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Estimates for degenerate Schrödinger operators and hypoellipticity for infinitely degenerate elliptic operators

By

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Introduction and main theorems

In Chapter II of [1] Fefferman and Phong estimated the eigenvalues of Schrödinger operators $-\Delta+V(x)$ on $\boldsymbol{R}^n$ by using the uncertainty principle. Inspired by their idea, in the present paper we give some L^2-estimates for degenerate Schrödinger operators of higher order, which are versions and extensions of Theorem 4 in Chapter II of [1]. As applications, we consider the hypoellipticity of infinitely degenerate elliptic operators of second order. Some parts of the present paper (Theorem 1, 2 and 6 below) are announced in [13].

Consider a symbol of the form

$$a(x, \xi)=\sum_{k=1}^{n} a_k(x)|\xi_k|^{2\mu_k}+V(x), \qquad x\in\boldsymbol{R}^n, \tag{1}$$

where μ_k are positive rational numbers, $V(x)\geqq 0$ belongs to $L_1^{\mathrm{loc}}(\boldsymbol{R}^n)$ and

$$\begin{cases} a_1(x)=1, \\ a_k(x)=\sum_{j=1}^{k-1}|x_j|^{2\kappa(k,j)} \qquad \text{for } k\geqq 2. \end{cases} \tag{2}$$

Here $\kappa(k, j)$ are non-negative rational numbers. If $x_0\in\boldsymbol{R}^n$ and if $\delta=(\delta_1, \cdots, \delta_n)$ for $\delta_j>0$, we denote by $B_\delta(x_0)$ a box

$$\{(x,\xi);\ |x_j-x_{0j}|\leqq\delta_j/2,\ |\xi_j|\leqq\delta_j^{-1}/2\}. \tag{4}$$

Clearly the volume of $B_\delta(x_0)$ is equal to 1. Let C denote a set of boxes $B_\delta(x_0)$ for all x_0 and all δ. We denote by $\boldsymbol{m}_l(\cdot)$ the Lebesgue measure in R^l. We set $m_k=\mu_k-1$ if μ_k is integer and $m_k=[\mu_k]$ otherwise. Set $m_0=\sum_{k=1}^{n} m_k$.

Theorem 1. *Let $a(x, \xi)$ be the above symbol and let $W(x)$ be a real-valued continuous function in $\boldsymbol{R}^n$. Assume that there exists a constant $1-2^{-m_0}<c\leqq 1$ such that for any $B=B_\delta(x_0)\in C$*

$$\boldsymbol{m}_{2n}(\{(x, \xi)\in B\,;\ a(x, \xi)\geqq \max_{\pi(B^{**})} W(x)\})\geqq c, \tag{4}$$

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where π is a natural projection from $\boldsymbol{R}^{2n}_{x,\xi}$ to $\boldsymbol{R}^n_x$ and B^{**} denotes a suitable dilation of B whose modulus depends only on μ_k and $\kappa(k, j)$. Then for any compact set K of $\boldsymbol{R}^n_x$ there exists a constant $c_K>0$ such that

(5) $$(a(x, D)u, u)\geqq c_K(W(x)u, u) \quad \text{for any } u\in C_0^\infty(K).$$

where (,) denotes the L^2 inner product. (cf. Theorem B in [11]).

Remark. The lower bound of c in (4) is 0 when all $\mu_k\leqq 1$. If all $a_k(x)\equiv 1$ then the constant c_K in (5) can be taken independent of K. The theorem holds even if each variable x_j is replaced by the vector $\boldsymbol{x}_j=(x_1^j, \cdots, x_{l_j}^j)$. The rationality assumption of μ_k and $\kappa(k, j)$ can be removed.

In the polynomial potential case the theorem becomes fairly simple. In order to explain this fact, for a $0<h\leqq 1$ we redefine a set C_h of boxes

(6) $$B_{\delta,h}(x_0)\equiv\{(x, \xi)\,;\ |x_j-x_{0j}|\leqq\delta_j/2,\ |\xi_j|\leqq h\delta_j^{-1}/2\}$$

for all x_0 and all δ.

Theorem 2. *Let $a(x, \xi)$ be the symbol of the form* (1) *with $V(x)$ replaced by a polynomial $U(x)$ in $\boldsymbol{R}^n$ of order d, which is not always non-negative. Then for any compact set K of $\boldsymbol{R}^n$ there exists a positive $h=h_K\leqq 1$ satisfying the following property: If the estimate*

(7) $$\max_{B_h} a(x, \xi)\geqq 0$$

holds for any $B_h=B_{\delta,h}(x_0)\in C_h$ then we have

(8) $$(a(x, D)u, u)\geqq 0 \quad \text{for any } u\in C_0^\infty(K).$$

Here the positive h depends only on d, n, μ_k and $\kappa(k, j)$ except K.

Remark 1. When all $a_k(x)\equiv 1$ then we can take $h>0$ independent of K. Furthermore, if all $\mu_k=1$ then Theorem 2 is nothing but one part of Theorem 4 in Chapter II of [1].

2. When $V(x)$ and $W(x)$ in Theorem 1 are polynomials, Theorem 1 follows from Theorem 2 by putting $U(x)=V(x)-h^{2\mu_0}W(x)$, where $\mu_0=\max_{1\leq k\leq n}\mu_k$. In fact, this is obvious if we note that for $0<h\leqq 1$

$$\max_{B_h}\{a(x, \xi)-h^{2\mu_0}W(x)\}\geqq h^{2\mu_0}\{\max_{B_1} a(x, \xi)-\max_{\pi(B_1)} W(x)\}.$$

Next we consider the case that the potential $V(x)$ depends on a large parameter $M>0$. Assume that $V(x)=V(x\,;M)\in C^\infty$ satisfies the following: There exists a $\kappa>0$ such that for any multi-index α the estimate

(9) $$\sup_{R^n}|\partial_x^\alpha V(x)|\leqq C_\alpha M^\kappa$$

holds with some constant C_α. Then we have the intermediate between Theorem 1 and 2 as follows:

Theorem 3. *Let* $a(x, \xi)$ *be the same as in Theorem* 1. *Assume that* $V(x)=V(x; M) \geqq 0$ *satisfies* (9). *Set*

$$w(x_0)=\inf_{\delta} \max_{B_\delta(x_0)} a(x, \xi). \tag{10}$$

If $w(x)$ *satisfies with* $\sigma>0$ *and* $c_0>0$

$$\inf_{R^n} w(x) \geqq c_0 M^\sigma \tag{11}$$

then for any compact set K *of* $\boldsymbol{R}_x^n$ *there exists a constant* $c_K>0$ *such that*

$$(a(x, D)u, u) \geqq c_K(w(x)u, u) \qquad \text{for any } u \in C_0^\infty(K),$$

provided that M *is sufficiently large.*

We remark that the function $w(x)$ defined by (10) is upper semi-continuous. If all coefficients $a_k(x)$ of $a(x, \xi)$ are constants, by analyzing the right hand side of (10) Theorem 3 becomes more clear as follows:

Theorem 4. *Assume that* $a(x, \xi)=\sum_{k=1}^{n} |\xi_k|^{2\mu_k}+V(x)$ *and that* $V(x)=V(x; M) \geqq 0$ satisfies (9). Let $r_0, r_1, \cdots, r_n$ be positive integers satisfying $r_j \mu_j = r_0$ for any $j=1, \cdots, n$ and set for an integer $d>0$

$$m_d(x)=\sum_{|\alpha: r|<d} |\partial_x^\alpha V(x)|^{2r_0/(|\alpha: r|+2r_0)}. \tag{12}$$

where $|\alpha: r|=\sum_{j=1}^{n} \alpha_j r_j$. *If there exist a* $d>0$, *a* $c_0>0$ *and a* $\sigma>0$ *such that*

$$\inf_{R^n} m_d(x) \geqq c_0 M^{2r_0\sigma} \tag{13}$$

then with a $c>0$ *we obtain*

$$(a(x, D)u, u) \geqq c(m_d(x)u, u) \qquad \text{for } u \in \mathcal{S}, \tag{14}$$

provided that M *is sufficiently large.*

We remark that if $V(x) \geqq 0$ is polynomial then Theorem 4 still holds without assumptions (9) and (13). Theorem 4 in the case of all $\mu_k=1$ is due to Mohamed-Nourrigat [6] (see Proposition 3 of [6]). In [6], they also studied the lower bound for the Schrödinger operator with magnetic vector fields of the form

$$a(x, D)=\sum_{j=1}^{n} (D_j - A_j(x))^2 + V(x) \tag{15}$$

with $V(x) \geqq 0$ and $A_j(x)$ real-valued. Their interesting result (Proposition 4 and Theorem 7 of [6]) leads us to the following extention of Theorem 1:

Theorem 5. *Let $a(x, D)$ be the operator of the form* (15) *and let $V(x)\in L_{\mathrm{loc}}^{1}(\boldsymbol{R}^{n})$ and $A_{j}(x)\in C^{1}(\boldsymbol{R}^{n})$. If $B_{jk}(x)=(\partial A_{j}/\partial x_{k}-\partial A_{k}/\partial x_{j})$ we assume that*

$$B_{jk}(x)\geqq 0 \quad or \quad \leqq 0 \qquad in\ \boldsymbol{R}^{n}. \tag{16}$$

Let $W(x)$ be a real-valued continuous function in $\boldsymbol{R}^{n}$ and let

$$\tilde{a}(x, \xi)=\sum_{j=1}^{n}\xi_{j}^{2}+V(x)+\sum_{j,k=1}^{n}|B_{jk}(x)|. \tag{17}$$

If there exists a $0<c\leqq 1$ satisfying (4) *with $a(x, \xi)$ replaced by $\tilde{a}(x, \xi)$ then we have with a coustant $c'>0$*

$$(a(x, D)u, u)\geqq c'(W(x)u, u) \qquad for\ any\ u\in\mathcal{S}. \tag{18}$$

As an application of Theorem 1 we consider a second order elliptic operator with infinite degeneracy as follows:

$$L=D_{1}^{2}+x_{1}^{2l}D_{2}^{2}+x_{1}^{2k}x_{2}^{2m}D_{3}^{2}+f(x)D_{4}^{2} \qquad \text{in } \boldsymbol{R}^{4}, \tag{19}$$

where l, k and m are positive integers and

$$f(x)=\exp(-1/x_{1}|^{\tau}-1/|x_{2}|^{\kappa})+\exp(-1/|x_{1}|^{\lambda}-1/|x_{2}|^{\sigma}).$$

Here we assume that

$$\tau>0, \qquad 0<\kappa<1, \qquad 0<\lambda<\min(l+1, k+1). \tag{20}$$

Theorem 6. *Let L be the operator of the form* (19) *with* (20).

(i) *Suppose that $l\geqq k$. If $0<\sigma<m+(k+1)/(l+1)$ then L is hypoelliptic in $\boldsymbol{R}^{4}$ and moreover for any $u\in\mathcal{D}'(\boldsymbol{R}^{4})$ we have*

$$\mathrm{WF}\, Lu=\mathrm{WF}\, u. \tag{21}$$

(ii) *Suppose that $k>l$. If $0<\sigma<m+1$ then we have* (21).

The proof of Theorem 6 will be carried out by using the L^{2}-apriori estimate method as in [8-10]. Theorem 1 will be used in order to derive some fundamental estimates (see Lemma 5.1 and 5.2). In the case of (i) of Theorem 6, the assumption of σ is optimal under the additional condition on τ as follows:

Theorem 7. *Let L be the operator of the form* (19). *Assume that*

$$0<\kappa<1, \qquad 0<\lambda<k+1, \tag{22}$$

$$\tau\geqq k+1+m(l+1). \tag{23}$$

If $\sigma\geqq m+(k+1)/(l+1)$ then L is not hypoelliptic in any neighborhood of the origin in $\boldsymbol{R}^{4}$.

This non-hypoellipticity result follows from the analogous method as in Theorem 1 of Hoshiro [4] (see Lemma 6.1 in Section 6, where we also use Theorem 1 requiring the condition (22)). The assumption (20) of Theorem 6 on λ (resp. κ) seems to be necessary because it is necessary for the operator frozen with respect to the variable $x_2 \neq 0$ (resp. $x_1 \neq 0$) (see the remark in the end of Section 6).

The operator discussed in Theorem 6 and 7 is fairly complicated. As a simple example of the operator of the form $-(X_1^2+X_2^2)$ with X_j real vector fields, we consider the following:

Theorem 8. *Let* $\alpha(t) \in C^\infty(\boldsymbol{R}^1)$ *satisfy* $\alpha(t) \neq 0$ *for* $t \neq 0$ *and be monotone in* $(-\infty, 0]$ *and* $[0, \infty)$, *respectively. Let* L *be a differential operator of the form*

$$L = D_1^2 + \alpha(x_1)^2 (D_2 + f(x_1) g(x_2) D_3)^2 \quad \text{in } \boldsymbol{R}^3, \tag{24}$$

where $f, g \in C^\infty$ *satisfy* $f'(t) \neq 0$, $g(t) > 0$ *for* $t \neq 0$. *Assume that*

$$\lim_{t \to 0} t \log g(t) = 0, \tag{25}$$

$$\lim_{t \to 0} t\alpha(t) \log |f'(t)| = 0 \qquad \text{(cf., (1.7) of [4])}. \tag{26}$$

Then L *is hypoelliptic in* $\boldsymbol{R}^3$ *and moreover for any* $u \in \mathcal{D}'(\boldsymbol{R}^3)$ *we have* (21).

The plan of this paper is as follows: In Section 1 we state an inequality of Poincaré type (see Lemma 1.1). Though it seems to be known but we prove it because the suitable reference can not be found. In Section 2, using the inequality given in Section 1 we prove Theorem 1 following the spirit of Fefferman-Phong [1-3]. The proof of Theorem 2 in Section 3 is almost parallel to the one of Theorem 4 of [1] except that we repeatedly use the polynomial property (b) in Lemma 3.1. In Section 4 we first prove Theorem 3 by reducing to the polynomial potential case, and prove Theorem 4 by using an elementary lemma (see Lemma 4.1) together with Lemma 3.1. In Section 4 we also prove Theorem 5. Section 5 is devoted to the proof of Theorem 6, where Theorem 1 really works. In Section 6, we prove Theorem 7 by solving an eigenvale problem (see Lemma 6.1) similarly as in [7] and [4]. In Section 7 we prove Therem 8 using the idea of proof of Theorem 1 and Theorem 5.

Though Theorem 1 is stated for operators of higher order, its application in the present paper is restricted to the second order hypoelliptic operators because the hypoellipticity in higher order case seems to be more delicate (cf., [8]). The author wishes to treat higher order case in the future.

1. Inequality of Poincaré type

In this section we give a simple but important estimate to the proof of Theorem 1, which is an extension of the one given in [1, p. 148]. Let k be a positive integer and write $k-1 = \sum_{j \geq 0} b_j 2^j$ for $b_j = 0$ or 1. We denote $(-1)^{\Sigma b_j}$ by sgn k and for a non-negative integer m and $v \in C_0^\infty(\boldsymbol{R}^n)$ we set

$$G_v^m(x, y) = \sum_{k=1}^{N} (\operatorname{sgn} k)\{v(x + k(y-x)/N) - v(x + (k-1)(y-x)/N)\},$$

where $N = 2^m$.

Lemma 1.1. *Let Q be a convex set in $\boldsymbol{R}^n$. If m is a non-negative integer, there exists a constant $c_{m,n}>0$ depending only on m and n such that*

$$\sum_{|\alpha|=m+1}\int_Q |D^\alpha v(x)|^2 dx \leqq c_{m,n}\frac{(\mathrm{diam}\, Q)^{-2(m+1)}}{|Q|}\int_{Q\times Q}|G_v^m(x, y)|^2 dx dy\,. \tag{1.1}$$

Furthermore, if $0<\lambda<1$ there exists a constant $c'_{m,n}>0$ depending only on m and n such that

$$\sum_{|\beta|=m}\int_{Q\times Q}\frac{|D^\beta v(x)-D^\beta v(y)|^2}{|x-y|^{n+2\lambda}}dx dy \tag{1.2}$$

$$\geqq c'_{m,n}\frac{(\mathrm{diam}\, Q)^{-2(m+\lambda)}}{|Q|}\int_{Q\times Q}|G_v^m(x, y)|^2 dx dy.$$

The estimates (1.1) and (1.2) seem to be known but we shall give the proof because the suitable reference can not be found. We prepare the following:

Lemma 1.2. *Let m be a non-negative integer and let $F\in C^\infty(\boldsymbol{R}^1)$. Then we have*

$$\sum_{k=1}^{2m}(\mathrm{sgn}\, k)\{F(k2^{-m})-F((k-1)2^{-m})\} \tag{1.3}$$

$$=(-1)^{m-1}2^{-m(m+3)}\int_0^1\cdots\int_0^1 F^{(m+1)}(\varphi(\theta))d\theta\,,$$

where $\theta=(\theta_0, \theta_1, \cdots, \theta_m)\in\boldsymbol{R}^{m+1}$ and

$$\varphi(\theta)=2^{-m}\theta_0+2^{-m}\theta_1+2^{-m+1}\theta_2+\cdots+2^{-1}\theta_m\,.$$

Proof. If I denotes the left hand side of the above formula we have

$$I=2^{-m}\int_0^1\left\{\sum_{k=1}^{2m}(\mathrm{sgn}\, k)F'(\theta_0 2^{-m}+(k-1)2^{-m})\right\}d\theta_0 \tag{1.4}$$

$$=2^{-m}\int_0^1\left\{\sum_{k=1}^{2m-1}(\mathrm{sgn}\, 2k)(F'(\theta_0 2^{-m}+(2k-1)2^{-m})\right.$$

$$\left.-F'(\theta_0 2^{-m}+(2k-2)2^{-m}))\right\}d\theta_0$$

because $\mathrm{sgn}\,(2k-1)=-\mathrm{sgn}\, 2k$. Since $\mathrm{sgn}\, 2k=-\mathrm{sgn}\, k$ we have

$$I=2^{-2m}\int_0^1\int_0^1\left\{\sum_{k=1}^{2m-1}(\mathrm{sgn}\, k)F''((\theta_0+\theta_1)2^{-m}+(k-1)2^{-m+1})\right\}d\theta_0 d\theta_1\,. \tag{1.5}$$

Note that the integrand in the right hand side of (1.5) is similar as the one in the middle member of (1.4). The repetition of the preceding procedure yields the desired formula (1.3). Q.E.D.

Proof of Lemma 1.1. For $v\in C_0^\infty$ set $F(t)=v(x+t(y-x))$. Then it follows from Lemma 1.2 that

$$G_v^m(x, y)=(-1)^{m-1}2^{-m(m+3)}\int_0^1\cdots\int_0^1\sum_{|\alpha|=m+1}D^\alpha v(x+\varphi(\theta)(y-x))\cdot(y-x)^\alpha d\theta \tag{1.6}$$

In view of the Schwartz inequality, we have

$$|G_v^m(x, y)|^2 \leqq C_m \int_0^1 \cdots \int_0^1 \sum_{|\alpha|=m+1} |D^\alpha v(x+\varphi(\theta)(y-x))|^2 |y-x|^{2(m+1)} d\theta$$

Integrating with respect to x and y over $Q\times Q$ and noting that $|y-x| \leqq \mathrm{diam}\, Q$ and Q is convex we see that

$$(\mathrm{diam}\, Q)^{-2(m+1)} \int_{Q\times Q} |G_v^m(x, y)|^2 dxdy$$

$$\leqq C_m \sum_{|\alpha|=m+1} \int_J \int_{Q\times Q} |D^\alpha v(x+\varphi(\theta)(y-x))|^2 dxdyd\theta ,$$

where $J=[0, 1]^{m+1}$. If $J_1=\{\theta \in J ; \varphi(\theta) \geqq 1/2\}$ and $J_2=J\setminus J_1$, by means of change of variables we have

$$\int_J \int_{Q\times Q} |D^\alpha v(x+\varphi(\theta)(y-x))|^2 dxdyd\theta$$

$$\leqq \int_Q dx \int_{J_1} \left\{ \int_Q |D^\alpha v(z)|^2 dz/\varphi(\theta)^n \right\} d\theta$$

$$+\int_Q dy \int_{J_2} \left\{ \int_Q |D^\alpha v(w)|^2 dw/(1-\varphi(\theta))^n \right\} d\theta .$$

Since $\varphi(\theta) \geqq 1/2$ on J_1 and $1-\varphi(\theta) \geqq 1/2$ on J_2 we obtain (1.1). We shall prove the estimate (1.2). It follows from the formula in Lemma 1.2 with the right hand side integrated by θ_m that

$$G_v^m(x, y)=C_m \int_0^1 \cdots \int_0^1 \sum_{|\beta|=m} (D^\beta v(w)-D^\beta v(z))(y-x)^\beta d\theta_0 \cdots d\theta_{m-1} ,$$

where $w=x+\varphi(\theta', 1)(y-x)$ and $z=x+\varphi(\theta', 0)(y-x)$. Here $(\theta', \theta_m)=\theta$. When (x, y) varies on $Q\times Q$, (w, z) belongs to $Q\times Q$ because Q is convex. Since $w-z=(y-x)/2$ and $|\partial(w, z)/\partial(x, y)|=(1/2)^n$ we have

$$(\mathrm{diam}\, Q)^{-2(m+\lambda)-n} \int_{Q\times Q} |G_v^m(x, y)|^2 dxdy$$

$$\leqq C_{m,n} \sum_{|\beta|=m} \int_{Q\times Q} \frac{|D^\beta v(w)-D^\beta v(z)|^2}{|w-z|^{n+2\lambda}} dwdz ,$$

which is our desired estimate (1.2). Q. E. D.

2 Proof of Theorem 1

Write $\mu_k=q_k/p_k$ and $\kappa(j, k-j)=q(j, k-j)/p(j, k-j)$, respectively, by using relatively prime integers p_k, $q_k>0$ and $p(j, k-j)>0$, $q(j, k-j) \geqq 0$, respectively. We take a convention with $p(j, k-j)=1$ if $q(j, k-j)=0$. Set $r_0=\prod_{k=1}^{n} (q_k \prod_{j=1}^{k-1} p(j, k-j))$. If K is a compact set of $\boldsymbol{R}^n$, we take a sufficiently large integer l_0 such that

(2.1) $$K \subset \{x ; 2|x_j| \leqq 2^{l_0 r_0/\mu_j}, j=1, \cdots, n\}.$$

If $I_0=\prod_{j=1}^{n} Q_j^0$ denotes the rectangle in the right hand side and if Q_j^0 are intervals in $\boldsymbol{R}_{x_j}$, we cut I_0 into $\prod_{j=1}^{n} 2^{r_0/\mu_j}$ congruent smaller rectangles by cutting Q_j^0 into $2^{r_0/\mu_j}$ subintervals with equal length. It should be noted that all r_0/μ_j are integers. Furthermore we cut each small rectangle by the same way. Starting with I_0, we repeat this procedure l_0 times. If $I_\nu=\prod_{j=1}^{n} Q_j^\nu$ is one of rectangles in some step of this procedure, we have

$$(\mathrm{diam}\, Q_j^\nu)^{-2\mu_j}=(\mathrm{diam}\, Q_1^\nu)^{-2\mu_1}, \qquad j=2, \cdots, n. \tag{2.2}$$

Hence, if $C_K=\max_{1\leq j\leq n} \max_{x\in I_0} a_j(x)$, we have

$$a_j(x)(\mathrm{diam}\, Q_j^\nu)^{-2\mu_j}\leqq C_K R_\nu, \qquad j=2, \cdots, n. \tag{2.3}$$

Here and in what follows R_ν denotes $(\mathrm{diam}\, Q_1^\nu)^{-2\mu_1}$. Furthermore, noting that $\mathrm{diam}\, Q_j^\nu \geqq 1$, for any $0<\varepsilon\leqq 1$ and any $j\in\{2, \cdots, n\}$ we have

$$a_j(x)(\mathrm{diam}\, Q_j^\nu)^{-2\mu_j}\geqq \varepsilon^{2\kappa_0}R_\nu \quad \text{on } \{x\in I_\nu;\ |x_l|\geqq\varepsilon\, \mathrm{diam}\, Q_l^\nu,\ l=1, \cdots, n-1\}, \tag{2.4}$$

where $\kappa_0=\max_{2\leq k\leq n} \prod_{j=1}^{k-1} \kappa(j, k-j)$.

On and after l_0+1 step, we modify the way how to cut a rectangle I_ν as follows; in order that (2.4) remains valid. For $l=1, \cdots, n-1$, let Σ_l denote a hyperplane $x_l=0$ and let $\sigma(\nu, l)=1$ if $\Sigma_l\cap\partial I_\nu\neq\emptyset$, $=0$ otherwise. We cut $I_\nu=\prod_{j=1}^{n} Q_j^\nu$ into $\prod_{j=1}^{n} 2^{r(\nu, j)/\mu_j}$ congruent smaller rectangles $I_{\nu'}$ $(I_\nu=\sum_{\nu'} I_{\nu'})$ by cutting Q_j^ν into $2^{r(\nu, j)/\mu_j}$ subintervals with equal length. Here $r(\nu, j)$ are determined succesively by $r(\nu, 1)=r_0$ and for $2\leqq j\leqq n$

$$r(\nu, j)=r_0+\sum_{l=1}^{j-1} \sigma(\nu, l)\kappa(l, j-l)r(\nu, l)\mu_l^{-1}. \tag{2.5}$$

It should be noted again that all $r(\nu, j)/\mu_j$ are integers. For $l\in\{1, \cdots, n-1\}$ let $I_{\nu''(l)}$ be the smallest rectangle in the cutting procedure such that $I_{\nu''(l)}\supset I_\nu$ and $\partial I_{\nu''(l)}\cap\Sigma_l \neq\emptyset$. Clealy, $\nu''(l)=\nu$ if $\sigma(\nu, l)=1$. It is easy to see that for a patch $I_{\nu'}=\prod_{j=1}^{n} Q_j^{\nu'}$ of I_ν and $j\in\{2, \cdots, n\}$ we have

$$a_j(x)(\mathrm{diam}\, Q_j^{\nu'})^{-2\mu_j}=R_{\nu'}. \tag{2.6}$$

if $x=(x_1, \cdots, x_n)$ satisfies

$$|x_l|=\min(d_l, 1) \qquad \text{for } l\in\{1, \cdots, n-1\},$$

where $d_l=2^{-r(\nu''(l), l)/\mu_l}\mathrm{diam}\, Q_l^{\nu''(l)}$. Here we take a convention with $r(\nu''(l), l)=r_0$ if $\mathrm{diam}\, Q_l^{\nu''(l)}>1$. Note that

$$I_{\nu'}\subset\bigcap_{l\in\mathcal{J}_\nu} \{x;\ |x_l|\geqq\min\{d_l, 1)\}, \tag{2.7}$$

where $\mathcal{J}_\nu$ is a subset of $\{1, \cdots, n-1\}$ such that $l\in\mathcal{J}_\nu$ means $\sigma(\nu, l)=0$. If follows from (2.6) and (2.7) that (2.4) still holds for $I_{\nu'}$ because $\operatorname{diam} Q_l^{\nu'}=d_l$ for $l\in\{1, \cdots, n-1\}\setminus\mathcal{J}_\nu$. Since we have

(2.8) $$I_{\nu'}\subset\bigcap_{l=1}^{n-1}\{x\,;\,|x_l|\leqq 2^{r(\nu''(l),l)/\mu_l}d_l\},$$

it follows from (2.6) that

(2.9) $$a_j(x)(\operatorname{diam} Q_j^{\nu'})^{-2\mu_j}\leqq\Big(\prod_{l=1}^{j-1}M_{j,l}\Big)\boldsymbol{R}_{\nu'}\,,$$

where $M_{j,l}=\max(2^{2\kappa(l,j-l)r(\nu''(l),l)/\mu_l}, \max_{I_0}|x_l|^{2\kappa(l,j-l)})$. If r_l for $l\in\{1, \cdots, n-1\}$ are defined succesively by (2.5) with all $\sigma(\nu, l)=1$ then we get $r_l\geqq r(\nu, l)$. So, with another constant C_K' we have (2.3) for any $I_{\nu'}$ in the cutting procedure. In the last of this paragraph we remark that if $r^*=1+\max_{1\leqq j\leqq n-1} r_j/\mu_j$ and if $(I_{\nu'})^*$ denotes the 2^{r^*} times dilation of $I_{\nu'}$ then we have

(2.10) $$I_\nu\subset(I_{\nu'})^*\,.$$

We may assume that

(2.11) $$\frac{1}{A}\max_{x\in I_0}W(x)\geqq R_0\ (\equiv(\operatorname{diam} Q_1^0)^{-2\mu_1}),$$

where A is a large number that will be chosen later on. Indeed, the theorem is trivial otherwise because of the usual Poinceré's inequality. We repeat the above cutting process and stop the cutting whenever we arrive at I_ν satisfying

(2.12) $$\frac{1}{A}\max_{x\in I_\nu}W(x)\leqq R_\nu\ (\equiv(\operatorname{diam} Q_1^\nu)^{-2\mu_1}).$$

This will eventually happen, since each time we cut I_ν the left hand side shrinks, while the right hand side grows. Consequently, the rectangle I_0 in (2.1) is patitioned into subrectangles $\{I_\nu\}$ each of which satisfies (2.12) and

(2.13) $$\frac{2^{2r_0}}{A}\max_{x\in(I_\nu)^*}W(x)\geqq R_\nu\,.$$

Indeed, in view of (2.10), the estimate (2.13) holds because I_ν in (2.12) arose by cutting a rectangle for which (2.12) fails. By means of (2.3), (2.4) and the arguments in the preceding paragraph, for each $I_\nu=\prod_{j=1}^{n}Q_j^\nu$ of the partition $I_0=\bigcup_\nu I_\nu$ we have

(2.14) $$a_j(x)(\operatorname{diam} Q_j^\nu)^{-2\mu_j}\leqq C_K' R_\nu\,,\qquad j=2, \cdots, n\,.$$

for a constant C_K' independent of ν and moreover we see that for any $0<\varepsilon\leqq 1$ there exists a subset I_ν^ε of I_ν satisfying

(2.15) $$\boldsymbol{m}_n(I_\nu^\varepsilon)\geqq(1-\varepsilon)^{n-1}|I_\nu|$$

and where

(2.16) $$a_j(x)(\operatorname{diam} Q_j^\nu)^{-2\mu_j}\geqq\varepsilon^{2\kappa_0}R_\nu\,,\qquad j=2, \cdots, n\,.$$

For $j=1, \cdots, n$ and $u\in C_0^\infty(K)$ we set

$$\Gamma_j(x_j, y_j; y_1, \cdots, y_{j-1}, x_{j+1}, \cdots, x_n)=G^{m_j}_{v_j}(x_j, y_j).$$

where $v_j(\cdot)=u(y_1, \cdots, y_{j-1}, \cdot, x_{j+1}, \cdots, x_n)$. In what follows we write $y_1^{j-1}=(y_1, \cdots, y_{j-1})$ and $x_n^{n-j}=(x_{j+1}, \cdots, x_n)$. Under this notation $a_j(y)=a_j(y_1^{j-1})$.

Lemma 2.1. *When μ_j is integer, there exists a $c_1>0$ such that*

$$\int_{I_\nu} a_j(x)|D_j^{\mu_j}u(x)|^2dx \geqq c_1\frac{(\mathrm{diam}\, Q_j^\nu)^{-2\mu_j}}{|I_\nu|} \times\int_{I_\nu\times I_\nu} a_j(y_1^{j-1})|\Gamma_j(x_j, y_j; y_1^{j-1}, x_n^{n-j})|^2dxdy \tag{2.17}$$

If μ_j is not integer, we have the same estimate as above with the left hand side replaced by

$$\int_{I_\nu\times Q_j^\nu} a_j(x)\frac{|D_j^{m_j}u(x)-D_j^{m_j}u(x_1^{j-1}, y_j, x_n^{n-j})|^2}{|x_j-y_j|^{1+2(\mu_j-m_j)}}dxdy_j. \tag{2.18}$$

Proof. Apply Lemma 1.1 in the case of the dimension $n=1$. From the estimate (1.1) of Lemma 1.1 with $v=v_j$, we obtain

$$\int_{Q_j^\nu}|D_j^{\mu_j}u(y_1^{j-1}, x_j, x_n^{n-j})|^2dx_j$$
$$\geqq c_1\frac{(\mathrm{diam}\, Q_j^\nu)^{-2\mu_j}}{|Q_j^\nu|}\int_{Q_j^\nu\times Q_j^\nu}|\Gamma_j(x_j, y_j; y_1^{j-1}, x_n^{n-j})|^2dx_jdy_j.$$

Multiply $a_j(y_1^{j-1})|Q_j^\nu|\,|I_\nu|^{-1}$ in both sides and integrate with respect to (x_1^{j-1}, y_n^{n-j}) and (y_1^{j-1}, x_n^{n-j}) over $I_\nu^j\equiv\prod_{k\neq j}Q_k^\nu$. Then we obtain (2.17) because of

$$\int_{I_\nu^j}dx_1^{j-1}dy_n^{n-j}=|I_\nu|\,|Q_j^\nu|^{-1}.$$

The rest of the lemma also follows from the estimate (1.2) of Lemma 1.1. Q.E.D.

When all μ_j are integers, we have

$$(a(x, D)u, u)=\sum_{j=1}^n\int_{I_0}a_j(x)|D_j^{\mu_j}u(x)|^2dx+\int_{I_0}V(x)|u(x)|^2dx \tag{2.19}$$
$$=\sum_\nu\left\{\sum_{j=1}^n\int_{I_\nu}a_j(x)|D_j^{\mu_j}u(x)|^2dx+\int_{I_\nu}V(x)|u(x)|^2dx\right\}.$$

Consequently it follows from (2.17) of Lemma 2.1 that

$$(a(x, D)u, u)\geqq c\sum_\nu\left\{\sum_{j=1}^n\Omega_j^\nu+\int_{I_0}V(x)|u(x)|^2dx\right\} \tag{2.20}$$
$$\equiv c\sum_\nu S_\nu,$$

where Ω_j^ν denote the right hand side of (2.17). If some μ_j is not integer, we see that for a constant C

$$C\int a_j(x)||D_j|^{\mu_j}u(x)|^2dx$$

$$\geqq\int_{I_0\times R^1}a_j(x)\frac{|D_j^{m_j}u(x)-D_j^{m_j}u(x_1^{j-1},\ y_j,\ x_n^{n-j})|^2}{|x_j-y_j|^{1+2(\mu_j-m_j)}}dx\,dy_j\,.$$

$$=\sum_\nu\int_{I_\nu\times R^1}\{\cdot\}dx\,dy_j\geqq\sum_\nu\int_{I_\nu\times Q_j^\nu}\{\cdot\}dx\,dy_j\,.$$

From this and the second part of Lemma 2.1 we also have (2.20) in the general case.

For $x,\ y\in I_\nu$ we consider $z(y\,;\,x)=(z_1(y\,;\,x),\ \cdots,\ z_n(y\,;\,x))\in I_\nu$ such that each component $z_j(y\,;\,x)$ equals one of $x_j+k(y_j-x_j)2^{-m_j}$ for $k\in\{1,\ \cdots,\ 2^{m_j}\}$. The number of considerable $z(y\,;\,x)$ is equal to $N_0\equiv 2^{m_0}$ $(m_0=\sum_{j=1}^n m_j)$ and so we denote them by $z^r(y\,;\,x)$, $r=0,\ \cdots,\ N_0-1$ with a convention $z^0(y\,;\,x)=y$.

Lemma 2.2. *For any $x\in I_\nu$ there exists a subset $I_{\nu,x}^0$ of I_ν satisfying*

$$m_n(I_{\nu,x}^0)\geqq c_0|I_\nu|\ ,\qquad c_0>0\,, \tag{2.21}$$

such that

$$V(z^r(y))\geqq c_1R_\nu\qquad \textit{for}\ \ y\in I_{\nu,x}^0\,. \tag{2.22}$$

Here c_0 and c_1 are positive constants independent of ν, r and x.

Proof. Set $J_\nu^r=\{z^r(y\,;\,x);\ y\in I_\nu\}$ for a fixed $x\in I_\nu$, which is a rectangle contained in I_ν, and so write $J_\nu^r=\prod_{j=1}^n Q_j^{\nu,r}$ for intervals $Q_j^{\nu,r}$ in $\boldsymbol{R}_{x_j}$. Setting $\delta_j=\operatorname{diam} Q_j^{\nu,r}$, we have

$$\delta_j\leqq\operatorname{diam} Q_j^\nu\leqq N_1\delta_j\,, \tag{2.23}$$

where $N_1=\max_j 2^{m_j}$. Consider a box $B\in C$ such that

$$B=J_\nu^r\times\Big(\prod_{j=1}^n\{\xi_j;\ |\xi_j|\leqq\delta_j^{-1}/2\}\Big)\,.$$

It follows from (2.14) that

$$\max_B|a-V|\leqq N_1^{2\mu_0}\{(n-1)C'_K+1\}R_\nu\,,$$

where $\mu_0=\max_j\mu_j$. If we choose A in (2.11) is large enough to satisfy $A\geqq 2^{2r_0+1}N_1^{2\mu_0}\{(n-1)C'_K+1\}$ we have

$$\max_B|a-V|\leqq 2^{-1}\max_{(I_\nu)^*}W\,. \tag{2.24}$$

If the modulus of the dilation $(\cdot)^{**}$ in the theorem is larger than $N_12^{r^*+1}$ times then we get

$$\max_{(I_\nu)^*}W\leqq\max_{\pi(B^{**})}W\,. \tag{2.25}$$

In view of (2.23) and (2.24), it follows from the hypothesis of the theorem that there exists a subset $J_{\nu,x}^{r,0}$ of J_{ν}^{r} with $m_n(J_{\nu,x}^{r,0})\geqq c|J_{\nu}^{r}|$ such that

$$V(z)\geqq 2^{-1}\max_{\pi(B^{**})} W\geqq 2^{-2r_0-1}AR_{\nu} \qquad \text{if } z\in J_{\nu,x}^{r,0}, \tag{2.26}$$

where the last estimate follows from (2.13) and (2.25). Let $I_{\nu,x}^{r,0}$ be the preimage of $J_{\nu,x}^{r,0}$ of the mapping $y\to z^r(y;x)$. Since $m_n(I_{\nu,x}^{r,0})\geqq c|I_{\nu}|$ and $c>1-N_0^{-1}$ we see that

$$m_n(I_{0,x}^{r,0}\cap I_{\nu,x}^{t,0})\geqq c'|I_{\nu}| \qquad \text{for } r, t\in\{1, \cdots, N_0\},$$

where $c'>1-2N_0^{-1}$. By induction we obtain (2.21) for $c_0>0$ if we set $I_{\nu,x}^{0}=\bigcap_{r=1}^{N_0} I_{\nu,x}^{r,0}$, for whose point we have (2.22) because of (2.26). Q.E.D.

Let $z^r(y_1^j;x_1^j)\in \boldsymbol{R}^j$ denote the first j components of $z^r(y, x)$. Set $\omega_{j,r}(x, y)=|\Gamma_j(x_j, y_j; z^r(y_1^{j-1}; x_1^{j-1}), x_n^{n-j})|^2$. Then, in view of change of variables $(x, z^r(y_1^{j-1}; x_1^{j-1}), y_n^{n-j+1})\to(x, z_1^{j-1}, y_n^{n-j+1})$, we have

$$\begin{aligned}\frac{(\operatorname{diam} Q_j^{\mu})^{-2\mu_j}}{|I_{\nu}|}&\int_{I_{\nu}\times I_{\nu}} a_j(z^r(y_1^{j-1}, x_1^{j-1}))\omega_{j,r}(x, y)dxdy\\ &\leqq N_1\Omega_j^{\nu},\end{aligned} \tag{2.27}$$

because $|\partial(x, z_1^{j-1}, y_n^{n-j+1})/\partial(x, y)|\geqq 1/N_1$. For a $z^r(y;x)$ we also have

$$\begin{aligned}&\int_{I_{\nu}\times I_{\nu}} V(z^r(y;x))|u(z^r(y, x)|^2dxdy\\ &\qquad\leqq N_1\int_{I_{\nu}\times I_{\nu}} V(z)|u(z)|^2dxdz\end{aligned} \tag{2.28}$$

because $|\partial(x, z)/\partial(x, y)|\geqq 1/N_1$.

Note that J_{ν}^{r} $(\subset I_{\nu})$ is the contraction of I_{ν} whose modulus is not smaller than $1/N_1$. In view of (1.6) and (1.16), we see that for any fixed $x\in I_{\nu}$ and any $z^r(y^{j-1}, x^{j-1})$ we get

$$\begin{aligned}&a_j(z^r(y^{j-1}, x^{j-1}))(\operatorname{diam} Q_j^{\nu})^{-2\mu_j}\\ &\qquad\geqq(\varepsilon/N_1)^{2\kappa_0}R_{\nu} \qquad \text{if } y\in I_{\nu}^{\varepsilon},\end{aligned} \tag{2.29}$$

where I_{ν}^{ε} is the same subset of I_{ν} as in (2.14) and (2.15). Indeed, this follows easily if we remined the way how to find I_{ν}^{ε}.

Choose an ε satisfying $(1-\varepsilon)^{n-1}\geqq 1-c_0/2$ and set $\tilde{I}_{\nu,x}^{0}=I_{\nu,x}^{0}\cap I_{\nu}^{\varepsilon}$. Then we have

$$m_n(\tilde{I}_{\nu,x}^{0})\geqq 2^{-1}c_0|I_{\nu}|. \tag{2.30}$$

In view of (2.29) and (2.22), it follows from (2.27) and (2.28) that for a constant C independent of ν we have

$$CS_{\nu}\geqq\frac{R_{\nu}}{|I_{\nu}|}\int_{I_{\nu}}dx\int_{\tilde{I}_{\nu,x}^{0}}\left\{\sum_{j=1}^{n}\sum_{r}\varphi_{j,r}(x, y)+A_0\right\}dy, \tag{2.31}$$

where we have set $A_0=\sum_{r}|u(z^r(y;x)|^2$.

We shall show that

$$\sum_{j=1}^{n}\sum_{r}\omega_{j,r}(x, y)+A_0\geqq 2^{-4n}|u(x)|^2 . \tag{2.32}$$

Note that the first term of Γ_n equals $u(y_1^{n-1}, x_n)$. Since $\omega_{n,0}(x, y)=|\Gamma_n|^2$, we have

$$2\omega_{n,0}(x, y)\geqq|u(y_1^{n-1}, x_n)|_2$$
$$-16\sum_{k\geqq 1}|u(y_1^{n-1}, k(y_n-x_n)2^{-m_n})|^2 .$$

Since similar estimates also hold for $\omega_{n,r}(x, y)$, we get

$$2\sum_{r}\omega_{n,r}(x, y)\geqq\sum_{r}|u(z^r(y_1^{n-1}, x_1^{n-1}), x_n)|^2-16A_0$$
$$\equiv A_1-16A_0 .$$

Here for $j\geqq 1$ we have set

$$A_j=\sum_{r}|u(z^r(y_1^{n-1}; x_1^{n-j}), x_n^{n-j+1})|^2 .$$

Noting the first term of Γ_{n-1} and so on, we get

$$2\sum_{r}\omega_{n-1,r}(x, y)\geqq A_2-16A_1 ,$$

Repeating this procedure, finally we obtain

$$2\omega_{1,0}(x, y)\geqq|u(x)|^2-16A_{n-1} .$$

Hence we obtain (2.32).

It follows from (2.31) and (2.32) that for a constant $c>0$ independent of ν we obtain

$$S_\nu\geqq cR_\nu\int_{I_\nu}|u(x)|^2dx \tag{2.33}$$

because of (2.30). Noting (2.12), from this and (2.20) we get the estimate (5) of Theorem 1.

In the rest of this section, we shall show remarks stated in Introduction. The constant c_K in (5) can be taken independent of K if all $a_k(x)\equiv 1$. Indeed, if we partition $\boldsymbol{R}^n$ into congrusnt large rectangles like I_0 and apply the above argument to each rectangle, we can easily see this fact because another dependence of K derives only from the constant C'_K in (2.14). Theorem 1 holds even if each variable x_j is replaced by the vector $\boldsymbol{x}_j=(x_1^j, \cdots, x_{l_j}^j)$. Actually, the preceding argument is still valid for a cube Q_ν^j in $\boldsymbol{R}_{x_j}^{l_j}$. The rationality assumption of μ_k and $\kappa(k, j)$ can be removed. In order to find this we consider one of the simplest case; $a(x, \xi)=\xi_1^2+\xi_2^{2\mu}+V(x)$. In the cutting procedure, instead of (2.2) it is only required that with a $C>0$

$$C^{-1}(\text{diam } Q_1^\nu)^{-2}\leqq(\text{diam } Q_2^\nu)^{-2\mu}\leqq C(\text{diam } Q_1^\nu)^{-2} . \tag{2.2$'$}$$

The modification of the ratio of cutting numbers in each step enables us to obtain (2.2)$'$. The general case can be also found by this modification of cutting intervals.

3. Proof of Theorem 2

The properties in the following lemma can be seen essentially in Chapter II of [1], but we state and prove them for the convinience of the reader.

Lemma 3.1. *Let I be a rectangle in $\boldsymbol{R}^n$ and let $P(x)$ be a polynomial in $\boldsymbol{R}^n$ of degree d.*

(a) *If $I^\dagger$ denotes the dilation of I of the modulus $N>1$ then there exists a constant $C>0$ depending only on N, d and n such that*

$$\max_{I^\dagger}|P(x)|\leqq C\max_{I}|P(x)|. \tag{3.1}$$

Furthermore, there exists a constant C' depending only on N, d and n such that

$$\max_{I^\dagger} P(x)-\min_{I^\dagger} P(x)\leqq C'\{\max_{I} P(x)-\min_{I} P(x)\}. \tag{3.2}$$

(b) *In addition, assume $P\geqq 0$ on I. Then there exists a similar rectangle $I'\subset I$ with* $\operatorname{diam} I'=c\operatorname{diam} I$ *on which we have*

$$\min_{I'} P\geqq\frac{1}{2}\max_{I} P. \tag{3.3}$$

Here c is a positive constant depending only on d and n.

(c) *If P and I are the same as in* (b) *then for any $0<\beta\leqq 1$ there exists another similar rectangle $I''\subset I$ with* $\operatorname{diam} I''=\beta\operatorname{diam} I$ *on which*

$$\max_{I''} P-\min_{I} P\leqq C''\beta\max_{I} P. \tag{3.4}$$

Here C'' is a constant depending only on n and d.

Proof. If T_0 is a unit cube in $\boldsymbol{R}^n$ and if $F(y)$ is a polynomial $\sum_{|\alpha|\leqq d} a_\alpha y^\alpha$ in $\boldsymbol{R}^n$ then for a $C_{d,n}>0$ we have

$$C_{d,n}^{-1}\max_{\alpha}|a_\alpha|\leqq\max_{T_0}|F(y)|\leqq C_{d,n}\max_{\alpha}|a_\alpha|. \tag{3.5}$$

Indeed, this follows from the equivalence of two norms of a finite dimensional vector space. Let I be centered at x_0 and let $I=\{x=x_0+ty\,;\,y\in T_0\}$ for a $t=(t_1,\cdots,t_n)\in R^n$, where $ty=(t_1y_1,\cdots,t_ny_n)$. If $P(x)=\sum b_\alpha(x-x_0)^\alpha$ then it follows from (3.5) that

$$C_{d,n}^{-1}\max_{\alpha}|b_\alpha t^\alpha|\leqq\max_{I}|P(x)|\leqq C_{d,n}\max_{\alpha}|b_\alpha t^\alpha|.$$

Since we have the analogous estimate for $\max_{I^\dagger}|P(x)|$ we obtain (3.1) because $\max_{\alpha}|b_\alpha t^\alpha|N^{|\alpha|}\leqq N^d\max_{\alpha}|b_\alpha t^\alpha|$. Set

$$Q(x)=P(x)-\min_{I} P(x).$$

Then by (3.1) we see that

$$\max_{I\dagger} P(x)-\min_{I} P(x) \leqq \max_{I\dagger}|Q(x)|$$
$$\leqq C \max_{I}|Q(x)|$$
$$=C\{\max_{I} P(x)-\min_{I} P(x)\}.$$

Note that we have the same estimate for $-P(x)$. Then we get (3.2) with $C'=2C-1$ because $\max -P=-\min P$ and so on. We shall prove (b) and (c). Since similar estimates as (3.5) hold for $\partial_j F(y)$, we see that for a $C''=C''_{d,n}$

$$\max_{T_0} |\nabla F| \leqq C'' \max_{T_0} |F|$$

So, if $F \geqq 0$ we have

$$|F(y)-F(y_0)| \leqq C''(\max_{T_0} F) \times |y-y_0| \tag{3.6}$$

If $F(y_0)=\max_{T_0} F$ then we get

$$F(y) \geqq \max_{T_0} F/2 \qquad \text{for } |y-y_0| \leqq C''^{-1}/2 .$$

Applying this to $F(y)=P(x_0+ty)$ we have (3.3). The estimate (3.4) also follows from (3.6) if $F(y_0)=\min_{T_0} F(y)$. Q.E.D.

We shall prove Theorem 2 by the almost similar way as in the proof of Theorem 4 in [1]. For a compact set K we take the same rectangle I_0 as in the section 2. If $U(x)$ is a constant the theorem is trivial because we see $U \geqq 0$ by means of the hypothesis (7). So we may assume that U is not constant. Taking a sufficiently large I_0 we may assume that

$$\max_{I_0} U(x)-\min_{I_0} U(x) \geqq R_0 .$$

We cut I_0 by the same way as in the section 2 and repeat the cutting procedure. However, we stop the cutting whenever we arrive at I_ν satisfying

$$\max_{I_\nu} U(x)-\min_{I_\nu} U(x) \leqq R_\nu , \tag{3.7}$$

instead of (2.12). In view of (2.10) and (3.2), it follows from the same reason as in the section 2 that for a constant $C_1>0$ depending only on d and n we have

$$C_1\{\max_{I_\nu} U(x)-\min_{I_\nu} U(x)\} \geqq R_\nu . \tag{3.8}$$

In place of Lemma 2.2 we prepare the following:

Lemma 3.2. *Set* $P(x)=U(x)-\min_{I_\nu} U(x)$. *Then for any* $x \in I_\nu$ *there exist a subset* $I^0_{\nu,x}$ *of* I_ν *satisfying*

$$m_n(I^0_{\nu,x}) \geqq c_0 |I_\nu| . \qquad c_0>0 , \tag{3.9}$$

such that for any $r=0, 1, \cdots, N_0-1$

(3.10) $$P(z^r(y\,;\,x))\geqq c_1 R_\nu \qquad \textit{if}\ \ y\in I^0_{\nu,x}\,.$$

Here c_0 *and* c_1 *are positive constants independent of* ν, r *and* x.

Proof. Note that $P\geqq 0$. If follows from (3.8) that

(3.11) $$\max_{I_\nu} P\geqq C_1^{-1}R_\nu\,.$$

Apply the part (b) of Lemma 3.1. Then there exists a subrectangle $I_\nu(0)$ similar to I_ν with diam $I_\nu(0)=c$ diam I_ν such that

(3.12) $$\min_{I_\nu(0)} P\geqq (2C_1)^{-1}R_\nu\,.$$

Set $J_\nu(0, r)$ be the image of $I_\nu(0)$ by the mapping $y\to z^r(y\,;\,x)$, where $r\neq 0$. If $(\cdot)^\dagger$ denotes the $4N_1/c$ times dilation, we see that $(J_\nu(0, r))^\dagger\supset I_\nu$. By means of (3.1) we have

(3.13) $$\max_{J_\nu(0,r)} P\geqq (CC_1)^{-1}R_\nu$$

for an absolute constant C. Apply the part (b) of Lemma 3.1 again. Then there exists a subrectangle $J'_\nu(0, r)$ similar to $J_\nu(0, r)$ with diam $J'_\nu(0, r)=c$ diam $J_\nu(0, r)$ such that

(3.14) $$\min_{J'_\nu(0,r)} P\geqq (2CC_1)^{-1}R_\nu\,.$$

Let $I_\nu(0, r)$ be the preimage of $J'_\nu(0, r)$ by the mapping $y\to z^r(y\,;\,x)$ and let $J_\nu(0, r, r')$ be the image of $I_\nu(0, r)$ by the mapping $y\to z^{r'}(y\,;\,x)$, where $r'\neq 0, r$. Note that $(J_\nu(0, r, r'))^{\dagger\dagger}\supset I_\nu$ if $(\cdot)^{\dagger\dagger}=((\cdot)^\dagger)^\dagger$. The repetition of (3.1) yields

(3.15) $$\max_{J_\nu(0,r,r')} P\geqq (C^2C_1)^{-1}R_\nu\,.$$

By the part (b) of Lemma 3.1, for $J_\nu(0, r, r')$ we have the similar estimate as (3.14) with C in the right hand side replaced by C^2. Recall $z^0(y\,;\,x)=y$ and repeat the above procedure for $r\in\{1, 2, \cdots, N_0-1\}$. If we set $I^0_{\nu,x}=I_\nu(0, 1, \cdots, N_0-1)$, we obtain (3.9) and (3.10) with $c_0=(c/4N_1)^{1-N_0}$ and $c_1=C^{1-N_0}/2C_1$, respectively, because

$$\begin{aligned}(I_\nu(0, 1, \cdots, k))^\dagger&\supset I_\nu(0, 1, \cdots, k-1)\\ &\supset I_\nu(0, 1, \cdots, k)\,.\end{aligned}$$

Q.E.D.

Using Lemma 3.2 instead of Lemma 2.2, by means of the arguments in Section 2 after Lemma 2.2 we have (2.33) with $V(x)$ replaced by $P(x)$. That is, we obtain

$$\begin{aligned}&\sum_{j=1}^n \Omega_j^\nu+\int_{I_\nu} U(x)|u(x)|^2dx\\ &\qquad\geqq\{c'R_\nu+\min_{I_\nu} U(x)\}\int_{I_\nu}|u(x)|^2dx\,,\end{aligned}$$

where the constant c' is independent of K because the constant c_0 and c_1 in Lemma 3.2 are independent of K.

Hence, for the proof of the theorem it only remains to prove

$$c'R_\nu+\min_{I_\nu} U(x)\geqq 0 \tag{3.16}$$

provided that h is sufficiently small. To prove (3.16) we use the part (c) of Lemma 3.1. Note that $\min_{I_\nu} P=0$. Then in view of (3.7) we see that for any $0<\beta\leqq 1$ there exists a rectangle I_ν'' similar to I_ν with $\operatorname{diam} I_\nu''=\beta \operatorname{diam} I_\nu$ on which

$$\max_{I_\nu''} U-\min_{I_\nu} U\leqq C''\beta R_\nu. \tag{3.17}$$

Set $\delta_j=\beta(\operatorname{diam} Q_j^\nu)$ for $I_\nu=\prod_{j=1}^n Q_j^\nu$ and consider a box $B_h\in\mathcal{C}_h$ such that

$$B_h=I_\nu''\times\Big(\prod_{j=1}^n \{\xi_j\,;\,|\xi_j|\leqq h\delta_j^{-1}/2\}\Big)\,.$$

It follows from (2.14) that

$$a(x,\xi)-U(x)\leqq h^{2\mu'}\beta^{-2\mu_0}C_K''R_\nu \qquad \text{on } B_h\,,$$

where $C_K''=(n-1)C_K'+1$, $\mu'=\min_{1\leqq j\leqq n}\mu_j$ and $\mu_0=\max_{1\leqq j\leqq n}\mu_j$. Using the assumption (7) for the above B_h we have

$$\max_{I_\nu''} U(x)\geqq -h^{2\mu'}\beta^{-2\mu_0}C_K''R_\nu\,.$$

From this and (3.17) we get

$$\min_{I_\nu} U\geqq -(C''\beta+h^{2\mu'}\beta^{-2\mu_0}C_K'')R_\nu\,. \tag{3.18}$$

Fix a small β satisfying $C''\beta\leqq c'/2$. Then, if h is sufficiently small such that $h^{2\mu'}\beta^{-2\mu_0}C_K''\leqq c'/2$ we have (3.16). The proof of Theorem 2 is completed.

To end this section we remark that the upper bound of h is independent of K if all $a_j(x)=1$. Indeed, the constant in (3.18) that depends on K is only C_K'', which derives from (2.14).

4. Proofs of Theorem 3-5

The proof of Theorem 3 is carried out in the almost same way as in Section 2. For a compact set K take a rectangle I_0 and divide I_0 into $\bigcup_\nu I_\nu$ by the same way as in Section 2 such that (2.14)-(2.16) hold. The proof will be completed if we show that Lemma 2.2 still holds under the assumption of Theorem 3.

It follows from (10) that with $\delta_j=\operatorname{diam} Q_\nu^j$ we have

$$\begin{aligned}\max_{x_0\in I_\nu^*} W(x_0) &\leqq \max_{x_0\in I_\nu^*}\max_{B_\delta(x_0)} a(x,\xi)\\ &\leqq \max_{B^\dagger} a(x,\xi)\,,\end{aligned} \tag{4.1}$$

where $B^\dagger=I_\nu^{*\dagger}\times\{\xi\,;\,|\xi_j|\leqq\delta_j^{-1}/2\}$ and $I_\nu^{*\dagger}$ denotes four times dilatoion of I_ν^*. By means of (2.14) we have $|a(x,\xi)-V(x)|\leqq C_K R_\nu$ on $B^\dagger$. If A is chosen sufficiently large then if follows from (2.13) and (4.1) that

(4.2) $$\max_{I_\nu^{*\dagger}} V \geqq cR_\nu .$$

In view of (11) and (2.12), we get diam $Q_1^\nu \leqq CM^{-\sigma'}$ with $\sigma'=\sigma/2\mu_1$. Using (2.16) we have for some $\tilde{\sigma}>0$

$$\text{diam } Q_j^\nu \leqq CM^{-\tilde{\sigma}}, \qquad j=1, \cdots, n .$$

Taking $P(x)$ to be the part of the Taylor expansion of $V(x)$ about the center of I_ν up to $[\kappa/\tilde{\sigma}]+2$, by (9) we have

(4.3) $$|V(x)-P(x)| \leqq CM^{-\tilde{\sigma}} \qquad \text{on } I_\nu^{*\dagger} .$$

If M is large enough, we see $P+1\geqq 0$ on $I_\nu^{*\dagger}$ because $V\geqq 0$. Applying Lemma 3.1-(a) to $P+1$ and noting that $R_\nu \geqq cM^\sigma$, from (4.2) and (4.3) we get $\max_{I_\nu} V \geqq c'R_\nu$. Since we can utilize Lemma 3.1 for $V(x)$ in the help of (4f3), by the same way as in the proof of Lemma 3.2 we see that Lemma 2.2 still holds. Now the proof of Theorem 3 is accomplished.

The proof of Theorem 4 requires an elementary lemma.

Lemma 4.1. *Let* $f(t)=t^{-r}+\sum_{j=1}^d a_j t^j$ *with* $r>0$ *and* $a_j\geqq 0$. *Then there exists a constant* $C=C(d, r)>0$ *such that*

(4.4) $$C^{-1}\sum_{j=0}^d a_j^{r/(j+r)} \leqq \inf_{t>0} f(t) \leqq C \sum_{j=0}^d a_j^{r/(j+r)} .$$

Proof. We may assume that $a_0=0$ and $a_d\neq 0$. Note that $f'(t)=t^{-r-1}(\sum_{j=1}^d j a_j t^{j+r}-r)$. Let τ be a simple positive root of $f'(t)=0$. Let τ_j be a positive root of $j a_j t^{j+r}-r=0$ if $a_j\neq 0$ and $=\infty$ otherwise. Set $\tau_* = \min_{0\leqq j\leqq n} \tau_j$. Since $\tau_*\geqq\tau$, we have

$$\inf_{t>0} f(t) \leqq f(\tau_*)=\tau_*^{-r}+\sum_{j=1}^d a_j \tau_*^j ,$$

thus we get the second estimate of (4.4) becausə $a_j=r\tau_j^{-(j+r)}/j$. The first estimate is also obvious in view of $f(t)\geqq t^{-r}+a_j t^j$ and the Hölder inequality. Q.E.D.

Remark. Set $g(t)=\sum_{j=1}^d a_j t^j$ and set $g_*(t)=g(t)$ for $t\in(0, \tau_*]$ and $g_*(t)=g(\tau_*)$ for $t>\tau_*$. Since $f(\tau)\leqq f(\tau_*)\leqq(d/r+1)g(\tau_*)$ we see that

(4.5) $$\inf_{t>0} f(t) \leqq (d/r+1) \inf_{t>0}(t^{-r}+g_*(t)) .$$

We shall prove Theorem 4. We may assume that $d\gg\kappa/\sigma$. Since $a(x, \xi)$ is non-degenerate, each rectangle I_ν of the partition $I_0=\bigcup_\nu I_\nu$ satisfies (2.2). Hence, in order to derive (4.1) we need only (10) with $\delta=(\delta_1, \cdots, \delta_n)$ satisfying $\delta_j=\delta_1^{r_j/r_1}$ if r_j are integers given in Theorem 4. For the proof of Theorem 4 it suffices to show that $m_d(x_0)$ is equivalent to $w(x_0)$ defined by (10) with the above restriction. That is, if we set $\delta_1=t^{r_1}$, the proof of Theorem 4 is reduced to the following:

Lemma 4.2. *Let $V\geqq 0$ satisfy (9) for $\kappa>0$ and let $m_d(x_0)$ be defined by (12) for a large integer $d\gg\kappa/\sigma$. If $m_d(x)$ satisfies (13) then there exists a constant $C=C(d, n)$ independent of M such that for any $x_0\in \boldsymbol{R}^n$*

$$C^{-1}m_d(x_0)\leqq \inf_{t>0}\{t^{-2r_0}+\max_{y\in T_0} V(x_0+\delta y)\}\leqq Cm_d(x_0), \tag{4.6}$$

provided that M is large enough. Here $\delta y=(t^{r_1}y_1, \cdots, t^{r_n}y_n)$ and T_0 is unit cube.

Proof. Set $t_*=\min(t, \rho M^{-\sigma})$ for a large constant ρ which will be fixed later on. If $\tilde{w}(x_0)$ denotes the middle member of (4.6) then we have

$$\tilde{w}(x_0)\geqq \inf_{t>0}\{t^{-2r_0}+\max_{y\in T_0} V(x_0+\delta_* y)\}, \tag{4.7}$$

where $\delta_* y$ is defined by the formula for δy with t replaced by t_*. Setting $P(y)=\sum_{|\alpha:r|<d}\partial_x^\alpha V(x_0)t_*^{|\alpha:r|}y^\alpha$ we have

$$\max_{y\in T_0}|V(x_0+\delta_* y)-P(y)|\leqq 1 \tag{4.8}$$

if $M\geqq M_\rho$ for a sufficiently large $M_\rho>0$ because of (9) and $d\gg\kappa/\sigma$. Since $P(y)+1\geqq 0$ on T_0 it follows from the equivalence of the norm that for a $c=c(d, n)>0$

$$\max_{y\in T_0}(P(y)+1)\geqq c\{V(x_0)+1+\sum_{j=1}^{d-1}(\sum_{|\alpha:r|=j}|\partial_x^\alpha V(x_0)|)t_*^j\}\equiv g(t_*).$$

In view of (4.7) and (4.8) we have

$$\tilde{w}(x_0)+2\geqq \inf_{t>0}\{t^{-2r_0}+g(t_*)\}.$$

We shall prove the first mequality of (4.6). We may assume $|D_x^{\alpha_0}V(x_0(|)\neq 0$ for some α_0 with $0<|\alpha_0: r|<d$. In fact, otherwise, the first inequality of (4.6) is obvious. Apply Lemma 4.1 to $f(t)=t^{-2r_0}+g(t)$ and note that the infinimum is attained at τ with

$$0<\tau\leqq\tau_*\equiv \min_{0<j<d}(ja_j/2r_0)^{-1/(j+2r_0)},$$

where $a_j=\sum_{|\alpha:r|=j}|D_x^\alpha V(x_0)|$. It follows from (13) that $\tau\leqq\tau_*\leqq\rho M^{-\sigma}$ if we choose a sufficiently large ρ. In view of Remark of Lemma 4.1 we see with a constant $c>0$

$$\inf_{t>0}\{t^{-2r_0}+g(t_*)\}\geqq c\inf_{t>0}\{t^{-2r_0}+g(t)\}.$$

By means of (4.4) of Lemma 4.1 we get $\tilde{w}(x_0)+2\geqq cm_d(x_0)$ for a $c>0$. The first inequality of (4.6) is proved in the help of (13). We shall prove the second inequality of (4.6). Note that

$$\max_{y\in T_0}|V(x_0+\delta y)-P(y)|\leqq C_d M^\kappa t^d. \tag{4.9}$$

Since $P(y)+C_d M^\kappa t^d\geqq 0$ on T_0 it follows that

$$\max_{y\in T_0}(P(y)+C_d M^\kappa t^N)\leqq C\{V(x_0)+\sum_{j=1}^{d-1}(\sum_{|\alpha:r|=j}|\partial_x^\alpha V(x_0)|)t^j+C_d M^\kappa t^d\}$$

$$\equiv h(t).$$

By Lemma 4.1 we see that

$$\tilde{w}(x_0)\leqq \inf_{t>0}(t^{-2r_0}+h(t))$$

$$\leqq C'(m_d(x_0)+M^{2\kappa r_0/(d+2r_0)}).$$

In view of (13) we obtain the second inequality of (4.6). Q.E.D.

As stated in Introduction, if $V(x)$ is polynomial then Theorem 4 is valid without assumptions (9) and (13). In fact, those assumptions were employed only on the polynomial approximation of $V(x)$.

In the rest of this section we shall prove Theorem 5. If $Y_j=D_j-A_j(x)$ then $(Y_j-iY_k)(Y_j+iY_k)=Y_j^2+Y_k^2-B_{jk}(x)$. Hence we have $\|Y_ju\|^2+\|Y_ku\|^2\geqq(B_{jk}u, u)$. Exchanging j and k, if necessary, in view of (16) we obtain

$$n(a(x, D)u, u)\geqq\sum_{j=1}^n\|Y_ju\|^2+(\{V(x)+\sum_{j,k=1}^n|B_{jk}(x)|\}u, u), \tag{4.10}$$

where $a(x, D)$ is the operator of the form (15). Let $\tilde{A}_j(x)$ be a primitive function of $A_j(x)$ with respect to x_j, that is, $\partial_{x_j}\tilde{A}_j(x)=A_j(x)$. Substituting $v(\cdot)=u(y_1^{j-1}, \cdot, x_n^{n-j})\cdot\exp\{i\tilde{A}_j(y_1^{j-1}, \cdot, x_n^{n-j})\}$ into (1.1) with $m=0$ and $n=1$ as in the proof of Lemma 2.1, we obtain for a rectangle $I=\prod_{j=1}^n Q_j$ $(Q_j\subset \boldsymbol{R}_{x_j})$

$$\int_I|Y_ju(x)|^2dx\leqq c_0\frac{(\operatorname{diam}Q_j)^{-2}}{|I|} \tag{4.11}$$

$$\times\int_{I\times I}|\tilde{u}(y_1^{j-1}, x_n^{n+1-j})-\tilde{u}(y_1^j, x_n^{n-j})|^2dxdy,$$

where $\tilde{u}(x)=u(x)\exp\{-i\tilde{A}_j(x)\}$. Use (4.11) instead of (2.17) by regarding Y_j as D_j. Then in view of (4.10) we can proceed the proof of Theorem 5 by the same way as in Section 2 because $\int_I|\tilde{u}(x)|^2dx=\int_I|u(x)|^2dx$.

5. Proof of Theorem 6

Throught this section let L denote a differential operator defined by (19) that satisfies (20) and $0<\sigma<\min\{m+1, m+(k+1)/(l+1)\}$. It follows from the usual Poincaré inequality that for any compact $K\subset\boldsymbol{R}^4$ there exists a $C_K>0$ such that

$$\|u\|^2\leqq C_K(Lu, u)\quad \text{for } u\in C_0^\infty(K), \tag{5.1}$$

because of $(Lu, u)\geqq\|D_1u\|^2$. For a real $\eta>0$ set

$$L_\eta=D_1^2+x_1^{2l}D_2^2+x_1^{2k}x_2^{2m}D_3^2+f(x)\eta^2 \qquad \text{in } \boldsymbol{R}^3 .$$

Then we have

$$(5.2)\qquad (L_\eta v, v)=\|D_1v\|^2+\|x_1^lD_2v\|^2 +\|x_1^kx_2^mD_3v\|^2+(f(x)\eta^2v, v), \qquad \text{for } v\in C_0^\infty(\boldsymbol{R}^3).$$

Lemma 5.1. *For any $s>0$ and any compact $K\subset\boldsymbol{R}^2$ there exists a $\eta(s, K)\geqq 1$ such that*

$$(5.3)\qquad \|x_1^kx_2^m(\log\eta^s)v\|^2\leqq(L_\eta v, v) \qquad \textit{for } v\in C_0^\infty(K\times\boldsymbol{R}^1)$$

if $\eta\geqq\eta(s, K)$.

Proof. For $\eta>0$ and $s>0$ set

$$a(x, \xi)=\xi_1^2+x_1^{2l}\xi_2^2+\exp(-1/|x_1|^\lambda-1/|x_2|^\sigma)\eta^2 ,$$

$$W(x)=x_1^{2k}x_2^{2m}(\log\eta^s)^2 .$$

For the proof of (5.2) it suffices to show that the estimate (5) of Theorem 1 holds if $\eta\geqq\eta_{s,K}$ for a sufficiently large $\eta_{s,K}$. We shall check the condition of (4) in the case of $l\geqq k$. If K is a compact set of $\boldsymbol{R}^2$ and if $p=\{k+1+m(l+1)\}^{-1}$ and $q=(l+1)p$ we set $\Omega_1=\{x\in K;\ |x_1|\leqq\rho_1(\log\eta)^{-p},\ |x_2|\leqq\rho_2(\log\eta)^{-q}\}$. Here ρ_j are small positives and in what follows we require that

$$(5.4)\qquad \rho_2\ll\rho_1\ll 1/s , \qquad \rho_1\ll 1/r^* ,$$

where r^* denotes the modulus of the dilation of $(\cdot)^{**}$. Suppose that $B\in C$ satisfies $\pi(B)\subset\Omega_1$. Then it follows from (5.4) that $\max_{\pi(B^{**})}W(x)\leqq(\log\eta^s)^{2p}$. Noting that $\xi_1^2\geqq(4\rho_1)^{-2}(\log\eta)^{2p}$ on a half of B, we get (4) in view of (5.4). If $\pi(B)$ is contained in $\{|x_1|\leqq\rho_1(\log\eta)^{-1/(k+1)}\}\cap K$ then we obtain (4) because we see that $\max_{\pi(B^{**})}W(x)\leqq C_K(\log\eta^s)^{2/(k+1)}$ and $\xi_1^2\geqq(4\rho_1)^{-2}(\log\eta)^{2/(k+1)}$ on a half of B. If B satisfies

$$(5.5)\qquad \pi(B)\subset\{|x_2|\leqq\rho_2(\log\eta)^{-q}\}\cap K ,$$

$$(5.6)\qquad b\equiv\max_{\pi(B)}|x_1|\geqq\rho_1(\log\eta)^{-p} ,$$

then we see that $\max_{\pi(B^{**})}W(x)\leqq(br^*)^{2k}(\log\eta^s)^{2-2qm}$ and $x_1^{2l}\xi_2^2\geqq(b/4)^{2l}(8\rho_2)^{-2}(\log\eta)^{2q}$ on a quater of B. In view of $l\geqq k$ and (5.6), we obtain (4) for this B. The condition (4) for other $B\in C$ is also obvious because we see that $\exp(-1|x_1|^\lambda-1/x_2|^\sigma)\eta^2\geqq\eta$ on

$$(5.7)\qquad \{|x_1|\geqq(\rho_1/2)(\log\eta)^{-1/(k+1)},\ |x_2|\geqq(\rho_2/2)(\log\eta)^{-q}\}$$

if η is large enough that $(2/\rho_1)^\lambda(\log\eta)^{\lambda/(k+1)}$ and $(2/\rho_2)^\sigma(\log\eta)^{q\sigma}$ are less than $\log\eta^{1/2}$. In the case of $k>l$, the condition (4) is checked by the same way as above if we replace q only in (5.5) and (5.7) by $1/(m+1)$. Q.E.D.

Lemma 5.2. *Let $\chi(t)\in C^\infty(\boldsymbol{R}^1)$ satisfy* $\operatorname{supp}\chi\subset\{|t|\geqq 1\}$. *For any $\delta>0$, any $s>0$ and any compact $K\subset R^2$ there exists a $\eta(\delta, s, K)\geqq 1$ such that if $\eta\geqq\eta(\delta, s, K)$ then we have*

$$\|x_1^l(\log\eta^s)\chi(x_2/\delta)v\|^2\leqq(L_\eta v, v)\qquad \textit{for } v\in C_0^\infty(K\times\boldsymbol{R}^1), \tag{5.8}$$

and moreover

$$\|(\log\eta^s)\chi(x_1/\delta)v\|^2\leqq(L_\eta v, v)\qquad \textit{for } v\in C_0^\infty(K\times\boldsymbol{R}^1). \tag{5.9}$$

Proof. Setting

$$a(x_1, \xi_1)=\xi_1^2+\exp(-\delta^{-\sigma})\exp(-1/|x_1|^\lambda)\eta^2,$$

$$W(x_1)=x_1^{2l}(\log\eta^s)^2,$$

we see that the estimate (5.8) is reduced to (5) because $(a(x_1, D_1)\chi(x_2/\delta)a, \chi(x_2/\delta)v)\leqq(L_\eta v, v)$. The condition (4) is fulfilled. In fact, when $B\in\mathcal{C}$ is contained in $\{|x_1|\leqq\rho(\log\eta)^{-1/(l+1)}\}$ the condition (4) holds with a sufficiently small $\rho\ll s^{-1}$. In other case, (4) is obvious because $\lambda<l+1$. The estimate (5.9) is also reduced to (5) by setting

$$a(x_2, \xi_2)=\xi_2^2+\delta^{-2l}\exp(-\delta^{-\tau})\exp(-1/|x_2|^\kappa)\eta^2,$$

$$W(x_2)=(\log\eta^s)^2,$$

because $(L_\eta v, v)\geqq\delta^{2l}(a(x_2, D_2)\chi(x_1/\delta)v, \chi(x_1/\delta)v)$. Q.E.D.

We shall prove that if $v\in\mathcal{D}'(\boldsymbol{R}^4)$ and $\rho_0=(0, (0, 0, 0, \pm 1))$ then $\rho_0\notin\mathrm{WF}\,Lv$ implies $\rho_0\notin\mathrm{WF}\,v$. Let $h(t)$ be a $C_0^\infty(R^1)$ function such that $h=1$ in $|t|\leqq 1$ and $h=0$ in $|t|\geqq 3/2$. For a $\delta>0$ let $\phi_\delta(\xi)\in C^\infty(\boldsymbol{R}^4\setminus 0)$ satisfy $\phi_\delta=1$ in $\{\pm\delta\xi_4\geqq|\xi'|\}\cap\{|\xi|\geqq 3/2\delta\}$ and $\phi_\delta=0$ in $\{\pm 3\delta\xi_4\leqq 2|\xi'|\}\cup\{|\xi|\leqq\delta)^{-1}\}$. Here $\xi=(\xi', \xi_4)$ and we choose one of $\pm$ signs according to $\rho_0=(0, (0, 0, 0, 1))$ or $(0, (0, 0, 0, -1))$. Set $\varphi(x)=\prod_{k=1}^4 h(x_k)$ and set $\varphi_\delta(x)=\varphi(x/\delta)$. If we set $\Psi_\delta(\xi)=h((M^{-1}|\xi_4|-3)/\delta)\phi_\delta(\xi)$ for a parameter $M\geqq 1$, then for any α there exists a C_α such that

$$|D_\xi^\alpha\Psi_\delta|\leqq C_\delta M^{-s}\langle\xi\rangle^{-|\alpha|+s} \tag{5.10}$$

with any real $0\leqq s\leqq|\alpha|$ because with a $C>0$ we have $C^{-1}\leqq M/\langle\xi\rangle\leqq C$ on $\operatorname{supp}D_\xi^\alpha\Psi_\delta$.

Fix an integer $N>0$. Take a sequence $\{\Psi_j(\xi)\}_{j=0}^N\subset S_{1,0}^0$ such that

$$\Psi_\delta=\Psi_0\Subset\Psi_1\Subset\Psi_2\Subset\cdots\Subset\Psi_{N-1}\Subset\Psi_N=\Psi_{2\delta}$$

and for any α the estimate

$$|D_\xi^\alpha\Psi_j|\leqq C_\alpha N^{|\alpha|}M^{-s}\langle\xi\rangle^{-|\alpha|+s},\qquad 0\leqq s\leqq|\alpha|, \tag{5.11}$$

holds with a constant C_α independent of N and j. It should be noted that Ψ_j can be taken of the form $\Psi_j=h_j(\xi_4; M)\phi_j(\xi)$ with $\phi_j=1$ in $\{\pm\delta\xi_4\geqq|\xi'|\}\cap\{|\xi|\geqq 3/2\delta\}$. Here one of $\pm$ signs is chosen following the above convention. Similarly, take a sequence $\{\varphi_j(x)\}_{j=0}^N\subset C_0^\infty(\boldsymbol{R}^4)$ such that

$$\varphi_\delta=\varphi_0\Subset\varphi_1\Subset\varphi_2\Subset\cdots\Subset\varphi_{N-1}\Subset\varphi_N=\varphi_{2\delta}$$

and for any α the estimate

(5.12) $$|D_x^\alpha \varphi_j| \leqq C'_\alpha N^{|\alpha|}$$

holds with a constant C'_α independent of N and j. We may assume that φ_j can be written as in $\varphi_j(x)=\prod_{k=1}^4 h_j(x_k)$. Here $\psi \Subset \phi$ means that $\phi \equiv 1$ in a neighborhood of supp φ.

Lemma 5.3. i) *Let $g(x)$ be a C^∞-function satisfying the similar estimates as* (5.12). *Then there exists a constant C_0 independent of N, M and j such that for any real $s>0$ the estimate*

(5.13) $$\mathrm{Re}\,([g(x), \Psi_j(D)]u, u) \leqq (C_0 N)^2 M^{-1}(1+C_s N^{2s+8} M^{-s})\|u\|^2, \qquad u \in \mathcal{S},$$

holds with a constant C_s independent of N, M and j.

ii) *Let K be a fixed compact set in $\boldsymbol{R}^4$ and $g(x)$ be a polynomial of degree d with coefficients independent of N and M. Then there exists a constant $C_0=C_{0,K}$ independent of N, M and j such that for any real $s \geqq d$ the estimate*

(5.13)′ $$\mathrm{Re}\,([g(x), \Psi_j(D)]u, u) \leqq (C_0 N)^2 M^{-1}(1+C_s N^{2s} M^{-s})\|u\|^2, \qquad u \in C_0^\infty(K),$$

holds with a constant C_s independent of N, M and j.

Proof. If $g(x)$ is C^∞-function, in view of (5.11) and (5.12), it follows from the Calderón-Vaillancourt theorem that for any integer $q>0$

(5.14) $$\mathrm{Re}\,([g(x), \Psi_j(D)]u, u) \leqq \{\sum_{j=1}^{q-1} C_j N^{2j} M^{-j} + C_q N^{2q+6} M^{-q}\}\|u\|^2, \qquad u \in \mathcal{S}.$$

If $q=[s]+1$, for the proof of (5.13) it suffices to show that for some C_s we have

(5.15) $$\sum_{j=1}^{[s]} C_j N^{2j} M^{-j} \leqq C_1 N^2 M^{-1}(2+C_s N^{2s} M^{-s}).$$

In fact, if $N^2 M^{-1} \leqq \min_{2 \leqq j \leqq q-1} \min(C_1/2C_j, 1/2) \equiv R$ then we have

$$\sum_{j=1}^{[s]} C_j N^{2j} M^{-j} < C_1 N^2 M^{-1}(1+\sum 2^{-j}) = 2C_1 N^2 M^{-1}.$$

If $N^2 M^{-1} \geqq R$ then we have

$$\sum_{j=1}^{[s]} C_j N^{2j} M^{-1} = N^2 M^{-1} \sum_{j=1}^{[s]} C_j (N^2/M)^{j-1-s} (N^2/M)^s$$
$$\leqq N^2 M^{-1} \sum_{j=1}^{[s]} C_j R^{j-1-s} (N^2/M)^s.$$

Thus we have (5.15) with $C_s = \sum_{j=1}^{[s]} C_j R^{j-1-s}/C_1$. When $g(x)$ is polynomial it follows from (5.11) that

(5.14)′ $$\mathrm{Re}\,([g(x), \Psi_j(D)]u, u) \leqq \sum_{j=1}^{d} C_j N^j M^{-j} \|u\|^2, \qquad u \in C_0^\infty(K).$$

By means of (5.15) we have (5.13)′ for any $s\geqq d$. Q.E.D.

Lamme 5.4. *Let K be a fixed compact set satisfying $K\supseteq \operatorname{supp}\varphi_{2\delta}$. There exist a constant C_0 independent of M and N such that for any $s>0$ and some $C_s>0$ we have*

$$(\log M^s)^2 \operatorname{Re}([L, \varphi_j(x)\Psi_j(D)]u, \varphi_j(x)\Psi_j(D)u) \leqq (C_0N)^2\{(Lu, u)+C_sN^{2s+10}M^{-s}\|u\|^2\}, \quad u\in C_0^\infty(K), \tag{5.16}$$

provided that $\log M^s\geqq C_0N$ *and* $M\geqq M_s$ *for a sufficiently large* $M_s>0$.

Proof. Note that

$$[L, \varphi_j(x)\Psi_j(D)]=[L, \varphi_j(x)]\Psi_j(D)+\varphi_j(x)[L, \Psi_j(D)]\ . \tag{5.17}$$

We see that

$$\operatorname{Re}([x_1^{2k}x_2^{2m}D_3^2, \varphi_j(x)]u, \varphi_j(x)u)\leqq (CN)^2\|x_1^kx_2^mu\|^2 \qquad \text{for } u\in\mathcal{S}\,.$$

Here and in what follows we denote different constants independent of N, M and s by the same notation C. From this we have

$$\begin{aligned}(\log M^s)^2 &\operatorname{Re}([x_1^{2k}x_2^{2m}D_3^2, \varphi_j(x)]\Psi_j(D)u, \varphi_j(x)\Psi_j(D)u)\\ &\leqq (CN)^2\|(\log M^s)x_1^kx_2^m\Psi_j(D)u\|^2\\ &\leqq (CN)^2\{\|(\log M^s)\Psi_j(D)x_1^kx_2^mu\|^2+(\log M^s)^2\|[x_2^kx_2^m, \Psi_j(D)]u\|^2\}\end{aligned} \tag{5.18}$$

Using (5.3) of Lemma 5.1, for any $s>0$ we have

$$\begin{aligned}\|(\log M^s)\Psi_j(D)x_1^kx_2^mu\|^2 &\leqq C\|(\log|D_4|^s)h((M^{-1}|D_4|-3)/2\delta)x_1^kx_2^mu\|^2\\ &\leqq C(Lu, u) \qquad \text{for } u\in C_0^\infty(K)\,,\end{aligned}$$

if $M\geqq M_s$ for a large $M_s>0$. It follows from (5.13)′ and (5.1) that

$$\begin{aligned}(\log M^s)^2&\|[x_1^kx_2^m, \Psi_j(D)]u\|^2\\ &\leqq (\log M^s)^4M^{-1}\{C_K(Lu, u)+C_sN^{2s+8}M^{-s}\|u\|^2\}, \qquad u\in C_0^\infty(K)\,,\end{aligned}$$

if $\log M^s\geqq C_0N$, where $C_0=C_{0,K}$ is the same as in (5.13)′. Therefore, if $\log M^s\geqq C_0N$ and $M\geqq M_s'$ for another large $M_s'>0$ such that $5!s^4/(\log M_s')\leqq 1$, we have

$$\begin{aligned}(\log M^s)^2 &\operatorname{Re}([x_1^{2k}x_2^{2m}D_3^2, \varphi_j(x)]\Psi_j(D)u, \varphi_j(x)\Psi_j(D)u)\\ &\leqq (CN)^2\{(Lu, u)+C_sN^{2s+8}M^{-s}\|u\|^2\}\equiv\Omega\,, \qquad u\in C_0^\infty(K)\,.\end{aligned} \tag{5.19}$$

Note that

$$\begin{aligned}(\log M_s)^2 &\operatorname{Re}([x_1^{2l}D_2^2, \varphi_j(x)]\Psi_j(D)u, \varphi_j(x)\Psi_j(D)u)\\ &\leqq (CN)^2(\log M^s)^2\|\chi(x_2/\delta)x_1^l\Psi_j(D)u\|^2\\ &\leqq (CN)^2\{\|(\log|D_4|^s)h)(M^{-1}|D_4|-3)/2\delta)x_1^l\chi(x_2/\delta)u\|^2\\ &\qquad +(\log M^s)^2\|[\chi(x_2/\delta)x_1^l, \Psi_j(D)]u\|^2\},\end{aligned}$$

where $\chi(t)$ is the same as in Lemma 5.2. Using (5.8) and (5.13) (and also (5.13)′) to estimate the first term and second one, respectively, we obtain

$$(5.20)\qquad (\log M^s)^2\,\mathrm{Re}\,([x_1^{2l}D_2^2,\ \varphi_j(x)]\Psi_j(D)u,\ \varphi_j(x)\Psi_j(D)u)\leqq\Omega\,,\qquad u\in C_0^\infty(K)\,,$$

if M satisfies the same condition as in (5.19). Similarly, using (5.9) we get

$$(5.21)\qquad (\log M^s)^2\,\mathrm{Re}\,([D_1^2,\ \varphi_j(x)]\Psi_j(D)u,\ \varphi_j(x)\Psi_j(D)u)\leqq\Omega\,,\qquad u\in C_0^\infty(K)\,.$$

Since $f^{1/2}\leqq C_{k,m}x_1^kx_2^m$ for a constant $C_{k,m}$ we have

$$\begin{aligned}\mathrm{Re}\,([f(x)D_4^2,\ \varphi_j(x)]u,\ \varphi_j(x)u)&\leqq(CN)^2\|f(x)^{1/2}u\|^2\\&\leqq(CN)^2\|x_1^kx_2^mu\|^2\qquad\text{for } u\in\mathcal{S}\,.\end{aligned}$$

Noting the middle term of (5.18), we have

$$\begin{aligned}(5.22)\qquad (\log M^s)^2\,\mathrm{Re}\,([f(x)D_4^2,\ \varphi_j(x)]\Psi_j(D)u,\ \varphi_j(x)\Psi_j(D)u)&\\ \leqq\Omega\,,\qquad u\in C_0^\infty(K)\,.&\end{aligned}$$

Summing up (5.19)-(5.22) we obtain with a constant $C\geqq C_0$

$$\begin{aligned}(5.23)\qquad \mathrm{Re}\,([L,\ \varphi_j(x)]\Psi_j(D)u,\ \varphi_j(x)\Psi_j(D)u)&\\ \leqq(CN)^2\{(Lu,\ u)+C_sN^{2s+8}M^{-s}\|u\|\},\qquad u\in C_0^\infty(K)\,.&\end{aligned}$$

if $\log M^s\geqq CN$ and $M\geqq M_s'$. On the other hand, since coefficients of L are independent of x_4, by noting the form of Ψ_j we see that

$$\begin{aligned}(5.24)\qquad (\log M^s)^2\,\mathrm{Re}\,(\varphi_j(x)[L,\ \Psi_j(D)]u,\ \varphi_j(x)\Psi_j(D)u)&\\ \leqq(\log M^s)^2CN^4(\|\chi_0(D)u\|^2+C_sN^{2s+10}M^{-s-1}\|u\|^2)\,,&\end{aligned}$$

where $\chi_0\in S_{1,0}^0$ satisfies

$$\mathrm{supp}\,\chi_0\subset\{2\delta|\xi_4|\geqq|\xi'|\geqq\delta|\xi_4|\}\cap\{2\leqq|\xi_4|/M\leqq4\}.$$

Note that with some $0<\mu\leqq1/2$ we have

$$\begin{aligned}(5.25)\qquad M^{2\mu}\|\chi_0(D)u\|^2&\leqq C\|\,|D'|^\mu u\|^2\\&\leqq C_K(Lu,\ u)\qquad\text{for } u\in C_0^\infty(K)\,.\end{aligned}$$

which follows from the well-known Hörmander theorem (and also Theorem 2). If $\log M^s\geqq CN$ then we have

$$(\log M^s)^2(CN)^2M^{-2\mu}\leqq(\log M^s)^4M^{-2\mu}\leqq1$$

provided that $M\geqq M_s''$ for a large $M_s''>0$. So under this condition we have

$$\begin{aligned}(5.26)\qquad (\log M^s)^2\,\mathrm{Re}\,(\varphi_j(x)[L,\ \Psi_j(D)]u,\ \varphi_j(x)\Psi_j(D)u)&\\ \leqq(CN)^2\{(Lu,\ u)+C_sN^{2s+10}M^{-s}\|u\|^2\},\qquad u\in C_0^\infty(K)\,.&\end{aligned}$$

Together with (5.23) we obtain (5.16) in view of (5.17). Q.E.D.

Lemma 5.5. *For any integer $N\geqq1$ there exists a constant C_0 independent of N*

and M such that for any real $s>0$ and some constant C_s independent of N and M we have

$$(\log M^s)^{2N}\|\varphi_\delta\Psi_\delta u\|^2\leqq C_0N(\log M^s)^{2N}\|\Psi_{2\delta}\varphi_{2\delta}Lu\|\|u\| + \{(C_0N)^{2N+2}M^2+C_s(\log M^s)^{2N}N^{2s+10}M^{-s}\}\|u\|^2, \qquad u\in\mathcal{S}, \tag{5.27}$$

provided that $M\geqq M_s$ for M_s the same as in Lemma 5.4.

Proof. We may assume that $\log M^s\geqq C_0N$ because of the term $(C_0N)^{2N+2}M^2\|u\|^2$ in the right hand side of (5.27). It follows from the expansion formula of pseudo-differential operators that for any $s>0$ we have with a $C_s>0$

$$(L\varphi_j\Psi_ju,\ \varphi_j\Psi_ju)\leqq \mathrm{Re}\,(\varphi_j\Psi_jLu,\ \varphi_j\Psi_ju) + \mathrm{Re}\,([L,\ \varphi_j\Psi_j]\varphi_{j+1}\Psi_{j+1}u,\ \varphi_j\Psi_j\varphi_{j+1}\Psi_{j+1}u) + C_sN^{2s+10}M^{-s}\|u\|^2, \qquad u\in\mathcal{S}, \tag{5.28}$$

In what follows we denote by $R(s)$ the last term of the right hand side. We see that

$$|(\varphi_j\Psi_jLu,\ \varphi_j\Psi_ju)\leqq\|\varphi_j\Psi_jLu\|\|u\| \leqq\|\Psi_{2\delta}\varphi_{2\delta}Lu\|\|u\|+R(s) \tag{5.29}$$

because $\varphi_j\Psi_j=\varphi_j\Psi_j\Psi_{2\delta}\varphi_{2\delta}+\varphi_j\Psi_j(1-\varphi_{2\delta})$. In view of $\varphi_\delta=\varphi_0$, $\Psi_\delta=\Psi_0$, it follows from (5.1) and (5.28) that with $C_0\geqq C_K$ we have

$$\begin{aligned}\|\varphi_\delta\Psi_\delta u\|^2&\leqq C_0(L\varphi_0\Psi_0u,\ \varphi_0\Psi_0u)\\ &\leqq C_0\{\mathrm{Re}\,(\varphi_0\Psi_0Lu,\ \varphi_0\Psi_0u)\\ &\quad+\mathrm{Re}\,([L,\ \varphi_0\Psi_0]\varphi_1\Psi_1u,\ \varphi_0\Psi_0\varphi_1\Psi_1u)+R(s)\}, \qquad u\in\mathcal{S}.\end{aligned} \tag{5.30}$$

Using (5.29) and (5.16) to estimate the first term and the second one we have

$$\|\varphi_\delta\Psi_\delta u\|^2\leqq C_0\{\|\Psi_{2\delta}\varphi_{2\delta}Lu\|\|u\|+R(s)\} + C_0(\log M^s)^{-2}(C_0N)^2\{(L\varphi_1\Psi_1u,\ \varphi_1\Psi_1u)+R(s)\}.$$

Apply (5.28) to estimate the term $(L\varphi_1\Psi_1u,\ \varphi_1\Psi_1u)$ in the right hand side and repeat this procedure N times. Then we have

$$\begin{aligned}\|\varphi_\delta\Psi_\delta u\|^2&\leqq C_0\sum_{j=0}^{N-1}(\log M^s)^{-2j}(C_0N)^{2j}(\|\Psi_{2\delta}\varphi_{2\delta}Lu\|\|u\|+R(s))\\ &\quad+C_0(\log M^s)^{-2N}(C_0N)^{2N}\{(L\varphi_N\Psi_Nu,\ \varphi_N\Psi_Nu)+R(s)\}\\ &\leqq C_0N(\|\Psi_{2\delta}\varphi_{2\delta}Lu\|\|u\|+R(s))\\ &\quad+(C_0N)^{2N+2}(\log M^s)^{-2N}M^2\|u\|^2, \qquad u\in\mathcal{S},\end{aligned}$$

because of $\Psi_N=\Psi_{2\delta}$ and $M^{-2}\|\Psi_{2\delta}u\|_1^2\leqq C\|u\|^2$. Q.E.D.

It follows from (5.27) that for any $N\geqq1$ and any $s>0$

$$(5.31)\qquad (\log M^s)^N \|\varphi_\delta \Psi_\delta u\| \leqq C_0(N+1)(\log M^s)^N(\|\Psi_{2\delta}\varphi_{2\delta}Lu\|\|u\|)^{1/2} + \{(C_0e)^N M(N+1) + C_s N^{s+6}2^N\} N!\|u\|, \qquad u\in\mathcal{S},$$

where we used $N^N \leqq e^N N!$ and $(\log t)^N t^{-1} \leqq N!$. We may assume that (5.30) holds with $N=0$. Multiply both sides of (5.31) by $\varepsilon_0^N/N!$ with a fixed $0<\varepsilon_0<1$ satisfying $\min\{C_0e\varepsilon_0, 2\varepsilon_0\}<1$ and sum up with respect to $N=0, 1, 2, \cdots$. Since $\log M^s \leqq M^{\varepsilon_0 s}/\varepsilon_0$ we have

$$(5.32)\qquad \begin{aligned} M^{\varepsilon_0 s}\|\varphi_\delta\Psi_\delta u\| &\leqq 2\varepsilon_0^{-1}C_0 M^{2\varepsilon_0 s}(\|\Psi_{2\delta}\varphi_{2\delta}Lu\|\|u\|)^{1/2} + C_sM\|u\| \\ &\leqq 2\varepsilon_0^{-1}C_0M^{4\varepsilon_0 s}\|\Psi_{2\delta}\varphi_{2\delta}Lu\| + C_sM\|u\|, \qquad u\in\mathcal{S}, \end{aligned}$$

where C_s denotes the different constant depending on s but independent of M. Since ε_0 is independent of $s>0$, by setting $\varepsilon_0 s = s'+s''+1$ for any $s', s''\geqq 1$ we obtain with a constant C independent of s', s'' and M

$$(5.33)\qquad M^{2s'}\|\varphi_\delta\Psi_\delta u\|^2 \leqq CM^{8(s'+s'')}\|\Psi_{2\delta}\varphi_{2\delta}Lu\|^2 + C_{s',s''}M^{-2s''}\|u\|^2, \qquad u\in\mathcal{S},$$

if $M\geqq M(s', s'')$ for a sufficiently large $M(s', s'')>0$. Even if $1\leqq M\leqq M(s', s'')$ the estimate (5.33) holds if we choose another sufficiently large $C_{s',s''}$. Since $\Psi_{\delta/2}\varphi_\delta = \Psi_{\delta/2}\varphi_\delta\Psi_\delta + \Psi_{\delta/2}\varphi_\delta(1-\Psi_\delta)$, it follows from (5.10) that

$$(5.34)\qquad M^{2s'}\|\Psi_{\delta/2}\varphi_\delta u\|^2 \leqq M^{2s'}\|\varphi_\delta\Psi_\delta u\|^2 + C_{s',s''}\|u\|^2_{-s''}.$$

Substituting $\Psi_{4\delta}u$ into (5.33) and noting that $\Psi_{2\delta}\varphi_{2\delta}L\Psi_{4\delta} = \Psi_{2\delta}\varphi_{2\delta}L + \Psi_{2\delta}\varphi_{2\delta}L(1-\Psi_{4\delta})$, by means of (5.34) we have

$$(5.35)\qquad M^{2s'}\|\Psi_{\delta/2}\varphi_\delta u\|^2 \leqq CM^{8s'+8s''}\|\Psi_{2\delta}\varphi_{2\delta}Lu\|^2 + C_{s',s''}\|u\|^2_{-s''}, \qquad u\in\mathcal{S}.$$

Here we estimated $M^{-2s''}\|\Psi_{4\delta}u\|^2$ by $C\|u\|^2_{-s''}$ because of (5.10). If $\varepsilon, l>0$ then it follows from (5.35) that

$$\begin{aligned} &(1+\varepsilon M)^{-2l}M^{2s'-2}\|\Psi_{\delta/2}\varphi_\delta u\|^2 \\ &\qquad \leqq C_{s',s''}\{(1+\varepsilon M)^{-2l}M^{8s'+8s''-2}\|\Psi_{2\delta}\varphi_{2\delta}Lu\|^2 + M^{-2}\|u\|^2_{-s''}\}. \end{aligned}$$

Note that M and the symbol of $\Lambda=(1+|D|^2)^{1/2}$ are equivalent on $\operatorname{supp}\Psi_{2\delta}$. Replacing s' by $s'+1$ we have for any $M>0$

$$(5.36)\qquad \begin{aligned} &\|(1+\varepsilon\Lambda)^{-l}\Psi_{\delta/2}\varphi_\delta u\|^2_{s'} \\ &\qquad \leqq C_{s',s''}\{\|(1+\varepsilon\Lambda)^{-l}\Psi_{2\delta}\varphi_{2\delta}Lu\|^2_{4s'+4s''+3} + M^{-2}\|u\|^2_{-s''}\}. \end{aligned}$$

We prepare the following:

Lemma 5.6. *If $h(t)$ is as above and $v\in\mathcal{S}$ then we have*

$$(5.37)\qquad \begin{aligned} \log\{(3-\delta)/(3+\delta)\}\int_1^\infty |v(t)|^2dt &\leqq \int_1^\infty\left\{\int_1^\infty |h((M^{-1}t-3)/\delta)v(t)|^2dt\right\}dM/M \\ &\leqq \log\{(2+\delta)/(2-\delta)\}\int_0^\infty |v(t)|^2dt \end{aligned}$$

Proof. The estimate follows from the exchange of the order of integration.

Integrate with respect to $M\in[1, \infty)$ after dividing both sides of (5.34) by M. Then we obtain by Lemma 5.6

$$\|(1+\varepsilon\Lambda)^{-l}\psi_{\delta/2}\varphi_\delta u\|_{s'}^2 \leq C_{s',s''}\{\|(1+\varepsilon\Lambda)^{-l}\psi_{2\delta}\varphi_{2\delta}Lu\|_{4s'+4s''+3}^2+\|u\|_{-s''}^2, \qquad u\in\mathcal{S}, \tag{5.38}$$

for any real s', $s''>0$.

We are now ready to prove that $\rho_0=(0, (0, 0, 0, \pm 1))\notin \mathrm{WF}\, Lv$ implies $\rho_0\notin \mathrm{WF}\, v$ for any $v\in\mathcal{D}'(\boldsymbol{R}^4)$. Without of loss of generality we may assume that $v\in\mathcal{E}'(\boldsymbol{R}^4)$ and hence $v\in H_{-s''}$ for a large $s''>0$. Choose $l>0$ such that $l\geqq 4s'+4s''+5$. Then by taking a sequence $\{w_j\}_{j=1}^\infty\subset\mathcal{S}$ such that

$$w_j \longrightarrow v \quad \text{in } H_{-s''},$$

from (5.38) we see that

$$\begin{aligned}\|(1+\varepsilon\Lambda)^{-l}\psi_{\delta/2}\varphi_\delta v\|_{s'}^2 &\leq C_{s',s''}\{\|(1+\varepsilon\Lambda)^{-l}\psi_{2\delta}\varphi_{2\delta}Lv\|_{4s'+4s''+3}^2+\|v\|_{-s''}^2\} \\ &\leq C_{s',s''}\{\|\psi_{2\delta}\varphi_{2\delta}Lv\|_{4s'+4s''+3}^2+\|v\|_{-s''}^2\}\end{aligned} \tag{5.39}$$

if $\delta>0$ is sufficiently small such that $\psi_{2\delta}\varphi_{2\delta}Lv\in\mathcal{S}$. Letting ε tend to 0 in (5.39), we have $\psi_{\delta/2}\varphi_\delta v\in H_{s'}$. Since s' is arbitrary, we have $\rho_0\notin \mathrm{WF}\, v$.

We consider the case where $\rho_0=\{x_0, (0, 0, 0, \pm 1))$ with $x_0=(x_{01}, x_{02}, x_{03}, x_{04})\neq 0$. If $\delta>0$ is the minimum of $|x_{0j}|/2$ for j with $x_{0j}\neq 0$, we have (5.39) with $\varphi_\delta(x)$ replaced by $\tilde{\varphi}_\delta(x)=\prod_{j=1}^4 h((x_j-x_{0j})/\delta)$. In fact, Lemma 5.4 still holds with the corresponding φ_j to $\tilde{\varphi}_\delta$ because $\mathrm{supp}\, h'((x_j-x_{0j})/\delta)\cap\{x_j=0\}=\emptyset$. Therefore, we also see that $\rho_0\notin \mathrm{WF}\, Lv$ implies $\rho_0\notin \mathrm{WF}\, v$. The case where $\rho_0=(x_0, (\eta', \eta_4))$ with $\eta'\neq 0$ is reduced to Corollary 2 of [10] because we have with $\mu>0$

$$\||D'|^\mu u\|^2 \leqq C_K(Lu, u) \qquad \text{for } u\in C_0^\infty(K).$$

It should be noted that the microlocal version of Theorem 1 of [10] holds (see Lemma 1.1 of [10]). Now the proof of Theorem 6 is completed.

6. Proof of Theorem 7

For an $a>0$ we set $\Omega_a=\{x\in\boldsymbol{R}^2; |x_j|\leqq a, j=1, 2\}$. As in [7] and [4], we consider the eigenvalue problem with a parameter $\eta>0$ as follows:

$$\begin{cases} (\mathcal{A}+f(x)\eta^2)v=\mu g(x)v & \text{in } \Omega_a \\ v|_{\partial\Omega_a}=0, \end{cases} \tag{6.1}$$

where $\mathcal{A}=D_1^2+x_1^{2l}D_2^2+D_2h(3(|x_2|-a)/a)D_2$, $g(x)=x_1^{2k}x_2^{2m}$ and

$$f(x)=\exp(-1/|x_1|^\tau-1/x_2|^\kappa)+\exp(-1/|x_1|^\lambda-1/|x_2|^\sigma). \tag{6.2}$$

Here $h(t)\in C_0^\infty(\boldsymbol{R}^1)$ is the same as in the biginning of Section 5. Throughout this section, we assume that

(6.3) $$0<\kappa<1\,,\qquad 0<\lambda<k+1\,,$$

(6.4) $$\tau\geqq k+1+m(l+1)\,,\qquad \sigma\geqq m+(k+1)/(l+1)\,.$$

Lemma 6.1. *The eigenvalue problem* (6.1) *can be solved. The smallest eigenvalue* $\mu(\eta)$ *and the corresponding eigenfunction* $v(x\,;\,\eta)$ *with* $\int_{\Omega_a}|v(x\,;\,\eta)|^2dx=1$ *satisfy the following*:

(I) *For any* $a>0$ *there exists a constant* C_1 *independent of* a *and* η *such that*

(6.5) $$\mu(\eta)\leqq C_1(\log\eta)^2\qquad \textit{if}\ \ \eta\geqq\eta_a$$

for a sufficiently large $\eta_a>0$.

(II) *For any fixed positive* $b<a$ *we see that*

(6.6) $$\lim_{\eta\to\infty}\int_{\Omega_b}|v(x\,;\,\eta)|^2dx=1\,.$$

Proof. Consider the Dirichlet problem

(6.7) $$\mathcal{L}_\eta v=F\,,\qquad v|_{\partial\Omega_a}=0\,,$$

where $\mathcal{L}_\eta=\mathcal{A}+f(x)\eta^2$. For $u,\,v\in C_0^\infty(\Omega_a)$ we have

(6.8) $$(\mathcal{L}_\eta u,\,v)=(D_1u,\,D_1v)+(x_1^lD_2u,\,x_1^lD_2v)+(hD_2u,\,D_2v)+(f\eta^2u,\,v)\,.$$

Let $\mathcal{H}$ be the Hlibert space that is the completion of $C_0^\infty(\Omega_a)$ by the norm $\|u\|_{\mathcal{H}}=\sqrt{(u,\,u)_{\mathcal{H}}}$. Here $(u,\,v)_{\mathcal{H}}$ denotes the right hand side of (6.8) and it is the positive Hermitian form. It follows from the Poincaré inequality that $\|u\|_{L^2(\Omega_a)}\leqq C_a\|u\|_{\mathcal{H}}$ for any $u\in\mathcal{H}$. Since $\mathcal{L}_\eta$ is elliptic in a neighborhood of $\partial\Omega_a$ and subelliptic in Ω_a, there exists a Green operator $\mathcal{G}_\eta$ from $\mathcal{H}'$ onto $\mathcal{H}$ such that $\mathcal{L}_\eta\mathcal{G}_\eta=I$ in $\mathcal{H}'$ and $\mathcal{G}_\eta\mathcal{L}_\eta=I$ in $\mathcal{H}$, where $\mathcal{H}'$ denotes the dual space of $\mathcal{H}$. Furthermore, $\mathcal{G}_\eta$ is a compact positive Hermitian operator in $L^2(\Omega_a)$ (see Mizohata [5, Chapter 3]). We shall show that the smallest eigenvalue $\mu(\eta)$ is given by

(6.9) $$\mu(\eta)=\inf_{\substack{v\in C_0^\infty(\Omega_a)\\ v\neq 0}}(\mathcal{L}_\eta v,\,v)/(gv,\,v)>0\,.$$

The positivity of the right hand side follows from the Poincaré inequality. Since $C_0^\infty(\Omega_a)$ is dense in $L^2(\Omega_a)$ we have

$$\mu(\eta)^{-1}=\sup_{\substack{u\in C_0^\infty(\Omega_a)\\ u\neq 0}}(\mathcal{G}_\eta g\mathcal{G}_\eta u,\,u)/(\mathcal{G}_\eta u,\,u)\,.$$

If $H=\mathcal{G}_\eta^{1/2}g\mathcal{G}_\eta^{1/2}$ then we have

$$\mu(\eta)^{-1}=\sup_{\substack{w\in C_0^\infty(\Omega_a)\\ \|w\|=1}}(Hw,\,w)\,,$$

because the image of $\mathcal{G}_\eta^{1/2}$ from $C_0^\infty(\Omega_a)$ is dense in $L^2(\Omega_a)$. Take a sequence $\{w_j\}\subset$

$C_0^\infty(\Omega_a)$ such that $\|w_j\|=1$ and $(Hw_j, w_j)\to\mu(\eta)^{-1}$. Note that

$$0\leqq\|Hw_j-\mu(\eta)^{-1}w_j\|^2$$
$$=\|Hw_j\|^2-2\mu(\eta)^{-1}(Hw_j, w_j)+\mu(\eta)^{-2}\longrightarrow 0 \qquad (j\to\infty).$$

We see that $\mathcal{G}_\eta g\mathcal{G}_\eta^{1/2}w_j-\mathcal{G}_\eta^{1/2}w_j/\mu(\eta)\to 0$. Since $\mathcal{G}_\eta$ is compact and $\{g\mathcal{G}_\eta^{1/2}w_j\}$ is a bounded set in $L^2(\Omega_a)$, there exists a subsequence $\{w_{j_k}\}$ such that $\{\mathcal{G}_\eta g\mathcal{G}_\eta^{1/2}w_{j_k}\}$ is convergent and so $\{\mathcal{G}_\eta^{1/2}w_{j_k}\}$ is also convergent. If $v_0=\lim_{k\to\infty}\mathcal{G}_\eta^{1/2}w_{j_k}\in L^2(\Omega_a)$ then we have $\mu(\eta)\mathcal{G}_\eta gv_0=v_0$ and $v_0\neq 0$ because $\mathcal{G}_\eta$ is positive. Therefore, $\mathcal{L}_\eta v_0=\mu(\eta)gv_0$ and $v_0|_{\partial\Omega_a}=0$. For the proof of (6.5) we set

(6.10) $$\Omega_\eta=\{x\in\boldsymbol{R}^2\,;\,1/2\leqq x_1(\log\eta^2)^p\leqq 1,\ 1/2\leqq x_2(\log\eta^2)^q\leqq 1\},$$

where $p=\{k+1+m(l+1)\}^{-1}$ and $q=(l+1)p$. We see that $\Omega_\eta\subset\Omega_{a/4}$ for a large $\eta>0$. It follows from (6.4) that

$$f(x)\eta^2\leqq 2,\quad g(x)\geqq 4^{p-(k+m+1)}(\log\eta)^{2p-2}\qquad \text{in } \Omega_\eta\,.$$

Since $h(3(|x_2|-a)/a)=0$ on Ω_η, the right hand side of (6.9) is estimated above from the constant times of $(\log\eta)^{2-2p}$ multiplied by

$$\inf_{\substack{v\in C_0^\infty(\Omega_\eta)\\ v\neq 0}}((D_1^2+(\log\eta^2)^{-2pl}D_2^2+2)v,\, v)/(v,\, v)=O((\log\eta)^{2p}),$$

so that we obtain (6.5). Since $v_\eta=v(x\,;\,\eta)$ belongs to $C_0^\infty(\Omega_a)$ we have

$$(\mathcal{L}_\eta,\, v_\eta,\, v_\eta)\geqq\int_{\Omega_a}\{|D_1v_\eta|^2+|x_1^lD_2v_\eta|^2+f(x)\eta^2|v_\eta|^2\}\,dx$$
$$\geqq c_b\int_{\Omega_1}\{|D_2v_\eta|^2+\exp(-1/x_2|^\kappa)\eta^2|v_\eta|^2\}\,dx\,,$$

where $\Omega_1=\Omega_a\cap\{|x_1|\geqq b\}$. Since $a(x_2,\xi_2)=\xi_2^2+\exp(-1/|x_2|^\kappa)\eta^2$ and $W(x_2)=2^{-2}(\log\eta)^{2/\kappa}$ satisfy the condition (4) of Theorem 1 we have

$$C_a\mu(\eta)\geqq\mu(\eta)(gv_\eta,\, v_\eta)\geqq c_b'(\log\eta)^{2/\kappa}\int_{\Omega_1}|v_\eta|^2dx\,.$$

In view of $\kappa<1$, it follows from (6.5) that $\lim_{\eta\to\infty}\int_{\Omega_1}|v_\eta|^2dx=0$. If $\Omega_2=\Omega_a\cap\{|x_2|\geqq b\}$ and $\Omega_{2,1}=\Omega_2\cap\{|x_1|\geqq(\log\eta)^{-1/(k+1)}\}$ then we have

$$(\mathcal{L}_\eta v_\eta,\, v_\eta)\geqq c_b''\int_{\Omega_2}\{|D_1v_\eta|^2+\exp(-1/|x_1|^\lambda)\eta^2|v_\eta|^2\}\,dx$$
$$\geqq c_b''\eta\int_{\Omega_{2,1}}|v_\eta|^2dx\,,$$

so that $\lim_{\eta\to\infty}\int_{\Omega_{2,1}}|v_\eta|^2dx=0$. If $\chi(x_1)=h(x_1(\log\eta)^{1/(k+1)})$ then

$$(\mathcal{L}_\eta \chi v_\eta, \chi v_\eta) \geqq c_b'' \int_{\Omega_2} \{|D_1 \chi v_\eta|^2 + \exp(-1/|x_1|^\lambda)\eta^2 |\chi v_\eta|^2\} dx$$

$$\geqq \tilde{c}_b (\log \eta)^{2/\lambda} \int_{\Omega_2} |\chi v_\eta|^2 dx .$$

Here the last inequality follows from Theorem 1. Since

$$(\mathcal{L}_\eta \chi v_\eta, \chi v_\eta) = \mu(\eta)(g v_\eta, \chi^2 v_\eta) + \mathrm{Re}([D_1^2, \chi] v_\eta, \chi v_\eta)$$

$$\leqq C_a' \mu(\eta)(\log \eta)^{-2k/(k+1)} + C_a''(\log \eta)^{2/(k+1)} ,$$

we see that $\lim_{\eta\to\infty} \int_{\Omega_2} |\chi v_\eta|^2 dx = 0$. In view of $\Omega_a \setminus \Omega_b = \Omega_1 \cup \Omega_2$ we obtain (6.6). Q.E.D.

Proof of Theorem 7. Suppose that L is hypoelliptic in some neighborhood Ω of the origin in $\boldsymbol{R}^4$. It follows from the Banach closed graph theorem that for any integer $r>0$ and for any open sets $\omega \Subset \omega' \subset \Omega$ there exists an integer $r'>0$ and a constant C satisfying

$$\|D_4^r u\|_{L^2(\omega)} \leqq C\{ \sum_{|\alpha| \leqq r'} \|D^\alpha L u\|_{L^2(\omega')} + \|u\|_{L^2(\omega')}\} \qquad \text{for any } u \in C^\infty(\bar{\omega}'). \tag{6.11}$$

If $\omega_a = \{x \in \boldsymbol{R}^4 ;\ |x_j| < a\}$ for a sufficiently small $a>0$ and if

$$u_\eta(x) = \exp\{\sqrt{\mu(\eta)}\, x_3 + i\eta x_4\} v(x_1, x_2 ; \eta) .$$

for $v(x_1, x_2 ; \eta)$ in the above lemma, we have $Lu_\eta = 0$ in $\omega_{a/2}$. Substituting u_η into (6.11) with $\omega = \omega_{a/4} \cap \{x_3 > 0\}$ and $\omega' = \omega_{a/2}$, by means of (6.5) and (6.6) we have $0 < c_a \eta^r \leqq C' \eta^\rho$ with $\rho = C_1^{1/2} a/2$ if η is sufficiently large. If we choose $r \geqq \rho$ then the estimate is absurd for large η. The proof of Theorem 7 is completed. Q.E.D.

Remark. As stated in Introduction, the other hypothesis $0 < \lambda < \min(k+1, l+1)$ (resp. $0 < \kappa < 1$ under the condition $\sigma \geqq 1$) seems to be necessary because it is necessary for the operator frozen with respect to the variable $x_2 \neq 0$ (resp. $x_1 \neq 0$). In fact, for example, the operator frozen with respect to $x_2 \neq 0$ is equal to $D_1^2 + x_1^{2l} D_2^2 + x_1^{2k} D_3^2 + \exp(-1/|x_1|^\lambda) D_4^2$ after the change of the scale. We can construct the solution $u_\eta(x) = \exp(\sqrt{\mu(\eta)}\, x_j + i\eta x_4) v(x_1)$ contradictory to (6.11) by considering the eigenvalue problem

$$\begin{cases} \{D_1^2 + \exp(-1/|x_1|^{1/\lambda})\eta^2\} v = x_1^{2s} \mu(\eta) v & \text{in } (-a, a) , \\ v = 0 \quad \text{on } x_1 = \pm a , \end{cases}$$

where $s = \min(l, k)$ and $j=2$ or 3 according to $s=l$ or $=k$.

7. Proof of Theorem 8

In the proof of Theorem 8 we may assume that $f(0)=0$ by taking the change of varibles, otherwise,

$$x_j = x_j' \quad (j=1, 2), \qquad x_3 = x_3' + f(0) \int_0^{x_2'} g(t) dt .$$

We may also assume that α, f and g are bounded because our consideration is local. At first we shall prove the theorem in the case when α vanishes infinitely at the origin. Then we may assume that $\alpha\geqq 0$. As in Section 5 we set for a real η (not always positive)

$$Y_\eta=D_2+f(x_1)g(x_2)\eta\,,\qquad L_\eta=D_1^2+\alpha(x_1)^2Y_\eta^2\,.$$

Noting that for an integer $k>0$

$$\begin{aligned}(7.1)\qquad P_\eta^*P_\eta&\equiv(D_1+ix_1^k\alpha^2Y_\eta)(D_1-ix_1^k\alpha^2Y_\eta)\\&=D_1^2+x_1^{2k}\alpha^4Y_\eta^2+ix_1^k\alpha^2[Y_\eta,D_1]-(kx_1^{k-1}\alpha^2+2x_1^k\alpha\alpha')Y_\eta\,,\end{aligned}$$

for any compact set $K\subset \boldsymbol{R}^2$ we have

$$(\{\alpha^2(x_1^kf')g\eta\}v,v)\leqq C_K(L_\eta v,v)\,,\qquad v\in C_0^\infty(K)\,.$$

In fact, this follows from

$$|((kx_1^{k-1}\alpha^2+2x_1^k\alpha\alpha')Y_\eta v,v)|\leqq C_K(\|\alpha Y_\eta v\|^2+\|v\|^2)\,,\qquad v\in C_0^\infty(K)$$

and the Poincaré inequality

$$(7.2)\qquad \|v\|^2\leqq C_K\|D_1v\|^2\leqq C_K(L_\eta v,v)\,,\qquad v\in C_0^\infty(K)\,.$$

Here and in what follows we denote by C_K different constants depending on a fixed compact set K. If we also consider (7.1) with P_η replaced by P_η^* then we have with $k=1$ or 2

$$(7.3)\qquad (|\alpha^2x_1^kf'g\eta|v,v)\leqq C_K(L_\eta v,v)\,,\qquad v\in C_0^\infty(K)\,,$$

because $x_1^kf'(x_1)$ has the definite sign if we choose k even or odd, suitably. If follows from (7.3) that

$$(7.4)\qquad C_K(L_\eta v,v)\geqq\|D_1v\|^2+\|\alpha Y_\eta v\|^2+(|\alpha^2x_1^kf'g\eta|v,v)\,,\qquad v\in C_0^\infty(K)\,.$$

From now on, for the proof of the theorem we shall show that for any $s>0$ and any compact $K\subset\boldsymbol{R}^2$ the estimate

$$(7.5)\qquad \|\alpha(x_1)(\log|\eta|^s)v\|^2\leqq(L_\eta v,v)\qquad\text{for } v\in C_0^\infty(K)$$

holds if $|\eta|\geqq\eta(s,K)$ for a large $\eta(s,K)$ (cf., Lemma 5.1).

In order to make the idea clear, at first we shall prove (7.5) assuming $g(0)>0$. Since (26) still holds with f' replaced by $\alpha^2(t)t^kf'(t)$, in view of (7.4) the estimale (7.5) is a direct consequence of

Lemma 7.1 (cf., Proposition 3.1 of [4]). *Let $\alpha,\gamma\in C^\infty(\boldsymbol{R}^1)$ satisfy $\gamma(0)=0$ and*

$$(7.6)\qquad \alpha(t)>0,\qquad \gamma(t)>0,\qquad t\alpha'(t)\geqq 0\qquad \textit{if } t\neq 0\,.$$

Furthermore, assume that

$$(7.7)\qquad \lim_{t\to 0}|t\alpha(t)|\,|\log\gamma(t)|=0\,.$$

Then for any $s>0$ *there exists a* $\zeta_s>0$ *such that for any* $u\in C_0^\infty(\boldsymbol{R}^1)$ *with* $\operatorname{supp} u\subset \{|x|\leqq 1\}$ *we have*

$$(7.8)\qquad (\{D^2+\zeta^2\gamma(x)\}u,\ u)\geqq s(\alpha(x)^2(\log\zeta)^2u,\ u)\qquad \textit{if } \zeta\geqq\zeta_s\,.$$

Proof. Set $a(x, \xi)=\xi^2+V(x)$ with $V(x)=\zeta^2\gamma(x)$ and $W(x)=s\alpha(x)^2(\log\zeta)^2$ for $s>0$. The direct application of Theorem 1 does not work when α vanishes infinitely at $x=0$ (see Remark 1 below). We have to return to its proof. It follows from (7.7) that for any $s>0$ there exists a $\delta(s)>0$ such that

$$(7.9)\qquad 0\leqq -|x|\alpha(x)\log\gamma(x)<1/s\qquad \text{if } |x|<\delta(s)\,.$$

For the brevity we assume that $\alpha(x)$ is even function. Since $\alpha(x)$ is monotone in $[0, \infty)$, for any $\zeta>0$ there exists a unique positive root x_ζ such that

$$(7.10)\qquad s\alpha(x_\zeta)\log\zeta=x_\zeta^{-1}\,.$$

We may assume that x_ζ is smaller than $\delta(s)$ if ζ is sufficiently large. It follows from (7.9) that if $x_\zeta\leqq|x|<\delta(s)$ then

$$\begin{aligned}\gamma(x)\zeta&=\exp\{\log\zeta+\log\gamma(x)\}\\ &\geqq\exp\{\log\zeta-(s|x|\alpha(x))^{-1}\}\geqq 1\,.\end{aligned}$$

Since $\gamma(x)\geqq c_s>0$ on $\{\delta(s)\leqq|x|\leqq 1\}$, we see that

$$(7.11)\qquad \gamma(x)\zeta\geqq 1\qquad \text{on } \{x\in\boldsymbol{R}^1;\ x_\zeta\leqq|x|\leqq 1\},$$

if $\zeta\geqq\zeta_s$ for a sufficiently large ζ_s. Divide $J=[-1, -x_\zeta]\cup[x_\zeta, 1]$ into four congruent intevals J_k $(k=1, \cdots, 4)$ and divide each J_k into two congruent intervals. We repeat this cutting until the decomposition $J=\sum_\nu I_\nu$ satisfies

$$(7.12)\qquad \zeta^{1/2}\leqq(\operatorname{diam} I_\nu)^{-2}\,.$$

Then we have $\zeta^{1/2}\geqq(2\operatorname{diam} I_\nu)^{-2}$. If follows from (7.11) that

$$(7.13)\qquad V(x)\geqq\zeta\qquad \text{on } I_\nu \text{ if } \zeta \text{ is sufficiently large.}$$

If $K_0=[-x_\zeta, x_\zeta]$ and if $u\in C_0^\infty(\{|x|\leqq 1\})$ then we have

$$\begin{aligned}(7.14)\qquad 2(a(x, D)u, u)&\geqq\int_{K_0^*}|Du(x)|^2dx+\int_{K_0^*}V(x)|u(x)|^2dx\\ &\quad+\sum_\nu\int_{I_\nu}|Du(x)|^2dx+\int_{I_\nu}V(x)|u(x)|^2dx\\ &\equiv\Omega_0+\sum_\nu\Omega_\nu\,,\end{aligned}$$

where K_0^* is four times dilation of K_0. It follows from Lemma 1.1 that

$$\Omega_0 \geqq c\int_{K_0}\Big[\int_{K_0^*\setminus K_0}\{(\mathrm{diam}\ K_0)^{-2}|u(x)-u(y)|^2+V(y)|u(y)|^2\}dy\Big]/|K_0|\,dx$$

$$\geqq c' s\alpha(x_\zeta)^2(\log\zeta)^2\int_{K_0}|u(x)|^2dx$$

because of (7.10) and (7.13) with I_ν replaced by $K_0^*\setminus K_0$. By means of (7.12) and (7.13) and Lemma 1.1 we have

$$\Omega_\nu \geqq c''\zeta^{1/2}\int_{I_\nu}|u(x)|^2dx\,.$$

Summing up above two estimates, in view of (7.14) we get the desired estimate (7.8). Q. E. D.

Remark 1. We can apply Theorem 1 directly if the condition (7.7) is strengtheened to

(7.7)′ $$\lim_{t\to 0}|t\alpha(\lambda t)|\,|\log\gamma(t)|=0$$

with a sufficiently large $\lambda>1$ which depends on the modulus of the dilation B^{**} in the condition (4).

2. The lemma still holds with $\gamma(x)$ replaced by $\gamma(x)\sin^2 1/x$. In fact, since $\zeta^{1/2}\geqq(2\,\mathrm{diam}\,I_\nu)^{-2}$ we see that $\sin^2(1/x)\geqq C\zeta^{-1/2}$ on a half of I_ν. Consequently, it follows from (7.11) that

(7.13)′ $$m_1(\{x\in I_\nu;\ V(x)\geqq\zeta^{1/2}\})\geqq 1/2\,|I_\nu|$$

Using this instead of (7.13) we get the same conclusion.

In the case when $g(0)=0$, the estimate (7.5) is obtained from the following lemma because Y_η can be regard as if D_2, as stated in the proof of Theorem 5 (see (4.11) in Section 4).

Lemma 7.2. *Let α, γ be the same as in Lemma 7.1 and let $g(t)\in C^\infty(\boldsymbol{R}^1)$ satisfy (25), $g(0)=0$ and $g(t)>0$ if $t\neq 0$. If $V(x)=\zeta^4\gamma(x_1)g(x_2)$ and if $I_0=\{x\in\boldsymbol{R}^2;\ |x_j|\leqq 1\}$ then for any $s>0$ there exists a $\zeta_s>0$ such that for any $u\in C_0^\infty(I_0)$ we have*

(7.15) $$(\{D_1^2+\alpha(x_1)^2D_2^2+V(x)\}u,\ u)\geqq s(\alpha(x_1)^2(\log\zeta)^2u,\ u)\qquad \text{if } \zeta\geqq\zeta_s\,.$$

Proof. It follows from (25) that for any $s>0$ there exists a $\zeta_s>0$ such that if $\zeta\geqq\zeta_s$ then

(7.16) $$g(x_2)\zeta\geqq 1\qquad \text{on } \{(s\log\zeta)^{-1}\leqq|x_2|\leqq 1\}.$$

If x_ζ is the same as in the proof of Lemma 7.1 and if $y_\zeta=(s\log\zeta)^{-1}$ we set

$$\omega_1=\{x\in I_0;\ |x_1|<x_\zeta\}$$

and

$$\omega_2=\{x\in I_0;\ |x_2|<y_\zeta\}.$$

Then $I_0\setminus(\omega_1\cup\omega_2)$ is composed of four congruent rectangles. We divide each rectangle into four smaller congruent rectangles. We repeat this cutting procedure. Let $I_\nu=Q_1^\nu\times Q_2^\nu$ $(\subset \boldsymbol{R}_{x_1}\times\boldsymbol{R}_{x_2})$ denote one of congruent rectangles on some step (, that is, $I_0\setminus(\omega_1\cup\omega_2)=\bigcup_\nu I_\nu$). We repeat the cutting and stop it if I_ν satisfies

$$\zeta^{1/2}\leqq(\operatorname{diam} I_\nu)^{-2}\,. \tag{7.12$'$}$$

Then we have $\zeta^{1/2}\geqq(2\operatorname{diam} I_\nu)^{-2}$. Noting that $\operatorname{diam} I_\nu$ is equivalent to $\operatorname{diam} Q_j^\nu$ with $j=1, 2$, by means of (7.16) and (7.11) we have

$$V(x)\geqq\zeta^2 \quad \text{on } I_\nu \text{ if } \zeta \text{ is sufficiently large.} \tag{7.17}$$

We also divide $\bar{\omega}_1\setminus\omega_2$ (and $\bar{\omega}_2\setminus\omega_1$) into congruent smaller rectangles as follows:

$$\bar{\omega}_1\setminus\omega_2=\bigcup_{\nu'} J_{1\nu'}\,, \qquad J_{1\nu'}=[-x_\zeta, x_\zeta]\times Q_2^{\nu'}$$

$$\bar{\omega}_2\setminus\omega_1=\bigcup_{\nu''} J_{2\nu''}\,, \qquad J_{2\nu''}=Q_1^{\nu''}\times[-y_\zeta, y_\zeta]\,,$$

where the diameter of $Q_2^{\nu'}$ (resp. $Q_1^{\nu''}$) is equal to that of Q_2^ν (resp. Q_1^ν). Set $K_0=\omega_1\cap\omega_2$ and let K_0^* denote four times dilation of K_0. If $u\in C_0^\infty(I_0)$ then we have

$$\begin{aligned}
&4(\{D_1^2+\alpha(x_1)^2D_2^2+V(x)\}u, u)\\
&\quad\geqq\int_{K_0^*}\{|D_1u|^2+|\alpha(x_1)D_2u|^2+V(x)|u|^2\}dx\\
&\qquad+\sum_\nu\int_{I_\nu}\{\cdot\}dx+\sum_{\nu'}\int_{J_{1\nu'}^\dagger}\{\cdot\}dx+\sum_{\nu''}\int_{J_{2\nu''}^\dagger}\{\cdot\}dx\\
&\quad\equiv\Omega_0+\sum_\nu\Omega_\nu+\sum_{\nu'}\Omega_{\nu'}+\sum_{\nu''}\Omega_{\nu''}\,,
\end{aligned} \tag{7.18}$$

where $J_{1\nu'}^\dagger=[-2x_\zeta, 2x_\zeta]\times Q_2^{\nu'}$ and $J_{2\nu''}^\dagger=Q_1^{\nu''}\times[-2y_\zeta, 2y_\zeta]$. If follows from Lemma 1.1 and (2.17) of Lemma 2.1 that

$$\begin{aligned}
\Omega_0\geqq c\int_{K_0}\Big[\int_{K_0^*\setminus(\omega_1\cup\omega_2)}&\{x_\zeta^{-2}|u(x)-u(y_1, x_2)|^2\\
&+\alpha(y_1)^2y_\xi^{-2}|u(y_1, x_2)-u(y)|^2+V(y)|u(y)|^2\}dy\Big]/|K_0|dx\\
\geqq c's\alpha(x_\zeta)^2&(\log\zeta)^2\int_{K_0}|u(x)|^2dx
\end{aligned} \tag{7.19}$$

because of (7.10) and (7.17) with I_ν replaced by $K_0^*\setminus(\omega_1\cup\omega_2)$. Exchanging the order of D_1^2 and $\alpha^2D_2^2$ and noting that $(\operatorname{diam} Q_1^{\nu''})^{-2}\sim\zeta^{1/2}$ we also have

$$\begin{aligned}
\Omega_{\nu''}\geqq c\int_{J_{2\nu''}}\Big[\int_{J_{2\nu''}^\dagger\setminus\omega_2}&\{\alpha(x_1)^2y_\zeta^{-2}|u(x)-u(x_1, y_2)|^2\\
&+\zeta^{1/2}|u(x_1, y_2)-u(y)|^2+V(y)|u(y)|^2\}dy\Big]/|J_{2\nu''}|dx\\
\geqq c's(\log\zeta)^2&\int_{J_{2\nu''}}|\alpha(x_1)u(x)|^2dx\,.
\end{aligned} \tag{7.20}$$

Similarly we have

$$\Omega_{\nu'} \geqq c' s\alpha(x_\zeta)^2(\log \zeta)^2 \int_{J_{1\nu'}} |u(x)|^2 dx . \tag{7.21}$$

$$\Omega_\nu \geqq c'' \zeta^{1/2} \int_{I_\nu} |u(x)|^2 dx . \tag{7.22}$$

Summing up (7.19-22), in view of (7.18) we obtain the desired estimate (7.15). Q.E.D.

Let $\chi(t)$ be $C^\infty(\boldsymbol{R}^1)$ function such that $\operatorname{supp}\chi \subset \{|t| \geqq 1\}$. Then, by substituting $\chi(x_1/\delta)v$ into (7.5), in view of (7.2) we see that for any $\delta>0$, any $s>0$ and any compact $K \subset \boldsymbol{R}^2$ there exists a $\eta(\delta, s, K) \geqq 1$ such that

$$\|(\log|\eta|^s)\chi(x_1/\delta)v\|^2 \leqq (L_\eta v, v) \qquad \text{for} \quad v \in C_0^\infty(K), \tag{7.23}$$

provided that $|\eta| \geqq \eta(\delta, s, K)$ (cf., Lemma 5.2). We remark that if compact set $\tilde{K}$ of $\boldsymbol{R}^3$ is contained in $\{|x_1| \geqq \delta\}$ for a $\delta>0$, then for any $\varepsilon>0$ there exists a constant $C=C(\varepsilon, \tilde{K})$ such that

$$\|(\log \Lambda)u\|^2 \leqq \varepsilon(Lu, u) + C\|u\|^2, \qquad u \in C_0^\infty(\tilde{K}). \tag{7.24}$$

In fact, it follows from (7.23) that

$$\|(\log(|D_3|+1)u\|^2 \leqq \varepsilon(Lu, u) + C\|u\|^2, \qquad u \in C_0^\infty(\tilde{K}).$$

This yeilds (7.24) because we have with a $c_\delta>0$

$$\begin{aligned} 2(Lu, u) &\geqq \|D_1 u\|^2 + \|\alpha D_2 u\|^2 - (\sup|g|)^2 \|\alpha f D_3 u\|^2 \\ &\geqq \|D_1 u\|^2 + c_\delta \|D_2 u\|^2 - C'_{\tilde{K}} \|D_3 u\|^2, \qquad u \in C_0^\infty(\tilde{K}). \end{aligned} \tag{7.25}$$

The formula (21) in the region $\{|x_1| \neq 0\}$ is clear by means of (7.24) and Corollary 2 in [10].

To consider (21) in the region near $x_1=0$ we prepare the following:

Lemma 7.3. *Let $\tilde{\chi}(\xi) \in S^0_{1,0}$ satisfy $0 \leqq \tilde{\chi} \leqq 1$ and $\operatorname{supp}\tilde{\chi} \subset \{|\xi'| \geqq \delta_0|\xi_3|\}$ for a $\delta_0>0$, where $\xi'=(\xi_1, \xi_2)$. If K is a compact set in $\boldsymbol{R}^3$ and if $\delta>0$ is sufficiently small then there exists a C_K such that*

$$\|\alpha(x_1)|D|\tilde{\chi}(D)u\|^2 \leqq C_K(Lu, u) \tag{7.26}$$

for $u \in C_0^\infty(K)$ satisfying

$$\operatorname{supp} u \subset \{|x_1| \leqq 4\delta\}. \tag{7.27}$$

Proof. Let $\chi_2(\xi) \in S^0_{1,0}$ satisfy $0 \leqq \chi_2 \leqq 1$ and $\operatorname{supp}\chi_2 \subset \{|\xi_2| \geqq \delta_0|\xi_3|/2\}$. Since the first inequality of (7.25) holds for any $u \in \mathcal{S}$ and f vanishes infinitely at the origin, by substituting $\chi_2(D)h(x_1/4\delta)u$ into (7.25) we have

$$\begin{aligned} 2(Lu, u) \geqq & \|D_1\chi_2(D)u\|^2 + \|\alpha D_2\chi_2(D)u\|^2 \\ & - C\{\delta\|\alpha h(x_1/4\delta)D_3\chi_2(D)u\|^2 + \|u\|^2\}. \end{aligned}$$

Here $h(t)$ is the same as in Section 5. If $\delta>0$ is sufficiently smaller than δ_0 then we have

$$2(Lu,\ u)\geqq 1/2\{\|D_1\chi_2(D)u\|^2+\|\alpha D_2\chi_2(D)u\|^2\}-C'\|u\|^2$$

for $u\in\mathcal{S}$ satisfying (7.27). Since (5.1) still holds, from (7.28) we obtain the desired estimate (7.14) because of $\|D_1u\|^2\leqq(Lu,\ u)$. Q.E.D.

We shall prove that if $\rho_0=(0,\ (0,\ 0,\ \pm 1)$ and if $v\in\mathcal{E}'$ then

$$\rho_0\notin \mathrm{WF}\ Lv \quad \text{implies} \quad \rho_0\notin \mathrm{WF}\ v. \tag{7.29}$$

As in Section 5, for a sufficiently small $\delta>0$ we define $\varphi_\delta(x)$ and $\Psi_\delta(\xi)$ with $x\in\boldsymbol{R}^4$ and $\xi=(\xi',\ \xi_4)\in\boldsymbol{R}^4$ replaced by $x\in\boldsymbol{R}^3$ and $\xi=(\xi',\ \xi_3)\in\boldsymbol{R}^3$, respectively. Then the implication (7.29) is obvious, if we show Lemma 5.4 for the corresponding $\{\varphi_j\}$, $\{\Psi_j\}$ to those φ_δ, Ψ_δ.

We shall derive (5.16) in the present case, assuming $K=\{|x_j|\leqq 4\delta\}$. Recall (5.17), that is,

$$[L,\ \varphi_j(x)\Psi_j(D)]=[L,\ \varphi_j(x)]\Psi_j(D)+\varphi_j(x)[L,\ \Psi_j(D)]\,.$$

We see that

$$\begin{aligned}&\mathrm{Re}([\alpha^2(D_2+fgD_3)^2,\ \varphi_j(x)]u,\ \varphi_j(x)u)\\&\leqq(CN)^2\|\alpha u\|^2 \quad \text{for} \quad u\in\mathcal{S}.\end{aligned}$$

As in the proof of Lemma 5.4, for a moment we denote by the same notation C different constants independent of N, M and s. Therefore,

$$\begin{aligned}&(\log M^s)^2\,\mathrm{Re}([\alpha^2(D_2+fgD_3)^2,\ \varphi_j(x)]\Psi_j(D)u,\ \varphi_j(x)\Psi_j(D)u)\\&\leqq(CN)^2\{\|(\log M^s)\Psi_j(D)\alpha u\|^2+(\log M^s)^2\|[\alpha,\ \Psi_j(D)\|u\|^2\}\,.\end{aligned}$$

Using (7.5), for any $s>0$ we have

$$\begin{aligned}\|(\log M^s)\Psi_j(D)\alpha u\|^2&\leqq C\|(\log|D_3|^s)h((M^{-1}|D_3|-3)/2\delta)\alpha u\|^2\\&\leqq C(Lu,\ u) \quad \text{for} \quad u\in C_0^\infty(K),\end{aligned}$$

if $M\geqq M_s$ for a large $M_s>0$. Since (5.1) still holds (cf., (7.2)), by means of (5.13) we see that

$$\begin{aligned}&(\log M^s)^2\|[\alpha,\ \Psi_j(D)]u\|^2\\&\leqq(\log M^s)^4M^{-1}\{C_K(Lu,\ u)+C_sN^{2s+8}M^{-s}\|u\|^2\},\quad u\in C_0^\infty(K),\end{aligned}$$

if $\log M^s\geqq CN$. Therefore, if $\log M^s\geqq CN$ and M is sufficiently large such that $(\log M^s)^4M^{-1}\leqq 1$ then we have

$$\begin{aligned}&(\log M^s)^2\,\mathrm{Re}([\alpha^2(D_2+fgD_3)^2,\ \varphi_j]\Psi_ju,\ \varphi_j\Psi_ju)\\&\leqq(CN)^2\{(Lu,\ u)+C_sN^{2s+8}M^{-s}\|u\|^2\}\equiv\Omega,\quad u\in C_0^\infty(K).\end{aligned} \tag{7.30}$$

Note that

$$(\log M^s)^2\,\mathrm{Re}([D_1^2,\ \varphi_j(x)]\Psi_j(D)u,\ \varphi_j(x)\Psi_j(D)u)$$
$$\leqq(CN)^2(\log M^s)^2\|\chi(x_1/\delta)\Psi_j(D)u\|^2$$
$$\leqq(CN)^2\{\|(\log|D_3|^s)h((M^{-1}|D_3|-3)/2\delta)\chi(x_1/\delta)u\|^2$$
$$+(\log M^s)^2\|[\chi(x_1/\delta),\ \Psi_j(D)]u\|^2\},$$

where $\chi(t)$ is the same as in (7.23). Using (7.23) and (5.13) to estimate the first term and second one, respectively, we obtain

$$(7.31)\qquad (\log M^s)^2\,\mathrm{Re}([D_1^2,\ \varphi_j(x)]\Psi_j(D)u,\ \varphi_j(x)\Psi_j(D)u)\leqq\Omega,\quad u\in C_0^\infty(K),$$

if M satisfies the same condition as in (7.30). From (7.30) and (7.31) we obtain (5.23). On the other hand, since coefficients of L are independent of x_3, by noting the form of Ψ_j we see that

$$(7.32)\qquad (\log M^s)^2\,\mathrm{Re}(\varphi_j(x)[L,\ \Psi_j(D)]u,\ \varphi_j(x)\Psi_j(D)u)$$
$$\leqq C(\log M^s)^2\{N^4\|\alpha\chi_0(D)u\|^2+N^3(|\alpha\alpha'|\chi_0(D)u,\ \chi_0(D)u)$$
$$+N^2((\alpha|\alpha''|+\alpha'^2)\chi_0(D)u,\ \chi_0(D)u)$$
$$+N^6M^{-1}\|u\|^2+C_sN^{2s+10}M^{-s-1}\|u\|^2\},$$

where $\chi_0\in S_{1,0}^0$ satisfies

$$\mathrm{supp}\,\chi_0\subset\{2\delta|\xi_3|\geqq|\xi'|\geqq\delta|\xi_3|\}\cap\{2\leqq|\xi_3|/M\leqq 4\}.$$

Note that the assumption $\alpha\geqq 0$ implies $|\alpha'|\leqq C\sqrt{\alpha}$ and that $(\sqrt{\alpha}N)^3\leqq\alpha N^2+(\alpha N^2)^2$. If $\log M^s\geqq CN$ then it follows from (7.32) that

$$(7.33)\qquad (\log M^s)^2\,\mathrm{Re}(\varphi_j(x)[L,\ \Psi_j(D)]u,\ \varphi_j(x)\Psi_j(D)u)$$
$$\leqq CN^2\{(\log M^s)^4\|\alpha\chi_0(D)u\|^2$$
$$+(1+(\log M^s)^4M^{-1})\|u\|^2+C_sN^{2s+10}M^{-s-1}\|u\|^2\},$$

because we have

$$(\alpha\chi_0u,\ \chi_0u)\leqq(\log M^s)^2\|\alpha\chi_0u\|^2+(\log M^s)^{-2}\|u\|^2.$$

By means of Lemma 7.3 and (5.1) we have

$$(7.34)\qquad \|\chi_0\alpha u\|^2\leqq CM^{-2}\||D|\chi_0\alpha u\|^2\leqq CM^{-2}(Lu,\ u)\qquad \text{for}\quad u\in C_0^\infty(K).$$

Using this to estimate the first term of the right hand side of (7.33) we get (5.26) if $\log M^s\geqq CN$ and M is sufficiently large such that $(\log M^s)^4M^{-1}\leqq 1$. Since (5.23) and (5.26) still holds we obtain (5.16). Therefore, we get (7.29) if $\rho_0=(0,\ (0,\ 0,\ \pm 1))$.

The implication (7.29) for $\rho_0=((0,\ x_{02},\ x_{03}),\ (0,\ 0,\ \pm 1))$ with $(x_{02},\ x_{03})\neq(0,\ 0)$ is obvious. In fact, Lemma 5.4 still holds for $\varphi_j(x)$ corresponding to $\tilde\varphi_\delta(x)=\prod_{j=1}^{3}h((x_j-x_{0j})/\delta)$, where $x_{01}=0$. In view of Lemma 7.3, the preceding argument also yields (7.29) for $\rho_0=(x_0,\ \xi_0)$ with $\xi_0\neq(0,\ 0,\ \pm 1)$ if we modify $\Psi_\delta(\xi)$ to correspond to the direction ξ_0. Thus the proof of Theorem 8 is accomplished when $\alpha(x_1)$ vanishes infinitely at the origin.

In the finite vanishing case, the above arguments can be carried out until (7.32) (without serious change). Instead of (7.32), we employ

$$(7.32)' \qquad (\log M^s)^2 \operatorname{Re}(\varphi_j(x)[L, \Psi_j(D)]u, \varphi_j(x)\Psi_j(D)u)$$
$$\leqq C(\log M^s)^2 N^4\{\|\chi_0(D)u\|^2 + C_s N^{2s+10} M^{-s-1}\|u\|^2\}.$$

If α vanishes of order l at $x_1=0$, by the well-knowun Hörmander theorem we have

$$\||D'|^{1/(l+1)}u\|^2 \leqq C_K(\|D_1 u\|^2 + \|\alpha(x_1)D_2 u\|^2) \qquad \text{for} \quad u\in C_0^\infty(K).$$

This and (7.26) give

$$(7.24)' \qquad \|\chi_0(D)u\|^2 \leqq CM^{-2/(l+1)}\||D'|^{1(l+1)}\chi_0(D)u\|^2$$
$$\leqq CM^{-2/(l+1)}(Lu, u) \qquad \text{for} \quad u\in C_0^\infty(K).$$

By (7.32)′ and (7.34)′ we get (5.26) and hence (5.16) in the finite vanishing case. The rest of the proof is the same as in the infinite vanishing case. Now the proof of Theorem 8 is completed.

To end this paper we state a conjecture about the assumptions (25) and (26). That is, (25) and (26) seem to be close to necessary under the additional condition that f' and g are monotone in $(-\infty, 0]$ and $[0, \infty)$. For instance, as for (26) we consider a little weaker condition as follows: For a positive $\kappa<1$ we have

$$(26)' \qquad \lim_{t\to 0} t\alpha(\kappa t)\log|f'(t)| = 0.$$

Suppose that (26)′ does not hold. Then, without loss of generality we may assume that there exist $\varepsilon_0>0$ and a sequence of positive numbers $1>t_1>t_2>\cdots>t_j\to 0$ such that

$$(7.35) \qquad |f'(t_j)| \leqq \exp\{-\varepsilon_0/t_j|\alpha(\kappa t_j)|\} \qquad (\text{cf., (1.5) of [4]}).$$

If we take the change of variables $x_j=y_j$ $(j=1, 2)$ and $x_3=y_3+f(y_1)\int_0^{y_2} g(t)dt$ then the operator L of Theorem 8 becomes

$$(7.36) \qquad \alpha(x_1)^2 D_2^2 + (D_1 - f'(x_1)\int_0^{x_2} g(t)dtD_3)^2,$$

where x denotes the new variables instead of y. Let ζ_j be a positive such that

$$(7.37) \qquad t_j|\alpha(\kappa t_j)|\log\zeta_j = \varepsilon_0.$$

Then ζ_j tends to ∞ as $j\to\infty$. For each ζ_j we consider a small box in $T^*(\boldsymbol{R}_{x_1}\times\boldsymbol{R}_{x_3})$

$$B_j = \{\kappa t_j \leqq x_1 \leqq t_j,\ |x_3|\leqq 1/2,\ |\xi_1|\leqq 1/2(1-\kappa)t_j,\ |\xi_3-\zeta_j|\leqq 1/2\}.$$

Since f' and α are monotone in $[0, \infty)$, it follows from (7.35) and (7.37) that on $\tilde{B}_j \equiv \{x_2;\ |x_2|\leqq 1\}\times B_j$ we have

$$(7.38) \qquad |\xi_1 - f'(x_1)\int_0^{x_2} g(t)dt\xi_3|/|\alpha(x_1)|$$
$$\leqq \{t_j^{-1} + C|f'(t_j)|\zeta_j\}/|\alpha(\kappa t_j)|$$
$$\leqq C'\log\zeta_j.$$

In view of (7.38), the operator L of the form (7.36) might be seen "hyperbolic" with respect to D_2 on $\tilde{B}_j$ in a certain microlocal sense (see also Introduction of [12]). We might expect the propagation of wave front set along the null-bicharateristic curve of D_2 passing $(0, (0, 0, \zeta_j)) \in T^*(\boldsymbol{R}^3)$, and hence L might be not hypoelliptic in a neighborhood of the origin. The similar consideration can be done to the assumption (25) without the change of variables.

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