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On evaluation of L-functions over real quadratic fields

By

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§0. Introduction

Our purpose is to give an effective algorithm to compute special values of L-functions over real quadratic fields and relative class numbers of totally imaginary quadratic extensions of real quadratic fields.

In his paper [3], T. Shintani developped a method for evaluating L-functions over totally real algebraic number fields at non-positive integers and wrote a class number formula for totally imaginary quadratic extensions of totally real algebraic number fields. His method is based on evaluation of certain kind of partial zeta functions. He also described the detail of evaluation of such partial zeta functions over real quadratic fields.

But no method for computing the values of characters at ideals was explaind. Moreover his class number formula involves a group index of form $[E_F : N_{K/F}E_K]$ (see §1 for the definition) which has not been determined.

In this paper, we restrict ourselves to the case of L-functions associated with quadratic characters and give a complete algorithm for computing special values of L-functions associated with arbitrary quadratic extensions over real quadratic fields by filling those two missing details. We describe an algorithm for computing the values of quadratic characters at ideals and the unit indices $[E_F : N_{K/F}E_K]$. The computation of characters at ideals are reduced to that at quadratic integers. The value of the characters at quadratic integers are written by the Legendre symbols and the Hilbert symbols over real quadratic fields and the Hilbert symbols are determined by propisition 4 in §2. The unit indices are written by the Hasse's unit indices which are determined by proposition 14 in §3.

After the above two jobs are done, we give some tables of special values and relative class numbers. For this purpose, we give an algorithm for enumurating quadratic extensions of real quadratic fields in §4. The tables are given in §5.

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§1. Preliminaries

We review some results of [3] in the case of real quadratic fields with modifications. The following notations are used throughout the paper except in the argument on the

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Hasses's unit index in §3. Let $F=\boldsymbol{Q}(\sqrt{d})$ be a real quadratic field and d its discriminant, $o=o_F$ its ring of integers, $E=E_F$ the group of units in F, E_+ the group of totally positive units in F. We denote the non-trivial conjugation of F by $\alpha\to\alpha'$, the trace by $\mathrm{tr}=\mathrm{tr}_{F/Q}$ and the norm of ideals or numbers by $N=N_{F/Q}$. We write $\alpha\gg 0$ when α is totally positive, and $\alpha\ll 0$ when α is totally negative. Let ε be the fundamental unit of F which is greater than 1 and set

$$\varepsilon_+=\begin{cases}\varepsilon & \text{if } \varepsilon\gg 0,\\ \varepsilon^2 & \text{otherwise.}\end{cases}$$

Let h be the class number of F, h^+ the narrow ideal class number of F and $\boldsymbol{a}_1,\cdots,\boldsymbol{a}_{h^+}$ a system of representatives for narrow ideal classes consisting of integral ideals. Let χ be an ideal character of conductor ϑ. The L-function $L_F(s,\chi)$ associated with χ is defined by

$$L_F(s,\chi)=\sum_{\substack{\boldsymbol{a}:\text{ integral-}\\ \text{ideal of } F}}\chi(\boldsymbol{a})N(\boldsymbol{a})^{-s}.$$

In particular,

$$\zeta_F(s)=L_F(s,1)$$

is the zeta function of F.

T. Shintani has obtained the following formula.

Theorem 1 (T. Shintani). *The special values of the L-function at non-positive integers are evaluated by the following formula:*

$$L_F(1-m,\chi)=\frac{1}{m^2}\sum_{\mu=1}^{h^+}N(\boldsymbol{a}_\mu\vartheta)^{m-1}\sum_{j=1}^{2}(-1)^{r_j}\sum_{\boldsymbol{x}\in R_j((\boldsymbol{a}_\mu\vartheta)^{-1})}\chi((\boldsymbol{x}\boldsymbol{v}_j)\boldsymbol{a}_\mu\vartheta)B_m(A_j,\boldsymbol{x}),$$

where

$$r_1=2,\qquad r_2=1,$$

$$A_1=(\boldsymbol{v}_1,\boldsymbol{v}_1')=\begin{pmatrix}1 & 1\\ \varepsilon_+ & \varepsilon_+'\end{pmatrix},$$

$$A_2=(\boldsymbol{v}_2,\boldsymbol{v}_2')=(1,1),$$

$$R_j(S)=\{\boldsymbol{x}\in(\boldsymbol{Q}\cap(0,1])^{r_j}\qquad(\text{row vectors})\,|\,\boldsymbol{x}\boldsymbol{v}_j\in S\},$$

and $B_m(A_j,*)$ are generalizations of Bernoulli polynomials which are written as

$$B_m(A_1,(x,y))=\frac{1}{2}\binom{2m}{m}^{-1}\sum_{j=0}^{m-1}\binom{2m-1}{m-j-1}(-1)^j\,\mathrm{tr}\,(\varepsilon_+^{2j+1})(B_{2m}(x)+B_{2m}(y))$$

$$+\frac{1}{2}\binom{2m}{m}^{-1}\sum_{i=1}^{2m-1}\binom{2m}{i}\sum_{j=0}^{i-1}\binom{i-1}{j}\binom{2m-i-1}{m-j-1}\mathrm{tr}\,(\varepsilon_+^{i-2j-1})B_i(x)B_{2m-i}(y),$$

$$B_m(A_2,x)=\frac{m^2}{2m-1}B_{2m-1}(x).$$

This formula is slightly modified from (2.1) in [3] (see also theorem 1 and 2 of that paper). The modification is in the formula for $B_m(A_1,*)$ which is proved by changes of variables in the second summation.

We note that the ideal $(\boldsymbol{x}\boldsymbol{v}_j)\boldsymbol{a}_\mu\vartheta$ in this theorem is an integral ideal. Thus, this theorem gives an effective method for evaluating $L_F(1-m, \chi)$ when the sets $R_j((\boldsymbol{a}_\mu\vartheta)^{-1})$ are given extensionally and an effective method for computing the values of the character χ at integral ideals is given. The sets $R((\boldsymbol{a}_\mu\vartheta)^{-1})$ are given by proposition 2 below. We treat the zeta function and L-functions associated with quadratic characters over F in this paper. The values of quadratic characters χ are discussed in § 2.

Let $\boldsymbol{a}_\mu\vartheta=D_\mu[q_\mu,(\rho_\mu+\sqrt{d})/2]$ with $D_\mu, q_\mu, \rho_\mu\in\boldsymbol{Z}$ and let $[x, y]$ denote the $\boldsymbol{Z}$-module generated by x, y. Then

$$(\boldsymbol{a}_\mu\vartheta)^{-1}=\frac{1}{D_\mu q_\mu}\left[q_\mu, \frac{-\rho_\mu+\sqrt{d}}{2}\right]=\frac{1}{D_\mu}[1, \omega_\mu]$$

with

$$\omega_\mu=\frac{-\rho_\mu+\sqrt{d}}{2q_\mu}.$$

Hence,

$$R_2((\boldsymbol{a}_\mu\vartheta)^{-1})=\left\{\frac{1}{D_\mu}, \frac{2}{D_\mu}, \cdots, \frac{D_\mu}{D_\mu}\right\}.$$

Write

$$\varepsilon_+=\frac{A+B\sqrt{d}}{2} \qquad (A, B\in\boldsymbol{Z})$$

and put

$$c_\mu=Bq_\mu, \qquad a_\mu=\frac{A+B\rho_\mu}{2}.$$

Then

$$\varepsilon_+=a_\mu+c_\mu\omega_\mu.$$

Hence

$$(\boldsymbol{a}_\mu\vartheta)^{-1} \bmod [1, \varepsilon_+]=\left\{\frac{1}{D_\mu}i+\frac{\omega_\mu}{D_\mu}j \,\middle|\, i=1, 2, \cdots, D_\mu;\ j=1, 2, D_\mu c_\mu\right\}.$$

Therefore we obtain:

Proposition 2. *Notations being as above, the sets $R_j((\boldsymbol{a}_\mu\vartheta)^{-1})$ are given explicity by*

$$R_2((\boldsymbol{a}_\mu\vartheta)^{-1})=\left\{\frac{1}{D_\mu}, \frac{2}{D_\mu}, \cdots, \frac{D_\mu}{D_\mu}\right\},$$

$$R_1((\boldsymbol{a}_\mu\vartheta)^{-1})=\left\{\left(\left\langle\frac{ic_\mu-a_\mu j}{D_\mu c_\mu}\right\rangle, \left\langle\frac{j}{D_\mu c_\mu}\right\rangle\right) \,\middle|\, i=1, 2, \cdots, D_\mu;\ j=1, 2, \cdots, D_\mu c_\mu\right\}.$$

where $\langle x\rangle\in(0, 1]$ and $x-\langle x\rangle\in\boldsymbol{Z}$.

We give some remarks on the values $L_F(1-m, \chi)$. We denote the infinite primes of F by ∞, ∞'. Let k be a non negative rational integer. We see

$$L_F(-2k, \chi)=0 \Longleftrightarrow \chi \text{ is unramified at } \infty \text{ or } \infty',$$

$$L_F(1-2k, \chi)=0 \Longleftrightarrow \chi \text{ is ramified at } \infty \text{ or } \infty',$$

from the functional equation. Hence we have

$$\begin{cases} L_F(-2k, \chi)=0, \ L_F(1-2k, \chi)=0 \text{ if } \chi \text{ is ramified at only one of } \infty \text{ and } \infty', \\ L_F(-2k, \chi)\neq 0, \ L_F(1-2k, \chi)=0 \text{ if } \chi \text{ is ramified at both } \infty \text{ and } \infty', \\ L_F(-2k, \chi)=0, \ L_F(1-2k, \chi)\neq 0 \text{ if } \chi \text{ is unramified at both } \infty \text{ and } \infty'. \end{cases}$$

When χ is associated with a totally imaginary quadratic extension $K=F(\sqrt{\Delta})$, the class number H of K is written as

$$\frac{H}{h}=\frac{w_K}{[E_F : N_{K/F}E_K]}L_F(0, \chi)$$

where w_K is the number of roots of unity in K and $N_{K/F}E_K$ is the group of norms of units in K. The constants w_K and $[E_F : N_{K/F}E_K]$ are discusses in §3.

After quadratic characters χ, constants w_K and unit indiced $[E_F : N_{K/F}E_K]$ are determined effectively, we can compute the values of $L_F(1-m, \chi)$ and the relative class numbers H/h. We give some tables of the values of $L_F(1-m, \chi)$ and the relative class numbers H/h in §5.

§2. Computation of the character associated with a quadratic extension

In this section, we give a method to compute the values of quadratic characters at integral ideals. We determine the character $\chi=\chi_\Delta$ associated with arbitrary quardratic extension $K=F(\sqrt{\Delta})$ of F with an algebraic integer $\Delta\in F$. We determine the value of the character in the following way. The computation of the value of the character at an ideal is reduced to that at a quadratic integer. The character at the integer is factored to the infinite part and the $\boldsymbol{q}$-parts with prime ideals $\boldsymbol{q}$ dividing the conductor of χ. The infinite part is written by the sign of the integer. A prime ideal is called even when it divides (2) or called odd when it does not divide (2). Odd prime ideals $\boldsymbol{q}$ dividing the conductor are easily deterimined and the $\boldsymbol{q}$-parts of χ are written by the Legendre symbols. Even prime ideals $\boldsymbol{q}$ dividing the conductor and their indices in the conductor are determined by proposition 3 and the $\boldsymbol{q}$-parts of χ are determined by the Hilbert symbols. We sum up the result of this section in theorem 6.

Let ϑ be the conductor of the character χ. Let $\boldsymbol{a}$ be an integral ideal in F. Then there is an odd prime ideal $\boldsymbol{p}$ prime to (Δ) such that $\boldsymbol{ap}$ is a principal ideal. This prime ideal $\boldsymbol{p}$ is also prime to the conductor ϑ since an odd prime ideal divides ϑ if and only if its index in Δ is odd. Let α be a generator of $\boldsymbol{ap}$. Then $\chi(\boldsymbol{a})$ is decomposed as

$$\chi(\boldsymbol{a})=\chi((\alpha))\chi(\boldsymbol{p})^{-1}.$$

The value $\chi(\boldsymbol{p})$ is given by the Legendre symbol over the quadratic field F, that is,

$$\chi(\boldsymbol{p})=\left(\frac{\Delta}{\boldsymbol{p}}\right)_F.$$

Here the Legendre symbol is the unique character of order 2 modulo $\boldsymbol{p}$ and is written by the usual Legendre symbol

$$(2.1)\qquad \left(\frac{\Delta}{\boldsymbol{p}}\right)_F=\begin{cases}\left(\dfrac{N\Delta}{p}\right) & \text{if } N(\boldsymbol{p})=p^2,\\[2ex] \left(\dfrac{\delta}{p}\right) & \text{with } \delta\equiv\Delta \bmod \boldsymbol{p},\ \delta\in\boldsymbol{Z} \text{ if } N(\boldsymbol{p})=p.\end{cases}$$

where p is the prime number in $\boldsymbol{p}$. The value $\chi((\alpha))$ factors as

$$\chi_\Delta((\alpha))=\chi_{\Delta,\infty}(\alpha)\chi_{\Delta,\infty'}(\alpha)\prod_{q|\vartheta}\chi_{\Delta,\boldsymbol{q}}(\alpha)$$

according to the factorization of the conductor. We omit the subscript Δ in the followings. χ_∞, $\chi_{\infty'}$ are the infinite parts of χ and $\chi_{\boldsymbol{q}}$ are the $\boldsymbol{q}$-parts of χ. The infinite parts of χ are given by

$$(2.2)\qquad \begin{aligned}\chi_\infty(\alpha)&=\begin{cases}1 & \text{if } \Delta>0,\\ \operatorname{sgn}(\alpha) & \text{if } \Delta<0,\end{cases}\\ \chi_{\infty'}(\alpha)&=\begin{cases}1 & \text{if } \Delta'>0,\\ \operatorname{sgn}(\alpha') & \text{if } \Delta'<0.\end{cases}\end{aligned}$$

An odd prime ideal $\boldsymbol{q}$ divides ϑ if and only if its index in Δ is odd. The $\boldsymbol{q}$-part of χ is given by

$$\chi_{\boldsymbol{q}}(\alpha)=\left(\frac{\alpha}{\boldsymbol{q}}\right)_F,$$

When $\boldsymbol{q}$ divides (2), the $\boldsymbol{q}$-component of the conductor is given by the following proposition (porbably due to D. Hibert).

Proposition 3. *Let F be an algebraic number field, $\boldsymbol{q}$ an even prime ideal and e the ramification index of $\boldsymbol{q}$ in $F/\boldsymbol{Q}$. Let ϑ be the conductor of $F(\sqrt{\Delta})/F$ for an integer $\Delta\in o_F$ which is not a square in F. Assume $\boldsymbol{q}^2$ does not divide Δ. And let f be the largest integer $f\leqq 2e+1$ such that $\gamma^2\equiv\Delta \pmod{\boldsymbol{q}^f}$ has a solution. Then, the behavior of $\boldsymbol{q}$ in K/F is as follows:*

$$\begin{cases}\boldsymbol{q}\text{ splits} & \text{if } f>2e,\\ \boldsymbol{q}\text{ remains} & \text{if } f=2e,\\ \boldsymbol{q}\text{ ramifies} & \text{if } f<2e.\end{cases}$$

In particular,

$$\begin{cases}\boldsymbol{q}^{2e+1}\,\|\,\vartheta & \text{if } \boldsymbol{q}\mid\Delta,\\ \boldsymbol{q}^{2e-[f/2]}\,\|\,\vartheta & \text{if } \boldsymbol{q}\nmid\Delta.\end{cases}$$

where $[f/2]$ is the Gauss symbol.

Proof. Let $F_{\boldsymbol{q}}$ be the completetion of F at $\boldsymbol{q}$, π a generator of $\boldsymbol{q}$. Then for every integer α in $F_{\boldsymbol{q}}$, we have

$$(1+2\pi\alpha)^2\equiv 1+4\pi\alpha \qquad (\bmod\ \boldsymbol{q}^{2e+2}).$$

Thus

$$1+\boldsymbol{q}^{2e+1}\subset(F_{\boldsymbol{q}}^\times)^2.$$

Therefore $\boldsymbol{q}$ splits if $f>2e$ and $\sqrt{\Delta}$ generates a quadratic extension over F_q if $f\leqq 2e$. Assume $f\leqq 2e$. Every integer in $F_q(\sqrt{\Delta})$ is written as

$$\frac{\alpha+\beta\sqrt{\Delta}}{2}$$

with integers α, β in F_q. Let e_1 qe the least index such that

$$\frac{\alpha+\pi^{e_1}\sqrt{\Delta}}{2}$$

is an integer for some α. Then $e_1=e-[f/2]$. Firstly $e_1\leqq e-[f/2]$ since

$$\frac{\gamma\pi^{e-[f/2]}+\pi^{e-[f/2]}\sqrt{\Delta}}{2}$$

is an integer in $F_q(\sqrt{\Delta})$ for a solution γ of $\gamma^2\equiv\Delta \pmod{\boldsymbol{q}^f}$. Conversely, if

$$\frac{\alpha+\pi^{e_1}\sqrt{\Delta}}{2}$$

is an integer, $\alpha^2\equiv\pi^{2e_1} \pmod 4$. Since $e_1\leqq e$, $\gamma=\alpha\pi^{-e_1}$ is an integer. This integer satisfies, $\gamma^2\equiv\Delta \pmod{\boldsymbol{q}^{2e-2e_1}}$. From the definition of f, we have $2e-2e_1\leqq f$. Hence $e_1\geqq e-[f/2]$. We have proved $e_1=e-[f/2]$. The different of $F_q(\sqrt{\Delta})/F_q$ is $\boldsymbol{q}^{e_1}(\sqrt{\Delta})=\boldsymbol{q}^{e-[f/2]}(\sqrt{\Delta})$. Therefore the discriminant ϑ_q of $F_q(\sqrt{\Delta})/F_q$ is

$$\vartheta_q=\boldsymbol{q}^{2e-2[f/2]}(\Delta).$$

If $f=2e$, there is a solution for $\gamma^2\equiv\Delta \pmod{\boldsymbol{q}^2}$. From the assumption on Δ, we see that $\boldsymbol{q}\nmid\Delta$. Thus $\vartheta_q=(1)$ and $\boldsymbol{q}$ remains prime. If $f<2e$, clearly $\boldsymbol{q}\mid\vartheta_q$ and $\boldsymbol{q}$ ramifies. This proves the first assertion.

The second assertion in the case of $\boldsymbol{q}\nmid\Delta$ is already shown. If $\boldsymbol{q}\mid\Delta$, the congruence $\gamma^2\equiv\Delta \pmod{\boldsymbol{q}^2}$ has no solution. Hence $f<2$. The discriminant ϑ_q is $\boldsymbol{q}^{2e}(\Delta)=\boldsymbol{q}^{2e+1}$. This completes the proof. ∎

The characters χ_q for $\boldsymbol{q}$ dividing (2) are computed by the Hilbert symbol $(\alpha,\Delta)_{F_q}$ in the completion F_q of F at $\boldsymbol{q}$ (see [2] for the definition and the properties of the Hilbert symbol). That is,

$$\chi_q(\alpha)=\begin{cases}(\alpha,\Delta)_{F_q} & \text{if } \boldsymbol{q}\nmid\alpha,\\ 0 & \text{if } \boldsymbol{q}\mid\alpha.\end{cases} \tag{2.3}$$

when $\boldsymbol{q}$ is ramified in K/F and

$$\chi_q(\alpha)=1$$

when $\boldsymbol{q}$ is unramified in K/F. By straightforward calculations, we obtain:

Proposition 4. *The Hilbert symbol* $(\alpha, \beta)_{F_q}$ *is determined by following tables.*

(i) $d\equiv 1 \pmod 8$

$F_q/(F_q^\times)^2$ *is generated by* 5, -1, 2.

	5	-1	2
5	$+1$	$+1$	-1
-1	$+1$	-1	$+1$
2	-1	$+1$	$+1$

(ii) $d\equiv 5 \pmod 8$

$F_q/(F_q^\times)^2$ *is generated by* $3+2\sqrt{d}$, -1, $\frac{(d-7)/2+\sqrt{d}}{2}$, 2.

	$3+2\sqrt{d}$	-1	$\frac{(d-7)/2+\sqrt{d}}{2}$	2
$3+2\sqrt{d}$	$+1$	$+1$	$+1$	-1
-1	$+1$	$+1$	-1	$+1$
$\frac{(d-7)/2+\sqrt{d}}{2}$	$+1$	-1	-1	$+1$
2	-1	$+1$	$+1$	$+1$

(iii) $d=4d_1$, $d_0\equiv 2 \pmod 4$

$F_q/(F_q^\times)^2$ *is generated by* 5, -1, $1-\sqrt{d_0}$, π *with*

$$\pi=\begin{cases} 2+\sqrt{d_0} & \text{if } d_0\equiv 2 \pmod 8, \\ \sqrt{d_0} & \text{if } d_0\equiv 6 \pmod 8). \end{cases}$$

	5	-1	$1-\sqrt{d_0}$	π
5	$+1$	$+1$	$+1$	-1
-1	$+1$	$+1$	-1	$+1$
$1-\sqrt{d_0}$	$+1$	-1	-1	$+1$
π	-1	$+1$	$+1$	$+1$

(iv) $d=4d_0$, $d_0\equiv 3 \pmod 4$

$F_q/(F_q^\times)^2$ *is generated by* 5, $2+\sqrt{d_0}$, $7+2\sqrt{d_0}$, $1+\sqrt{d_0}$.

	5	$2+\sqrt{d_0}$	$7+2\sqrt{d_0}$	$1+\sqrt{d_0}$
5	$+1$	$+1$	$+1$	-1
$2+\sqrt{d_0}$	$+1$	$+1$	-1	$+1$
$7+2\sqrt{d_0}$	$+1$	-1	$+1$	$+1$
$1+\sqrt{d_0}$	-1	$+1$	$+1$	-1

(v) $d=4d_0$, $d_0\equiv 7 \pmod 4$

$F_q/(F_q^\times)^2$ is generated by 5, $2+\sqrt{d_0}$, $7+2\sqrt{d_0}$, $1+\sqrt{d_0}$.

	5	$2+\sqrt{d_0}$	$7+2\sqrt{d_0}$	$1+\sqrt{d_0}$
5	$+1$	$+1$	$+1$	-1
$2+\sqrt{d_0}$	$+1$	$+1$	-1	$+1$
$7+2\sqrt{d_0}$	$+1$	-1	$+1$	$+1$
$1+\sqrt{d_0}$	-1	$+1$	$+1$	$+1$

Remark 5. When $N\varepsilon=-1$, the character χ_q for $q|(2)$ can be determined by examining its values at units. For $\eta=\pm 1, \pm\varepsilon$, let $\chi_{\eta\Delta,\mathrm{inf}}=\chi_{\eta\Delta,\infty}\chi_{\eta\Delta,\infty'}$, $\chi_{\eta\Delta,\mathrm{odd}}=\prod_{q|\vartheta, q\nmid(2)}\chi_{\eta\Delta,q}$ and $\chi_{\eta\Delta,\mathrm{even}}=\prod_{q|\vartheta, q|(2)}\chi_{\eta\Delta,q}$. Then $\chi_{\eta\Delta}((\alpha))=\chi_{\eta\Delta,\mathrm{inf}}(\alpha)\chi_{\eta\Delta,\mathrm{odd}}(\alpha)\chi_{\eta\Delta,\mathrm{even}}(\alpha)$ for $\alpha\in o_F$. The four characters $\chi_{\eta\Delta,\mathrm{odd}}$ are the same and the four pairs $(\chi_{\eta\Delta,\mathrm{inf}}(-1), \chi_{\eta\Delta,\mathrm{inf}}(\varepsilon))$ are distinct. On the other hand, the ideal factorization of $(\Delta)=(\eta\Delta)$ determines the conductor of $\chi_{\eta\Delta,\mathrm{even}}$ up to a square. And there are exactly 4 characters with such conductors. Therefore the system of two equations

$$\begin{cases} 1=\chi_{\eta\Delta,\mathrm{inf}}(-1)\chi_{\eta\Delta,\mathrm{odd}}(-1)\chi_{\eta\Delta,\mathrm{even}}(-1), \\ 1=\chi_{\eta\Delta,\mathrm{inf}}(\varepsilon)\chi_{\eta\Delta,\mathrm{odd}}(\varepsilon)\chi_{\eta\Delta,\mathrm{even}}(\varepsilon), \end{cases}$$

determines $\chi_{\eta\Delta,\mathrm{even}}$.

The conductor of the character and the value of the character at an ideal is determined by the following theorem.

Theorem 6. *Let χ be the character associated with a quadratic extension $K=F(\sqrt{\Delta})$ and assume Δ is an integer in F. Then the conductor ϑ is written as $\vartheta=\vartheta_{\mathrm{odd}}\vartheta_{\mathrm{even}}$ where ϑ_{odd} is the product of all odd prime ideals which have odd index in Δ and $\vartheta_{\mathrm{even}}$ is the product of powers of even prime ideals determined by proposition* 3. *Moreover when $\boldsymbol{a}$ is a given integral ideal, one can choose an odd prime ideal $\boldsymbol{p}$ such that $\boldsymbol{ap}$ is a principal ideal. Let α be a generator of $\boldsymbol{ap}$. Then the value $\chi(\boldsymbol{a})$ is written as follows:*

$$\chi(\boldsymbol{a})=\left(\frac{\Delta}{\boldsymbol{p}}\right)_F\chi_\infty(\alpha)\chi_{\infty'}(\alpha)\prod_{q|\vartheta_{\mathrm{odd}}}\left(\frac{\alpha}{\boldsymbol{q}}\right)_F\prod_{q|\vartheta_{\mathrm{even}}}\chi_q(\alpha).$$

where the symbol $(-)_F$ is the Legendre symbol over F determined by (2.1), *the characters χ_∞ and $\chi_{\infty'}$ are given by* (2.2) *and the character χ_q for an even prime ideal $\boldsymbol{q}$ is given by* (2.3) *and proposition* 4.

We note that the choice of $\boldsymbol{p}$ can be carried out by succesively applying continued fraction algorithm to prime ideals (see [4]).

§ 3. Unit groups of totally imaginary quadratic extensions of real quadratic fields

We determine the number w_K and $[E_F : N_{K/F}E_K]$ related to the unit group of a totally imaginary quadratic extension over a real quadratic field. Let K be a totally imaginary quadratic extension over F. Let E_F be the group of units in F, E_K the group of units in K, μ_K the group of roots of unity in K, $w_K=\#\mu_K$ and ζ_k a primitive k-th root of unity.

Proposition 7. *The notations being above, we have*

$$w_K=\begin{cases} \text{(i) } 12 & \text{if } K=F(\sqrt{-1}) \text{ and } F=\boldsymbol{Q}(\sqrt{3}), \\ \text{(ii) } 10 & \text{if } K=F\left(\sqrt{-\frac{5+\sqrt{5}}{2}}\right) \text{ and } F=\boldsymbol{Q}(\sqrt{5}), \\ \text{(iii) } 8 & \text{if } K=F(\sqrt{-1}) \text{ and } F=\boldsymbol{Q}(\sqrt{2}), \\ \text{(iv) } 6 & \text{if } K=F(\sqrt{-3}) \text{ and } F\neq\boldsymbol{Q}(\sqrt{3}), \\ \text{(v) } 4 & \text{if } K=F(\sqrt{-1}) \text{ and } F\neq\boldsymbol{Q}(\sqrt{2}), Q(\sqrt{3}), \\ \text{(vi) } 2 & \text{otherwise.} \end{cases}$$

Proof. Since K is quartic, $w_K=2, 4, 6, 8, 10$ or 12. If $w_K=8, 10$ or 12, ζ_{w_K} generates a quartic extension over $\boldsymbol{Q}$ therefore $K=\boldsymbol{Q}(\zeta_{w_K})$ which are fields given in (i), (ii), (iii). If $w_K=4$ or 6, $K=F(\zeta_{w_K})$ since ζ_{w_K} generates a quadratic extension over a real quadratic field. And $K\neq\boldsymbol{Q}(\zeta_8)$, $\boldsymbol{Q}(\zeta_{10})$, $\boldsymbol{Q}(\zeta_{12})$ in these cases. This proves the proposition. ■

In the rest of this section, we assume that F is a totally real algebraic number field and that K is a totally imaginary quadratic extension of F. Let E_F be the group of units in F, E_+ the group of totally positive units in F, E_K the group of units in K, μ_K the group of roots of unity in K. The computation of the index $[E_F: N_{K/F}E_K]$ is reduced to that of the Hasse's unit index of K. We note that the following argument does not require K to be abelian over $\boldsymbol{Q}$ although the original proof of H. Hasse in [1] does.

Definition 8 (Hasse's unit index). For a totally imaginary quadratic extension K of a totally real algebraic number field F, we define

$$Q_K=[E_K: \mu_K E_F].$$

Let σ denote the non-trivial conjugation of K/F. Then $1+\sigma=N_{K/F}$ defines a homomorphism $\psi=1+\sigma: E_K\to E_F$ whose kernel is μ_K since $\eta^{1+\sigma}=1$ implies $(\eta^\tau)^{1+\tau^{-1}\sigma\tau}=(\eta^{1+\sigma})^\tau=1$ for each embedding τ of K into $\boldsymbol{C}$. Hence, $1-\sigma$ defines a homomorphism $\phi=1-\sigma: E_K\to\mu_K$. The kernel of ϕ is clearly E_F.

Proposition 9. *For $\xi\in E_K$, the following conditions are equivalent to each other.*

(i) $\xi\in\mu_K E_F$;

(ii) $\xi^\psi\in E_F^2$;

(iii) $\xi^\phi\in\mu_K^2$.

Proof. It is easy to see (i) implies (ii) and (iii). If (ii) is satisfied, $\xi^\psi=\eta^{-2}$ for some $\eta\in E_F$. Then $(\xi\eta)^{\phi}=1$, that is, $\xi\eta\in\mu_K$. Hence, (ii) implies (i). We can prove that (iii) implies (i) in the same way. ■

Corollary 10. *The map ϕ induces an isomorphism*

$$\phi: E_K/\mu_K E_F \longrightarrow N_{K/F}E_K/E_F^2.$$

In particular,

$$[N_{K/F}E_K : E_F^2] = Q_K$$

and

$$\frac{1}{[E_F : N_{K/F}E_K]} = \frac{Q_K}{2^{[F:Q]}}.$$

Proof. The first assertion is clear from the proposition. The second assertion follows from the fact $2^{[F:Q]}=[E_F : N_{K/F}E_K][N_{K/F}E_K : E_F^2]$. ■

Corollary 11. $E_+\neq E_F^2$ if $Q_K=2$.

Proof. This assertion is clear from the inequality $[E_+ : E_F^2]\geqq[N_{K/F}E_F : E_F^2]=Q_K$. ■

Corollary 12. *The map ϕ induces an injection*

$$\phi: E_K/\mu_K E_F \longrightarrow \mu_K/\mu_K^2.$$

In particular,

$$\begin{cases} Q_K=2 & \text{if } \phi \text{ is surjective,} \\ Q_K=1 & \text{otherwise.} \end{cases}$$

Theorem 13. *Let ζ be a generator of the 2-sylow subgroup of μ_K. Then, one has*

$$\text{(i)}\quad Q_K=\begin{cases} 2 & K=F(\sqrt{-\eta}) \text{ with } \eta\in E_+\setminus E_F^2 \\ 1 & \text{otherwise} \end{cases} \qquad \text{when } \sqrt{-1}\notin\mu_K;$$

$$\text{(ii)}\quad Q_K=\begin{cases} 2 & K=F(\sqrt{-1}) \text{ and } (1-\zeta)(1-\zeta)^{\sigma}\in(F^{\times})^2E_F \\ 1 & \text{otherwise} \end{cases} \qquad \text{when } \sqrt{-1}\in\mu_K.$$

Proof. $Q_K=2$ if ϕ is surjective to μ_K and otherwise $Q_K=1$. Since μ_K is cyclic, $\mu_K=\mu_K^2\cap\zeta\mu_K^2$. Therefore, $Q_K=2$ if and only if $\zeta=\xi^{\phi}$ for some $\xi\in E_K$.

In case (i), $\zeta=-1$. Suppose $\xi^{\phi}=\zeta=-1$. Put $\eta=\xi^{\phi}\in E_F$. Then $\eta\gg0$. From proposition 9, $\eta\notin E_F^2$. Since $\xi^{1-\sigma}=\xi^{\phi}=-1$, $\xi^2=-\xi^{1+\sigma}=-\eta$. Therefore $K=F(\sqrt{-\eta})$ for some $\eta\in E_F$ such that $\eta\gg0$, $\eta\notin E_F^2$. Conversely. if these conditions are satisfied, the unit $\xi=\sqrt{-\eta}$ satisfies $\xi^{\phi}=-1$.

In case (ii), $K=F(\sqrt{-1})$ and $\sqrt{-1}\in E_K$. Suppose $\zeta=\xi^{\phi}=\xi^{1-\sigma}$. Then we see $\alpha=\sqrt{-1}(\xi-\xi^{\sigma})=\sqrt{-1}(1-\zeta)\xi^{\sigma}$ is preserved by σ and hence is in F. Taking the norm, we have $\alpha^2=(1-\zeta)(1-\zeta)^{\sigma}\xi^{\phi}$. Since $\xi^{\phi}\in E_F$, $(1-\zeta)(1-\zeta)^{\sigma}\in(F^{\times})^2E_F$. Conversely if this condition is satisfied, there is an integer $\alpha\in F$ such that, $(\alpha^2)=((1-\zeta)(1-\zeta)^{\sigma})$. Put $\xi=\sqrt{-1}\alpha/(1-\zeta^{\sigma})$. Then $\zeta=\xi^{1-\sigma}=\xi^{\phi}$. ■

As a specialization of this theorem we obtain the following proposition.

Proposition 14. *Assume F is a real quadratic field. Then one has*

$$Q_K=\begin{cases} 2 & \text{if } K=F(\sqrt{-\varepsilon}) \text{ and } \varepsilon\gg 0, \\ 2 & \text{if } K=F(\sqrt{-1}) \text{ and } 2\varepsilon\in(F^\times)^2, \\ 1 & \text{otherwise.}\end{cases}$$

Proof. Since K is a quartic field, $\zeta=-1$, $\sqrt{-1}$ or ζ_8. If $\zeta=\zeta_8$, $K=\boldsymbol{Q}(\zeta_8)=\boldsymbol{Q}(\sqrt{2}, \sqrt{-1})$. $Q_K=1$ in this case, since $N(\varepsilon)=-1$ for $F=\boldsymbol{Q}(\sqrt{2})$. The proposition follows since $2=(1-\zeta)(1-\zeta)^\sigma$ for $\zeta=\sqrt{-1}$. ■

§ 4. Description of quardratic extensions

In this section, we describe the set of quadratic extensions over a real quadratic field. For convenience, we write $F(\sqrt{\Delta})=F$ for $\Delta\in(F^\times)^2$. There are many choices of field generators of form $\sqrt{\Delta}$ for a given quadratic extension. For example, $\boldsymbol{Q}(\sqrt{3}, \sqrt{-2})=\boldsymbol{Q}(\sqrt{3}, \sqrt{-2-\sqrt{3}})$ and $\boldsymbol{Q}(\sqrt{6}, \sqrt{-3})=\boldsymbol{Q}(\sqrt{6}, \sqrt{-5-2\sqrt{6}})$. It is important to know whether given $\sqrt{\Delta_1}$ and $\sqrt{\Delta_2}$ generate the same field or not, when one applies proposition 4 or proposition 7. Clearly $F(\sqrt{\Delta_1})=F(\sqrt{\Delta_2})$ if and only if $\Delta_1\Delta_2\in(F^\times)^2$.

Proposition 15. *Let $\Delta\in F^\times$. If*

$$\Delta=\left(\frac{x+y\sqrt{d}}{2}\right)^2$$

for x, $y\in\boldsymbol{Z}$,

$$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\pm\sqrt{\operatorname{tr}\Delta+2\sqrt{N\Delta}}\\ \pm\sqrt{(\operatorname{tr}\Delta-2\sqrt{N\Delta})/d}\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\pm\sqrt{\operatorname{tr}\Delta-2\sqrt{N\Delta}}\\ \pm\sqrt{(\operatorname{tr}\Delta+2\sqrt{N\Delta})/d}\end{pmatrix}.$$

Proof. The assertions can be easily verified by calculations. ■

We describe the set

$$\{F(\sqrt{\Delta})\mid\Delta\in F^\times\}$$

by constructing a system of representatives for $F^\times/(F^\times)^2$. We introduce some notations. Let $\boldsymbol{I}_F$ be the ideal group of F, $\boldsymbol{P}_F$ the group of principal ideals of F in the wide sense, $\boldsymbol{S}_F=\boldsymbol{I}_F^2\boldsymbol{P}_F$ the group of ideals in square ideal classes of F and $\boldsymbol{A}_F=\{\boldsymbol{a}\in\boldsymbol{I}_F:\boldsymbol{a}^2\in\boldsymbol{P}_F\}$. Put $g=\#\boldsymbol{A}_F/\boldsymbol{P}_F$ and let integral ideals $\boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_g$ be a system of representatives for $\boldsymbol{A}_F/\boldsymbol{P}_F$. We consider the following diagram, in which all homomorphisms are surjections, and construct systems of representatives from the right, using auxiliary functions ρ, M, G and τ which are described later.

$$F^\times/(F^\times)^2 \xrightarrow{\iota} \boldsymbol{P}_F/\boldsymbol{P}_F^2 \xrightarrow{\kappa} \boldsymbol{P}_F/\boldsymbol{A}_F^2 \xrightarrow{\lambda} \boldsymbol{S}_F/\boldsymbol{I}_F^2$$

The homomorphism ι is defined by $\iota(\Delta(F^\times)^2)=(\Delta)\boldsymbol{P}_F^2$. The homomorphisms κ and λ are natural homomorphisms. The set

$$\widetilde{\boldsymbol{S}_F/\boldsymbol{I}_F^2}=\{\boldsymbol{a}\in\boldsymbol{S}_F\mid\boldsymbol{a}\text{ is a square free integral ideal}\}$$

is a system of representatives for $\boldsymbol{S}_F/\boldsymbol{I}_F^2$. The homomorphism λ is an isomorphism since $\boldsymbol{A}_F^2=\boldsymbol{P}_F\cap\boldsymbol{I}_F^2$ and $\boldsymbol{S}_F=\boldsymbol{P}_F\boldsymbol{I}_F^2$. There exists an ideal function $\rho:\boldsymbol{S}_F\to\boldsymbol{I}_F$ such that $\boldsymbol{a}\rho(\boldsymbol{a})^2\in\boldsymbol{P}_F$. And the set

$$\widetilde{\boldsymbol{P}_F/\boldsymbol{A}_F^2}=\{\boldsymbol{a}\rho(\boldsymbol{a})^2\mid \boldsymbol{a}\in\widetilde{\boldsymbol{S}_F/\boldsymbol{I}_F^2}\}$$

is a system of representatives for $\boldsymbol{P}_F/\boldsymbol{A}_F^2$. Clearly, the set $\{\boldsymbol{b}_1^2, \boldsymbol{b}_2^2, \cdots, \boldsymbol{b}_g^2\}$ is a system of representatives for $\boldsymbol{A}_F^2/\boldsymbol{P}_F^2$ which is the kernel of the homomorphism at the middle. Let $M(\boldsymbol{a})$ be the representative of $\boldsymbol{a}$ in a fixed system of representatives of ideal classes. Then the set

$$\widetilde{\boldsymbol{P}_F/\boldsymbol{P}_F^2}=\{\boldsymbol{a}M(\rho(\boldsymbol{a})\boldsymbol{b}_i)^2\mid \boldsymbol{a}\in\widetilde{\boldsymbol{S}_F/\boldsymbol{I}_F^2},\ i=1, 2, \cdots, g\}$$

is a system of representatives for $\boldsymbol{P}_F/\boldsymbol{P}_F^2$. Let $G(\boldsymbol{a})$ be a generator of $\boldsymbol{a}$ for a principal ideal $\boldsymbol{a}$. Then, the set

$$\widetilde{F^\times/(F^\times)^2E_F}=\{G(\boldsymbol{a}M(\rho(\boldsymbol{a})\boldsymbol{b}_i)^2)\mid \boldsymbol{a}\in\widetilde{\boldsymbol{S}_F/\boldsymbol{I}_F^2},\ i=1, 2, \cdots, g\}$$

is a system of representatives for $F^\times/(F^\times)^2E_F$. Clearly, the set $\{\pm1, \pm\varepsilon\}$ is a system of representatives for $(F^\times)^2E_F/(F^\times)^2$ which is the kernel of the homomorphism at the left. Let $\tau(\alpha)$ be the representative of α in a fixed system of representatives for $F^\times/E_F^2$. Then, the set

$$\widetilde{F^\times/(F^\times)^2}=\{\tau(\pm\Delta), \tau(\pm\varepsilon\Delta)\mid \Delta=G(\boldsymbol{a}M(\rho(\boldsymbol{a})\boldsymbol{b}_i)^2) \text{ for } \boldsymbol{a}\in\widetilde{\boldsymbol{S}_F/\boldsymbol{I}_F^2},\ i=1, 2, \cdots, g\}$$

is a system of representatives for $F^\times/(F^\times)^2$.

We now give choices of auxiliary functions so that the functions are computable. We note that the class group of a real quadratic field is effectively determined by the continued fraction algorithm (see [4]). Firstly, the function $\rho(\boldsymbol{a})$ can be constructed as an ideal class funcion on $\boldsymbol{S}_F$. For each ideal $\boldsymbol{a}\in\boldsymbol{S}_F$, there is an ideal $\boldsymbol{b}$ such that $\boldsymbol{a}\boldsymbol{P}_F=(\boldsymbol{b}\boldsymbol{P}_F)^{-2}$ in the ideal class group. This time, $\boldsymbol{a}\boldsymbol{b}^2\in\boldsymbol{P}_F$. One can find such $\boldsymbol{b}$ by examining the class group of F. We can put $\rho(\boldsymbol{a})=\boldsymbol{b}$. The set $\widetilde{F^\times/(F^\times)^2}$ is independent of the choice of the function ρ because we multiply by $\boldsymbol{b}_i$ for $i=1, 2, \cdots, g$ and apply M. Secondly, we let $M(\boldsymbol{a})=[a, (b+\sqrt{d})/2]$ with $a>0$ and $0<b<2a$ be the representative of the ideal class $\boldsymbol{a}\boldsymbol{P}_F$ which is the minimum in the lexicographical order in a and b. Next, the generator $G(\boldsymbol{a})$ of $\boldsymbol{a}$ can be found by the continued fraction algorithm (see [4]). The set $\widetilde{F^\times/(F^\times)^2}$ is independent of the function G because we finally apply τ. Lastly, we define $\tau(\alpha)$. We embed F in $\boldsymbol{R}$ and assume $\sqrt{d}>0$. We define $\tau(\alpha)$ to be the unique element β_0 of αE_F^2 which satisfies

$$\begin{cases} \text{(i)} & |\mathrm{tr}\,\beta|\geqq|\mathrm{tr}\,\beta_0|\,; \\ \text{(ii)} & |\beta|\leqq|\beta_0| \quad \text{if } |\mathrm{tr}\,\beta_0|=|\mathrm{tr}\,\beta|\,; \end{cases}$$

for any $\beta\in\alpha E_F^2$. The number $\tau(\alpha)=\beta_0$ is well defined, since there is clearly the minimum $a=|\mathrm{tr}\,\beta|$, there are at most two solutions for the equation $X^2-aX+c=0$ with $c=N(\alpha)=N(\beta)$ and elements of αE_F^2 have different magnitudes.

§5. The table of special values and relative class numbers

We present some special values of L-functions associated with quadratic extensions of real quadratic fields and of zeta functions of real quadratic fields. Let d_0 be a square free rational positive integer, $F=\boldsymbol{Q}(\sqrt{d_0})$ a real quadratic field, d its discriminant, ε the fundamental unit in F which is greater than 1, ε_+ the generator of the group of totally positive units in F which is greater than 1, h the class number of F in the wide sense, h_+ the class number of F in the narrow sense. We consider a quadratic extension $K=F(\sqrt{\Delta})$ of F. Let χ be the character associated with the extension K/F, ϑ its conductor, w_K the number of roots of unity in K, H the class number of K. And let Q_K be the Hasse's unit index (see definition 8) of K when K is a totally imaginary field. For convenience, we treat the case of $\chi=1$ as the case of $\Delta=1$. We note that $L_F(1-m, \chi)$ is non-zero only when m is odd and χ is associcated with a totally imaginary extension or m is even and χ is associcated with a totally real extension.

Firstly, the tables of special values $L_F(0, \chi)$, $L_F(-1, \chi)$, $L_F(-2, \chi)$, $L_F(-3, \chi)$, with $N(\vartheta)<50$ over base fields $\boldsymbol{Q}(\sqrt{2})$, $\boldsymbol{Q}(\sqrt{3})$, $\boldsymbol{Q}(\sqrt{5})$ and $\boldsymbol{Q}(\sqrt{6})$ are listed. Next, the tables of class numbers of totally imaginary quadratic extensions with $N(\vartheta)<100$ are listed. The base fields listed in the tables of class numbers are $\boldsymbol{Q}(\sqrt{2})$, $\boldsymbol{Q}(\sqrt{3})$, $\boldsymbol{Q}(\sqrt{5})$, $\boldsymbol{Q}(\sqrt{6})$, $\boldsymbol{Q}(\sqrt{7})$, $\boldsymbol{Q}(\sqrt{10})$, $\boldsymbol{Q}(\sqrt{11})$, $\boldsymbol{Q}(\sqrt{13})$, $\boldsymbol{Q}(\sqrt{14})$, $\boldsymbol{Q}(\sqrt{15})$, $\boldsymbol{Q}(\sqrt{17})$, $\boldsymbol{Q}(\sqrt{19})$, $\boldsymbol{Q}(\sqrt{21})$, $\boldsymbol{Q}(\sqrt{22})$, $\boldsymbol{Q}(\sqrt{23})$, $\boldsymbol{Q}(\sqrt{26})$, $\boldsymbol{Q}(\sqrt{28})$, $\boldsymbol{Q}(\sqrt{29})$, $\boldsymbol{Q}(\sqrt{30})$, $\boldsymbol{Q}(\sqrt{31})$, $\boldsymbol{Q}(\sqrt{33})$, $\boldsymbol{Q}(\sqrt{34})$, $\boldsymbol{Q}(\sqrt{35})$, $\boldsymbol{Q}(\sqrt{37})$, $\boldsymbol{Q}(\sqrt{38})$, $\boldsymbol{Q}(\sqrt{39})$, $\boldsymbol{Q}(\sqrt{79})$ and $\boldsymbol{Q}(\sqrt{229})$. The last two base fields are chosen because their class numbers have odd prime factors. The generator Δ is normalized as described in §4. To eliminate a duplication due to a conjugate pair of Δ, we have adapted the one which has the greater magnitude in a fixed embedding of F in $\boldsymbol{R}$ and excluded the other. Entries for $w_K=2$ and $Q_K=1$ are left blank. The symbol $[x, y]$ denotes the ideal which is generated by x, y as a $\boldsymbol{Z}$-module.

$F=\boldsymbol{Q}(\sqrt{2})$, $d=8$, $\varepsilon=1+\sqrt{2}$, $\varepsilon_+=3+2\sqrt{2}$, $h=1$, $h_+=1$

Δ	ϑ	$L_F(0,\chi)$	$L_F(-2,\chi)$	$L_F(-4,\chi)$
$-2-\sqrt{2}$	$[8, 4\sqrt{2}]$	2	274	1651234
-3	$[3, 3\sqrt{2}]$	2/3	92/9	15940/3
-7	$[7, 7\sqrt{2}]$	4	6336/7	11594880
$-5-2\sqrt{2}$	$[17, 11+\sqrt{2}]$	2	64	98944
-1	$[2, 2\sqrt{2}]$	1/2	3/2	285/2

Δ	ϑ	$L_F(-1,\chi)$	$L_F(-3/\chi)$
$2+\sqrt{2}$	$[8, 4\sqrt{2}]$	10	15898
3	$[6, 6\sqrt{2}]$	12	24012
5	$[5, 5\sqrt{2}]$	28/5	6308
$7+2\sqrt{2}$	$[41, 24+\sqrt{2}]$	12	35652
1	$[1, \sqrt{2}]$	1/12	11/120

$F=\boldsymbol{Q}(\sqrt{3})$, $d=12$, $\varepsilon=2+\sqrt{3}$, $\varepsilon_+=2+\sqrt{3}$, $h=1$, $h_+=2$

Δ	ϑ	$L_F(0,\chi)$	$L_F(-2,\chi)$	$L_F(-4,\chi)$
$-9-4\sqrt{3}$	$[33, 27+\sqrt{3}]$	4	928	12136768
-5	$[5, 5\sqrt{3}]$	4	480	3493824
-7	$[7, 7\sqrt{3}]$	4	16832/7	71596160
-1	$[1, \sqrt{3}]$	1/6	1/9	5/3
$-2-\sqrt{3}$	$[4, 4\sqrt{3}]$	2	138	454290

Δ	ϑ	$L_F(-1,\chi)$	$L_F(-3,\chi)$
$9+4\sqrt{3}$	$[33, 27+\sqrt{3}]$	16	68944
5	$[5, 5\sqrt{3}]$	48/5	25776
7	$[7, 7\sqrt{3}]$	32	278432
1	$[1, \sqrt{3}]$	1/6	23/60
$2+\sqrt{3}$	$[4, 4\sqrt{3}]$	6	5742

$F=\boldsymbol{Q}(\sqrt{5})$, $d=5$, $\varepsilon=(1+\sqrt{5})/2$, $\varepsilon_+=(3+\sqrt{5})/2$, $h=1$, $h_+=1$

Δ	ϑ	$L_F(0,\chi)$	$L_F(-2,\chi)$	$L_F(-4,\chi)$
-3	$[3, (3+3\sqrt{5})/2]$	2/3	32/9	1984/3
$(-5-\sqrt{5})/2$	$[5, (5+\sqrt{5})/2]$	2/5	4/5	1172/25
-7	$[7, (7+7\sqrt{5})/2]$	2	1728/7	1355904
$(-13-\sqrt{5})/2$	$[41, (13+\sqrt{5})/2]$	2	160	608320
-1	$[4, 2+2\sqrt{5}]$	1	15	8805

Δ	ϑ	$L_F(-1,\chi)$	$L_F(-3,\chi)$
$(15+3\sqrt{5})/2$	$[15, (15+3\sqrt{5})/2]$	8	10088
$(11+\sqrt{5})/2$	$[29, (11+\sqrt{5})/2]$	4	2164
1	$[1, (1+\sqrt{5})/2]$	1/30	1/60

$F=\boldsymbol{Q}(\sqrt{6})$, $d=24$, $\varepsilon=5+2\sqrt{6}$, $\varepsilon_+=5+2\sqrt{6}$, $h=1$, $h_+=2$

Δ	ϑ	$L_F(0,\chi)$	$L_F(-2,\chi)$	$L_F(-4,\chi)$
$-3-\sqrt{6}$	$[12, 4\sqrt{6}]$	4	11716	1436217220
$-25-10\sqrt{6}$	$[5, 5\sqrt{6}]$	4	2528	78409664
-7	$[7, 7\sqrt{6}]$	4	13376	1618987136
-1	$[2, 2\sqrt{6}]$	1	23	19925
$-5-2\sqrt{6}$	$[1, \sqrt{6}]$	1/3	2/3	38

Δ	ϑ	$L_F(-1,\chi)$	$L_F(-3,\chi)$
$3+\sqrt{6}$	$[12, 4\sqrt{6}]$	100	3076132
5	$[5, 5\sqrt{6}]$	136/5	291608
$35+14\sqrt{6}$	$[7, 7\sqrt{6}]$	80	3083696
1	$[1, \sqrt{6}]$	1/2	87/20
$5+2\sqrt{6}$	$[2, 2\sqrt{6}]$	2	506

$F=\boldsymbol{Q}(\sqrt{2})$, $d=8$, $\varepsilon=1+\sqrt{2}$, $\varepsilon_+=3+2\sqrt{2}$, $h=1$, $h_+=1$

Δ	ϑ	H/h	w_K	Q_K
$-2-\sqrt{2}$	$[8, 4\sqrt{2}]$	1		
-3	$[3, 3\sqrt{2}]$	1	6	
-5	$[10, 10\sqrt{2}]$	2		
-7	$[7, 7\sqrt{2}]$	2		
$-5-2\sqrt{2}$	$[17, 11+\sqrt{2}]$	1		
$-9-2\sqrt{2}$	$[73, 41+\sqrt{2}]$	1		
$-11-4\sqrt{2}$	$[89, 25+\sqrt{2}]$	1		
$-13-6\sqrt{2}$	$[97, 83+\sqrt{2}]$	3		
-1	$[2, 2\sqrt{2}]$	1	8	

$F=\boldsymbol{Q}(\sqrt{3})$, $d=12$, $\varepsilon=2+\sqrt{3}$, $\varepsilon_+=2+\sqrt{3}$, $h=1$, $h_+=2$

Δ	ϑ	H/h	w_K	Q_K
$-3-\sqrt{3}$	$[24, 12+4\sqrt{3}]$	2		
$-9-4\sqrt{3}$	$[33, 27+\sqrt{3}]$	2		
-5	$[5, 5\sqrt{3}]$	2		
-7	$[7, 7\sqrt{3}]$	2		
$-5-2\sqrt{3}$	$[26, 18+2\sqrt{3}]$	2		
$-11-4\sqrt{3}$	$[73, 21+\sqrt{3}]$	4		
$-17-8\sqrt{3}$	$[97, 87+\sqrt{3}]$	2		
-1	$[1, \sqrt{3}]$	1	12	2
$-2-\sqrt{3}$	$[4, 4\sqrt{3}]$	2		2

$F=\boldsymbol{Q}(\sqrt{5})$, $d=5$, $\varepsilon=(1+\sqrt{5})/2$, $\varepsilon_+=(3+\sqrt{5})/2$, $h=1$, $h_+=1$

Δ	ϑ	H/h	w_K	Q_K
-2	$[8, 4+4\sqrt{5}]$	1		
-3	$[3, (3+3\sqrt{5})/2]$	1	6	
$(-5-\sqrt{5})/2$	$[5, (5+\sqrt{5})/2]$	1	10	
-7	$[7, (7+7\sqrt{5})/2]$	1		
$(-13-\sqrt{5})/2$	$[41, (13+\sqrt{5})/2]$	1		
$(-17-3\sqrt{5})/2$	$[61, (87+\sqrt{5})/2]$	1		
-1	$[4, 2+2\sqrt{5}]$	1	4	

$F=\boldsymbol{Q}(\sqrt{6})$, $d=24$, $\varepsilon=5+2\sqrt{6}$, $\varepsilon_+=5+2\sqrt{6}$, $h=1$, $h_+=2$

Δ	ϑ	H/h	w_K	Q_K
$-9-2\sqrt{6}$	$[57, 33+\sqrt{6}]$	4		
$-3-\sqrt{6}$	$[12, 4\sqrt{6}]$	2		
-5	$[10, 10\sqrt{6}]$	4		
$-25-10\sqrt{6}$	$[5, 5\sqrt{6}]$	2		
-7	$[7, 7\sqrt{6}]$	2		
$-17-6\sqrt{6}$	$[73, 15+\sqrt{6}]$	6		
$-31-12\sqrt{6}$	$[97, 43+\sqrt{6}]$	4		
-1	$[2, 2\sqrt{6}]$	2	4	2
$-5-2\sqrt{6}$	$[1, \sqrt{6}]$	1	6	2

$F=\boldsymbol{Q}(\sqrt{7})$, $d=28$, $\varepsilon=8+3\sqrt{7}$, $\varepsilon_+=8+3\sqrt{7}$, $h=1$, $h_+=2$

Δ	ϑ	H/h	w_K	Q_K
$-3-\sqrt{7}$	$[8, 4+4\sqrt{7}]$	2		
-3	$[3, 3\sqrt{7}]$	2	6	
$-13-4\sqrt{7}$	$[57, 46+\sqrt{7}]$	6		
$-43-16\sqrt{7}$	$[57, 49+\sqrt{7}]$	2		
$-7-2\sqrt{7}$	$[42, 28+2\sqrt{7}]$	4		
-5	$[5, 5\sqrt{7}]$	2		
-1	$[1, \sqrt{7}]$	1	4	2
$-8-3\sqrt{7}$	$[4, 4\sqrt{7}]$	4		2

$F=\mathbf{Q}(\sqrt{10})$, $d=40$, $\varepsilon=3+\sqrt{10}$, $\varepsilon_{+}=19+6\sqrt{10}$, $h=2$, $h_{+}=2$

Δ	ϑ	H/h	w_K	Q_K
$-4-\sqrt{10}$	$[24, 16+4\sqrt{10}]$	6		
$-8-2\sqrt{10}$	$[24, 16+4\sqrt{10}]$	2		
-3	$[3, 3\sqrt{10}]$	2	6	
-6	$[3, 3\sqrt{10}]$	2		
$-15-4\sqrt{10}$	$[65, 20+\sqrt{10}]$	6		
$-30-8\sqrt{10}$	$[65, 20+\sqrt{10}]$	2		
-7	$[7, 7\sqrt{10}]$	2		
-14	$[7, 7\sqrt{10}]$	4		
$-9-2\sqrt{10}$	$[41, 25+\sqrt{10}]$	2		
$-18-4\sqrt{10}$	$[41, 25+\sqrt{10}]$	4		
$-27-8\sqrt{10}$	$[89, 59+\sqrt{10}]$	2		
$-54-16\sqrt{10}$	$[89, 59+\sqrt{10}]$	8		
-1	$[2, 2\sqrt{10}]$	1	4	
-2	$[2, 2\sqrt{10}]$	1		

$F=\mathbf{Q}(\sqrt{11})$, $d=44$, $\varepsilon=10+3\sqrt{11}$, $\varepsilon_{+}=10+3\sqrt{11}$, $h=1$, $h_{+}=2$

Δ	ϑ	H/h	w_K	Q_K
-3	$[3, 3\sqrt{11}]$	2	6	
-5	$[5, 5\sqrt{11}]$	4		
$-7-2\sqrt{11}$	$[10, 2+2\sqrt{11}]$	2		
$-4-\sqrt{11}$	$[20, 16+4\sqrt{11}]$	2		
-7	$[7, 7\sqrt{11}]$	4		
$-67-20\sqrt{11}$	$[89, 79+\sqrt{11}]$	4		
$-41-12\sqrt{11}$	$[97, 60+\sqrt{11}]$	8		
-1	$[1, \sqrt{11}]$	1	4	2
$-10-3\sqrt{11}$	$[4, 4\sqrt{11}]$	2		2

$F=\boldsymbol{Q}(\sqrt{13})$, $d=13$, $\varepsilon=(3+\sqrt{13})/2$, $\varepsilon_+=(11+3\sqrt{13})/2$, $h=1$, $h_+=1$

Δ	ϑ	H/h	w_K	Q_K
-2	$[8, 4+4\sqrt{13}]$	3		
-3	$[3, (3+3\sqrt{13})/2]$	2	6	
$(-17-\sqrt{13})/2$	$[69, (17+\sqrt{13})/2]$	2		
$(-5-\sqrt{13})/2$	$[12, 10+2\sqrt{13}]$	2		
-7	$[7, (7+7\sqrt{13})/2]$	1		
$(-13-3\sqrt{13})/2$	$[13, (13+\sqrt{13})/2]$	1		
$(-9-\sqrt{13})/2$	$[17, (9+\sqrt{13})/2]$	1		
$-9-2\sqrt{13}$	$[29, (19+\sqrt{13})/2]$	1		
-1	$[4, 2+2\sqrt{13}]$	1	4	

$F=\boldsymbol{Q}(\sqrt{14})$, $d=56$, $\varepsilon=15+4\sqrt{14}$, $\varepsilon_+=15+4\sqrt{14}$, $h=1$, $h_+=2$

Δ	ϑ	H/h	w_K	Q_K
$-4-\sqrt{14}$	$[8, 4\sqrt{14}]$	2		
-3	$[3, 3\sqrt{14}]$	2	6	
$-45-12\sqrt{14}$	$[6, 6\sqrt{14}]$	4		
-5	$[10, 10\sqrt{14}]$	4		
$-75-20\sqrt{14}$	$[5, 5\sqrt{14}]$	2		
$-53-14\sqrt{14}$	$[65, 27+\sqrt{14}]$	2		
$-31-8\sqrt{14}$	$[65, 12+\sqrt{14}]$	6		
-1	$[2, 2\sqrt{14}]$	4	4	2
$-15-4\sqrt{14}$	$[1, \sqrt{14}]$	1		2

$F=\boldsymbol{Q}(\sqrt{15})$, $d=60$, $\varepsilon=4+\sqrt{15}$, $\varepsilon_{+}=4+\sqrt{15}$, $h=2$, $h_{+}=4$

Δ	ϑ	H/h	w_K	Q_K
$-9-2\sqrt{15}$	$[42, 30+2\sqrt{15}]$	4		
$-12-2\sqrt{15}$	$[42, 12+2\sqrt{15}]$	4		
-7	$[7, 7\sqrt{15}]$	4		
$-56-14\sqrt{15}$	$[7, 7\sqrt{15}]$	4		
-1	$[1, \sqrt{15}]$	1	4	
$-4-\sqrt{15}$	$[4, 4\sqrt{15}]$	4		2
-2	$[4, 4\sqrt{15}]$	2		
$-8-2\sqrt{15}$	$[1, \sqrt{15}]$	1	6	

$F=\boldsymbol{Q}(\sqrt{17})$, $d=17$, $\varepsilon=4+\sqrt{17}$, $\varepsilon_{+}=33+8\sqrt{17}$, $h=1$, $h_{+}=1$

Δ	ϑ	H/h	w_K	Q_K
-2	$[8, 4+4\sqrt{17}]$	2		
$(-5-\sqrt{17})/2$	$[8, (9+\sqrt{17})/2]$	1		
-3	$[3, (3+3\sqrt{17})/2]$	1	6	
-7	$[7, (7+7\sqrt{17})/2]$	5		
$-9-2\sqrt{17}$	$[13, (11+\sqrt{17})/2]$	1		
$-6-\sqrt{17}$	$[76, (25+\sqrt{17})/2]$	4		
$-19-4\sqrt{17}$	$[89, (27+\sqrt{17})/2]$	7		
-1	$[4, 2+2\sqrt{17}]$	2	4	

$F=\mathbf{Q}(\sqrt{19})$, $d=76$, $\varepsilon=170+39\sqrt{19}$, $\varepsilon_+=170+39\sqrt{19}$, $h=1$, $h_+=2$

Δ	ϑ	H/h	w_K	Q_K
$-109-25\sqrt{19}$	$[24, 4+4\sqrt{19}]$	10		
$-5-\sqrt{19}$	$[24, 20+4\sqrt{19}]$	2		
-3	$[3, 3\sqrt{19}]$	2	6	
$-19-4\sqrt{19}$	$[57, 19+\sqrt{19}]$	10		
-5	$[5, 5\sqrt{19}]$	8		
$-9-2\sqrt{19}$	$[10, 4+2\sqrt{19}]$	2		
$-48-11\sqrt{19}$	$[20, 12+4\sqrt{19}]$	6		
-7	$[7, 7\sqrt{19}]$	2		
$-279-64\sqrt{19}$	$[17, 11+\sqrt{19}]$	4		
$-53-12\sqrt{19}$	$[73, 47+\sqrt{19}]$	6		
-1	$[1, \sqrt{19}]$	1	4	2
$-170-39\sqrt{19}$	$[4, 4\sqrt{19}]$	6		2

$F=\mathbf{Q}(\sqrt{21})$, $d=21$, $\varepsilon=(5+\sqrt{21})/2$, $\varepsilon_+=(5+\sqrt{21})/2$, $h=1$, $h_+=2$

Δ	ϑ	H/h	w_K	Q_K
-2	$[8, 4+4\sqrt{21}]$	2		
$-5-\sqrt{21}$	$[8, 4+4\sqrt{21}]$	4		
$(-25-5\sqrt{21})/2$	$[5, (5+5\sqrt{21})/2]$	2		
$(-37-7\sqrt{21})/2$	$[85, (151+\sqrt{21})/2]$	2		
$-13-2\sqrt{21}$	$[85, (49+\sqrt{21})/2]$	2		
$(-13-\sqrt{21})/2$	$[37, (13+\sqrt{21})/2]$	2		
-1	$[4, 2+2\sqrt{21}]$	2	4	
$(-5-\sqrt{21})/2$	$[1, (1+\sqrt{21})/2]$	1	6	2

$F=\mathbf{Q}(\sqrt{22})$, $d=88$, $\varepsilon=197+42\sqrt{22}$, $\varepsilon_+=197+42\sqrt{22}$, $h=1$, $h_+=2$

Δ	ϑ	H/h	w_K	Q_K
-3	$[3, 3\sqrt{22}]$	4	6	
$-591-126\sqrt{22}$	$[6, 6\sqrt{22}]$	4		
$-61-13\sqrt{22}$	$[12, 4+4\sqrt{22}]$	6		
$-319-68\sqrt{22}$	$[33, 11+\sqrt{22}]$	8		
$-5-\sqrt{22}$	$[12, 8+4\sqrt{22}]$	2		
-5	$[10, 10\sqrt{22}]$	12		
$-985-210\sqrt{22}$	$[5, 5\sqrt{22}]$	4		
-7	$[7, 7\sqrt{22}]$	4		
$-441-94\sqrt{22}$	$[89, 17+\sqrt{22}]$	10		
$-85-18\sqrt{22}$	$[97, 64+\sqrt{22}]$	2		
-1	$[2, 2\sqrt{22}]$	2	4	2
$-197-42\sqrt{22}$	$[1, \sqrt{22}]$	1		2

$F=\mathbf{Q}(\sqrt{23})$, $d=92$, $\varepsilon=24+5\sqrt{23}$, $\varepsilon_+=24+5\sqrt{23}$, $h=1$, $h_+=2$

Δ	ϑ	H/h	w_K	Q_K
$-5-\sqrt{23}$	$[8, 4+4\sqrt{23}]$	2		
-3	$[3, 3\sqrt{23}]$	4	6	
-5	$[5, 5\sqrt{23}]$	2		
-7	$[7, 7\sqrt{23}]$	8		
$-29-6\sqrt{23}$	$[26, 14+2\sqrt{23}]$	4		
$-77-16\sqrt{23}$	$[41, 33+\sqrt{23}]$	2		
$-21-4\sqrt{23}$	$[73, 60+\sqrt{23}]$	10		
-1	$[1, \sqrt{23}]$	3	4	2
$-24-5\sqrt{23}$	$[4, 4\sqrt{23}]$	4		2

$F=\boldsymbol{Q}(\sqrt{26})$, $d=104$, $\varepsilon=5+\sqrt{26}$, $\varepsilon_+=51+10\sqrt{26}$, $h=2$, $h_+=2$

Δ	ϑ	H/h	w_K	Q_K
-3	$[3, 3\sqrt{26}]$	2	6	
-6	$[3, 3\sqrt{26}]$	4		
-5	$[10, 10\sqrt{26}]$	4		
-10	$[10, 10\sqrt{26}]$	8		
$-13-2\sqrt{26}$	$[65, 39+\sqrt{26}]$	10		
$-26-4\sqrt{26}$	$[65, 39+\sqrt{26}]$	2		
-7	$[7, 7\sqrt{26}]$	6		
-14	$[7, 7\sqrt{26}]$	4		
$-11-2\sqrt{26}$	$[34, 28+2\sqrt{26}]$	4		
$-22-4\sqrt{26}$	$[34, 28+2\sqrt{26}]$	4		
-1	$[2, 2\sqrt{26}]$	3	4	
-2	$[2, 2\sqrt{26}]$	1		

$F=\boldsymbol{Q}(\sqrt{29})$, $d=29$, $\varepsilon=(5+\sqrt{29})/2$, $\varepsilon_+=(27+5\sqrt{29})/2$, $h=1$, $h_+=1$

Δ	ϑ	H/h	w_K	Q_K
-2	$[8, 4+4\sqrt{29}]$	1		
$(-21-3\sqrt{29})/2$	$[15, (21+3\sqrt{29})/2]$	2		
-3	$[3, (3+3\sqrt{29})/2]$	3	6	
$(-41-7\sqrt{29})/2$	$[65, (43+\sqrt{29})/2]$	4		
$(-17-\sqrt{29})/2$	$[65, (17+\sqrt{29})/2]$	2		
$(-7-\sqrt{29})/2$	$[20, 14+2\sqrt{29}]$	2		
-7	$[7, (7+7\sqrt{29})/2]$	2		
$(-9-\sqrt{29})/2$	$[13, (9+\sqrt{29})/2]$	1		
$(-29-5\sqrt{29})/2$	$[29, (29+\sqrt{29})/2]$	1		
$-13-2\sqrt{29}$	$[53, (33+\sqrt{29})/2]$	1		
-1	$[4, 2+2\sqrt{29}]$	3	4	

$F=\boldsymbol{Q}(\sqrt{30})$, $d=120$, $\varepsilon=11+2\sqrt{30}$, $\varepsilon_+=11+2\sqrt{30}$, $h=2$, $h_+=4$

Δ	ϑ	H/h	w_K	Q_K
$-6-\sqrt{30}$	$[24, 4\sqrt{30}]$	8		
$-12-2\sqrt{30}$	$[24, 4\sqrt{30}]$	4		
-7	$[7, 7\sqrt{30}]$	4		
$-77-14\sqrt{30}$	$[7, 7\sqrt{30}]$	4		
-1	$[2, 2\sqrt{30}]$	2	4	
$-11-2\sqrt{30}$	$[2, 2\sqrt{30}]$	4		2
-2	$[1, \sqrt{30}]$	1		
$-22-4\sqrt{30}$	$[1, \sqrt{30}]$	1	6	

$F=\boldsymbol{Q}(\sqrt{31})$, $d=124$, $\varepsilon=1520+273\sqrt{31}$, $\varepsilon_+=1520+273\sqrt{31}$, $h=1$, $h_+=2$

Δ	ϑ	H/h	w_K	Q_K
$-39-7\sqrt{31}$	$[8, 4+4\sqrt{31}]$	4		
-3	$[3, 3\sqrt{31}]$	2	6	
$-3697-664\sqrt{31}$	$[33, 25+\sqrt{31}]$	10		
$-23-4\sqrt{31}$	$[33, 14+\sqrt{31}]$	2		
-5	$[5, 5\sqrt{31}]$	4		
$-6-\sqrt{31}$	$[20, 4+4\sqrt{31}]$	4		
$-657-118\sqrt{31}$	$[10, 8+2\sqrt{31}]$	4		
-7	$[7, 7\sqrt{31}]$	4		
$-45-8\sqrt{31}$	$[41, 21+\sqrt{31}]$	2		
$-401-72\sqrt{31}$	$[97, 15+\sqrt{31}]$	22		
-1	$[1, \sqrt{31}]$	3	4	2
$-1520-273\sqrt{31}$	$[4, 4\sqrt{31}]$	8		2

$F=\boldsymbol{Q}(\sqrt{33})$, $d=33$, $\varepsilon=23+4\sqrt{33}$, $\varepsilon_+=23+4\sqrt{33}$, $h=1$, $h_+=2$

Δ	ϑ	H/h	w_K	Q_K
-2	$[8, 4+4\sqrt{33}]$	4		
$-46-8\sqrt{33}$	$[8, 4+4\sqrt{33}]$	2		
$(-121-21\sqrt{33})/2$	$[88, (33+\sqrt{33})/2]$	8		
$-81-14\sqrt{33}$	$[93, (39+\sqrt{33})/2]$	4		
$-6-\sqrt{33}$	$[12, (9+\sqrt{33})/2]$	2		
$-115-20\sqrt{33}$	$[5, (5+5\sqrt{33})/2]$	4		
-7	$[7, (7+7\sqrt{33})/2]$	6		
$-13-2\sqrt{33}$	$[37, (25+\sqrt{33})/2]$	2		
$-47-8\sqrt{33}$	$[97, (115+\sqrt{33})/2]$	14		
-1	$[4, 2+2\sqrt{33}]$	2	4	
$-23-4\sqrt{33}$	$[1, (1+\sqrt{33})/2$	1	6	2

$F=\boldsymbol{Q}(\sqrt{34})$, $d=136$, $\varepsilon=35+6\sqrt{34}$, $\varepsilon_+=35+6\sqrt{34}$, $h=2$, $h_+=4$

Δ	ϑ	H/h	w_K	Q_K
$-6-\sqrt{34}$	$[8, 4\sqrt{34}]$	4		
-3	$[3, 3\sqrt{34}]$	2	6	
$-105-18\sqrt{34}$	$[3, 3\sqrt{34}]$	2		
$-47-8\sqrt{34}$	$[33, 10+\sqrt{34}]$	6		
$-13-2\sqrt{34}$	$[33, 23+\sqrt{34}]$	10		
-5	$[10, 10\sqrt{34}]$	12		
$-175-30\sqrt{34}$	$[10, 10\sqrt{34}]$	4		
-7	$[7, 7\sqrt{34}]$	4		
$-245-42\sqrt{34}$	$[7, 7\sqrt{34}]$	20		
$-17-2\sqrt{34}$	$[17, \sqrt{34}]$	2		
-1	$[2, 2\sqrt{34}]$	4	4	2
$-35-6\sqrt{34}$	$[2, 2\sqrt{34}]$	4		2

$F=\boldsymbol{Q}(\sqrt{35})$, $d=140$, $\varepsilon=6+\sqrt{35}$, $\varepsilon_{+}=6+\sqrt{35}$, $h=2$, $h_{+}=4$

Δ	ϑ	H/h	w_K	Q_K
-3	$[3, 3\sqrt{35}]$	4	6	
$-36-6\sqrt{35}$	$[3, 3\sqrt{35}]$	4		
$-25-4\sqrt{35}$	$[65, 55+\sqrt{35}]$	8		
$-20-2\sqrt{35}$	$[65, 10+\sqrt{35}]$	4		
-1	$[1, \sqrt{35}]$	1	4	
$-6-\sqrt{35}$	$[4, 4\sqrt{35}]$	8		2
-2	$[4, 4\sqrt{35}]$	2		
$-12-2\sqrt{35}$	$[1, \sqrt{35}]$	1		

$F=\boldsymbol{Q}(\sqrt{37})$, $d=37$, $\varepsilon=6+\sqrt{37}$, $\varepsilon_{+}=73+12\sqrt{37}$, $h=1$, $h_{+}=1$

Δ	ϑ	H/h	w_K	Q_K
-2	$[8, 4+4\sqrt{37}]$	5		
-3	$[3, (3+3\sqrt{37})/2]$	4	6	
$(-13-\sqrt{37})/2$	$[33, (13+\sqrt{37})/2]$	2		
$(-7-\sqrt{37})/2$	$[12, 2+2\sqrt{37}]$	2		
$-13-2\sqrt{37}$	$[21, (17+\sqrt{37})/2]$	2		
-7	$[7, (7+(\sqrt{37})/2]$	2		
$(-81-13\sqrt{37})/2$	$[77, (101+\sqrt{37})/2]$	2		
$-37-6\sqrt{37}$	$[37, (37+\sqrt{37})/2]$	1		
$(-33-5\sqrt{37})/2$	$[41, (23+\sqrt{37})/2)$	3		
$(-45-7\sqrt{37})/2$	$[53, (67+\sqrt{37})/2]$	3		
$(-25-3\sqrt{37})/2$	$[73, (57+\sqrt{37})/2]$	7		
-1	$[4, 2+2\sqrt{37}]$	1	4	

$F=\boldsymbol{Q}(\sqrt{38})$, $d=152$, $\varepsilon=37+6\sqrt{38}$, $\varepsilon_+=37+6\sqrt{38}$, $h=1$, $h_+=2$

Δ	ϑ	H/h	w_K	Q_K
-3	$[3, 3\sqrt{38}]$	4	6	
$-111-18\sqrt{38}$	$[6, 6\sqrt{38}]$	4		
-5	$[10, 10\sqrt{38}]$	4		
$-185-30\sqrt{38}$	$[5, 5\sqrt{38}]$	8		
-7	$[7, 7\sqrt{38}]$	10		
$-25-4\sqrt{38}$	$[34, 4+2\sqrt{38}]$	4		
$-13-2\sqrt{38}$	$[17, 15+\sqrt{38}]$	2		
$-99-16\sqrt{38}$	$[73, 29+\sqrt{38}]$	4		
-1	$[2, 2\sqrt{38}]$	6	4	2
$-37-6\sqrt{38}$	$[1, \sqrt{38}]$	1		2

$F=\boldsymbol{Q}(\sqrt{39})$, $d=156$, $\varepsilon=25+4\sqrt{39}$, $\varepsilon_+=25+4\sqrt{39}$, $h=2$, $h_+=4$

Δ	ϑ	H/h	w_K	Q_K
-5	$[5, 5\sqrt{39}]$	4		
$-125-20\sqrt{39}$	$[5, 5\sqrt{39}]$	8		
-7	$[7, 7\sqrt{39}]$	4		
$-175-28\sqrt{39}$	$[7, 7\sqrt{39}]$	4		
$-13-2\sqrt{39}$	$[26, 2\sqrt{39}]$	4		
-1	$[1, \sqrt{39}]$	2	4	
$-25-4\sqrt{39}$	$[1, \sqrt{39}]$	2	6	2
-2	$[4, 4\sqrt{39}]$	2		
$-50-8\sqrt{39}$	$[4, 4\sqrt{39}]$	6		

$F=\boldsymbol{Q}(\sqrt{79})$, $d=316$, $\varepsilon=80+9\sqrt{79}$, $\varepsilon_{+}=80+9\sqrt{79}$, $h=3$, $h_{+}=6$

Δ	ϑ	H/h	w_K	Q_K
$-9-\sqrt{79}$	$[8, 4+4\sqrt{79}]$	8		
-3	$[3, 3\sqrt{79}]$	6	6	
$-267-30\sqrt{79}$	$[42, 8+2\sqrt{79}]$	4		
$-89-10\sqrt{79}$	$[42, 22+2\sqrt{79}]$	12		
-5	$[5, 5\sqrt{79}]$	8		
$-43-4\sqrt{79}$	$[65, 27+\sqrt{79}]$	24		
$-19-2\sqrt{79}$	$[10, 4+2\sqrt{79}]$	4		
$-249-28\sqrt{79}$	$[65, 53+\sqrt{79}]$	4		
$-98-11\sqrt{79}$	$[20, 12+4\sqrt{79}]$	12		
-7	$[7, 7\sqrt{79}]$	4		
$-409-46\sqrt{79}$	$[26, 24+2\sqrt{79}]$	16		
$-107-12\sqrt{79}$	$[73, 15+\sqrt{79}]$	34		
$-145-16\sqrt{79}$	$[89, 48+\sqrt{79}]$	14		
$-77-8\sqrt{79}$	$[97, 46+\sqrt{79}]$	4		
-1	$[1, \sqrt{79}]$	5	4	2
$-80-9\sqrt{79}$	$[4, 4\sqrt{79}]$	8		2

$F=\boldsymbol{Q}(\sqrt{229})$, $d=229$, $\varepsilon=(15+\sqrt{229})/2$, $\varepsilon_+=(227+15\sqrt{229})/2$, $h=3$, $h_+=3$

Δ	ϑ	H/h	w_K	Q_K
-2	$[8, 4+4\sqrt{229}]$	13		
$-93-6\sqrt{229}$	$[15, (9+3\sqrt{229})/2]$	4		
-3	$[3, (3+3\sqrt{229})/2]$	6	6	
$(-321-21\sqrt{229})/2$	$[57, (37+\sqrt{229})/2]$	12		
$-16-\sqrt{229}$	$[12, 2+2\sqrt{229}]$	10		
$(-57-3\sqrt{229})/2$	$[33, (41+\sqrt{229})/2]$	10		
$(-49-3\sqrt{229})/2$	$[85, (73+\sqrt{229})/2]$	16		
$-41-2\sqrt{229}$	$[85, (63+\sqrt{229})/2]$	2		
$-31-2\sqrt{229}$	$[20, 6+2\sqrt{229}]$	6		
-7	$[7, (7+7\sqrt{229})/2]$	3		
$(-29-\sqrt{229})/2$	$[17, (29+\sqrt{229})/2]$	5		
$(-21-\sqrt{229})/2$	$[53, (21+\sqrt{229})/2]$	5		
$(-89-5\sqrt{229})/2$	$[61, (91+\sqrt{229})/2]$	7		
$(-61-\sqrt{229})/2$	$[97, (61+\sqrt{229})/2]$	7		
-1	$[4, 2+2\sqrt{229}]$	5	4	

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