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All valuations on $K(X)$

By

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This work is a natural continuation of our previous works [1], [2], [3]. We intend here to describe all types of valuations on $K(X)$. This possibility is given by our main result in [2] which give a description of so-called residual transcendental extension of a valuation on K to $K(X)$. Following an ideea of MacLane (see [7]) we define the notion of "ordered system of valuations on $K(X)$" (see § 2) and the limit of such a system. The main result given in section 5 shows that every r.a.t. extension w to $K(X)$ of a valuation v on K may be defined as a limit of a suitable ordered system of r.t. extensions of v to $K(X)$.

In the last sections we are concerned with the existence of r.a.t. extensions of v to $K(X)$ with a given residue field, or with a given value group, or both.

Sometimes there exist some similarity between a lot of our results and results of MacLane [7] (and even with some results of Ostrowski [9]). However, we remark that all our considerations and method of proofs are based on our notion of "minimal pair of definition of an r.t. extension of a valuation v on K to $K(X)$" and on the results we obtained in [1], [2] and [3].

1. Notation and definitions.

1. Let K be a field and v a valuation on K. We emphasize sometimes this situation saying that (K, v) is a valuation pair. Denote by k_v the residue field, by G_v the value group and by O_v the valuation ring of v. If $x \in O_v$, we denote by x^* the image of x in k_v. We refer the reader to [5], [6] or [10] for general notions and definitions.

Let K'/K be an extension of fields. A valuation v' on K' will be called an *extension* of v if $v'(x)=v(x)$ for all $x \in K$. If v' is an extension of v, we identify canonically k_v with a subfield of $k_{v'}$ and G_v with a subgroup of $G_{v'}$.

In what follows we consider a fixed valuation pair (K, v). Let us denote by $\bar{K}$ a fixed algebraic closure of K and by $\bar{v}$ a fixed extension of v to $\bar{K}$. It is easy to see that $G_{\bar{v}}$ is a divisible group, i.e., for every $\delta \in G_{\bar{v}}$ and $n \in \mathbf{N}$, there exists an element $\gamma \in G_{\bar{v}}$ such that $n\gamma=\delta$. Moreover, $G_{\bar{v}}=\mathbf{Q}G_v$, i.e., $G_{\bar{v}}$ is the smallest divisible group which contains G_v.

As usual, by $K(X)$ we shall denote the field of rational functions of an indeterminate X over K.

2. Let w be an extension of v to $K(X)$. Denote by $\bar{w}$ a common extension of w

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and $\bar{v}$ to $\bar{K}(X)$, i.e., $\bar{w}$ is a valuation of $\bar{K}(X)$ which extends simultaneously w and $\bar{v}$. In [3, Proposition 3.1] it is proved that there always exists such a common extension. Let us set

(1) $$M_{\bar{w}}=\{\bar{w}(\bar{X}-a)\mid a\in\bar{K}\}\subseteqq G_{\bar{w}}.$$

According to [8] (see also [1], [2]) w is called a *residual transcendental* (r.t.) extension of v if k_w/k_v is a transcendental extension. According to [2, Proposition 1.1] w is an r.t. extension of v if and only if: i) $G_{\bar{v}}=G_{\bar{w}}$, ii) the set (1) is upper bounded in $G_{\bar{w}}$ and iii) $G_{\bar{w}}$ contains its upper bound. Let δ be the upper bound of the set (1). Then there exists $a\in\bar{K}$ such that $\delta=\bar{w}(X-a)$, and thus (see [2]) $\bar{w}$ is an r.t. extension of $\bar{v}$ defined by $\bar{v}$, inf, a and δ (see [2]). Since $\bar{w}$ is defined by a and δ, we say that (a, δ) is a *pair of definition* of $\bar{w}$. Generally w has many pairs of definitions. In [1] it is proved that two pairs (a, δ), (a', δ') of $\bar{K}\times G_{\bar{v}}$ define the same r.t. extension of $\bar{v}$ to $\bar{K}(X)$ if and only if $\delta=\delta'$ and $\bar{v}(a-a')\geqq\delta$. According to [2], a pair of definition (a, δ) of $\bar{w}$ is called *minimal relative to* K if the number $[K(a): K]$ is the smallest possible one, i.e., if (b, δ) is another pair of definition of w, then $[K(b): K]\geqq[K(a): K]$. A (minimal) pair of definition of $\bar{w}$ (with respect to K) is also called a *(minimal) pair of definition* of w. In [2, Theorem 2.1] it is proved that an r.t. extension w is determined by v and a minimal pair of definition (a, δ). Later, we shall see that minimal pairs of definition are also useful to define other extensions of v to $K(X)$.

3. Let w_1, w_2 be two r.t. extensions of v to $K(X)$. According to [7] one says that w_2 *dominate* w_1 (written $w_1\leqq w_2$) if $w_1(f(X))\leqq w_2(f(X))$ for all polynomials $f\in K[X]$. This inequality may be understood in $QG_v=G_{\bar{v}}$ because G_{w_1} and G_{w_2} are of finite index over G_v (see [1], [2] or [3]), and they are canonically imbedded in QG_v. If $w_1\leqq w_2$ and there exists $f\in K[X]$ such $w_1(f)<w_2(f)$, then we write $w_1<w_2$.

Proposition 1.1. *Let K be algebraically closed and let w_1, w_2 be two r.t. extensions of v to $K(X)$. Let (a_i, δ_i) be a pair of definition of w_i, $i=1, 2$. The following statements are equivalent*:

1) $w_1\leqq w_2$

2) $\delta_1\leqq\delta_2$ *and* $v(a_1-a_2)\geqq\delta_1$.

Moreover, $w_1<w_2$ if and only if $\delta_1<\delta_2$ and $v(a_1-a_2)\geqq\delta_1$.

Proof. 1)⇒2) Since (a_i, δ_i) is a pair of definition of w_i, $w_i(X-a_i)=\delta_i$, $i=1, 2$. If $w_1\leqq w_2$, then $w_1(X-a_1)=\delta_1\leqq w_2(X-a_1)=\inf(\delta_2, v(a_1-a_2))$ and $\delta_1\leqq\delta_2$ and $\delta_1\leqq v(a_1-a_2)$.

2)⇒1) If $v(a_2-a_1)\geqq\delta_1$, then (see [1]) (a_2, δ_1) is also a pair of definition of w_1. Let $f(X)\in K[X]$ of the form $f(X)=\sum b_i(X-a_2)^i$. Then we have

$$w_1(f)=\inf_i(v(b_i)+i\delta_1)$$

$$w_2(f)=\inf_i(v(b_i)+i\delta_2)$$

Now since $\delta_1\leqq\delta_2$, one has $v(b_1)+i\delta_1\leqq v(b_i)+i\delta_2$, for all i, and $w_1(f)\leqq w_2(f)$, as claimed.

Furthermore, let us assume that $w_1<w_2$. Then there exists an element $a\in K$ such that

(2) $$w_1(X-a)=\inf(\delta_1,\ v(a_1-a))<w_2(X-a)=\inf(\delta_2,\ v(a_2-a)).$$

According to the above equivalence, this inequality is possible only if $\delta_1<\delta_2$.

Conversely, if $w_1\leqq w_2$, and $\delta_1<\delta_2$, then $w_1(X-a_2)=\delta_1<\delta_2=w_2(X-a_2)$, i.e., $w_1<w_2$.

4. Let K be algebraically closed and w_1, w_2 two r.t. extensions of v to $K(X)$. Let (a_i, δ_i) be a pair of definition of w_i, $i=1, 2$. We shall say that w_2 *well dominates* w_1 if $w_1<w_2$ and $v(a_1-a_2)=\delta_1$.

2. Ordered systems of valuations

1. By an *ordered system of r.t. extensions of v to K(X)* we mean a family $(w_i)_{i\in I}$ of r.t. extensions of v to $K(X)$, where I is a well ordered set without last element and such that w_j dominates w_i when $i<j$.

Let $(w_i)_{i\in I}$ be an ordered system of r.t. extensions of v to $K(X)$. For every $f\in K[X]$ let us define:

(3) $$w(f)=\sup_i w_i(f)$$

We remark that since w_i is an r.t. extension of v, G_{w_i}/G_v is a finite group and $G_v\subseteqq G_{w_i}\subseteqq G_{\bar{v}}$. Hence (3) must be understood in $G_{\bar{v}}$. However, the element in (3) may or may not be an element of $G_{\bar{v}}$. Therefore we say that the ordered system $(w_i)_{i\in I}$ of r.t. extensions of v to $K(X)$ has *a limit* if for every $f\in K[X]$, $w(f)$ defined by (3) is an element of $G_{\bar{v}}$. Then one easily sees that the assignment:

$$f\longrightarrow w(f)$$

defines a valuation w on $K[X]$ which is canonically extended to $K(X)$. This valuation w is an extension of v to $K(X)$, and will be called *the limit* of the given system $(w_i)_{i\in I}$. We write: $w=\sup_i w_i$.

Let K be algebraically closed and let $(w_i)_{i\in I}$ be an ordered system of r.t. extensions of v to $K(X)$. For every $i\in I$ we denote by (a_i, δ_i) a pair of definition of w_i. Then, according to Proposition 1.1, the set $(\delta_i)_i$ is a well ordered subset of G_v. Moreover, if for every $i, j\in I$, $i<j$, w_j well dominates w_i well dominates w_i, then $(a_i)_i$ is a pseudo-convergent sequence on K (see [10, p. 39]). Generally, $(a_i)_i$ contains a subset which is a pseudo-convergent sequence. However, we do not deal with this situation, because in our further consideration all dominations of valuations are well dominations. One has the following result:

Proposition 2.1. *Let K be algebraically closed and let $(w_i)_{i\in I}$ be an ordered system of r.t. extensions of v to $K(X)$. The following statements are equivalent:*

1) *The ordered system $(w_i)_i$ has a limit w which is an r.t. extension of v to $K(X)$.*

2) *There exists an element $a\in K$ such that $v(a-a_i)\geqq\delta_i$ for all $i\in I$. (If $(a_i)_i$ is a pseudo-convengent sequence of K this means that this sequence has a pseudo-limit in K). Also $\sup_i\delta_i$ is defined in G_v.*

Proof. 1)$\Rightarrow$2) Let (a, δ) be a pair of definition of w. According to (3) one sees that $w \geqq w_i$ for all i. Hence, by Proposition 1.1 one has:

(4) $$\delta \geqq \delta_i \text{ and } v(a_i - a) \geqq \delta_i, \qquad i \in I.$$

Therefore, according to (4) it follows that:

$$\delta = w(X-a) = \sup_i w_i(X-a) = \sup_i(\inf(\delta_i, v(a-a_i)) = \sup_i \delta_i .$$

(Also by (4) it follows that a is a pseudo-limit of $(a_i)_i$.)

2)$\Rightarrow$1) Let (a, δ) be such that $\delta = \sup \delta_i$ and such that $v(a-a_i) \geqq \delta_i$ for all $i \in I$. Let w be a valuation on $K(X)$ defined by inf, v, a and δ. Then it is clear that $w = \sup_i w_i$.

The following result (somewhat complementary to Proposition 2.1) is valid.

Proposition 2.2. *Let K be algebraically closed and let $(w_i)_i$ be an ordered system of r.t. extensions of v to $K(X)$. The following statements are equivalent*:

1) *The ordered system $(w_i)_i$ has a limit w which is not an r.t. extension of v to $K(X)$.*

2) *For every $a \in K$ there exists $i \in I$ such that $w_i(X-a) < \delta_i$.*

Proof. 1)$\Rightarrow$2) Let $w = \sup w_i$. Since by the hypothesis w is not an r.t. extension of v to $K(X)$, according to [2, Proposition 1.1], the set (see (2)):

$$M_w = \{w(X-b) \mid b \in K\} \subseteqq G_w$$

is unbounded in G_w or is bounded but does not contain its upper bound. Let $a \in K$. In both cases there exists $b \in K$ such that $w(X-a) < w(X-b)$. But $w(X-b) = \sup_i w_i(X-b)$ and there exists $i \in I$ such that $w(X-a) < w_i(X-b) \leqq \delta_i$. As $w(X-a) = \sup_i w_i(X-a)$, we have $w_i(X-a) \leqq w(X-a) < w_i(X-b) \leqq \delta_i$, as claimed.

2)$\Rightarrow$1) Let $a \in K$. Then since $w_j(X-a) < \delta_j$ for a suitable j, it results that $\sup_i w_i(X-a) = w_j(X-a)$. Since K is algebraically closed, it follows that for every $f \in K[X]$, $\sup_i w_i(f)$ exists and is in G_v, and $w = \sup_i w_i$ is defined. Now we must prove that w is not an r.t. extension of v. Indeed, let us assume that w is an r.t. extension of v and let (a, δ) be a pair of definition of w. Then by the hypothesis there exists $j \in I$ such that $w_j(X-a) < \delta_j$. According to (3), it follows that $w(X-a) < \delta_j$. Also by (3) one has that $w \geqq w_i$ for all $i \in I$. In particular, one has $w_j(X-a_j) = \delta_j \leqq w(X-a_j) = w(X-a+a-a_j) = \inf(w(X-a), v(a-a_j)) \leqq w(X-a)$, a contradiction. Hence w is not an r.t. extension of v.

Theorem 2.3. *Let K be a (not necessarily algebraically closed) field, and let $(\bar{w}_i)_{i \in I}$ be an ordered system of r.t. extensions of $\bar{v}$ to $\bar{K}(X)$. For every $i \in I$, we denote by (a_i, δ_i) a fixed minimal pair of definition of $\bar{w}_i$ with respect to K. Denote by w_i the restriction of $\bar{w}_i$ to $K(X)$ and by v_i the restriction of $\bar{v}$ to $K(a_i)$, $i \in I$. Then*

a) *For all $i, j \in I$, $i < j$, one has $w_i < w_j$, i.e., $(w_i)_{i \in I}$ is an ordered system of r.t. extensions of v to $K(X)$.*

b) *For all $i, j \in I$, $i < j$, one has $k_{v_i} \subseteqq k_{v_j}$ and $G_{v_i} \subseteqq G_{v_j}$.*

c) *Assume that* $\bar{w}=\sup_i \bar{w}_i$ *and* $\bar{w}$ *is not an r.t. extension of* $\bar{v}$ *to* $\bar{K}(X)$. *Let* w *be the restriction of* $\bar{w}$ *to* $K(X)$. *Then* $w=\sup_i w_i$. *Moreover one has*:

$$k_w=\bigcup_i k_{v_i} \quad \text{and} \quad G_w=\bigcup_i G_{w_i}$$

Proof. a) Let us denote by f_i the monic minimal polynomial of a_i relative to K, and let $n_i=\deg f_i=[K(a_i): K]$, $i\in I$. Since $\bar{w}_i<\bar{w}_j$ if $i<j$, it follows that $w_i\leqq w_j$. We note that in fact $w_i<w_j$. Indeed, if $w_i=w_j$ then since (a_i, δ_i) is a minimal pair of w_i by [3, Theorem 2.2], it follows that $\delta_i=\delta_j$, contrary to the assumption $\bar{w}_i<\bar{w}_j$, i.e. $\delta_i<\delta_j$. (A short computation shows that $w_i(f_j)<w_j(f_j)$, if $i<j$.)

Since (a_i, δ_i) is a minimal pair of definition of w_i, $i\in I$, by Proposition 1.1 we have:

(5) $$n_i\leqq n_j, \qquad \bar{v}(a_i-a_j)\geqq\delta_i, \qquad \text{if} \quad i<j,\ i,\ j\in I.$$

Therefore $(w_i)_{i\in I}$ is really an ordered system of r.t. extensions of v to $K(X)$.

b) Let $c\in K(a_i)$. Then $c=f(a_i)$, where $f(X)\in K[X]$ and $n=\deg f<n_i$. Since (a_i, δ_i) is a minimal pair of definition of w_i, for every root b of f one has $\bar{v}(a_i-b)<\delta_i$. Thus by (5) it follows:

(6) $$\bar{v}(f(a_j))=v_j(f(a_j))=\bar{v}(f(a_i))=v_i(f(a_i))=v_i(c).$$

Now let us assume that $v_i(c)=0$. Then $v_j(f(a_j))=0$ and the image c^* of c in k_{v_i}, coincides with the image of $f(a_j)$ in k_{v_j}. Indeed, let $b_1, \cdots, b_n$ be all roots of $f(X)$ in $\bar{K}$. For any t, $1\leqq t\leqq n$, let $d_t\in K$ be such that:

$$\bar{v}(a_i-b_t)=\bar{v}(a_j-b_t)=\bar{v}(d_t),\ 1\leqq t\leqq n.$$

Then one has $\bar{v}((a_i-b_t)/d_t)=\bar{v}((a_j-b_t)/d_t)=0$ and so

$$\bar{v}\left(\frac{(a_j-b_t)/d_t}{(a_i-b_t)/d_t}-1\right)=\bar{v}\left(\frac{a_j-b_t}{a_i-b_t}-1\right)=\bar{v}\left(\frac{a_j-a_i}{a_i-b_t}\right)>0.$$

Hence

$$((a_j-b_t)/d_t)^*=((a_j-b_t)/d_t)^*, \qquad 1\leqq t\leqq n.$$

By these equalities it follows that:

$$\frac{f(a_j)^*}{f(a_i)^*}=\left(\frac{f(a_j)}{f(a_i)}\right)^*=\left(\prod_{t=1}^{n}\frac{(a_j-b_t)/d_t}{(a_i-b_t)/d_t}\right)^*=\prod_{t=1}^{n}\frac{((a_j-b_t)/d_t)^*}{((a_i-b_t)/d_t)^*}=1,$$

i.e., $f(a_i)^*=f(a_j)^*\in k_{\bar{v}}$ as claimed.

The inclusion $G_{v_i}\subseteqq G_{v_j}$ follows easily from (6).

c) Since $\bar{w}=\sup_i \bar{w}_i$, it is easy to see that $w=\sup_i w_i$. Moreover, it is clear that w is not an r.t. extension of v.

Now we shall prove that $k_{v_i}\subseteqq k_w$ and $G_{v_i}\subseteqq G_w$ for all $i\in I$. For that, let $f(X)\in K[X]$ be such that $n=\deg f<n_i$, and let $b_1, \cdots, b_n$ be all roots of f in $\bar{K}$. Since (a_i, δ_i) is a minimal pair of definition of w_i, one has $\bar{v}(a_i-b_t)<\delta_i$, $1\leqq t\leqq n$, and $\bar{w}(X-b_t)=\bar{w}(X-a_i+a_i-b_t)=\bar{v}(a_i-b_t)$, $1\leqq t\leqq n$. Hemce we have:

(7) $$\bar{w}(f(X))=w(f(X))=\bar{v}(f(a_i)).$$

If $\bar{v}(f(a_i))=v_i(f(a_i))=0$, then $w(f(X))=0$ and by the proof of b) one obtains that $f(X)^*=f(a_i)^*$, i. e. $k_{v_i}\subseteqq k_w$. Relation (7) implies that $G_{v_i}\subseteqq G_w$. Hence one has

$$(8)\qquad \bigcup_i k_{v_i}\subseteqq k_w \quad\text{and}\quad \bigcup_i G_{v_i}\subseteqq G_w\,.$$

For proving that these inclusions are in fact equalities, let $r(X)=f(X)/g(X)\in K(X)$. Let $b_1, \cdots, b_n$ and $c_1, \cdots, c_m$ be all roots (not necessarily distinct) of f, g, respectively in $\bar{K}$. Since $\bar{w}$ is not an r. t. extension of $\bar{v}$ to $\bar{K}(X)$, by Proposition 2.2, 2) there exists an $i\in I$ such that:

$$(9)\qquad w(X-b_t)<\delta_i,\quad 1\leqq t\leqq n\,,\qquad w(X-c_s)<\delta_i,\quad 1\leqq s\leqq m\,.$$

According to (9), one has $\bar{v}(a_i-b_t)=\bar{w}(a_i-X+X-b_t)=\bar{w}(X-b_t)$, $1\leqq t\leqq n$, and analogously $\bar{v}(a_i-c_s)=\bar{w}(X-c_s)$, $1\leqq s\leqq m$. Therefore we have: $\bar{v}(f(a_i))=w(f(X))$, $\bar{v}(g(a_i))=w(g(X))$, and:

$$(10)\qquad \bar{v}(r(a_i))=v_i(r(a_i))=w(r(X))\,.$$

Now if $w(r(X))=0$, then by (10), $v_i(r(a_i))=0$ and as above we can easily prove that $(r(a_i))^*=(r(X))^*$, i. e.

$$(11)\qquad r(X)^*\in k_{v_i}\,.$$

Therefore by (8), (10) and (11) it follows that:

$$(12)\qquad \bigcup_{i\in I} k_{v_i}=k_w,\qquad \bigcup_{i\in I} G_{v_i}=G_w\,,$$

as claimed.

3. Types of valuations of $K(X)$.

It is natural to ask for the description of all valuations on $K(X)$. In this work we try to give an answer to this question. In this section we describe all types of valuations on $K(X)$.

A) Valuations on $K(X)$ which are trivial on K. These valuations are well known (see [10]): They are defined by the irreducible polynomials of $K[X]$ and also by the valuation at "infinity", defined by $1/X$. All these are of rank one and discrete. These valuations play a prominent part in algebraic theory of functions of one variable and elsewhere.

B) Valuations on $K(X)$ which extend non-trivial valuations on K. Since distinct valuations on K give distinct extensions to $K(X)$, we deal only with extensions of a fixed valuation v on K. We classify these extensions as follows:

(RT) Residual transcendent extensions w of v to $K(X)$. There are defined by the condition:

$$\deg\operatorname{tr}(k_w/k_v)=1\,.$$

R. t. extensions of v to $K(X)$ had been described in [2, Theorem 2.1]. According to this result, to describe an r. t. extension w of v to $K(X)$ we have to know an algebraic closure $\bar{K}$ of K, an extension $\bar{v}$ of v to $\bar{K}$, and a minimal pair of definition of w. Now, a minimal pair of definition (a, δ) of w is in fact a minimal pair of definition of a common extension $\bar{w}$ of w and $\bar{v}$ to $\bar{K}(X)$. Furthermore, one has $\bar{w}=w_{(a,\delta)}$, i.e. $\bar{w}$ is defined by inf, $\bar{v}$, a and δ. Finally, to know all r.t. extensions of v to $K(X)$, we have to know all pairs $(a, \delta)\in\bar{K}\times G_{\bar{v}}$ such that (a, δ) is a minimal pair of definition of $w_{(a,\delta)}$ with respect to K. This question is discussed in [3]. Although a complete solution is not given in [3], the answer is given in some important cases.

(RA) Residual algebraic (r.a.) extensions w of v to $K(X)$. These are defined by the condition:

k_w/k_v is an algebraic extension.

Furthermore, r.a. extensions are divided into two distinct classes according to the nature of the value group G_w relative to G_v:

(RAT) Residual algebraic torsion (r.a.t.) extensions w of v to $K(X)$. These are defined by the condition that the quotient group:

$$G_w/G_v$$

is a torsion group (i.e. every element is of finite order). It is clear that w is an r.a.t. extension of v to $K(X)$ if and only if $G_v\subseteqq G_w\subseteqq G_{\bar{v}}$.

(RAF) Residual algebraic extension w of v $K(X)$ which are not of torsion (r.a.f. extension). These are defined by the condition that the quotient group G_w/G_v is not a torsion group. Later, (see §4) we shall see that G_w/G_v is in fact a free abelian group; more precisely, it is isomorphic to $\boldsymbol{Z}$, the additive group of rational integers.

4. Residual algebraic extensions. The case K is algebraically closed.

Let K be an algebraically closed field, v a valuation on K, and w an r.a. extension of v to $K(X)$.

1. First, we consider the case when w is an r.a.t. extension of v to $K(X)$. According to the definition, this means that k_w/k_v is an algebraic extension, and G_w/G_v is a torsion group. Now since K is algebraically closed, k_v is also algebraically closed and so $k_w=k_v$. Moreover, $G_v=G_w$ because G_v is a divisible group. Then, according to [16, Ch. II], $(K(X), w)$ is an immediate extension of (K, v). Let us consider the set M_w defined in (1). Since w is not an r.t. extension of v, according to [2, Proposition 1.1], it follows that M_w has no upper bound, or it does not contain its upper bound. Furthermore, since M_w is a totally ordered set, according to [4, §2, Exercise 4], it contains a cofinal well ordered subset $\{\delta_i\}_{i\in I}$. Since M_w does not contain an upper bound, I has no last element. For every $i\in I$, we choose an element $a_i\in K$ such that:

$$w(X-a_i)=\delta_i, \qquad i\in I. \tag{13}$$

Consider $w_i=w_{(a_i,\delta_i)}$, i.e., w_i is the r.t. extension of v to $K(X)$ defined by inf, v, a_i and δ_i.

Theorem 4.1. *With above notation one has:*

a) *$w_i<w_j$ if $i<j$, i.e., $\{w_i\}_{i\in I}$ is an ordered system of r.t. extensions of v to $K(X)$. Moreover, for every $i<j$, w_j well dominates w_i.*

b) $w_i \leqq w$ *for all* $i \in I$ *and* $w = \sup_{i \in I} w_i$.

Proof. a) Let $i<j$. We shall prove that for every $b \in K$ one has:

$$w_i(X-b) \leqq w_j(X-b). \tag{14}$$

First, we note that, according to (13) and the inequality $\delta_i < \delta_j$, one has:

$$v(a_i-a_j)=w(a_i-a_j)=w(a_i-X+X-a_j)=w(a_i-X)=\delta_i. \tag{15}$$

Then, for every $b \in K$, one has:

$$w_i(X-b)=\inf(\delta_i, v(a_i-b)), \qquad w_j(X-b)=\inf(\delta_j, v(a_j-b)).$$

According to (15) we have that: $v(a_j-b)=v(a_j-a_i+a_i-b) \geqq \inf(\delta_i, v(a_i-b))=w_i(X-b)$. Hence:

$$w_j(X-b)=\inf(\delta_j, v(a_j-b)) \geqq \inf(\delta_i, v(a_i-b))=w_i(X-b),$$

i. e., $w_i \leqq w_j$. In particular:

$$w_j(X-a_j)=\delta_j > \inf(\delta_i, v(a_i-a_j))=w_i(X-a_j)$$

and one has $w_i < w_j$. Moreover, by (15) it follows that w_j well dominates w_i.

b) Let $b \in K$. Then one has: $w(X-b)=w(X-a_i+a_i-b) \geqq \inf(w(X-a_i), v(a_i-b))= \inf(\delta_i, v(a_i-b))=w_i(X-b)$. Hence $w_i \leqq w$ for all $i \in I$. In proving that $w=\sup_i w_i$ it is enough to show that for every $b \in K$ one has

$$w(X-b)=\sup_i w_i(X-b). \tag{16}$$

Indeed, since $w(X-b) \in M_w$, there exists $i \in I$ such that $w(X-b)<\delta_i$. Hencee $w(X-b) =w(X-a_i+a_i-b) \geqq \inf(\delta_i, v(a_i-b))$, and $w(X-b)=v(a_i-b)<\delta_i$. Thus $w_i(X-b)=v(a_i-b)$. If $j>i$, then

$$w_j(X-b)=w_i(X-b)=v(a_i-b)=w(X-b). \tag{17}$$

This shows that (16) is valid and $w=\sup_i w_i$.

Remark 4.2. According to (15), it follows that $\{a_i\}_{i \in I}$ is a pseudo-convergent sequence (see [11, Ch. II]). By (13), it follows that X is a pseudo-limit of $\{a_i\}_{i \in I}$ in $K(X)$. Moreover, since X is transcendental over K, $\{a_i\}_{i \in I}$ is a transcendental pseudo-convergent sequence. According to (17) it follows that for every $f(X) \in K[X]$, one has:

$$w(f(X))=\sup_i w_i(f(X))=\sup_i v(f(a_i)).$$

This remark enables us to reobtain (using our considerations) the classical results of Ostrowski (see [9, Teil III] and to give a new proof of [10, Ch. II, Lemma 11].

2. We consider now the r. a. f. extensions w of v to $K(X)$. Thus the quotient group G_w/G_v contains at least a free element (i. e., an element $\bar{\delta}$ such that $n\bar{\delta} \neq 0$ for all $n \in \boldsymbol{Z}$, $n \neq 0$). Hence in the group G_w there exists at least one element δ such that $\boldsymbol{Z}\delta \cap G_v=0$. It is clear that we may assume that there exists $a \in K$ such that:

$$\delta = w(X-a).$$

We assert that:

(18) $$G_w = G_v + \boldsymbol{Z}\delta.$$

Indeed, assume that there exists $\delta' \in G_w$ such that $\delta' \notin G_v + Z\delta$. Let $r \in K(X)$ be such that $w(r)=\delta'$. Write $r=f/g$, $f, g \in K[X]$, and $f=a\prod_i (X-a_i)$, $g=b\prod_j (X-b_j)$, one sees that $\delta' = w(r) = v(a)-v(b)+\sum_i w(X-a_i)-\sum_j w(X-b_j)$. Since $\delta' \notin G_v + \boldsymbol{Z}\delta$, then for at least one i or one j, we have $w(X-a_i) \notin G_v + \boldsymbol{Z}\delta$ or $w(X-b_j) \notin G_v + \boldsymbol{Z}\delta$. Suppose that $\delta_1 = w(X-a_1) \notin G_v + \boldsymbol{Z}\delta$. Then $v(a-a_1) = w(a-X+X-a_1) = \inf(\delta, \delta_1)$, a contradiction. Hence the equality (18) is valid.

Finally, the valuation w can be described easily. Let $f(X) \in K[X]$. Write:

$$f(X) = a_0 + a_1(X-a) + \cdots + a_n(X-a)^n.$$

Then according to (18), we have:

(19) $$w(f(X)) = \inf_i (v(a_i) + i\delta).$$

Theorem 4.3. *Let w be an r.t.f. extension of v to $K(X)$. Then there exists a pair $(a, \delta) \in K \times G_w$ such that $w(X-a)=\delta$. Moreover, $G_w = G_v \oplus \boldsymbol{Z}\delta$ and w is defined by* (19).

Conversely, let G be an ordered group which contains G_v as a subgroup, and $\delta \in G$ be such that $Z\delta \cap G_v = 0$. Let $a \in K$ and let w: $K(X) \to G$ be defined by the equality (19). *Then w is an r.a.f. extension of v to $K(X)$. Moreover, $G_w = G_v \oplus \boldsymbol{Z}\delta$, and $k_w = k_v$.*

The first part of the theorem follows from the above considerations. The proof of the last part is obvious.

Let w be an r. a. f. extension of v to $K(X)$. A pair $(a, \delta) \in K \times G_w$ as in the above theorem is also called a *pair of definition* of w. How many pairs of definition has w? One has the following result:

Remark 4.4. Let w be an r. a. f. extension of v to $K(X)$ and (a_1, δ_1), (a_2, δ_2) be two pairs of definition of w. Then

(20) $$\delta_1 = \delta_2 \quad \text{and} \quad v(a_1 - a_2) \geqq \delta_1.$$

Indeed, $w(X-a_1)=\delta_1$, $w(X-a_2)=\delta_2$. According to (19), $w(X-a_2)=w(X-a_1+a_1-a_2) = \inf(\delta_1, v(a_1-a_2)) = \delta_2$. Hence $\delta_1 \geqq \delta_2$, $v(a_1-a_2) \geqq \delta_2$. By symmetry, it follows that $\delta_1 \leqq \delta_2$ and $v(a_1-a_2) \geqq \delta_1$. Finally, $\delta_1 = \delta_2$ and $v(a_1-a_2) \geqq \delta_1$.

By the above considerations one sees that r. a. f. extensions of v to $K(X)$ are similar to r. t. extensions. They are defined by inf, v and a suitable pair $(a, \delta) \in K \times G_w$. Moreover, (20) shows that the relation between various pairs of definition of an r. a. f. extension is the same as the relation between various pairs of definition of an r. t. extension (see [1]). The only (but essential) difference is the nature of δ. For r. t. extensions $\delta \in G_{\bar{v}} = \boldsymbol{Q}G_v$, while for r. a. f. extensions δ belongs to an ordered group G which strictly contains G_v and $\boldsymbol{Z}\delta \cap G_v = 0$.

3. We define now a family of r. a. f. extensions of v to $K(X)$, namely, those extensions w of v whose rank (see [5, Ch. VI] or [10, Ch. I]) is different from the rank of v (of course, we assume that the rank of v is finite).

Let consider the lexicographically ordered group $G=G_v\times \boldsymbol{Z}$. Then one has $\mathrm{rg}(G)=\mathrm{rg}(G_v)+1$. Let $\delta=(0, 1)\in G$, and $a\in K$. Denote by w the valuation on $K(X)$ defined by inf, v, a and δ (see (19)). Since $\delta\notin G_v$, w is an r. a. f. extension of v. Denote by w_1 the r. t. extension of v to $K(X)$ defined by the pair $(a, 0)\in K\times G_v$ (i. e. w_1 is defined by inf, v, a and 0). It is easy to see that $O_w\subset O_{w_1}$, $G_w=G$, $G_{w_1}=G_v$. Let M_w and M_{w_1} be the maximal ideal of O_w and O_{w_1}, respectively. Then one has $M_{w_1}\subset M_w$ and O_{w_1} is the ring of quotients of O_w with respect to the complements of M_{w_1}.

Conversely, let a be an element of K and let w_1 be the r. t. extension of v to $K(X)$ defined by the pair $(a, 0)\in K\times G_v$. Let O_{w_1} the valuation ring of w_1 and M_{w_1} the maximal ideal of O_{w_1}. Denote $t=(X-a)^*$; then t is transcendental over k_v and $k_{w_1}=k_v(t)$, i. e. k_{w_1} is the field of rational functions of t over k_v. Denote v' the valuation on $k_v(t)$ (trivial on k_v) defined by the irreducible polynomial t. One has $k_{v'}=k_v$, $G_{v'}=\boldsymbol{Z}$. Let $\varphi\colon O_{w_1}\to k(t)$ be the canonical homomorphism. Denote $O_w=\varphi^{-1}(O_{v'})$, $M_w=\varphi^{-1}(M_{v'})$. Then one has $M_{w_1}\subset M_w\subset O_w\subset O_{w_1}$. It is easy to see that O_w is in fact the valuation ring of the valuation w on $K(X)$ defined by the pair (a, δ), where $\delta=(0, 1)\in G_v\times \boldsymbol{Z}$.

5. The r. a. extensions. The general case.

Now let K be a (not necessarily algebraically closed) field and v a valuation on K. We consider the r. a. extensions w of v to $K(X)$. As usual we denote by $\bar{K}$ a fixed algebraic closure of K and by v a fixed extension of $\bar{v}$ to $\bar{K}$. Let $\bar{w}$ be a fixed common extension of $\bar{v}$ and w to $\bar{K}(X)$.

1. First, we assume that w is an r. a. t. extension of v. Then it is easy to see that $\bar{w}$ is also an r. a. t. extension of $\bar{v}$. Consider the set $M_{\bar{w}}$ defined in (1). As in §4, 1., let $\{\delta_i\}_{i\in I}$ be a cofinal well ordered subset of $M_{\bar{w}}$. Since by the hypothesis $\bar{w}$ is not an r. t. extension, I has no last element. For every $i\in I$ we choose an element $a_i\in\bar{K}$ such that

(21) $$\bar{w}(X-a_i)=\delta_i \quad\text{and}\quad [K(a_i)\colon K] \text{ is the smallest possible}$$

(this means that if $\bar{w}(X-b)=\delta_i$ then $[K(b)\colon K]\geqq[K(a_i)\colon \bar{K}]$). Denote by $\bar{w}_i$ the r. t. extension of $\bar{v}$ to $\bar{K}(X)$ defined by the pair (a_i, δ_i). By (21) it follows that (a_i, δ_i) is a minimal pair of definition of w_i with respect to K. According to Theorem 4.1, we see that:

(22) $$\bar{w}_i<\bar{w}_j \ \text{ if } \ i<j, \quad \bar{w}_i<\bar{w} \ \text{ for all } \ i\in I \ \text{ and } \ \bar{w}=\sup_i \bar{w}_i .$$

For all $i\in I$, we denote by w_i the restriction of $\bar{w}_i$ to $K(X)$ and by v_i the restriction of $\bar{v}$ to $K(a_i)$. It is easy to see that (a_i, δ_i) is in fact a minimal pair of definition of w_i. Since $\{\bar{w}_i\}_{i\in I}$ is an ordered system of r. t. extensions of v to $K(X)$ and $\bar{w}=\sup \bar{w}_i$, according to Theorem 2.3, one has the following result:

Theorem 5.1. *Let w be an r.a.t. extension of v to $K(X)$. Then with above notation, we have:*

1) $w_i < w_j$, $k_{v_i} \subseteqq k_{v_j}$ *and* $G_{v_i} \subseteqq G_{v_j}$ *whenever* $i<j$.

2) $(w_i)_{i\in I}$ *is an ordered system of r.t. extensions of v to $K(X)$ and* $w=\sup_i w_i$. *Moreover, we have*

$$k_w = \bigcup_i k_{v_i}\,;\ G_w = \bigcup_i G_{v_i}$$

Corollary 5.2. *If w is an r.a.t. extension of v to $K(X)$ then:*

a) k_w/k_v *is an algebraic extension and is countably generated (i.e. k_w is obtained by adjoining to k_v at most countably mony algebraic elements).*

b) *The group G_w/G_v is countable.*

The proof follows from Theorem 5.1, because $\{k_{v_i}\}_i$ and $(G_{v_i})_i$ are totally ordered sets.

2. Now we consider the r.a.f. extensions of v to $K(X)$.

Let w be an r.a.f. extension of v to $K(X)$. Denote by $\bar{w}$ a common extension of w and $\bar{v}$ to $\bar{K}(X)$. It is easy to see that $\bar{w}$ is also an r.a.f. extension of $\bar{v}$ to $\bar{K}(X)$. According to Theorem 4.3, $\bar{w}$ is defined by a pair of definition (a, δ). We shall say that (a, δ) is a *minimal pair of definition of w with respect to K* if $[K(a): K]$ is the smallest possible. Hence if $[K(b): K]<[K(a): K]$, then according to Remark 4.4 one has: $\bar{v}(b-a)<\delta$.

Theorem 5.3. *Let w be an r.a.f. extension of v to $K(X)$ and (a, δ) a minimal pair of definition of w with respect to K. Denote by f the (monic) minimal polynomial of a over K ane let $\gamma=w(f)$. If $g\in K[X]$ and $g=g_0+g_1f+\cdots+g_nf^n$, where* $\deg g_i<\deg f$, $0\leqq i\leqq n$, *then:*

$$w(g)=\inf(v(g_i(a))+i\gamma).$$

Moreover, if v_1 is the restriction of $\bar{v}$ to $K(a)$, then

$$k_w=k_{v_1} \quad and \quad G_w=G_{v_1}\oplus \boldsymbol{Z}\gamma.$$

Proof. Let $a=a_1, \cdots, a_m$ be all roots of f in $\bar{K}$. Then $\gamma=w(f(X))=\bar{w}(\prod_{i=1}^{m}(X-a_i))$ $=\sum_i \bar{w}(X-a_i)$. But according to (19), we have $\bar{w}(X-a_1)=\delta$, $\bar{w}(X-a_i)=\inf(\delta, \bar{v}(a-a_i))$ $i=2, \cdots, m$. This means that $\gamma\notin G_{\bar{v}}$ and so $\boldsymbol{Z}\gamma\cap G_{\bar{v}}=0$. The proof follows now in a canonical manner.

6. Existence of extensions of v to $K(X)$ with a given residue field.

1. Let us assume that (K, v) is a valuation pair such that k_v is not algebraically closed. By Corollary 5.2 it follows that if w is an r.a.t. extension of v to $K(X)$ then k_w/k_v is a countably generated extension. There exists a somewhat converse result:

Theorem 6.1. *Let k/k_v be a countably generated infinite algebraic extension. Then there exists an r.a.t. extension w of v to $K(X)$ such that $k_w\cong k$. Moreover, w can be* *chosen such that* $G_w=G_v$.

Proof. Since $k_{\bar{v}}$ is in fact an algebraic closure of k_v we can assume that $k_v\subseteqq k\subseteqq k_{\bar{v}}$.

Since k/k_v is countably generated, there exists a tower $k_v\subseteqq k_1\subseteqq k_2\subseteqq\cdots$ of finite extensions of k such that $\cup k_n=k$. We shall prove that for every natural number n there exists an element $b_n\in\bar{K}$ such that:

1) b_n is separable over K and $[K(b_n): K]=[k_n: k_v]$.

2) If v_n is the restriction of $\bar{v}$ to $K(b_n)$, then $k_{v_n}=k_n$.

3) $K(b_n)\subseteqq K(b_{n+1})$, $n\geqq 1$.

The proof is given by induction on n. Indeed, according to [3, Lemma 4.2] there exists b_1 such that 1) and 2) are satisfied. Let us assume that $n\geqq 1$ and $b_1, \cdots, b_n$ are defined such that all conditions 1)-3) are safisfied. Again according to [3, Lemma 4.2] there exists an element $c\in\bar{K}$ such that c is separable over $K(b_n)$, $[K(b_n, c): K(b_n)]=[k_{n+1}: k_n]$ and $k_{v_{n+1}}=k_{n+1}$, where v_{n+1} is the restriction of $\bar{v}$ to $K(b_n, c)$. Since $K(b_n)/K$ and $K(b_n, c)/K(b_n)$ are separable extensions, $K(b_n, c)/K$ is also separable and $K(b_n, c)=K(b_{n+1})$ for a suitable element b_{n+1} of $\bar{K}$.

Furthermore, let (K', v') be the Henselization of (K, v) included in $(\bar{K}, \bar{v})$ (see [6, p. 131]). This means that $K\subseteqq K'\subseteqq\bar{K}$, v' is the restriction of $\bar{v}$ to K', v' is Henselian and (K', v') is an immediate extension of (K, v), i. e. $k_v=k_{v'}$ and $G_v=G_{v'}$ (see [10], Ch. II]).

We assert that $[K'(b_n): K']=[K(b_n): K]$. Indeed, one has $K(b_n)\subseteqq K'(b_n)$ and $k_{v_n}\subseteqq k_{v'_n}$, where v'_n is the restriction of $\bar{v}$ to $K'(b_n)$. Since $k_v=k_{v'}$, $k_{v_n}=k_{v'_n}$ and according to 1), it follows that $[K(b_n): K]=[K'(b_n): K']$. Moreover, by 3) it follows that for all n one has:

$$K'(b_n)\subseteqq K'(b_{n+1}). \tag{23}$$

Now, for every positive integer n we shall define a pair $(a_n, \delta_n)\in\bar{K}\times G_{\bar{v}}$ such that:

α) If we denote by $\bar{w}_n$ the r. t. extension of $\bar{v}$ to $\bar{K}(X)$ defined by inf, $\bar{v}$, a_n and δ_n, then (a_n, δ_n) is a minimal pair of definition of $\bar{w}_n$ with respect to K.

β) $\delta_n<\delta_{n+1}$ and $\bar{v}(a_{n+1}-a_n)\geqq\delta_n$, or equivalently $\bar{w}_n<\bar{w}_{n+1}$ (see Proposition 1.1).

γ) $K(a_n)=K(b_n)$ for all n.

The pair (a_n, δ_n) is taken by induction on n. Let us denote $a_1=b_1$. Since a_1 is separable over K', by [3, Theorem 3.9] it follows that there exists $\delta_1\in G_{\bar{v}}$ such that (a_1, δ_1) is a minimal pair of defintion of $\bar{w}_1$ with respect to K.

Let us assume that $n\geqq 1$ and that the pairs (a_i, δ_i), $i=1, \cdots, n$, satisfy the conditions α)-γ). Since $[K'(b_n): K']=[K(b_n): K]$, by γ) it follows that $K'(a_n)=K'(b_n)$ and by (23) and γ), we have

$$K'(b_n)=K'(a_n)\subseteqq K'(b_{n+1}). \tag{24}$$

Let $a\in K$ be such that

$$v(a)>\sup(\delta_n, \omega(a_n))-v(b_{n+1}) \tag{25}$$

with $\omega(a_n)=\sup(\bar{v}(a_n-a'_n))$, where a'_n runs over all conjugate elements of a_n in $\bar{K}$ over K and distinct from a_n. Let us denote:

$$a_{n+1}=ab_{n+1}+a_n .$$

Obviously, by (25) one has $\bar{v}(a_{n+1}-a_n)>\omega(a_n)$, and according to Krasner's Lemma (see [6, p. 22]) it follows that $K'(a_n)\subseteqq K'(a_{n+1})$. According to (24) and the inductive hypothesis γ) it follows that $K'(b_{n+1})=K'(a_{n+1})$ and $K(a_{n+1})=K(b_{n+1})$.

Let $\delta_{n+1}\in G_{\bar{v}}$ be such that

$$\delta_{n+1}>\sup(\delta_n, \omega(a_{n+1})). \tag{26}$$

Then, by [3, Proposition 3.2], it follows that (a_{n+1}, δ_{n+1}) is a minimal pair of definition of $\bar{w}_n$ with respect to K'. Moreover, since $[K'(a_{n+1}): K']=[K(a_{n+1}): K]$ and a_{n+1} is separable over both K' and K, by [3, Proposition 4.1], it follows that (a_{n+1}, δ_{n+1}) is a minimal pair of definition of $\bar{w}_{n+1}$ with respect to both K' and K. Therefore it is clear that conditions α)-γ) are satisfied by all pairs (a_i, δ_i), $i=1, \cdots, n+1$.

Finally, let us denote by w_n the restriction of $\bar{w}_n$ to $K(X)$. By β) it follows that $w_n\leqq w_{n+1}$ for all n and so $\{w_n\}_n$ is an ordered system of r. t. extensions of v to $K(X)$. We show that the ordered system $\{\bar{w}_n\}_n$ has a limit. To do this we shall prove that the condition 2) of Proposition 2.2 is verified. Indeed, let $c\in\bar{K}$ and assume that for every n one has $\bar{w}_n(X-c)\geqq\delta_n$. This means that $\bar{v}(a_n-c)=\bar{w}_n(a_n-X+X-c)\geqq\delta_n$. According to (26) it follows that $\bar{v}(a_n-c)>\omega(a_n)$ if $n\geqq 2$. Hence by Krasner's Lemma, it follows that $K'(a_n)\subseteqq K(c)$ for all $n\geqq 2$. But this is a contradiction, because the sequence $[K'(a_n): K]=[k_n: k_v]$ tends to infinity. Therefore by Proposition 2.2 it follows that $\{\bar{w}_n\}_n$ has a limit $\bar{w}$ which is not an r. t. extension of $\bar{v}$. Then, according to Theorem 2.4, it follows that w, the restriction of $\bar{w}$ to $K(X)$, is a limit of $\{w_n\}_n$ and $k_w=\bigcup k_{v_n}$ $=\bigcup k_n=k$. Moreover, according to [3, Lemma 4.2], we can choose δ_n such that $G_{w_n}=G_v$ for all n. Then by Theorem 5.1 one has $G_w=G_v$, as claimed.

Now, let us consider a finite extension k/k_v (assume also that $k_v\subset k\subset k_{\bar{v}}$). The existence of an r. a. t. extension w of v to $K(X)$ such that $k_w=k$ is proved under additional assumptions.

Theorem 6.2. *Let* k/k_v *be a finite algebraic extension. Let* $(\tilde{K}, \tilde{v})$ *be the completion of* (K, v) (see [5, Ch. VI, §5]). *Assume thkt* tr. deg $\tilde{K}/K>0$. *Then there exists an r. a. t. extension* w *of* v *to* $K(X)$ *such that* $k_w=k$. *Moreover, we can choose* w *such that* $G_v=G_w$.

Proof. Since k/k_v is finite, according to [3, Lemma 4.2] there exists an element $a\in\bar{K}$ such that a is separable over K, $[K(a): K]=[k: k_v]$ and $k_{v_1}=k$, where v_1 is the restriction of $\bar{v}$ to $K(a)$. Moreover, if (K', v') is the Henselization of (K, v) included in $(\bar{K}, \bar{v})$ (see [6, p. 131]) then

$$[K'(a): K']=[K(a): K]=[k: k_v]. \tag{27}$$

Since there exists an element $\tilde{a}\in\tilde{K}$ transcendental over K, there exists a well ordered set $\{\delta_i\}_{i\in I}$ of elements of G_v and a system $\{a_i\}_i$ of elements of K such that:

(28)
1) δ_i is a cofinal subset of G_v,
2) $v(a_i - a_j) = \delta_i$ whenever $i < j,\ i,\ j \in I$,
3) $v(a_i - \tilde{a}) = \delta_i$ for all $i \in I$.

Let $a = a^{(1)}, \cdots, a^{(n)}$ be all conjugates of a over K.

Set $\omega(a) = \sup\{\bar{v}(a - a^{(t)}),\ t = 2, \cdots, n\}$. According to (28), 1), there exists $i_0 \in I$ such that $\delta_{i_0} > \omega(a)$. By a suitable modification of the set I, we may assume that

$$\omega(a) < \delta_i \quad \text{for all} \quad i \in I. \tag{29}$$

Let $\bar{w}_i$ be the r. t. extension of $\bar{v}$ to $\bar{K}(X)$ defined by inf, $\bar{v}$, $a_i + a$ and δ_i. Since all conjugates of $a_i + a$ over K are obviously $a_i + a^{(1)}, \cdots, a_i + a^{(n)}$, it follows that $\omega(a_i + a) = \omega(a)$. Hence, according to (29) and [3, Proposition 3.2], it follows that $(a_i + a,\ \delta_i)$ is a minimal pair of definition of w_i with respect to K'. Now since $K(a) = K(a_i + a)$, by (27) and [3, Proposition 4.1], it follows that $(a_i + a,\ \delta_i)$ is also a minimal pair of definition of $\bar{w}_i$ with respect to K.

We show that in fact $\{\bar{w}_i\}_i$ is an ordered system of r. t. extensions of $\bar{v}$ to $\bar{K}(X)$. Indeed, one has: $\bar{v}(a_i + a - (a_j + a)) = \bar{v}(a_i - a_j) = \delta_i$ and $\delta_i < \delta_j$ if $i < j$ (see (28), 2)). Thus by Proposition 1.1 it follows that $\bar{w}_i < \bar{w}_j$.

Furthermore we show that the ordered system $\{\bar{w}_i\}_{i \in I}$ has a limit. For that we verify the condition 2) of Proposition 2.2. Indeed, let $b \in \bar{K}$. Assume that $\bar{w}_i(X - b) \geqq \delta_i$ for any $i \in I$. Then $\bar{v}(b - (a_i + a)) = \bar{w}_i(b - X + X - (a_i + a)) \geqq \delta_i$. Hence the element $b - a \in \bar{K}$ is also a limit of the Cauchy sequence $\{a_i\}_{i \in I}$, or equivatently $\tilde{a}$ is algebraic over K, a contradiction. Therefore the condition 2) of Proposition 2.2 is verified for all $b \in \bar{K}$, and $\{\bar{w}_i\}_i$ has a limit $\bar{w}$.

Let us denote by w_i the restriction of $\bar{w}_i$ to $K(X)$ for all $i \in I$, and let w be the restriction of $\bar{w}$ to $K(X)$. By Therem 5.1 we have $w = \sup_i w_i$ and $k_w = \bigcup_i k_{v_i} = k$. As usual v_i is the restriction of $\bar{v}$ to $K(a_i + a) = K(a) = K_1$. Finally $G_w = \bigcup_i G_{v_i} = G_v$, for the equality $[K(a): K] = [k: k_v]$ implies $G_{v_1} = G_v$ and $\delta_i \in G_v$ implies $G_{v_1} = G_v = G_{v_i}$.

2. If w is an r. a. f. extension of v to $K(X)$ then by Theorem 5.3 it fhllows that k_w/k_v is a finite extension. Now a somewhat converse result is valid:

Proposition 6.3. *Let k/k_v be a finite extension. Thee there exists an r.a.f. extension w of v to $K(X)$ such that $k_w = k$.*

Proof. Since k/k_v is finite, according to [3, Lemma 4.2], there exists an element $a \in \bar{K}$ such that a is separable over K, $[K(a): K] = [k: k_v]$ and $k_{v_1} = k$, where v_1 is the restiction of $\bar{v}$ to $K(a)$.

Order $G = \boldsymbol{Z} \times G_{\bar{v}}$ lexicographically and write $\delta = (1, 0) \in G$. Let $\bar{w}$ be the extension of $\bar{v}$ to $\bar{K}(X)$ defined by inf, $\bar{v}$, a and δ. It is clear that $\bar{w}$ is an r. a. f. extension of $\bar{v}$ to $\bar{K}(X)$ and so $\bar{w}$, the restriction of $\bar{w}$ to $K(X)$, is also an r. a. f. extension of v. Furthermore since $\delta > \gamma$ for all $\gamma \in G_{\bar{v}}$ (we remark that $G_{\bar{v}}$ is identified with $0 \times G_{\bar{v}}$) then (a, δ) is a minimal pair of definition of w with respect to K. Therefore, according to

Theorem 5.3, we have: $k_w = k_{v_1} = k$, as claimed.

7. Existence of extensions of v to $K(X)$ with given value group.

Let us assume that (K, v) is a valuation pair such that G_v is not divisible. By Corollary 5.2 it follows that if w is an r. a. t. extension of v to $K(X)$ then the group G_w/G_v is countable. There exists a somewhat converse result:

Theorem 7.1. *Let (K, v) be a valuation pair. Assume $G_v \subset G \subseteqq \boldsymbol{Q}G_v = G_{\bar{v}}$ and that G/G_v is an infinite but countable group. Then there exists an r.a.t. extension w of v to $K(X)$ such that $G_w = G$. Moreover one can choose w such that $k_w = k_v$.*

Proof. Since G/G_v is a countable torsion group, we can find a sequence of subgroups:

$$G_v \subset G_1 \subset G_2 \subset \cdots \subset G_n \subset \cdots G$$

such that $G_n \neq G_{n+1}$, G_n/G_v is finite for all n, and that $\bigcup_n G_n = G$.

Now we shall define, for each positive integer n, an element $a_n \in \bar{K}$, separable over K, such that:

a) $[K(a_n): K] = [G_n: G_v] (= |G_n/G_v|)$

b) $K(a_n) \subset K(a_{n+1})$

c) If we denote by v_n the restriction of $\bar{v}$ to $K(a_n)$ then $G_{v_n} = G_n$.

The element a_n can be defined by induction on n. Indeed, according to [3, Lemma 4.3], there exists an element a_1 such that a) and c) are satisfied. Let us assume that $n \geqq 1$ and that the elements $a_1, \cdots, a_n$ are defined such that a), b) and c) are satisfied. Again, according to [3, Lemma 4.3], there exists an element $b_{n+1} \in \bar{K}$ separable over $K(a_n)$ such that $[K(a_n)(b_{n+1}): K(a_n)] = [G_{n+1}: G_n]$ and $G_{v_{n+1}} = G_{n+1}$, where v_{n+1} is the restriction of $\bar{v}$ to $K(a_n, b_{n+1})$. Now, since b_{n+1} is separable over $K(a_n)$ and a_n is separable over K by hypotheses, there exists an element $a_{n+1} \in \bar{K}$ such that $K(a_n, b_{n+1}) = K(a_{n+1})$. It is clear that the elements $a_1, \cdots, a_n, a_{n+1}$ are such that the conditions a), b), c) are satisfied.

The rest of the proof is made in the same way as the proof of Theorem 6.1 and it is left to the reader.

In the same manner as we have proved Theorem 6.2, we can prove the following result:

Theorem 7.2. *Let (K, v) be a valuation pair and let G be an ordered group such that $G_v \subseteqq G$ and G/G_v is finite. Assume that* tr.deg$\tilde{K}/K > 0$, *where $(\tilde{K}, \tilde{v})$ is the completion of (K, v)* (see [4, Ch. V, § 5]). *Then there exists an r.a.t. extension w of v to $K(X)$ such that $G_w = G$. Moreover we can choose w such that $k_w = k_v$.*

By Theorems 6.1 and 7.1 one may derive in a canonical way the following result:

Corollary 7.3. *Let (K, v) be a valuation pair. Assume that there exist a countably generated infinite algebraic extension k/k_v and an ordered group G such that $G_v \subset G$ and G/G_v is a countably infinite torsion group. Then there exists an r.a.t. extension w of v*

to $K(X)$ *such that* $k_w \cong k$ *and* $G_w \cong G$.

Also by Theorems 6.2 and 7.2 it follows:

Corollary 7.4. *Let* (K, v) *be a valuation pair. Let* k/k_v *be a finite algebnaic extension and let* G *be an ordered group such that* $G_v \subset G$ *and* G/G_v *is finite. Assume that* tr.deg $\tilde{K}/K>0$, *where* $(\tilde{K}, \tilde{v})$ *is the completion of* (K, v) (see [4, Ch. V. § 5]). *Then there exists an r.a.t. extension* w *of* v *to* $K(X)$ *such that* $k_w \cong k$ *and* $G_w \cong G$.

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