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On elliptic cyclopean forms

By

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Introduction

Shimura [3] defined cyclopean forms and showed that they are closely related to the zeros of L-functions by giving the necessary and sufficient condition on the existance of cyclopean forms. The purpose of this paper is to give generators of the space of cyclopean forms in the elliptic modular case and to prove that the examples he gave in [3] exhaust all cyclopean forms.

Let Γ be a congruence modular group. An automorphic eigenform is a function on the upper half complex plane $\mathbf{H}$ which is an eigen function of the differential operator $L_k = -4y^2\dfrac{\partial^2}{\partial z\partial\bar{z}} + 2iky\dfrac{\partial}{\partial\bar{z}}$ and satisfies the automorphic condition (1.2*a*) and the magnitude condition (1.2*c*) at cusps. We denote by $\mathscr{A}_k(\Gamma, \lambda)$ the space of automorphic eigenforms belonging to the eigenvalue λ. We say f ($\in \mathscr{A}_k(\Gamma, \lambda)$) is a cusp form if it satisfies (1.2*d*) and denote by $\mathscr{S}_k(\Gamma, \lambda)$ the subspace of $\mathscr{A}_k(\Gamma, \lambda)$ consisting of all cusp forms. We also denote by $\mathscr{N}_k(\Gamma, \lambda)$ the orthogonal complement of $\mathscr{S}_k(\Gamma, \lambda)$ in $\mathscr{A}_k(\Gamma, \lambda)$ with respect to the Petersson inner product defined by (1.3). Any element $f(z)$ ($\in \mathscr{A}_k(\Gamma, \lambda)$) has a Fourier expansion of the form

$$\begin{aligned} f(z) &= cy^{s_0} + dy^{1-k-s_0} \\ &\quad + \sum_{n=1}^{\infty} a_n\omega(4\pi ny/T;\, k+s_0,\, s_0)e^{2\pi inz/T} \\ &\quad + y^{-k}\sum_{n=1}^{\infty} b_n\omega(4\pi ny/T;\, s_0,\, k+s_0)e^{-2\pi in\bar{z}/T}, \end{aligned}$$

where $\omega(t;\alpha,\beta)$ is the Whittaker function (see §1), $y = \mathrm{Im}(z)$ and s_0 is a complex number satisfying $\lambda = s_0(1-k-s_0)$. We call $f(z) \in \mathscr{N}_k(\Gamma, \lambda)$ a *cyclopean form* if

$$(c1) \qquad \lambda = s_0(1-k-s_0) \quad \text{with} \quad -\frac{k}{2} < \mathrm{Re}(s_0) < \frac{1-k}{2},$$

(c2) the term cy^{s_0} of the Fourier expansion of $f|_k\gamma$ vanishes for all $\gamma \in SL_2(\mathbf{Z})$.

Here "$|_k\gamma$" indicates the operation of γ to functions on $\mathbf{H}$ defined by (1.1). We denote by $\mathscr{C}_k(\Gamma, \lambda)$ the subspace of $\mathscr{N}_k(\Gamma, \lambda)$ consisting of all cyclopean forms. To discuss generators of $\mathscr{C}_k(\Gamma, \lambda)$, we have only to consider the case where $\Gamma = \Gamma(N)$, since $\mathscr{N}_k(\Gamma, \lambda) \subset \mathscr{N}_k(\Gamma', \lambda)$ if $\Gamma \supset \Gamma'$. For a Dirichlet character χ defined modulo

N, we define an Eisenstein series

$$E_k(z, s; \chi) = y^s \sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \chi(n)(mNz+n)^{-k}|mNz+n|^{-2s} \qquad (z\in \mathbf{H},\ s\in \mathbf{C}).$$

Then it is convergent for $\mathrm{Re}(s) > 1 - \frac{k}{2}$ and continued meromorphically to the whole s-plane. By $\ll \cdots \gg$, we denote the vector space over $\mathbf{C}$ generated by $\cdots$. For a Dirichlet character χ, $L(s, \chi)$ denotes the Dirichlet L-function. Now our main theorem is stated as follows:

Theorem. *Let λ be a complex number expressed as $\lambda = s_0(1-k-s_0)$ with $-\frac{k}{2} < \mathrm{Re}(s_0) < \frac{1-k}{2}$. Then*

$$\mathscr{C}_k(\Gamma(N), \lambda) = \ll E_k(z, s_0; \chi)|_k\gamma \gg.$$

Here χ runs over all Dirichlet characters defined modulo N satisfying $\chi(-1)=(-1)^k$ and $L(2s_0+k, \chi)=0$, and γ runs over a complete set of representatives of $\Gamma_0(N)\backslash\Gamma(1)$.

1. Automorphic eigenforms

For a positive integer N, we let $\Gamma(N)$, $\Gamma_0(N)$ and $\Gamma_1(N)$ be the modular groups defined by

$$\Gamma(N) = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \Big| a\equiv d\equiv 1,\ b\equiv c\equiv 0 \pmod N\right\},$$

$$\Gamma_1(N) = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \Big| a\equiv d\equiv 1,\ c\equiv 0 \pmod N\right\},$$

$$\Gamma_0(N) = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \Big| c\equiv 0 \pmod N\right\}.$$

We call a subgroup Γ of $SL_2(\mathbf{Z})$ a congruence subgroup if $\Gamma \supset \Gamma(N)$ for some N. Let $\mathbf{H}$ be the upper half complex plane. For $\alpha = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL_2(\mathbf{R})$, we put $j(\alpha, z) = cz + d$ as usual. Let k be an integer. For a function $f(z)$ on $\mathbf{H}$ we define an operation of α $(\in SL_2(\mathbf{R}))$ by

$$(1.1) \qquad (f|_k\alpha)(z) = f\left(\frac{az+b}{cz+d}\right) j(\alpha, z)^{-k}.$$

We also define a differential operator L_k on $\mathbf{H}$ by

$$L_k = -4y^2\frac{\partial^2}{\partial z\partial\bar{z}} + 2iky\frac{\partial}{\partial\bar{z}}.$$

Here and hereafter we write $z = x + iy$ for $z \in \mathbf{H}$. For a congruence subgroup Γ and a complex number λ, we denote by $\mathscr{A}_k(\Gamma, \lambda)$ the space of all complex valued real-analytic functions satisfying

(1.2*a*) $$f|_k\gamma = f \qquad (\gamma \in \Gamma),$$

(1.2*b*) $$L_k f = \lambda f,$$

(1.2*c*) for each $\gamma \in \Gamma(1)$, there exist positive constants A, B and c such that

$$y^{k/2}|(f|_k\gamma)(x+iy)| \leq Ay^c \qquad (y > B).$$

We call elements f of $\mathscr{A}_k(\Gamma, \lambda)$ *automorphic eigenforms* of weight k with respect to Γ belonging to an eigenvalue λ. Furthermore, if an element f of $\mathscr{A}_k(\Gamma, \lambda)$ satisfies the following condition, we call it a *cusp form*:

(1.2*d*) for each $\gamma \in \Gamma(1)$, there exist positive constants A, B and C such that

$$|(f|_k\gamma)(x+iy)| \leq A\exp(-Cy) \qquad (y > B).$$

We denote by $\mathscr{S}_k(\Gamma, \lambda)$ the subspace of $\mathscr{A}_k(\Gamma, \lambda)$ consisting of all cusp forms. For two elements f and g of $\mathscr{A}_k(\Gamma, \lambda)$, we put

(1.3) $$\langle f, g\rangle = \mu(\Gamma\backslash\mathbf{H})^{-1}\int_{\Gamma\backslash\mathbf{H}} \bar{f} g y^{k-2}\,dx\,dy,$$

where $\mu(\Gamma\backslash\mathbf{H})$ is the volume of $\Gamma\backslash\mathbf{H}$ with respect to the invariant measure $y^{-2}dx\,dy$. The right-hand side of (1.3) is convergent if either f or g is a cusp form. We call $\langle\,,\rangle$ the Petersson inner product of $\mathscr{A}_k(\Gamma, \lambda)$. We denote by $\mathscr{N}_k(\Gamma, \lambda)$ the orthogonal complement of $\mathscr{S}_k(\Gamma, \lambda)$ in $\mathscr{A}_k(\Gamma, \lambda)$ with respect to the Petersson inner product.

To explain the Fourier expansion of a function $f(z)$ in $\mathscr{A}_k(\Gamma, \lambda)$, we need the Whittaker function. We put $\mathbf{R}_+ = \{t \in \mathbf{R} \mid t > 0\}$. Then the Whittaker function $\omega(t; \alpha, \beta)$ is the function on $\mathbf{R}_+ \times \mathbf{C} \times \mathbf{C}$ which is holomorphic in α and β, real-analytic in t and has an integral expression

(1.4) $$\omega(t; \alpha, \beta) = t^\beta\Gamma(\beta)^{-1}\int_0^\infty e^{-ut}(1+u)^{\alpha-1}u^{\beta-1}\,du \qquad \text{for } \operatorname{Re}(\beta) > 0.$$

It has the following properties:

(1.5*a*) $$\omega(t; 1-\beta, 1-\alpha) = \omega(t; \alpha, \beta),$$

(1.5*b*) $$\omega(t; \alpha, 0) = 1,$$

(1.5*c*) $$\lim_{t\to\infty}\omega(t; \alpha, \beta) = 1.$$

For a complex number λ, let s_0 be a root of the quadratic equation

(1.6) $$X^2 - (1-k)X + \lambda = 0.$$

Then $f(z)(\in \mathscr{A}_k(\Gamma, \lambda))$ has a Fourier expansion of the form

$$
\begin{aligned}
f(z) &= cy^{s_0} + dy^{1-k-s_0} \\
&+ \sum_{n=1}^{\infty} a_n \omega(4\pi ny/T;\, k+s_0,\, s_0) e^{2\pi inz/T} \qquad (1.7)\\
&+ y^{-k} \sum_{n=1}^{\infty} b_n \omega(4\pi ny/T;\, s_0,\, k+s_0) e^{-2\pi in\bar{z}/T}
\end{aligned}
$$

with a positive constant T. Since the other root of (1.6) is $1-k-s_0$, $\omega(4\pi ny/T;\, k+s_0,\, s_0)$ and $\omega(4\pi ny/T;\, s_0,\, k+s_0)$ are independent of the choice of s_0. We call cy^{s_0} and dy^{1-k-s_0} the *constant terms* of the Fourier expansion of $f(z)$. We see easily that $f(z)$ is a cusp form if and only if the constant terms of the Fourier expansion of $f|_k\gamma$ vanish for all $\gamma \in \Gamma(1)$. We call $f(z) \in \mathscr{N}_k(\Gamma, \lambda)$ a *cyclopean form* if it satisfies the following two conditions:

(1.8*a*) $$\lambda = s_0(1-k-s_0) \quad \text{with} \quad -\frac{k}{2} < \mathrm{Re}(s_0) < \frac{1-k}{2},$$

(1.8*b*) the constant term cy^{s_0} of the Fourier expansion of $f|_k\gamma$ vanishes for all $\gamma \in \Gamma(1)$.

We denote by $\mathscr{C}_k(\Gamma, \lambda)$ the subspace of $\mathscr{N}_k(\Gamma, \lambda)$ consisting of all cyclopean forms.

2. Functional equation of Eisenstein series

We shall define Eisenstein series with characters. Let χ and ψ be Dirichlet characters defined modulo L and modulo M, respectively. We put

(2.1) $$E_k(z, s; \chi, \psi) = y^s \sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} \chi(m)\psi(n)(mz+n)^{-k}|mz+n|^{-2s} \qquad (z \in \mathbf{H},\ s \in \mathbf{C})$$

and

(2.2) $$E_k^*(z, s; \chi, \psi) = y^s \sum_{\substack{m,n=-\infty \\ (m,n)=1}}^{\infty} \chi(m)\psi(n)(mz+n)^{-k}|mz+n|^{-2s} \qquad (z \in \mathbf{H},\ s \in \mathbf{C}).$$

We see easily that these Eisenstein series are convergent for $\mathrm{Re}(s) > 1 - k/2$ and satisfy

(2.3) $$E_k(z, s; \chi, \psi) = L(2s+k, \chi\psi) E_k^*(z, s; \chi, \psi).$$

Here we consider $\chi\psi$ as a Dirichlet character defined modulo L.C.D. (L, M), and $L(s, \chi\psi)$ is the Dirichlet L-function with character $\chi\psi$. We also see that

$$E_k(z, s; \chi, \psi) = 0 \quad \text{if} \quad \chi(-1)\psi(-1) \neq (-1)^k.$$

The proof of the following theorem can be seen in [2].

Theorem 1. *Let χ and ψ be primitive Dirichlet characters of conductor L and of conductor M, respectively. If $\chi(-1)\psi(-1) = (-1)^k$, then the Eisenstein series $E_k(z, s; \chi, \psi)$ has the following Fourier expansion*:

$$\begin{aligned} E_k(z, s; \chi, \psi) = {} & C(s)y^s + D(s)y^{1-k-s} \\ & + A(s)\sum_{n=1}^{\infty} a_n(s)n^{-s}\omega(4\pi yn/M; k+s, s)e^{2\pi inz/M} \\ & + B(s)y^{-k}\sum_{n=1}^{\infty} a_n(s)n^{-k-s}\omega(4\pi yn/M; s, k+s)e^{-2\pi in\bar{z}/M}, \end{aligned}$$

where

$$C(s) = \begin{cases} 2L(2s+k, \psi) & (L = 1), \\ 0 & (L \neq 1), \end{cases}$$

$$D(s) = \begin{cases} 2\sqrt{\pi}\, i^{-k}\Gamma(s)^{-1}\Gamma(s+k)^{-1}\Gamma\left(\frac{2s+k-1}{2}\right)\Gamma\left(\frac{2s+k}{2}\right)L(2s+k-1, \chi) & (M = 1), \\ 0 & (M \neq 1), \end{cases}$$

$$A(s) = 2^{k+1}i^{-k}W(\psi)(\pi/M)^{k+s}\Gamma(k+s)^{-1},$$

$$B(s) = 2^{1-k}i^{-k}\psi(-1)W(\psi)(\pi/M)^s\Gamma(s)^{-1},$$

$$a_n(s) = \sum_{0<c|n} \chi(n/c)\bar{\psi}(c)c^{k+2s-1}.$$

Here $L(s, \psi)$ is the Dirichlet L-function with character ψ and $W(\psi)$ is the Gauss sum of ψ.

The summations in the right-hand side are uniformly convergent on any compact subset of $\mathbf{H} \times \mathbf{C}$. By defining the function $E_k(z, s; \chi, \psi)$ on $\mathbf{H} \times \mathbf{C}$ by the right-hand side of the equality of the theorem, $E_k(z, s; \chi, \psi)$ can be continued meromorphically to the whole s-plane. We call the Dirichlet character defined modulo 1 the *principal character* and denote it by ε_0. We see easily by Theorem 1 the following

Theorem 2. (1) *For primitive Dirichlet characters χ and ψ, $E_k(z, s; \chi, \psi)$ is holomorphic on the whole s-plane except for the case where $k = 0$ and $\chi = \psi = \varepsilon_0$.*

(2) *$E_0(z, s; \varepsilon_0, \varepsilon_0)$ is holomorphic on the whole s-plane except for a simple pole at $s = 1$.*

Let χ and ψ be any (not necessarily primitive) Dirichlet characters defined modulo L and defined modulo M, respectively. For a prime number p, we denote by χ' (resp. ψ') the Dirichlet character defined modulo pL (resp. modulo pM) induced from χ (resp. ψ). Then we see easily that

(2.4a) $$E_k(z, s; \chi, \psi') = E_k(z, s; \chi, \psi) - \psi(p)p^{-k-s}E_k\left(\frac{z}{p}, s; \chi, \psi\right)$$

and that

(2.4b) $$E_k(z, s; \chi', \psi) = E_k(z, s; \chi, \psi) - \chi(p)p^{-s}E_k(pz, s; \chi, \psi).$$

Therefore applying the equalities (2.4*a*, *b*) repeatedly, we obtain by Theorem 2 that

(2.5) for any (not necessarily primitive) Dirichlet characters χ and ψ, $E_k(z, s; \chi, \psi)$ is holomorphic on the whole s-plane (except for $s = 1$ if $k = 0$).

Now let χ and ψ be primitive Dirichlet characters of conductor L and of conductor M, respectively, and put

(2.6) $$F_k(z, s; \chi, \psi) = \left(\frac{\pi}{L}\right)^{-s/2}\left(\frac{\pi}{M}\right)^{(-s+k)/2}\Gamma(s+k)\,E_k(z, s; \chi, \psi).$$

Then using the functional equation of L-functions and properties of Γ-function, we can easily prove the following functional equation.

Theorem 3. *Let χ and ψ be the same as in Theorem* 1. *Then*

$$F_k(Mz, 1-k-s; \chi, \psi) = \sqrt{\frac{L}{M}}\frac{W(\psi)}{W(\bar{\chi})}F_k(Lz, s; \bar{\psi}, \bar{\chi}).$$

3. Spaces of Eisenstein series

For a Dirichlet character χ defined modulo N, we put

$$E_k(z, s; \chi) = N^{-s}E_k(Nz, s; \varepsilon_0, \chi)$$

and

$$E_k^*(z, s; \chi) = N^{-s}E_k^*(Nz, s; \varepsilon_0, \chi).$$

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, we put

$$\chi(\gamma) = \chi(d).$$

Then we see easily that for $\gamma \in \Gamma_0(N)$,

(3.1) $$E_k(z, s; \chi)|_k\gamma = \bar{\chi}(\gamma)E_k(z, s; \chi), \quad E_k^*(z, s; \chi)|_k\gamma = \bar{\chi}(\gamma)E_k^*(z, s; \chi).$$

We define the spaces of Eisenstein series $\mathscr{E}_k(N)$ and $\mathscr{E}_k^{(1)}(N)$ by

$$\mathscr{E}_k(N) = \left\langle\left\langle E_k(z, s; \chi)|_k\gamma \,\middle|\, \begin{matrix}\chi\text{: Dirichlet characters defined modulo } N \\ \gamma \in \Gamma_0(N)\backslash\Gamma(1)\end{matrix}\right\rangle\right\rangle$$

and

$$\mathscr{E}_k^{(1)}(N) = \left\langle\!\!\left\langle E_k\left(\frac{u}{v}z, s; \chi, \psi\right) \;\middle|\; \begin{array}{l} \chi : \textit{a primitive character of conductor } L \\ \psi : \textit{a primitive character of conductor } M \\ 0 < u,\ v \in \mathbf{Z},\ uL \mid N,\ vM \mid N \end{array} \right\rangle\!\!\right\rangle .$$

We shall define Eisenstein series of slightly different type. Let L and M be positive integers. For two integers a and b, we put

$$E_k(z, s; a \bmod L, b \bmod M) = y^s \sum_{\substack{m \equiv a \bmod L \\ n \equiv b \bmod M \\ (m,n) \neq (0,0)}} (mz+n)^{-k} |mz+n|^{-2s} \qquad (z \in \mathbf{H},\ s \in \mathbf{C}), \tag{3.2}$$

$$E_k^*(z, s; a \bmod L, b \bmod M) = y^s \sum_{\substack{m \equiv a \bmod L \\ n \equiv b \bmod M \\ (m,n) = 1}} (mz+n)^{-k} |mz+n|^{-2s} \qquad (z \in \mathbf{H},\ s \in \mathbf{C}). \tag{3.3}$$

It is easy to see that they are convergent for $\mathrm{Re}(s) > 1 - k/2$. We also see that for Dirichlet characters χ defined modulo L and ψ defined modulo M,

$$E_k(z, s; \chi, \psi) = \sum_{\substack{0 \le a < L \\ 0 \le b < M}} \chi(a)\psi(b) E_k(z, s; a \bmod L, b \bmod M) \tag{3.4}$$

and

$$E_k^*(z, s; \chi, \psi) = \sum_{\substack{0 \le a < L \\ 0 \le b < M}} \chi(a)\psi(b) E_k^*(z, s; a \bmod L, b \bmod M). \tag{3.5}$$

Conversely, we see that for integers a $(0 \le a < L)$ and b $(0 \le b < M)$,

$$E_k(z, s; a \bmod L, b \bmod M) = \frac{u^{-s} v^{-k-s}}{\varphi(L')\varphi(M')} \sum_{\chi,\psi} \bar{\chi}(c)\bar{\psi}(d) E_k\left(\frac{u}{v}z, s; \chi, \psi\right), \tag{3.6}$$

where the summation is taken over all Dirichlet characters χ defined modulo L' and ψ defined modulo M', φ is the Euler function and $u = (a, L)$, $c = a/u$, $L' = L/u$, $v = (b, M)$, $d = b/v$, $M' = M/v$. In particular, $E_k(z, s; a \bmod L, b \bmod M)$ is continued meromorphically to the whole s-plane. For a Dirichlet character χ defined modulo N, we see that

$$E_k(z, s; \chi) = \sum_{0 \le a < N} \chi(a) E_k(z, s; 0 \bmod N, a \bmod N), \tag{3.7}$$

and that if $(a, N) = 1$, then

$$E_k(z, s; 0 \bmod N, a \bmod N) = \frac{1}{\varphi(N)} \sum_{\chi} \bar{\chi}(a) E_k(z, s; \chi), \tag{3.8}$$

where χ runs over all Dirichlet characters defined modulo N. One of the basic properties of $E_k(z, s; a \bmod N, b \bmod N)$ is that for $\gamma \in \Gamma(1)$,

(3.9) $$E_k(z, s; a \bmod N, b \bmod N)|_k \gamma = E_k(z, s; a' \bmod N, b' \bmod N),$$

where

$$(a', b') \equiv (a, b)\gamma \bmod N.$$

We also put

$$\mathscr{E}_k^{(2)}(N) = \ll E_k(z, s; a \bmod N, b \bmod N) \mid 0 \le a, b < N \gg.$$

By (3.9), $\mathscr{E}_k^{(2)}(N)$ is stable under the operation of $\Gamma(1)$.

Now for a fixed complex number s_0, we put

$$\mathscr{E}_k[N, s_0] = \{f(z, s) \in \mathscr{E}_k(N) \mid f(z, s) \text{ is holomorphic at } s = s_0\},$$

and

$$\mathscr{E}_k(N, s_0) = \{f(z, s_0) \mid f(z, s) \in \mathscr{E}_k[N, s_0]\}.$$

The spaces $\mathscr{E}_k^{(i)}[N, s_0]$ and $\mathscr{E}_k^{(i)}(N, s_0)$ $(i = 1, 2)$ are similarly defined.

Theorem 4. *For any complex number s_0 (except for $s_0 = 1$ if $k = 0$), we have:*

(1) $\mathscr{E}_k[N, s_0] = \mathscr{E}_k(N)$, $\mathscr{E}_k^{(1)}[N, s_0] = \mathscr{E}_k^{(1)}(N)$, $\mathscr{E}_k^{(2)}[N, s_0] = \mathscr{E}_k^{(2)}(N)$.

(2) *If* $\mathrm{Re}(s_0) \neq -\frac{k}{2}$, *then*

$$\mathscr{E}_k(N, s_0) = \mathscr{E}_k^{(1)}(N, s_0) = \mathscr{E}_k^{(2)}(N, s_0).$$

Proof. The first assertion follows from (2.5) and (3.6). Let L and M be positive integers, and χ and ψ be any Dirichlet characters defined modulo L and modulo M, respectively. For positive integers u, v satisfying $uL|N$ and $vM|N$,

(3.10) $$E_k\left(\frac{u}{v}z, s; \chi, \psi\right) = \sum_{\substack{0 \le a < L \\ 0 \le b < M}} \chi(a)\psi(b) E_k\left(\frac{u}{v}z, s; a \bmod L, b \bmod M\right)$$

$$= u^s v^{k+s} \sum_{\substack{0 \le a < L \\ 0 \le b < M}} \chi(a)\psi(b) E_k(z, s; au \bmod uL, bv \bmod vM),$$

and

(3.11) $$E_k(z, s; au \bmod uL, bv \bmod vM)$$

$$= \sum_{\substack{0 \le l < c \\ 0 \le r < d}} E_k(z, s; au + luL \bmod N, bv + rvM \bmod N),$$

where $c = N/uL$, $d = N/vM$. In particular, $\mathscr{E}_k^{(1)}(N, s_0) \subset \mathscr{E}_k^{(2)}(N, s_0)$. Conversely, applying (2.4a, b) repeatedly, we see that $E_k\left(\frac{u}{v}z, s_0; \chi, \psi\right)$ belongs to $\mathscr{E}_k^{(1)}(N, s_0)$

for not necessarily primitive characters χ, ψ. Thus (3.6) implies

$$\mathscr{E}_k^{(1)}(N, s_0) = \mathscr{E}_k^{(2)}(N, s_0).$$

Next we see by (3.7) and (3.9) that $\mathscr{E}_k(N, s_0) \subset \mathscr{E}_k^{(2)}(N, s_0)$. Lastly we shall show the converse inclusion, or $\mathscr{E}_k^{(2)}(N, s_0) \subset \mathscr{E}_k(N, s_0)$. By (3.1), (3.7), (3.8) and (3.9), we see that

$$\mathscr{E}_k(N) = \ll E_k(z, s; a \text{ mod } N, b \text{ mod } N) \mid 0 \le a, b < N, (a, b, N) = 1 \gg.$$

For each positive divisor t of N, we put

$$\mathscr{E}_k(N)^t = \ll E_k(z, s; a \text{ mod } N, b \text{ mod } N) \mid 0 \le a, b < N, (a, b, N) = t \gg$$

and for each $s_0 \in \mathbf{C}$,

$$\mathscr{E}_k(N, s_0)^t = \{f(z, s_0) \mid f(z, s) (\in \mathscr{E}_k(N)^t)\text{: } \textit{holomorphic at } s = s_0\}.$$

Then $\mathscr{E}_k(N)^1 = \mathscr{E}_k(N)$ and $\mathscr{E}_k(N, s_0)^1 = \mathscr{E}_k(N, s_0)$. Since $\mathscr{E}_k^{(2)}(N, s_0) = \sum_{t|N} \mathscr{E}_k(N, s_0)^t$, we have only to show that $\mathscr{E}_k(N, s_0)^t \subset \mathscr{E}_k(N, s_0)$ for all positive divisors t of N. Let $t = (a, b, N)$ and put $a' = a/t$, $b' = b/t$ and $N' = N/t$. Then we see that

$$E_k(z, s; a \text{ mod } N, b \text{ mod } N) = t^{-2s-k} E_k(z, s; a' \textit{ mod } N', b' \text{ mod } N').$$

This implies that

$$\mathscr{E}_k(N, s_0)^t = \mathscr{E}_k(N/t, s_0)$$

for any divisor t of N. Therefore we have only to prove

$$\mathscr{E}_k(N/t, s_0) \subset \mathscr{E}_k(N, s_0) \tag{3.12}$$

for each prime divisor t of N. Put $N' = N/t$. First assume that N' is divisible by t. Since $(cN', 1 + dN', N) = 1$ for all integers c and d,

$$\begin{aligned} &E_k(z, s; 0 \text{ mod } N', 1 \text{ mod } N') \\ &\quad = \sum_{0 \le c, d < t} E_k(z, s; cN' \text{ mod } N, 1 + dN' \text{ mod } N) \in \mathscr{E}_k(N), \end{aligned}$$

and therefore, $\mathscr{E}_k(N') \subset \mathscr{E}_k(N)$. This implies (3.12). Next we assume that t is prime to N'. For integers u $(0 \le u < N)$ and v $(0 \le v < t)$, we denote by $\mu_{u,v}$ the integer satisfying

$$0 \le \mu_{u,v} < N, \ \mu_{u,v} \equiv u \text{ mod } N', \ \mu_{u,v} \equiv v \text{ mod } t.$$

Let a be an integer such that $0 \le a < N'$ and $(a, N') = 1$. For integers c, d $(0 \le c, d < t)$, we see easily that $(\mu_{0,c}, \mu_{a,d}, N) \ne 1$ if and only if $c = d = 0$. Now

$$\begin{aligned} &E_k(z, s; 0 \text{ mod } N', a \text{ mod } N') \\ &\quad = \sum_{0 \le c, d < t} E_k(z, s; \mu_{0,c} \text{ mod } N, \mu_{a,d} \text{ mod } N) \\ &\quad = E_k(z, s; \mu_{0,0} \text{ mod } N, \mu_{a,0} \text{ mod } N) \end{aligned}$$

$$+ \sum_{\substack{0 \le c, d < t \\ (c,d) \ne (0,0)}} E_k(z, s; \mu_{0,c} \bmod N, \mu_{a,d} \bmod N).$$

Since $E_k(z, s; \mu_{0,c} \bmod N, \mu_{a,d} \bmod N) \in \mathscr{E}_k(N)$ for $(c, d) \ne (0, 0)$ and

$$E_k(z, s; \mu_{0,0} \bmod N, \mu_{a,0} \bmod N) = t^{-k-2s} E_k(z, s; 0 \bmod N', a' \bmod N')$$

with the integer a' satisfying $0 \le a' < N'$ and $ta' \equiv a \bmod N$, the subtraction of Eisenstein series

$$(3.13) \qquad E_k(z, s; 0 \bmod N', a \bmod N') - t^{-k-2s} E_k(z, s; 0 \bmod N', a' \bmod N')$$

belongs to $\mathscr{E}_k(N)$. Let χ be a Dirichlet character defined modulo N'. Multiply (3.13) by $\chi(a)$, and take the summation over integers a such that $0 \le a < N'$ and $(a, N') = 1$. Then we see by (3.7) that $(1 - \chi(t)t^{-k-2s}) E_k(z, s; \chi)$ belongs to $\mathscr{E}_k(N)$. Since $\mathrm{Re}(k + 2s_0) \ne 0$,

$$\mathscr{E}_k(N', s_0) \subset \mathscr{E}_k(N, s_0).$$

Thus we complete the proof.

Theorem 5. *If* $\mathrm{Re}(s_0) > \dfrac{1-k}{2}$ *and* $s_0 \ne 1 - \dfrac{k}{2}$, *then*

$$\mathscr{N}_k(\Gamma(N), \lambda) = \mathscr{E}_k(N, s_0),$$

where $\lambda = s_0(1 - k - s_0)$.

Proof. Put

$$\Gamma(N)_\infty = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N) \,\middle|\, c = 0 \right\}.$$

Then

$$(3.14) \qquad E_k^*(z, s; 0 \bmod N, 1 \bmod N) = \delta \sum_{\gamma \in \Gamma(N)_\infty \backslash \Gamma(N)} y^s|_k \gamma$$

with $\delta = 2$ $(N \le 2)$ or $= 1$ $(N \ge 3)$. Therefore [3, Theorem 7.3] implies that

$$\mathscr{N}_k(\Gamma(N), \lambda) = \langle\!\langle E_k^*(z, s_0; 0 \bmod N, 1 \bmod N)|_k \gamma \mid \gamma \in \Gamma(1) \rangle\!\rangle.$$

Now we see

$$E_k^*(z, s; 0 \bmod N, 1 \bmod N) = \frac{1}{\varphi(N)} \sum_\chi E_k^*(z, s; \chi)$$

$$= \frac{1}{\varphi(N)} \sum_\chi L(2s + k, \chi)^{-1} E_k(z, s; \chi),$$

where χ runs over all Dirichlet characters defined modulo N. Since $L(2s_0 + k, \chi) \ne 0$, $E_k^*(z, s_0; 0 \bmod N, 1 \bmod N)$ belongs to $\mathscr{E}_k(N, s_0)$. As $\mathscr{E}_k(N, s_0)$ is stable under the operation of $\Gamma(1)$ by (3.9) and Theorem 4(2), $\mathcal{N}_k(\Gamma(N), \lambda) \subset \mathcal{E}_k(N, s_0)$. To see the converse inclusion, we take, for each a satisfying $0 < a < N$ and $(a, N) = 1$, an element $\gamma_a \in \Gamma(1)$ so that

$$\gamma_a \equiv \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \mod N.$$

Then

$$E_k(z, s; \chi) = L(2s + k, \chi) E_k^*(z, s; \chi) \tag{3.15}$$

$$= L(2s + k, \chi) \sum_{\substack{0<a<N \\ (a,N)=1}} \chi(a) E_k^*(z, s; 0 \bmod N, 1 \bmod N)|_k \gamma_a,$$

thus $E_k(z, s_0; \chi)$ belongs to $\mathcal{N}_k(\Gamma(N), \lambda)$. Since $\mathcal{N}_k(\Gamma(N), \lambda)$ is stable under the operation of $\Gamma(1)$, we see that

$$\mathcal{E}_k(N, s_0) \subset \mathcal{N}_k(\Gamma(N), \lambda).$$

Theorem 6. *If* $-\frac{k}{2} < \mathrm{Re}(s_0) < \frac{1-k}{2}$, *then*

$$\mathcal{E}_k(N, s_0) = \mathcal{N}_k(\Gamma(N), \lambda),$$

where $\lambda = s_0(1 - k - s_0)$.

Proof. By Theorem 4(1), all elements of $\mathcal{E}_k(N)$ are holomorphic at s_0 and at $1 - k - s_0$. Since $\Gamma(1 - s)/\Gamma(s + k)$ is holomorphic and nonzero at s_0, the functional equation in Theorem 3 implies

$$\mathcal{E}_k(N, s_0) = \mathcal{E}_k(N, 1 - k - s_0),$$

which is equal to $\mathcal{N}_k(\Gamma(N), \lambda)$ by Theorem 5, since $\frac{1-k}{2} < \mathrm{Re}(1 - k - s_0) < 1 - \frac{k}{2}$.

4. The Main Theorem

Before proceeding to prove the main theorem, we shall prove the following

Lemma 7. *For* $\gamma \in \Gamma(1)$, *we have*

$$E_k(z, s; \chi)|_k \gamma = \begin{cases} 2L(2s + k, \chi)\bar{\chi}(\gamma) y^s + D_\gamma(s) y^{1-k-s} + \sum \cdots & (\gamma \in \Gamma_0(N)), \\ D'_\gamma(s) y^{1-k-s} + \sum \cdots & (\gamma \notin \Gamma_0(N)), \end{cases}$$

where the summations are taken over non-constant terms and $D_\gamma(s)$ *and* $D'_\gamma(s)$ *are meromorphic functions of* s.

Proof. By (3.14) and [3, Proposition 5.2], we get

(4.1) $$E_k^*(z, s; 0 \bmod N, 1 \bmod N)|_k \gamma = \delta y^s + f_\gamma(s) y^{1-k-s} + \sum \cdots,$$

where $\delta = 1$ $(\gamma \in \{\pm 1\} \Gamma_1(N))$ or $= 0$ $(\gamma \notin \{\pm 1\} \Gamma_1(N))$, $f_\gamma(s)$ is a meromorphic function of s and the summation is taken over non-constant terms. Note that $\Gamma(N)\Gamma(1)_\infty = \{\pm 1\} \Gamma_1(N)$. Then by (3.15) and (4.1), we obtain the lemma.

We note by Theorem 4 that $D_\gamma(s)$, $D'_\gamma(s)$ and non-constant terms are holomorphic at s_0 $\left(-\frac{k}{2} < \mathrm{Re}(s_0) < \frac{1-k}{2}\right)$.

Now we can prove our main theorem mentioned in the introduction.

Theorem 8. *Let λ be a complex number expressed as $\lambda = s_0(1-k-s_0)$ with $-\frac{k}{2} < \mathrm{Re}(s_0) < \frac{1-k}{2}$. Then*

$$\mathscr{C}_k(\Gamma(N), \lambda) = \ll E_k(z, s_0; \chi)|_k \gamma \gg,$$

where χ runs over all Dirichlet characters defined modulo N such that $\chi(-1) = (-1)^k$ and $L(2s_0 + k, \chi) = 0$, and γ runs over a complete set of representatives of $\Gamma_0(N) \backslash \Gamma(1)$.

Proof. Let $f(z)$ be an element of $\mathscr{N}_k(\Gamma(N), \lambda)$. Since $E_k(z, s; \chi) = 0$ if $\chi(-1) \neq (-1)^k$, $f(z)$ can be expressed, by Theorem 6, as

$$f(z) = \sum_{\chi, i} a_{\chi, i} E_k(z, s_0; \chi)|_k \gamma_i,$$

where χ is taken over all Dirichlet characters defined modulo N satisfying $\chi(-1) = (-1)^k$, and γ_i is taken over a complete set of representatives of $\Gamma_0(N) \backslash \Gamma(1)$. We take $\gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. By definition, $f(z) \in \mathscr{C}_k(\Gamma(N), \lambda)$ if and only if the constant term containing y^{s_0} of the Fourier expansion of $f|_k \gamma$ vanishes for each $\gamma \in \Gamma(1)$. Take $\delta \in \Gamma(1)$ and fix it for a while. For each i, we let i' be the index given by $\gamma_i \delta = \delta' \gamma_{i'}$ with $\delta' \in \Gamma_0(N)$. Then

$$E_k(z, s_0; \chi)|_k \gamma_i \delta = \bar{\chi}(\delta') E_k(z, s_0; \chi)|_k \gamma_{i'}.$$

Therefore Lemma 7 implies that if $i' \neq 1$ then the constant term containing y^{s_0} does not appear in the constant terms of the Fourier expansion of $E_k(z, s_0; \chi)|_k \gamma_i \delta$. Since $\Gamma(1) = \bigcup_i \gamma_i^{-1} \Gamma_0(N)$, we may take $\delta = \gamma_i^{-1} \gamma$ $(\gamma \in \Gamma_0(N))$. Then by Lemma 7 the constant term containing y^{s_0} of the Fourier expansion of $f(z)|_k \delta$ is

$$2 \sum_\chi a_{\chi, i} L(2s_0 + k, \chi) \bar{\chi}(\gamma) y^{s_0}.$$

Therefore $f(z)$ is a cyclopean form if and only if

$$\sum_\chi a_{\chi, i} L(2s_0 + k, \chi) \bar{\chi}(\gamma) = 0$$

for any $\gamma \in \Gamma_0(N)$ and any i, where the summation is taken over all Dirichlet characters defined modulo N satisfying $\chi(-1) = (-1)^k$. Since we can take elements $\gamma \in \Gamma_0(N)$ so that the matrix $(\bar{\chi}(\gamma))$ is regular, we see that $f(z)$ is cyclopean if and only if $a_{\chi,i} L(2s_0 + k, \chi) = 0$ for any χ and i. Therefore if $a_{\chi,i} \neq 0$ then $L(2s_0 + k, \chi) = 0$. This implies the theorem.

Remark. We see easily that under the assumption in Theorem 8, the set $\{E_k(z, s_0; \chi)|_k \gamma\}$ is a basis of $\mathscr{N}_k(\Gamma(N), \lambda)$. Here χ runs over all Dirichlet characters defined modulo N such that $\chi(-1) = (-1)^k$ and γ runs over a complete set of representatives of $\Gamma_0(N)\backslash\Gamma(1)$. Therefore the set of generators of $\mathscr{C}_k(\Gamma(N), \lambda)$ given in Theorem 8 is a basis of the space.

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