j=1}^{r}\sin 2t_j \sin^{2(n-r+1)}t_j\right|,$

$$\omega=2^{(1/2)r(r-1)}\prod_{j<k}(\cos 2t_j-\cos 2t_k).$$

The Satake diagram of G/K is given by (4.2) or (4.3),

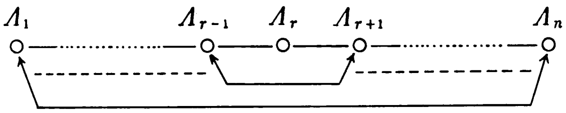
case A $n+1>2r$:

(4.2)



case B $n+1=2r$:

(4.3)



In the diagram (4.2) or (4.3), $\Lambda_1, \cdots, \Lambda_n$ denote the fundamental weights of $\mathfrak{g}$, corresponding to the simple roots of $\mathfrak{g}$.

Since rank $G/K=r$, there are r fundamental weights $M_1, \cdots, M_r$ of (G, K). By the diagram (4.2) or (4.3), $M_1, \cdots, M_r$ are given by

$$M_1=\Lambda_1+\Lambda_n, \cdots, M_{r-1}=\Lambda_{r-1}+\Lambda_{n-r+2}, \ M_r=\Lambda_r+\Lambda_{n-r+1}, \quad \text{(case A)},$$

$$M_1=\Lambda_1+\Lambda_n, \cdots, M_{r-1}=\Lambda_{r-1}+\Lambda_{n-r+2}, \ M_r=2\Lambda_r, \quad \text{(case B)}.$$

Then we have

$$M_k=\frac{1}{2(n+1)}\sum_{j=1}^{k}H_j, \qquad (1\leqq k\leqq r),$$

Let $Z(G, K)$ be the weight lattice, that is, $Z(G, K)=\{(1(4(n+1))(\mu_1H_1+\cdots+\mu_rH_r); \mu_1, \cdots, \mu_r\in \boldsymbol{Z}\}$. The highest weight of a spherical representation of (G, K) is of the form $m_1M_1+\cdots+m_rM_r$, where $m_1, \cdots, m_r$ are non-negative integers. Let $V(m_1, \cdots, m_r)$ denote the eigenspace of the Laplacian Δ_M on G/K that is an irreducible representation space with the highest weight $m_1M_1+\cdots+m_rM_r$.

The Weyl group $W(G, K)$ of (G, K) is the set of all maps $s: \mathfrak{a}\rightarrow\mathfrak{a}$ such that

$$s:(t_1, \cdots, t_r)\longmapsto(\varepsilon_1t_{\sigma(1)}, \cdots, \varepsilon_rt_{\sigma(r)}), \qquad \varepsilon_j=\pm1, \ \sigma\in\mathfrak{S}_r,$$

§ 5. Radial part of P

We calculate the radial part of the invariant differential operator P. The result in this section is used to calculate the eigenvalues of P in the next section.

To each invariant differential operator D on G/K, there corresponds a unique differential operator on Weyl chambers which is invariant under the action of the Weyl group $W(G, K)$. This operator is called a radial part of D, and we denote it by $\mathrm{rad}(D)$.

The following lemma is well-known. (See [10] Theorem 10.4.)

Lemma 5.1. *The radial part of the Laplacian Δ_M on M is given by*

$$\mathrm{rad}(\Delta_M)=-\frac{1}{4(n+1)}\sum_{j=1}^{r}\left(\frac{\partial^2}{\partial t_j^2}+\frac{\Omega_{t_j}}{\Omega}\frac{\partial}{\partial t_j}\right).$$

We define a differential operator Q_1 on $\boldsymbol{R}^r$ by

$$Q_1:=\frac{1}{\omega}\sum_{j=1}^{r}\left(\frac{\partial^2}{\partial t_j^2}+\frac{\sigma_{t_j}}{\sigma}\frac{\partial}{\partial t_j}\right)\circ\omega.$$

The next lemma is easily checked.

Lemma 5.2.

$$-4(n+1)\,\mathrm{rad}(\Delta_M)=Q_1-\sum_{j=1}^{r}4j(j+n+2-2r).$$

We consider the following conditions (A), (B), (C) and (D) on a differential operater Q on $\boldsymbol{R}^r$ that is regular in all Weyl chambers.

(A) $Q=\frac{1}{16}\sum_{j<k}\frac{\partial^2}{\partial t_j^2}\frac{\partial^2}{\partial t_k^2}+$lower order terms.

(B) Q is formally self-adjoint with respect to the density Ωdt.

(C) Q is $W(G, K)$-invariant.

(D) $[Q, \mathrm{rad}(\Delta_M)]:=Q\,\mathrm{rad}(\Delta_M)-\mathrm{rad}(\Delta_M)Q=0$.

Then the differential operator $\mathrm{rad}(P)$ satisfies the above four conditions (A), (B), (C), and (D). Indeed, the principal symbol of P is given by $\frac{1}{16}F_2(Z)$, which was defined in (1.1). (Notice that $\frac{\partial}{\partial z_{ij}}=\frac{1}{2}\frac{\partial}{\partial x_{ij}}-\frac{\sqrt{-1}}{2}\frac{\partial}{\partial y_{ij}}$ for $z_{ij}=x_{ij}+\sqrt{-1}y_{ij}$.) Therefore its restriction to $\mathfrak{a}$ $\frac{1}{16}F_2(H(t))$ is $\frac{1}{16}\sum_{j<k}t_j^2t_k^2$, and the condition (A) holds. The condition (B) follows from the self-adjointness of P. Since P is an invariant differential operator, the conditions (C) and (D) are easily verified.

We defined a differential operator Q_2 by

$$Q_2:=\frac{1}{16\omega}\sum_{j<k}\left(\frac{\partial^2}{\partial t_j^2}+\frac{\sigma_{t_j}}{\sigma}\frac{\partial}{\partial t_j}\right)\left(\frac{\partial^2}{\partial t_k^2}+\frac{\sigma_{t_k}}{\sigma}\frac{\partial}{\partial t_k}\right)\circ\omega.$$

Lemma 5.3. *The differential operator Q_2 satisfies the conditions (A), (B), (C), and (D).*

Proof. The condition (A) is obvious. The condition (D) follows from Lemma 5.2.

The conditions (B) and (C) are easily checked using the formula (4.3). ■

Lemma 5.4. *If a differential operator Q satisfies the conditions (A), (B), (C) and (D), then Q can be written in the form*

$$Q=Q_2+c_1\,\mathrm{rad}\,(\Delta_M)+c_2\,,$$

for suitable constants c_1, c_2.

Proof. Because of the conditions (A) and (B), $Q-Q_2$ is a second order differential operator and satisfies the conditions (B), (C), and (D). Therefore the proof is reduced to the following lemma.

Lemma 5.5. *If a second order differential operator*

$$Q:=\sum_{j=1}^{r}A_j\frac{\partial^2}{\partial t_j^2}+\sum_{j<k}B_{jk}\frac{\partial^2}{\partial t_j\partial t_k}+\sum_{j=1}^{r}C_j\frac{\partial}{\partial t_j}$$

satisfies the conditions (B), (C), and (D), then Q can be written in the form

$$Q=c\,\mathrm{rad}\,(\Delta_M)\,,$$

where c is a suitable constant.

Proof. By the condition (D), the third order terms of $[Q,\,\mathrm{rad}\,(\Delta_M)]$ vanish. Thus we have the following equations.

(5.1) $$A_{j,t_j}=0 \qquad (1\leqq j\leqq r);$$

(5.2) $$A_{k,t_j}+B_{jk,t_k}=0, \qquad A_{j,t_k}+B_{jk,t_j}=0, \qquad (j<k);$$

(5.3) $$B_{ij,t_k}+B_{jk,t_i}+B_{ik,t_j}=0, \qquad (1\leqq i<j<k\leqq r).$$

By the equations (5.1), (5.2), and (5.3), we obtain

(5.4) $$A_{j,t_k t_k t_k}=0, \qquad (j\neq k);$$

(5.5) $$B_{jk,t_j t_j}=B_{jk,t_k t_k}=0,$$

(5.6) $$B_{jk,t_i t_i t_i}=0, \qquad (i\neq j,\,k).$$

From the condition (C) and the equations (5.1-6), the coefficients A_j and B_{jk} are polynomials of the form

(5.7) $$A_j=\delta_1\sum_{k\neq j}t_k^2+\delta_2\,,$$

(5.8) $$B_{jk}=-2\delta_1 t_j t_k\,,$$

where δ_1 and δ_2 are some constants.

Using the conditions (B), we have

(5.9) $$C_j=\frac{1}{\Omega}(A_j\Omega)_{t_j}+\frac{1}{2\Omega}\sum_{j<k}(B_{jk}\Omega)_{t_k}+\frac{1}{2\Omega}\sum_{k<j}(B_{kj}\Omega)_{t_k}\,.$$

If $\delta_1=0$, then the coefficient $B_{jk}=0$, and the coefficient $C_j=\delta_2\Omega_{t_j}/\Omega$ by (5.9).

Therefore we obtain $Q=-4(n+1)\delta_2 \operatorname{rad}(\Delta_M)$, and the lemma holds.

Now, we suppose that $\delta_1 \neq 0$. Furthermore, we may suppose that $\delta_1=1$ and $\delta_2=0$. By the condition (D), the first order terms of $[Q, \operatorname{rad}(\Delta_M)]$ vanish. Thus we have

$$(5.10) \qquad Qa_j=-4(n+1)\operatorname{rad}(\Delta_M)C_j \qquad (1 \leqq j \leqq r),$$

where we put $a_j=\Omega_{t_j}/\Omega$.

We extend the both sides of (5.10) to $\boldsymbol{C}$ as meromorphic functions of $t_1=\mu_1+\sqrt{-1}\nu_1$. By the formula (4.1), we have

$$a_1=2\frac{\cos 2t_1}{\sin 2t_1}+2(n-r+1)\frac{\cos t_1}{\sin t_1}+2\sum_{j=2}^{r}\frac{-2\sin 2t_1}{\cos 2t_1-\cos 2t_j}.$$

As $\nu_1 \to +\infty$, we have $a_{1,t_j} \to 0$, $a_{1,t_j t_k} \to 0$ (rapidly decreasing), and $a_1=O(1)$. The same fact holds for a_j $(j=2, \cdots, r)$. Thus $Qa_1 \to 0$ (rapidly decreasing). Therefore we get $\operatorname{rad}(\Delta_M)C_1 \to 0$ (rapidly decreasing) by (5.10). However, when ν_1 tends to $+\infty$, we have

$$-4(n+1)\operatorname{rad}(\Delta_M)C_1=\frac{1}{2}\sum_{j,k=2}^{r}\left(\frac{\partial^2}{\partial t_k^2}+a_k\frac{\partial}{\partial t_k}\right)(B_{1j}a_j+B_{1j,t_j})+O(1)$$

$$=-t_1\sum_{k=2}^{r}a_k^2+O(1).$$

(In the above computation, we used (5.7), (5.8) and the fact that $a_j=O(1)$ and the derivatives of $a_j \to 0$ as $\nu_1 \to \infty$) Therefore, we have $\operatorname{rad}(\Delta_M)C_1 \to \infty$, for suitable $t_2, \cdots, t_r$, and μ_1. It is a contradiction. ■

Lemmas 5.1, 5.2, and 5.3 imply the following proposition.

Proposition 5.5. *The differential operator* $\operatorname{rad}(P)$ *can be expressed in the form*

$$\operatorname{rad}(P)=Q_2+c_1Q_1+c_2,$$

for some constants c_2, c_2.

§6. Proof of Theorem 1.2

We calculate the eigenvalue of P on $V(m_1, \cdots, m_r)$ to prove Theorem 1.2.

Let $a(m_1, \cdots, m_r)$ be the eigenvalue of P on $V(m_1, \cdots, m_r)$ and $\phi_{(m_1,\cdots,m_r)}$ the zonal spherical function which belongs to $V(m_1, \cdots, m_r)$. We denote by $u_{(m_1,\cdots,m_r)}$ the restriction of $\phi_{(m_1,\cdots,m_r)}$ to the Weyl chamber $\mathcal{A}^+$.

Lemma 6.1 ([10], Theorem 8.1). *The function* $u_{(m_1,\cdots,m_r)}$ *has a Fourier series expansion on* $\mathcal{A}^+$ *of the form*

$$u_{(m_1,\cdots,m_r)}(t_1, \cdots, t_r)=\sum_{\substack{\lambda \leqq m_1M_1+\cdots+m_rM_r \\ \lambda \in Z(G,K),\ \text{finite sum}}}\eta_\lambda \exp\sqrt{-1}(\lambda, t_1H_1+\cdots+t_rH_r),$$

where $\eta_{m_1M_1+\cdots+m_rM_r}>0$.

Let f_1 and f_2 be Fourier series on $\mathcal{A}^+$ of the form

$$f_1=\sum_{\lambda\leq\Lambda_1\lambda\in Z(G,K)}\zeta_\lambda\exp\sqrt{-1}(\lambda, t_1H_1+\cdots+t_rH_r),$$

$$f_2=\sum_{\lambda\leq\Lambda_2\lambda\in Z(G,K)}\xi_\lambda\exp\sqrt{-1}(\lambda, t_1H_1+\cdots+t_rH_r).$$

We denote $f_1\sim f_2$ when $\Lambda_1=\Lambda_2$ and $\zeta_{\Lambda_1}=\xi_{\Lambda_2}$ $(\neq 0)$. Obviously the relation $\sim$ is an equivalence relation.

Lemma 6.2. *We have the following relations.*

$$\sigma_{t_j}\sim 2\sqrt{-1}(n+2-2r)\sigma\,, \tag{6.1}$$

$$\omega_{t_j}\sim 2\sqrt{-1}(r-j)\omega\,, \tag{6.2}$$

$$\frac{\partial}{\partial t_j}u_{(m_1,\cdots,m_r)}\sim 2\sqrt{-1}(m_j+m_{j+1}+\cdots+m_r)u_{(m_1,\cdots,m_r)}\,. \tag{6.3}$$

Proof. The relations (6.1) and (6.2) are easily checked. The relation (6.3) follows from Lemma 6.1. ∎

Lemma 6.3.

$$a(m_1,\cdots,m_r)=\sum_{j<k}(l_j+r-j)(l_j+n+2-r-j)(l_k+r-k)(l_k+n+2-r-k)$$
$$-4c_1\sum_{j=1}^{r}(l_j+r-j)(l_j+n+2-r-j)+c_2\,, \tag{6.4}$$

where c_1 and c_2 are constants in Proposition 5.5, *and* $l_j=m_j+m_{j+1}+\cdots+m_r$.

Proof. Since $\phi_{(m_1,\cdots,m_r)}\in V(m_1,\cdots,m_r)$,

$$P\phi_{(m_1,\cdots,m_r)}=a(m_1,\cdots,m_r)\phi_{(m_1,\cdots,m_r)}\,. \tag{6.5}$$

We restrict the both sides of (6.5) to $\mathcal{A}^+$, and then we have

$$\mathrm{rad}\,(P)u_{(m_1,\cdots,m_r)}=a(m_1,\cdots,m_r)u_{(m_1,\cdots,m_r)}\,.$$

By Proposition 5.4, we get

$$\sigma^2\omega\sum_{j<k}\frac{1}{\omega}\Big(\frac{\partial^2}{\partial t_j^2}+\frac{\sigma_{t_j}}{\sigma}\frac{\partial}{\partial t_j}\Big)\Big(\frac{\partial^2}{\partial t_k^2}+\frac{\sigma_{t_k}}{\sigma}\frac{\partial}{\partial t_k}\Big)(\omega u_{(m_1,\cdots,m_r)})$$
$$+c_1\sigma^2\omega\sum_{j=1}\frac{1}{\omega}\Big(\frac{\partial^2}{\partial t_j^2}+\frac{\sigma_{t_j}}{\sigma}\frac{\partial}{\partial t_j}\Big)(\omega u_{(m_1,\cdots,m_r)})+c_2\sigma^2\omega u_{(m_1,\cdots,m_r)}$$
$$=a(m_1,\cdots,m_r)\sigma^2\omega u_{(m_1,\cdots,m_r)}\,.$$

Using Lemma 6.2, we have

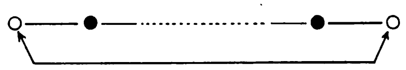
$$\sum_{j<k}\{(l_j+r-j)(l_j+n+2-r-j)(l_k+r-k)(l_k+n+2-r-k)\sigma^2\omega u_{(m_1,\cdots,m_r)}\}$$
$$-4c_1\sum_{j=1}^{r}(l_j+r-j)(l_j+n+2-r-j)\sigma^2\omega u_{(m_1,\cdots,m_r)}+c_2\sigma^2\omega u_{(m_1,\cdots,m_r)} \tag{6.6}$$
$$\sim a(m_1,\cdots,m_r)\sigma^2\omega u_{(m_1,\cdots,m_r)}\,.$$

Comparing the leading coefficients of the both sides of (6.6), we get (6.4). ■

Lemma 6.4. *$R: V_m \to V(m, 0, \cdots, 0)$ is an isomorphism.*

Proof. By Proposition 3.2, R is G-equivariant and one to one. Thus we have only to prove that the highest weight of V_m is equal to that of $V(m, 0, \cdots, 0)$. The Satake diagram of $\boldsymbol{P}^n\boldsymbol{C}$ is given by

(6.7)



Comparing (6.7) with the diagram (4.2) or (4.3), we find that V_m corresponds to mM_1. On the other hand, the highest weight of $V(m, 0, \cdots, 0)$ is mM_1 by definition. This completes our proof. ■

Now, we can calculate the eigenvalue of P by combining the above lemmas.

Theorem 6.5. *The eigenvalue $a(m_1, \cdots, m_r)$ of P on $V(m_1, \cdots, m_r)$ is given by*

$$a(m_1, \cdots, m_r) = \sum_{j<k} l_j l_k (l_j+n+2-2j)(l_k+n+2-2k) + \sum_{j=2}^{r} (j-1)(n+1-j) l_j (l_j+n+2-2j), \tag{6.8}$$

where $l_j = m_j + m_{j+1} + \cdots + m_r$.

Proof. By Proposition 2.2 and Lemma 6.4, we have $a(m, 0, \cdots, 0)=0$ for any non-negative integer m. Then, by Lemma 6.3, we have

$$\begin{aligned}
&(m+r-1)(m+n+1-r)\sum_{k=2}^{r}(r-k)(n+2-r-k) \\
&+\sum_{2\leq j<k<r}(r-j)(n+2-r-j)(r-k)(n+2-r-k) \\
&-4c_1(m^2+nm)-4c_1\sum_{j=1}^{r}(r-j)(n+2-r-j)+c_2 \\
&=0\,.
\end{aligned}$$

Therefore, we get

$$c_1 = \frac{1}{4}\sum_{k=2}^{r-1}(r-k)(n+2-r-k), \tag{6.9}$$

$$\begin{aligned} c_2 &= \sum_{j<k}(r-j)(n+2-r-j)(r-k)(n+2-r-k) \\ &\quad + \Big\{\sum_{j=1}^{r-1}(r-j)(n+2-r-j)\Big\}^2\,. \end{aligned} \tag{6.10}$$

Substituting (6.9) and (6.10) to (6.4), we obtain the formula (6.8). ■

The following corollary is now obvious.

Corollary 6.6. *$V(m_1, \cdots, m_r)$ is contained in* Ker P, *if and only if* $m_2 = \cdots = m_r = 0$.

Proof of Theorem 1.2. Let $V := \bigoplus_{m=0}^{\infty} V(m, 0, \cdots, 0)$ and $\tilde{V} := \bigoplus_{m=0}^{\infty} V_m$ (direct sums). Then, we have $R: \tilde{V} \to V$ and $S: V \to \tilde{V}$. Here S is the inversion map defined in (3.1). Moreover, we have $SR = Id$ on $\tilde{V}$ and $RS = Id$ on V by Proposition 3.2 and Lemma 6.4.

By Collorary 6.6, V is dense in Ker P in C^∞-topology. Since the inversion map $S: C^\infty(M) \to C^\infty(\boldsymbol{P}^n \boldsymbol{C})$ is continuous, we have $RS = Id$ on Ker P. This completes the proof. ■

Remark 6.7. If $\phi \in$ Ker P, the inverse image of ϕ is given by $S\phi$, this is, $R(S\phi) = \phi$.

Remark 6.8. The invariant differential operator P, which we constructed in Section 1, is of least degree in all the invariant differential operators on M that characterize the range of R. It follows from the fact that the principal symbol $(1/16)F_2(Z)$ of P is of least degree in all the Ad-K-invariant polynomials on $\mathfrak{m}$ except for the principal symbol of the Laplacian.

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