J. Math. Kyoto Univ. (JMKYAZ)
31-1 (1991) 259–264

A characterization of extended Schottky type groups with a remark to Ahlfors' conjecture

By

Katsuhiko MATSUZAKI

Introduction

The class of Schottky groups can be extended to a larger class of Kleinian groups which may contain parabolic elementary subgroups. These were defined by Chuckrow [4] and called extended Schottky type (E.S.T.) groups (Definition 2–1). In § 2 of this paper, after summarizing certain known results about E.S.T. groups, we give an algebraic characterization to them (Theorem 2–1). For Schottky groups, this was done by Maskit [6]. He proved that a Kleinian group is a Schottky group if and only if it is purely loxodromic and free. Theorem 2–1 is a generalization of this result and we call our characterization condition perfectly decomposable (Definition 2–2). We note that Gusevskii and Zindinova showed first our theorem in their paper [5] by considering 3-dimensional manifolds, but our proof is carried out without any argument on them.

Next we think about the measure of the limit set of some Kleinian group. Ahlfors conjectured that 2-dimensional Lebesgue measure of the limit set of a finitely generated Kleinian group is equal to zero [1]. Recently Bonahon [3] proved that this is true for Kleinian groups which satisfy a certain condition (Proposition 3–2). In this paper as an application of Theorem 2–1 and Proposition 3–2, we remark that Ahlfors' conjecture is valid for doubly generated torsion-free Kleinian groups (Theorem 3-1).

And this paper contains the author's gratitude to Prof. Kusunoki.

1. Preliminaries

We denote all the Möbius transformations by Möb. In this paper a *Kleinian group* is a *finitely generated torsion-free* discrete subgroup of Möb which acts discontinuously at some point of $\hat{C} = C \cup \{\infty\}$. (To make sure, abbreviations *f. g.* and *t. f.* are used at appropriate places.)

The set of points at which a Kleinian group G acts discontinuously is denoted by $\Omega = \Omega(G)$, and its complement $\Lambda = \Lambda(G)$ is the limit set. An *elementary group*

Received, July 3, 1989

is a Kleinian group whose limit set has a finite number of points. In our case an elementary group is either trivial, loxodromic cyclic, parabolic cyclic group, or parabolic abelian group of rank 2.

A connected component of $\Omega(G)$ is briefly said to be a component of G. A component Δ of G is invariant if $g(\Delta)=\Delta$ for all $g\in G$. A (*f.g.*) Kleinian group which has an invariant component is called *function group.* If G is a function group and Δ is an invariant component of G, then components of G other than Δ are all simply connected. Moreover a function group whose invariant component is simply connected is called *B-group.* We always exclude elementary groups from B-groups. It is known that a B-group has simply connected invariant components at most two. If there are two, we call that *quasi-Fuchsian.* A *totally degenerate group* is a B-group which has only one component.

For a component Δ of G, we define $\mathrm{stab}(\Delta)=\{g\in G;\, g(\Delta)=\Delta\}$ and call it the component subgroup for Δ. From Ahlfors' Finiteness Theorem we can see that a Riemann surface $\Delta/\mathrm{stab}(\Delta)$ is of finite type. The following proposition is almost a corollary to this theorem.

Proposition 1-1 (Ahlfors [2]). *Let G be a non-elementary (f.g.) Kleinian group. Then the component subgroup for each Δ is non-elementary (f.g.) Kleinian group and has Δ as its component.*

It is easily seen that every component subgroup of a function group for non-invariant component is quasi-Fuchsian.

Möb is uniquely extended from on $\hat{\mathrm{C}}$ to the group of orientation preserving isometries of hyperbolic 3-space H^3. A discrete subgroup G of Möb is called *geometrically finite* if it has a finite-sided Dirichlet polyhedron in H^3.

A (*f.g.*) Kleinian group is constructed from certain basic groups by using Combination Theorems I and II of Maskit [7]. Particularly for a function group, the next proposition is known.

Proposition 1–2 (Maskit [8] [10]). *A function group is constructed from elementary, quasi-Fuchsian, and totally degenerate groups by a finite number of applications of Combination Theorems I and II. Further it is geometrically finite if and only if it is constructed from elementary and quasi-Fuchsian groups.*

2. A characterization of E.S.T. groups

Definition 2–1. A Kleinian group G is called *E.S.T.* (*extended Schottky type*) *group* of type (r,s,p) if G has generators $f_1,\dots,f_r,\ a_1,\dots,a_s,\ u_1,\ v_1,\dots,u_p,\ v_p$ and defining curves $C_1,\ C'_1,\dots,C_r,\ C'_r,\ B_1,\ B'_1,\dots,B_s,\ B'_s,\ L_1,\dots,L_p$, where L_i is a topological rectangle with sides $I_i,\ I'_i,\ J_i,\ J'_i$, and they satisfy the following conditions:

(1) The curves are disjoint Jordan curves, except B_j, B'_j having a common point z_j, which bound a $2r+p+s$-ply connected domain D such that $f_i(D)\cap D=a_j(D)\cap D=u_k(D)\cap D=v_k(D)\cap D=\emptyset$.

(2) $f_i(C_i) = C'_i$, $a_j(B_j) = B'_j$, $u_k(I_k) = I'_k$, $v_k(J_k) = J'_k$.
(3) $\langle u_k, v_k \rangle$ is a parabolic abelian group.
(4) a_j is parabolic with fixed point z_j.

Definition 2–2. We say that a finitely generated subgroup G of Möb is *perfectly decomposable* if G has a free product decomposition such that

$$G = F_1 * \cdots * F_r * A_1 * \cdots * A_s * T_1 * \cdots * T_p$$

where F_i is loxodromic cyclic, A_i is parabolic cyclic, T_i is parabolic abelian of rank 2, and every parabolic transformation of G is in a conjugate subgroup of some A_i or T_i. Moreover if G is free, we call it *perfectly free*.

We can see that every E.S.T. group is constructed from elementary groups by Combination Theorems. Thus it is geometrically finite and perfectly decomposable. In this section we will prove conversely that if a Kleinian group is perfectly decomposable, then it is an E.S.T. group.

Proposition 2–1. *If G is a (f.g.& t.f.) Kleinian group, then the following three conditions are equivalent*:
(*a*) *G is an E.S.T. group.*
(*b*) *The limit set $\Lambda(G)$ is totally disconnected.*
(*c*) *G is a function group constructed from elementary groups by a finite number of applications of Combination Theorems.*

Proof. (a)→(b): This was remarked in Chuckrow [4]. (b)→(c): Since $\Omega(G)$ is only one component, G is a function group. If G has a quasi-Fuchsian or totally degenerate group as a subgroup, $\Lambda(G)$ has a continiuum. Thus G is constructed from elementary groups.

(c)→(a): In order to prove this, we need the signature of a function group (Maskit [9] [10]). It is well defined for each function group. The signature consists of both a geometric object K called the marked 2-complex, and a non-negative integer t, called the Schottky number.

The K contains a marked finite (disconnected) Riemann surface X and connectors. The connected components of X are called parts of K, and each part corresponds to the structure subgroup, from which a function group is constructed by Combination Theorems. The end points of each connector are special points of X and have the same ramification number.

We consider two signatures (K, t) and (K^*, t^*) to be the same if there is a homeomorphism from K onto K^* which preserves distinguished points with their order, and $t = t^*$.

Let G be a function group which satisfies Prop. 2–1(c) with the signature (K, t). Each part P in K is a finite Riemann surface of type $(0, 2; \infty, \infty)$ or $(1, 0)$, and respectively corresponds to a parabolic cyclic group or a parabolic abelian group of rank 2 as the structure subgroup. Further K has no connector because a part of the signature $(0, 2; \infty, \infty)$ cannot have an endpoint of a connector (Maskit [10]*X.D.*15).

Lemma 2–1 (Maskit [9]). *Two geometrically finite function groups G and G^* have the same signature if and only if G^* is a quasiconformal deformation (i.e. there is a global quasiconformal homeomorphism φ so that $G^* = \varphi \circ G \circ \varphi^{-1}$).*

Proof of (c) → (a): Let G be a function group which satisfies Prop. 2–1(c) with the signature (K, t). Then there exists an E.S.T. group G^* whose signature (K^*, t^*) is the same as (K, t). From Lemma 2–1. G is a quasiconformal deformation of G^*. Thus G is an E.S.T. group. □

Corollay 2–1 (Chuckrow [4]). *A finitely generated subgroup of an E.S.T. group is also an E.S.T. group.*

Proposition 2–2. *Let G be a finitely generated perfectly decomposable subgroup of Möb. Then a finitely generated subgroup H of G is also perfectly decomposable.*

Proof. Let $G = F_1 * \cdots * F_r * A_1 * \cdots * A_s * T_1 * \cdots * T_p$ be a perfect decomposition of G. We can make an E.S.T. group $\tilde{G} = \tilde{F}_1 * \cdots * \tilde{F}_r * \tilde{A}_1 * \cdots * \tilde{A}_s * \tilde{T}_1 * \cdots * \tilde{T}_p$ so that there exists a type-preserving isomeophism $\varphi: G \to \tilde{G}$. $\tilde{H} = \varphi(H)$ is a finitely generated subgroup of $\tilde{G}$, so from Corollary 2–1, $\tilde{H}$ is an E.S.T. group. Hence $\tilde{H}$ is perfectly decomposable and $H = \varphi^{-1}(\tilde{H})$ is also perfectly decomposable, for φ is type-preserving. □

Proposition 2–3. *If G is a perfectly decomposable Kleinian group, then G cannot have a (non-elementary) B-group as a subgroup.*

Proof. We assume G has a B-group H as a subgroup. Then from Proposition 2-2, H is perfectly decomposable. Because a B-group cannot have a parabolic abelian subgroup of rank 2, H is a perfectly free B-group.

Let Δ be a simply connected invariant component of H. By Ahlfors' Finiteness Theorem we see that $S = \Delta/H$ is a finite Riemann surface, i.e. $S = \hat{S} - \{p_1, \dots, p_n\}$, where $\hat{S}$ is a closed Reimann surface and p_i is a puncture on it. Since Δ is a universal covering surface of S, we can identify H to $\pi_1(S)$ by the canonical isomorphism.

For each puncture p_i, there exists the conjugacy class of primitive prarbolic elements $[w_i]$ such that the corresponding loop of $\pi_1(S)$ is freely homotopic to the simple loop round p_i.

Let $f_1, \dots, f_r, a_1, \dots, a_s$ be perfectly free generators of H and X be the set of these. Since every primitive paraboric element of H is conjugate to a_i or a_i^{-1}, for each w_i there exists a_{k_i} such that $[w_i]$ is equal to $[a_{k_i}]$. Hence the fundamental group of $S \cup \{p_i\}$ is presented by the generators X and a relation $a_{k_i} = id$. But since a_{k_i} is in X, $\pi_1(S \cup \{p_i\}) = \langle X; a_{k_i} \rangle = \langle X - \{a_{k_i}\} \rangle$ is a free group. Filling up all the punctures, we can see $\pi_1(\hat{S})$ is a free group with generators $X - \{a_{k_1}, \dots, a_{k_n}\}$. If $\hat{S}$ is not a sphere this cannot occur, because the fundamental group of a non-trivial closed surface doesn't have a non-trivial free product decomposition (Shenitzer [11]).

Therefore we know that $\hat{S}$ is a sphere, S is an n-punctured sphere, and $X - \{a_{k_1}, \dots, a_{k_n}\} = \emptyset$. This means $\pi_1(S) = \langle X \rangle = \langle a_{k_1}, \dots, a_{k_n} \rangle$ is a free group of rank n, which contradicts the fact that the fundamental group of an n-punctured sphere is a free group of rank $n-1$. Hence our first assumption is false and we have proved the proposition. □

Proposition 2–4. *If G is a perfectly decomposable Kleinian group, then it satisfies Prop.2–1(c); especially it is geometrically finite.*

Proof. Let Δ be a component of $\Omega(G)$. We assume $\Omega(G) \supsetneq \Delta$. The stabilizer of Δ is a function group H, and by Propositin 1–1, Δ is an invariant component of $\Omega(H)$. Since $\Omega(H) \supset \Omega(G) \supsetneq \Delta$, $\Omega(H)$ has a component other than Δ. Let it be Δ_1. It is simply connected. Further we consider the stabilizer of Δ_1 in H. This is a quasi-Fuchsian group with an invariant component Δ_1. But from Proposition 2–3, G cannot have a B-group as a subgroup. This contradiction asserts that $\Delta = \Omega(G)$, i.e. G has only one component $\Omega(G)$. Hence G is a function group and does not contain B-groups. This means G is constructed from elementary groups, and especially it is geometrically finite. □

From Proposition 2–1 and 2–4, we obtain the following theorem. It is a characterization of an E.S.T. group.

Theorem 2–1. *A Kleinian group is an E.S.T. group if and only if it is perfectly decomposable.*

3. Ahlfors' conjecture for doubly generated Kleinian groups

Ahlfors conjectured that for a finitely generated Kleinian group G 2-dimensional Lebesgue measure of its limit set $\Lambda(G)$ is equal to zero, and he proved that in the case G is geometrically finite it is true [1]. Later Thurston defined a class of finitely generated torsion-free discrete subgroup of Möb which is called geometrically tame, and proved that for a geometrically tame Kleinian group, Ahlfors' conjecture is valid [12]. The next theorem is due to Bonahon who gave a certain condition of geometric tameness [3]. Before the theorem we state this condition, namely,

Definition 3–1. For a finitely generated torsion-free discrete subgroup G of Möb, the following (*) is called Bonahon's condition.

(*) For each non-trivial free product decomposition $G = A * B$, there exists a parabolic element $j \in G$ such that j does not belong to any conjugate subgroup of either A or B.

Especially if G is indecomposable, then G satisfies (*).

Proposition 3–1 (Bonahon [3]). *If a finitely generated torsion-free discrete subgroup G of Möb satisfies Bonahon's condition, then G is geometrically tame.*

Applying Thurston's result, Bonahon noted the next proposition, which is now needed for us to prove that Ahlfors' conjecuture is valid for doubly generated $(t.f.)$ Kleinian groups.

Proposition 3–2. *If a $(f.g.\&t.f.)$ Kleinian group G satisfies Bonahon's condition, then the 2-dimensional Lebesgue measure of $\Lambda(G)$ is equal to zero.*

Theorem 3–1. *Let G be a doubly generated $(t.f.)$ Kleinian group, then the 2-dimensional Lebesgue measure of $\Lambda(G)$ is equal to zero.*

Proof. If G is indecomposable, then it satisfies Bonahon's condition, and from Proposition 3–2, meas $\Lambda(G)=0$. Hence we may assume that G has a non-trivial free product decomposition. Since G is doubly generated, each factor of the free product is a cyclic group.

It suffices to consider the case that G does not satisfy Bonahon's condition. In other words, there exist g_1 and g_2 such that $G=\langle g_1\rangle * \langle g_2\rangle$ and every parabolic element of G is in a conjugate of either $\langle g_1\rangle$ or $\langle g_2\rangle$. This means G is perfectly decomposable, and by Proposition 2–4, we can see G is geometrically finite. Thus meas $\Lambda(G)$ is equal to zero. □

DEPARTMENT OF MATHEMATICS
KYOTO UNIVERSITY

CURRENT ADDRESS
DEPARTMENT OF MATHEMATICS
TOKYO INSTITUTE OF TECHNOLOGY

References

[1] L. V. Ahlfors, Fundamental polyhedrons and limit point sets of Kleinian groups, Proc. Nat. Acad. Sci. USA, **55** (1966), 251–254.
[2] L. V. Ahlfors, The structure of a finitely generated Kleinian group, Acta Math., **122** (1969), 1–17.
[3] F. Bonahon, Bouts des variétés hyperboliques de dimension 3, Ann. of Math. (2), **124** (1986), 71–158.
[4] V. Chuckrow, Subgroups and automorphisms of extended Schottky type groups, Trans. Amer. Math. Soc., **150** (1970), 121–129.
[5] N. A. Gusevskii and N. S. Zindinova, Space of extended Schottky groups, Sibirsk. Mat. Zh., **27** (1986), 838–849.
[6] B. Maskit, A characterization of Schottky groups, J. d'Analyse Math., **19** (1967), 227–230.
[7] B. Maskit, Decomposition of certain Kleinian groups, Acta Math., **130** (1973), 243–263.
[8] B. Maskit, On the classification of Kleinian groups: I-Koebe groups, Acta Math., **135** (1975), 249–270.
[9] B. Maskit, On the classification of Kleinian groups: II-Signatures, Acta Math., **138** (1977), 17–42.
[10] B. Maskit, Kleinian Groups, Springer-Verlag 1988.
[11] A. Shenitzer, Decomposition of a group with a single defining relation into a free product, Proc. Amer. Math. Soc., **6** (1955) 273–279.
[12] W. Thurston, The geometry and topology of 3-manifolds, Lecture Notes, Dep't. of Math. Princeton Univ. 1978/79.