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On the homology of $BU(2n,\cdots,\infty)$

By

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§ 1. Introduction

We denote by $X(n, \cdots, \infty)$ the $(n-1)$-connective fibering over X. The cohomology ring $H^*(BU(2n, \cdots, \infty); \boldsymbol{F}_p)$ was determined by the work of Adams [1], of Stong [9] and of Singer [8]. The purpose of this paper is to calculate the image of

$$i_{n*}: H_*(BU(2n, \cdots, \infty)) \longrightarrow H_*(BU)$$

(where i_n is the fiber inclusion) in the stable range $*<4n$. Our method can be outlined as follows. First we construct a map t_n from the n-fold smash product CP^∞ to $BU(2n, \cdots, \infty)$. Using this map, we get a system of elements

$$\beta_I \in H_{2|I|}(BU(2n, \cdots, \infty))$$

where $I=(i_1, i_2, \cdots, i_n)$, $i_j>0$ and $|I|=\sum i_j$. We shall show that β_I generate $H_*(BU(2n, \cdots, \infty))/\text{Torsion}$ in the stable range. To prove this, we will use a result of Adams [2] on $H_*(\boldsymbol{bu})$.

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§ 2. Construction of maps and elements

We write $X^{\wedge n}$ for the n-fold smash product of a space X.

Let $t_n: BU^{\wedge n} \to BU$ be the n-fold tensor product of virtual bundles of virtual dimension zero. We use the same t_n for its restriction to $(CP^\infty)^{\wedge n}$. Since $(CP^\infty)^{\wedge n}$ is $(2n-1)$-connected, we have a map

$$\tilde{t}_n: (CP^\infty)^{\wedge n} \longrightarrow BU(2n, \cdots, \infty)$$

such that $i_n \circ \tilde{t}_n = t_n$.

Let $\boldsymbol{E}$ be a complex oriented theory. Then $\boldsymbol{E}_*(CP^\infty)$ is a free $\boldsymbol{E}_*(pt)$-module generated by $\beta_1, \beta_2, \cdots, \beta_k, \cdots$, where β_k is the dual element of the k-th power of the Euler class $x^E \in \boldsymbol{E}^2(CP_\infty)$. (See Adams [2], part II.) So $\boldsymbol{E}_*((CP^\infty)^{\wedge n})$ is a free $\boldsymbol{E}_*(pt)$-module generated by

$$\beta_{i_1} \otimes \beta_{i_2} \otimes \cdots \otimes \beta_{i_n} \quad \text{where} \quad i_1, i_2, \cdots, i_n > 0.$$

We denote $\tilde{t}_n(\beta_{i_1} \otimes \beta_{i_2} \otimes \cdots \otimes \beta_{i_n}) \in \boldsymbol{E}_*(BU(2n, \cdots, \infty))$ as

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$$\beta_{(i_1, i_2, \cdots, i_n)}.$$

Remark. In the case of $n=2$, $BU(4, \cdots, \infty)$ is BSU and $\{\beta_{(i,j)}\}$ generate $E_*(BSU)$ as algebra. This can be proved by the same way as in Baker [5] or Kozima [7]. (See also Kochman [6].)

§ 3. The proof of the main result

Let $\mu_n: (CP^\infty)^{\wedge n} \to MU(2n)$ be the map induced from the external product of line bundles. Then we have the following lemma.

Lemma 3.1. *The diagram*

$$\begin{array}{ccc} (CP^\infty)^{\wedge n} \xrightarrow{\mu_n} MU(2n) & \xrightarrow{\iota_n} & \Sigma^{2n}\boldsymbol{MU} \\ \tilde{t}_n \downarrow & & \Sigma^{2n}t \downarrow \\ BU(2n, \cdots, \infty) & \xrightarrow{\iota_n} & \Sigma^{2n}\boldsymbol{bu} \end{array}$$

commutes up to homotopy where t is the Thom map and ι_n the inclusion as the 2n-th term of spectrum.

Proof. Since $\boldsymbol{bu}^*((CP^\infty)^{\wedge n}) \to \boldsymbol{K}^*((CP^\infty)^{\wedge n})$ and $\boldsymbol{K}^*((CP^\infty)^{\wedge n}) \to \boldsymbol{K}^*((CP^\infty)^n)$ are monic, we show it in $\boldsymbol{K}^*((CP^\infty)^n)$. Let $\pi_i: (CP^\infty)^n \to CP^\infty$ be the i-th projection and ξ the canonical line bundle. Clearly $\iota_n \circ \tilde{t}_n$ represents

$$u^{-n} \cdot (\xi_1 - 1) \cdot (\xi_2 - 1) \cdots (\xi_n - 1)$$

Where $u \in \boldsymbol{K}_2(pt)$ is the generator and $\xi_i = \pi_i^* \xi$. On the other hand, by the multiplicativity of Thom class, we have

$$\begin{aligned} t(\xi_1 \times \xi_2 \times \cdots \times \xi_n) &= t(\xi_1) \cdot t(\xi_2) \cdots t(\xi_n) \\ &= u^{-n} \cdot (\xi_1 - 1) \cdot (\xi_2 - 1) \cdots (\xi_n - 1). \end{aligned}$$

Proposition 3.2. *The image of the composition*

$$H_*(\boldsymbol{MU}) \xrightarrow{t_*} H_*(\boldsymbol{bu}) \longrightarrow H_*(\boldsymbol{bu})/\text{Torsion}$$

is epic.

Proof. Let $\boldsymbol{Q}_p$ be the localization of $\boldsymbol{Z}$ at a prime p. Since $H_*(\boldsymbol{bu})/\text{Torsion} \subset H\boldsymbol{Q}_*(\boldsymbol{bu})$, it is sufficient to show

$$\text{Im}\, \rho \circ t_* = \text{Im}\, \rho$$

where $\rho: H_*(\boldsymbol{bu}) \to H\boldsymbol{Q}_{p*}(\boldsymbol{bu})$ for all prime p. By the result of Adams ([2], part III, proof of 16.5.), $H\boldsymbol{Q}_{p*}(\boldsymbol{bu})$ is the $\boldsymbol{Q}_p$-subalgebra of $\boldsymbol{Q}_p[u]$ generated by u and u^{p-1}/p. Since $H_*(\boldsymbol{MU}) = Z[b_1, b_2, \cdots, b_k, \cdots]$ where $b_k = \sigma^{-2}(\iota_{2*}\beta_{k+1})$ and an easy calculation shows $t_* b_1 = u/2$ and $t_* b_{p-1} = u^{p-1}/p!$, the result follows.

By (3.1), we have

$$\sigma^{-2n}\iota_{n*}\beta_{(i_1, i_2, \cdots, i_n)} = t_*(b_{i_1-1}b_{i_2-1}\cdots b_{i_n-1})$$

where $i_k>0$ and $b_0=1$. Since

$$\iota_{n*}\colon H_*(BU(2n, \cdots, \infty)) \longrightarrow H_*(\Sigma^{2n}\boldsymbol{bu})$$

is isomorphic for $*<4n$, the proof of the following theorem is clear.

Theorem 3.3. *$\{\beta_I\}$ generate*

$$H_*(BU(2n, \cdots, \infty))/\text{Torsion} \quad \textit{for} \quad *>4n.$$

Corollary 3.4. *$\{i_{n*}\beta_I\}$ generate* $\operatorname{Im} i_{n*}$ *for* $*<4n$.

Remark. The condition on the degree is needed in these cases. The first problem is $H_{12}(BU(6, \cdots, \infty))/\text{Torsion}$. Using an easy argument on the Serre spectral sequence, one can prove that $H_{12}(BU(6, \cdots, \infty))/\text{Torsion}$ generated $\beta_{(4,1,1)}$ and $\frac{1}{4}\cdot\beta^2_{(1,1,1)}$, (thus $H_*(BU(2n, \cdots, \infty))/\text{Torsion}$ is not a polynomial algebra). On the other hand, $\operatorname{Im}(t_{3*})$ is generated by $\beta_{(4,1,1)}$ and $\beta_{(2,2,2)}-3\cdot\beta_{(4,1,1)}=\frac{1}{2}\cdot\beta^2_{(1,1,1)}$ at this degree.

§4 Relations between $i_{n*}\beta_I$

In this section we will give some relations of $i_{n*}\beta_I$ in $\boldsymbol{E}_*(BU)$. We write β_I for $i_{n*}\beta_I$ in this section. Let

$$\beta(x_1, x_2, \cdots, x_n)$$

be the formal power series

$$1+\sum_{l(I)=n}\beta_I x^I \in \boldsymbol{E}_*(BU)[\![x_1, x_2, \cdots, x_n]\!]$$

where $l(I)$ is the length of I and $x^I=x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}$ if $I=(i_1, i_2, \cdots, i_n)$.

Let $+_E$ be the sum defined by the formal group of $\boldsymbol{E}$. Then we have the following formulae.

Theorem 4.1.

(1) $\beta(x_{\tau(1)}, x_{\tau(2)}, \cdots, x_{\tau(n)})=\beta(x_1, x_2, \cdots, x_n)$ *for any permutation* τ.

(2) $\beta(x_1, \cdots, x_{k-1}, x_k+_E x_{k+1}, x_{k+2}, \cdots, x_n)=\beta(x_1, \cdots, x_n)\cdot$
$\beta(x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_n)\cdot\beta(x_1, \cdots, x_k, x_{k+2}, \cdots, x_n)$ *for all* k.

Proof. We use the same notations as is the proof of (3.1) and put $\eta_i=\xi_{i-1}$. Then t_n classifies $\eta_1\eta_2\cdots\eta_n$ and the Boardman image of t_n is $\beta(x_1, x_2, \cdots, x_n)$. Since $\eta_{\tau(1)}\eta_{\tau(2)}\cdots\eta_{\tau(n)}=\eta_1\eta_2\cdots\eta_n$, (1) is clear. (2) follows from the fact that $\beta(x_1, \cdots, x_{k-1}, x_k+_E x_{k+1}, x_{k+2}, \cdots, x_n)$ is the image of the classifying map of

$$\begin{aligned}&\eta_1\cdots\eta_{k-1}(\xi_k\xi_{k+1}-1)\eta_{k+2}\cdots\eta_n\\ &\quad=\eta_1\cdots\eta_{k-1}\eta_k\eta_{k+1}\eta_{k+2}\cdots\eta_n+\eta_1\cdots\eta_{k-1}\eta_k\eta_{k+2}\cdots\eta_n+\eta_1\cdots\eta_{k-1}\eta_{k+1}\eta_{k+2}\cdots\eta_n\end{aligned}$$

and the Whitney sum of bundles corresponds to the product in $E_*(BU)[[x_1, x_2, \cdots, x_n]]$. (See Ravenel-Wilson [8] and Kozima [7].)

Example 4.2. For example, in H_*, we have

$$\beta(x_1, x_2+x_3)=\beta(x_1, x_2, x_3)\cdot\beta(x_1, x_2)\cdot\beta(x_1, x_3) \quad \text{by (4.1)(2)}.$$

So one can easily obtain

$$\beta_{(1,i,j)}=\binom{i+j}{i}\cdot\beta_{(1,i+j)} \qquad \text{for} \quad i, j>0$$

and

$$\beta_{(2,2,2)}=6\cdot\beta_{(2,4)}-6\cdot\beta_{(1,3)}\beta_{(1,1)}-\beta^2_{(1,2)}.$$

We have also

$$\beta(x_3, x_1+x_2)=\beta(x_3, x_1, x_2)\cdot\beta(x_3, x_1)\cdot\beta(x_3, x_2)$$

and, by (4.1)(1)

$$\beta(x_1, x_2+x_3)\cdot\beta(x_2, x_3)=\beta(x_1+x_2, x_3)\cdot\beta(x_1, \beta_2).$$

This last equation gives the relations between $\beta_{(i,j)}$.

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