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On the irreducible very cuspidal representations

By

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Introduction

The notion of 'very cuspidal representation' of the maximal compact modulo center subgroup of GL_n over a non-archimedean local field F was introduced by Carayol [C]. He has shown that the compact-induction to $GL_n(F)$ of an irreducible very caspidal representation is irreducible and supercuspidal. If an irreducible very cuspidal representation has an even level (cf. Definition 1.3.2), it has a simple description by a generic element u (cf. Definition 1.2.1) and a quasi-character θ of $F(u)^\times$. (See 5.6 in [C]). But if it has an odd level, there is no simple description. When n is a prime, any irreducible supercuspidal representation of $GL_n(F)$ is, up to twisting by quasi-character of $F^\times$, compactly-induced from an irreducible very cuspidal representation. In this case, Kutzko and Moy [K-M] have calculated the ε-factor of any irreducible supercuspidal representation of $GL_n(F)$ and proved the local Langlands conjecture ([L]) for $GL_n(F)$. In the so-called tame case when n is relatively prime to the residual characteristic p of F, Moy [M] has calculated the ε-factor of any irreducible supercuspidal representation and proved the local Langlands conjecture. (Remark that the irreducible supercuspidal representation is not always obtained from an irreducible very cuspidal representation unless n is a prime.)

The purpose of this paper is to study very cuspidal representations with odd levels in detail, in particular to calculate the ε-factors of the induced supercuspidal representations of $GL_n(F)$ for general n and p. Our main result is Theorem 3.3.2. We note that we shall not treat the level one very cuspidal representation since the ε-factor in this case is calculated in [G2].

In section 1, we review the definition and properties of very cuspidal representations according to [C]. Section 2 is devoted to the construction of irreducible very cuspidal representations. Our goal in this section is to show: after all, the irreducible very cuspidal representation with an odd level can be described by a generic element u and a quasi-character θ of $F(u)^\times$ not exactly but in a sense similar as in the even level case. For the purpose, we want to apply the method of Moy (see Sections 3.5-3.6 in [M]). But it is not directly applicable when p divides the order of the group $F(u)^\times/F^\times(1+P_{F(u)})$, so we find some quotient group $F(u)^\times/F^\times\langle u^h\rangle(1+P_{F(u)})$ (cf. Subsections 2.4 and 2.6) to which we apply Moy's method with slight modification. Thus we get information which is enough to calculate the ε-factor, but not yet enough to get a complete character formula. In section 3, we calculate the ε-factor of the

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compact-induction to $\mathrm{GL}_n(F)$ of the irreducible very cuspidal representation we construct in section 2.

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Notation. For a non-archimedean local field F, we denote by $\mathcal{O}_F$, P_F, ϖ_E, k_F, and q_F the maximal order of F, the maximal ideal of $\mathcal{O}_F$, a prime element of $\mathcal{O}_F$, the residue field of F and the order of k_F, respectively. Let v_F be the valuation of F normalized by $v_F(\varpi_F)=1$ and $|\ |_F=q_F^{-v_F(\ )}$ the absolute value of F.

The $n\times n$ zero and identity matrices are denoted by $\mathbf{0}_n$ and $\mathbf{1}_n$, respectively. If X is a matrix, $\det X$ and $\operatorname{tr} X$ stand for its determinant and trace, respectively. We denote by $|A|$ the order of the finite group A. Let G be a group, H be a normal subgroup of G and χ be a representation of H. We say χ is lifted to G if there exists a representation π of G whose restriction to H is equivalent to χ, and then we call π a lift of χ.

1. Review of very cuspidal representations

1.1. Let F be a non-archimedean local field of residual characteristic p and $G=\mathrm{GL}_n(F)$. We set $V_F=F^n$ so that $\mathrm{M}_n(F)=\mathrm{End}_F(V_F)$ and $G=\mathrm{Aut}_F(V_F)$. Suppose n is decomposed into two factors r and s i.e. $n=rs$.

Definition 1.1.1. Let $\{L_i \mid i\in \boldsymbol{Z}\}$ be the set of $\mathcal{O}_F$-lattices in V_F. $\{L_i \mid i\in \boldsymbol{Z}\}$ is said to be a lattice flag of length s if the following conditions hold for all integers i:

(1) $L_i\supset L_{i+1}$.
(2) $P_F L_i=L_{i+s}$.
(3) $\dim_{k_F}(L_i/L_{i+1})=r$.

Hereafter we fix the length s of lattice flags.

Definition 1.1.2. Let $\{L_i \mid i\in \boldsymbol{Z}\}$ be a lattice flag.

(1) We set $K_s=\{g\in G \mid gL_i=L_i \text{ for all } i\in \boldsymbol{Z}\}$. K_s is a compact subgroup in G.

(2) Let z_s be an element in G such that $(z_s)^s=\varpi_F\cdot\mathbf{1}_n$ and $z_sL_i=L_{i+1}$ for all i. Let Z_s be a cyclic group generated by z_s.

(3) For integers m, we set $A_s{}^m=\{f\in M_n(F) \mid fL_i\subset L_{i+m}\}$. $A_s{}^0$ is a ring and $A_s{}^m=(z_s)^m A_s{}^0=A_s{}^0(z_s)^m$.

(4) For positive integers m, we set $K_s{}^m=1+A_s{}^m$, which are normal subgroups of K_s.

Remark 1.1.3. We can construct all lattice flags explicitly by taking an appropriate basis.

We can take $e_1, e_2, \cdots, e_n\in L_0$ such that $\{e_{n-ri+1} \bmod L_i, \cdots, e_{n-(i-1)r} \bmod L_i\}$ makes a basis of L_i/L_{i+1} over k_F for $i=0, \cdots, s-1$.

By Nakayama's lemma, $\{e_1, \cdots, e_n\}$ forms an $\mathcal{O}_F$-basis for L_0/L_1. Then L_i ($i=0, \cdots, s-1$) is expressed as follows:

$L_0=\mathcal{O}_F e_1 \oplus \cdots \oplus \mathcal{O}_F e_n$

$L_1=\mathcal{O}_F e_1 \oplus \cdots \oplus \mathcal{O}_F e_{n-r} \oplus P_F e_{n-r+1} \oplus \cdots \oplus P_F e_n$

....

$L_i=\mathcal{O}_F e_1 \oplus \cdots \oplus \mathcal{O}_F e_{n-ir} \oplus P_F e_{n-ir+1} \oplus \cdots \oplus P_F e_n$

....

$L_{s-1}=\mathcal{O}_F e_1 \oplus \cdots \oplus \mathcal{O}_F e_r \oplus P_F e_{r+1} \oplus \cdots \oplus P_F e_n$

and $L_{i+ms}=\varpi_F^m L_i$ for all integers m.

For this basis, we can express K_s, z_s, A_s^0 and A_s^1 as follows:

$$K_s=\left\{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ a_{21} & a_{22} & \cdots & a_{2s} \\ & \cdots\cdots & & \\ a_{s1} & a_{s2} & \cdots & a_{ss} \end{pmatrix} \;\middle|\; \begin{array}{ll} a_{ij}\in M_r(\mathcal{O}_F) & \text{if } i<j \\ a_{ii}\in \mathrm{GL}_r(\mathcal{O}_F) & \\ a_{ij}\in M_r(P_F) & \text{if } i>j \end{array}\right\},$$

$$z_s=\begin{pmatrix} 0_r & 1_r & 0_r & \cdots\cdots & 0_r \\ 0_r & 0_r & 1_r & \cdots\cdots & 0_r \\ & \cdots\cdots\cdots\cdots & & & \\ 0_r & 0_r & 0_r & \cdots\cdots & 1_r \\ \varpi_F\cdot 1_r & 0_r & 0_r & \cdots\cdots & 0_r \end{pmatrix},$$

$$A_s^0=\left\{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ a_{21} & a_{22} & \cdots & a_{2s} \\ & \cdots\cdots & & \\ a_{s1} & a_{s2} & \cdots & a_{ss} \end{pmatrix} \;\middle|\; \begin{array}{ll} a_{ij}\in M_r(\mathcal{O}_F) & \text{if } i\leqq j \\ a_{ij}\in M_r(P_F) & \text{if } i>j \end{array}\right\},$$

$$A_s^1=\left\{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ a_{21} & a_{22} & \cdots & a_{2s} \\ & \cdots\cdots & & \\ a_{s1} & a_{s2} & \cdots & a_{ss} \end{pmatrix} \;\middle|\; \begin{array}{ll} a_{ij}\in M_r(\mathcal{O}_F) & \text{if } i<j \\ a_{ij}\in M_r(P_F) & \text{if } i\geqq j \end{array}\right\}.$$

1.2. The multiplication by ϖ_F induces a k_F-isomorphism between L_i/L_{i+1} and L_{i+s}/L_{i+s+1}. An element of A_s^0 induces an endomorphism of L_i/L_{i+1}, so we have a ring homomorphism:

$$R\colon A_s^0 \longrightarrow \prod_{i\in Z/sZ} \mathrm{End}_{k_F}(L_i/L_{i+1}).$$

Since $\mathrm{End}_{k_F}(L_i/L_{i+1})$ is identified with $\mathrm{M}_r(k_F)$ and $\mathrm{Ker}\, R=A_s^1$, R induces a ring isomorphism:

$$A_s^0/A_s^1 \longrightarrow \mathrm{M}_r(k_F)^{Z/sZ}.$$

We shall use the same symbol R for this isomorphism. The conjugate action of z_s on A_s^0/A_s^1 induces the cyclic permutation of $\mathrm{M}_r(k_F)^{Z/sZ}$:

If $R(a)=(\alpha_0, \alpha_1, \cdots, \alpha_{s-1})$ $(\alpha_i\in \mathrm{M}_r(k_F))$, then $R(z_s\, a\, z_s^{-1})=(\alpha_{s-1}, \alpha_0, \cdots, \alpha_{s-2})$.

We shall consider the map:

$$A_s^m/A_s^{m+1} \longrightarrow A_s^0/A_s^1.$$
$$u \longrightarrow \varpi_F^{-m}u^s$$

If we put $u=z_s{}^m u_0$ with $u_0 \in A_s{}^0$ and $R(u_0)=(\alpha_0, \alpha_1, \cdots, \alpha_{s-1})$ with $\alpha_i \in \mathrm{M}_r(k_F)$, then $R(\varpi_F{}^{-m}u^s)=(\beta_0, \beta_1, \cdots, \beta_{s-1})$ with:

$$\beta_i=\alpha_{i+(s-1)m} \cdots \alpha_{i+m}\alpha_i .$$

Definition 1.2.1. An element $u \in A_s{}^m/A_s{}^{m+1}$ is said to be generic if the following conditions hold:

(1) $(m, s)=1$ where '(,)' is the greatest common divisor function.

(2) If $R(\varpi_F{}^{-m}u^s)=(\beta_0, \beta_1, \cdots, \beta_{s-1})$, then the fields $k_F(\beta_i)$ are extensions of k_F of degree r for $i=0, \cdots, s-1$.

We also say that an element $u \in M_n(F)$ is generic of level m if $u \in A_s{}^m$ and u mod $A_s{}^{m+1}$ is generic.

Now we shall summarize the properties of generic elements without proofs. Proofs may be found in 3.3 and 3.5 of [C].

Proposition 1.2.2. *Let u be generic of level m.*

(1) *The field $F(u)$ is an extension of F of degree n and its ramification degree over F is s.*

(2) *$F(u) \subset Z_sK_s$ and $F(u)^\times \cap K_s=\mathcal{O}_{F(u)}^\times$. Moreover $F(u) \cap A_s{}^m=P_{F(u)}{}^m$ for all integers m and $F(u)^\times \cap K_s{}^m=1+P_{F(u)}{}^m$ for all integers $m \geqq 1$.*

(3) *Let x be an element of $A_s{}^l$. Suppose $ux-xu \in A_s{}^{m+l+1}$, then $x \in F(u)+A_s{}^{l+1}$.*

1.3. Fix an additive character ψ of F with conductor P_F.

Lemma 1.3.1. *Let l, m be integers such that $m \leqq l \leqq 2m$, $l \geqq 1$ and $m \geqq 1$. We shall define the function ψ_u on $K_s{}^m$ by:*

$$\psi_u(x)=\psi(\mathrm{tr}(u(x-1))).$$

Then the map $u \rightarrow \psi_u$ induces an isomorphism between $A_s{}^{-l+1}/A_s{}^{-m+1}$ and the complex dual, $(K_s{}^m/K_s{}^l)^\wedge$, of $K_s{}^m/K_s{}^l$.

Proof. See 2.7 and 2.8 of [C].

We call ψ_u a generic character if u is generic.

Definition 1.3.2. An admissible representation ρ of Z_sK_s is said to be very cuspidal of level N ($N \geqq 2$) if the following conditions hold:

(1) $K_s{}^N \subset \mathrm{Ker}\,\rho$.

(2) The restriction of ρ to $K_s{}^{N-1}$ is decomposed into a sum of generic characters.

Proposition 1.3.3 (Carayol). *Let ρ be an irreducible very cuspidal representation of Z_sK_s. Then the compact-induction of ρ to G (we denote this representation by $\mathrm{ind}_{Z_sK_s}^G(\rho)$) is an irreducible supercuspidal representation of G.*

Proof. This is contained in Theorem 4.2 of [C].

Remark 1.3.4. Let u be generic of level $1-N$ and $m=[(N+1)/2]$ where '[ ]' is the greatest integer function. Since K_s^m/K_s^N is an abelian group, the map $x\to\psi((\mathrm{tr}(u(x-1)))$ $(x\in K_s^m)$ is a character of K_s^m and a lift of ψ_u. (We shall use the same symbol ψ_u for this character.) Therefore we can replace the condition (2) of Definition 1.3.2 by the condition (2′):

(2′) The restriction of ρ to K_s^m contains a sum of characters of the form ψ_u, where u is generic of level $1-N$.

2. Construction of very cuspidal representations

2.1. From now on, we fix a generic element u of level $1-N$ and set $E=F(u)$. E is an extension of F of degree n whose ramification degree over F is s. We shall start with the following lemma.

Lemma 2.1.1. *Let H_u be a stability group of ψ_u in Z_sK_s, i.e. $H_u=\{g\in Z_sK_s \mid \psi_u{}^g=\psi_u\}$ where $\psi_u{}^g(x)=\psi_u(gxg^{-1})$. Then $H_u=E^\times\cdot K_s^{[N/2]}$.*

Proof. See 5.5 of [C].

First we shall treat the even level case $N=2m$. This case is essentially contained in 5.6 of [C]. We note that $H_u=E^\times\cdot K_s^m$ in this case.

Proposition 2.1.2. (1) $\mathrm{Ind}_{K_s^m}^{H_u}(\psi_u)$ *is decomposed into a sum of one-dimensional representations of H_u, each of which is a lift of ψ_u.*

(2) *Let η be any such lift of ψ_u to H_u. Then η is written in the form $\theta\cdot\psi_u$ where θ is a quasi-character of $E^\times$ with the property that $\theta(1+x)=\psi_u(1+x)$ for $x\in P_E^m$ and $\theta\cdot\psi_u$ is defined on H_u by $\theta\cdot\psi_u(t\cdot k)=\theta(t)\cdot\psi_u(k)$ for $t\in E^\times$ and $k\in K_s^m$.*

(3) *If we put $\sigma(\theta\,;u)=\mathrm{Ind}_{H_u}^{Z_sK_s}(\theta\cdot\psi_u)$, then $\sigma(\theta\,;u)$ is an irreducible very cuspidal representation of level N of Z_sK_s and every irreducible very cuspidal representation of level N of Z_sK_s is equivalent to some representation $\sigma(\theta\,;u)$.*

Proof. The proof of (1) and (2) follows immediately from the fact that $H_u/\mathrm{Ker}\,\psi_u$ is abelian. As for (3), $\mathrm{Ind}_{H_u}^{Z_sK_s}(\theta\cdot\psi_u)$ is evidently very cuspidal from the definition of very cuspidal representation and its irreducibility is a consequence of the application of the Clifford Theory (cf. 50.6 of [C-R]). The rest of (3) follows from the Frobenius reciprocity.

2.2. Now we shall treat the odd level case $N=2m-1$, which is more complicated. In this case, we cannot lift ψ_u to H_u since $H_u=E^\times\cdot K_s^{m-1}$. So we need to investigate the space $K_s^{m-1}/K_s^m\cong A_s^{m-1}/A_s^m$ more carefully. Let $W=A_s^{m-1}/A_s^m$ and write $x\to\bar{x}$ for the natural map from A_s^{m-1} to W. W is a vector space over k_F. We denote $I\in\mathrm{End}_{k_F}W$ by the conjugate action of u i.e. $I(\bar{x})=\overline{uxu^{-1}}$ for $\bar{x}\in W$. Let $s=p^l\cdot t$ and $(t,p)=1$. We note p is an odd prime since $(1-N,s)=1$. Set $h=\dfrac{q_F^r-1}{q_F-1}\cdot t$ and $J=I^h\in\mathrm{End}_{k_F}W$. Until otherwise stated, we omit the subscript F of q_F.

Lemma 2.2.1. (1) $(J-1)^{p^l}=0$.
(2) $\dim_{k_F}\mathrm{Ker}(J-1)=r^2t$.

Proof. Since $|E^\times/F^\times(1+P_E)|=\frac{q^r-1}{q-1}\cdot s$, $(u^h)^{p^l}$ belongs to $F^\times(1+P_E)$ and thus $J^{p^l}=1$. So $(J-1)^{p^l}=0$ from the fact k_F has characteristic p. For the proof of (2), we shall use the isomorphism R between A_s^0/A_s^1 and $\mathrm{M}_r(k_F)^{Z/sZ}$. Let ϖ_E be a uniformiser of E. We may and shall assume $m=1$ since the multiplication by ϖ_E^{1-m} induces a k_F-isomorphism between A_s^{m-1}/A_s^m and A_s^0/A_s^1 which is compatible with the conjugate action of $E^\times$. It is obvious that $\bar{x}\in\mathrm{Ker}(J-1)$ if and only if $u^hx-xu^h\in A_s^{1+h(1-N)}$. If we put $u=z_s^{1-N}\cdot u_0$, $R(u_0)=(\alpha_0, \cdots, \alpha_{s-1})$ and $R(x)=(\gamma_0, \cdots, \gamma_{s-1})$, the relation $u^hx-xu^h\in A_s^{1+h(1-N)}$ is equivalent to the following relations:

$$\begin{aligned}&\alpha_{i+(h-1)(1-N)}\alpha_{i+(h-2)(1-N)}\cdots\alpha_{i+1-N}\alpha_i\gamma_i\\&=\gamma_{i+h(1-N)}\alpha_{i+(h-1)(1-N)}\cdots\alpha_{i+(1-N)}\alpha_i\qquad(i=0, 1, \cdots, s-1).\end{aligned}$$

Each α_i is non-singular since u is generic, so:

$$\gamma_{i+h(1-N)}=C_i\gamma_iC_i^{-1}\qquad(i=0, 1, \cdots, s-1)$$

where $C_i=\alpha_{i+(h-1)(1-N)}\alpha_{i+(h-2)(1-N)}\cdots\alpha_{i+1-N}\alpha_i$.

If we determine $\gamma_0, \gamma_1, \cdots$, and γ_{t-1}, then $\gamma_t, \gamma_{t+1}, \cdots, \gamma_{s-1}$ are automatically determined by the above relations since $(1-N, s)=1$ and $(h, s)=t$. Therefore $\dim_{k_F}\mathrm{Ker}(J-1)\leqq r^2t$. But $\dim_{k_F}\mathrm{Ker}(J-1)\geqq r^2t$ since $(J-1)^{p^l}=0$ and $\dim_{k_F}W=r^2s$, whence our lemma.

We define a k_F-alternating form on W by:

$$\langle\bar{x}, \bar{y}\rangle=\mathrm{tr}(u(xy-yx)))\mod P_F.$$

Remark 2.2.2. We note that $\mathrm{rad}\langle\,,\rangle=\mathrm{Ker}(I-1)$ and the conjugate action of $E^\times$ on W preserves this alternating form. We also remark that $\dim_{k_F}\mathrm{rad}\langle\,,\rangle=r$ by Proposition 1.2.3 (3).

2.3. We shall define $T\in\mathrm{End}_{k_F}W$ by $T=I^{h-1}+\cdots+I+1$ and set $W_0=(J-1)^{(p^l-1)/2}W$, $W_1=(J-1)^{(p^l-1)/2}TW$. Now we shall investigate the spaces W_0 and W_1 in the following lemmas.

Lemma 2.3.1. (1) $\dim_{k_F}W_0=r^2t\cdot\frac{p^l+1}{2}$.
(2) $\dim_{k_F}W_1=r^2t\cdot\frac{p^l-1}{2}+r$.

Proof. We set $r_i=\dim_{k_F}(J-1)^iW-\dim_{k_F}(J-1)^{i+1}W$ $(i=0, 1, \cdots, p^l-1)$, then $r_i\leqq r^2t$ by Lemma 2.1.1. On the other hand:

$$\sum_{i=0}^{p^l-1}r_i=r^2s$$

from the definition of r_i. Hence $r_i=r^2t$ for all i. Therefore we have:

$$\dim_{k_F} W_0 = r^2 t \cdot \frac{p^l+1}{2}.$$

As for the proof of (2), it suffices to see that:

$$\dim_{k_F} \operatorname{Ker} T = r^2 t - r$$

since $W_1 = TW_0$ and $\operatorname{Ker} T \subset W_0$. The map $I-1$ induces an injective homomorphism from $\operatorname{Ker}(J-1)/\operatorname{Ker}(I-1)$ to $\operatorname{Ker} T$, so we have:

$$\dim_{k_F} \operatorname{Ker} T \geqq r^2 t - r.$$

(See Lemma 2.2.1 and Remark 2.2.2.) Since $\operatorname{Ker} T \cap \operatorname{Ker}(I-1) = 0$, $\operatorname{Ker} T \oplus \operatorname{Ker}(I-1) \subset \operatorname{Ker}(J-1)$. Therefore:

$$\dim_{k_F} \operatorname{Ker} T \leqq r^2 t - r.$$

Hence our lemma.

Lemma 2.3.2. *$W_1^{\perp} = W_0$ with respect to $(W, \langle\, ,\rangle)$ i.e. $\{\bar{x} \in W \mid \langle \bar{x}, \bar{y} \rangle = 0$ for all $\bar{y} \in W_1\} = W_0$.*

Proof. Using Remark 2.2.2 and the fact that $(J-1)^{(p^l-1)} T\bar{x} \in \operatorname{rad}\langle\, ,\rangle$, we can see that for $\bar{x}, \bar{y} \in W$:

$$\langle (J-1)^{(p^l-1)/2} T\bar{x}, (J-1)^{(p^l-1)/2} \bar{y} \rangle = 0.$$

Hence $W_0 \subset W_1^{\perp}$.

On the other hand:

$$\dim_{k_F}(W_0/\operatorname{rad}\langle\, ,\rangle) + \dim_{k_F}(W_1/\operatorname{rad}\langle\, ,\rangle) = \dim_{k_F}(W/\operatorname{rad}\langle\, ,\rangle)$$

by Lemma 2.3.1. Therefore $\dim_{k_F} W_1^{\perp} = \dim_{k_F} W_0$, whence our lemma.

Lemma 2.3.3. *Let $A_s^{m,0}$ (resp. $A_s^{m,1}$) be the total iuverse image in A_s^{m-1} of W_0 (resp. W_1). Set $K_s^{m,0} = 1 + A_s^{m,0}$, $K_s^{m,1} = 1 + A_s^{m,1}$ and define a function $\tilde{\psi}_u$ on $K_s^{m,1}$ by:*

$$\tilde{\psi}_u(1+x) = \psi\left(\operatorname{tr} u\left(x - \frac{x^2}{2}\right)\right)$$

for $1+x \in K_s^{m,1}$, which is equal to ψ_u on K_s^m. Then $K_s^{m,0}$ and $K_s^{m,1}$ are normal subgroups of $E^\times \cdot K_s^{m-1}$ and $\tilde{\psi}_u$ is a character of $K_s^{m,1}$ whose stability subgroup, $\tilde{H}_u$, in $E^\times \cdot K_s^{m-1}$ is $E^\times \cdot K_s^{m,0}$.

Proof. Since W_0 (resp. W_1) is invariant by the conjugate action of $E^\times$, it is clear that $K_s^{m,0}$ (resp. $K_s^{m,1}$) is normal in $E^\times \cdot K_s^{m-1}$. If x and y lie in $A_s^{m,1}$, then:

$$\tilde{\psi}_u((1+x)(1+y)) = \tilde{\psi}_u(1+x) \cdot \tilde{\psi}_u(1+y) \cdot \psi\left(\operatorname{tr} \frac{1}{2} u(xy - yx)\right).$$

Lemma 2.3.2 tells us $W_1 \subset W_0 = W_1^{\perp}$, so $\operatorname{tr} \frac{1}{2} u(xy-yx) \equiv 0 \mod P_F$. (We note $p \neq 2$.) Thus $\tilde{\psi}_u$ is a character of $K_s^{m,1}$. As for the normalizer $\tilde{H}_u$ of $\tilde{\psi}_u$ in $E^\times \cdot K_s^{m-1}$, we remark that $\tilde{H}_u \subset H_u = E^\times \cdot K_s^{m-1}$. If $x \in A_s^{m,1}$ and $y \in A_s^{m-1}$, then

$$\tilde{\phi}_u^{(1+y)}(1+x)=\tilde{\phi}_u(1+x)\cdot\psi(\mathrm{tr}\ u(xy-yx)).$$

Therefore $\tilde{H}_u=E^\times\cdot K_s^{m,0}$ since $W_1^\perp=W_0$.

2.4. We set $U=F^\times\langle u^h\rangle(1+P_E)$ where $\langle u^h\rangle$ is a group generated by u^h, $H_0=U\cdot K_s^{m,0}$ and $H_1=U\cdot K_s^{m,1}$. In the same way of Proposition 2.1.1 (3), we can lift $\tilde{\phi}_u$ to $E^\times\cdot K_s^{m,1}$ and any lift of $\tilde{\phi}_u$ to $E^\times\cdot K_s^{m-1}$ is written in the form $\theta\cdot\tilde{\phi}_u$, where θ is a quasi-character of $E^\times$. We denote a quasi-character $\theta\cdot\tilde{\phi}_u$ of $E^\times\cdot K_s^{m,1}$ by $\eta_{u,\theta}$ and a quasi-character $(\theta|_U)\cdot\tilde{\phi}_u$ of $U\cdot K_s^{m,1}$ by $\eta_{u,\bar{\theta}}$ where $\theta|_U$ is the restriction of θ to U.

Lemma 2.4.1. (1) *H_0 and H_1 are normal subgroups in $E^\times\cdot K_s^{m,0}$ and the stability subgroup of $\eta_{u,\bar{\theta}}$ is $E^\times\cdot K_s^{m,0}$.*

(2) *We set that:*

$$\langle x,\ y\rangle_{H_0}=\mathrm{tr}\ u(xyx^{-1}y^{-1}-1) \bmod P_F$$

for $x,\ y\in H_0$. Then $\langle\ ,\ \rangle_{H_0}$ induces a nondegenerate alternating from on H_0/H_1.

(3) *The induced representation $\mathrm{Ind}_{H_1}^{H_0}(\eta_{u,\bar{\theta}})$ is a homogeneous sum of an irreducible representation $\kappa_{u,\bar{\theta}}$ of degree $q^{(r^2t-r)/2}$.*

(4) *We can lift $\kappa_{u,\bar{\theta}}$ to $E^\times\cdot K_s^{m,0}$ and the number of those lifts is h.*

Proof. Part one of the above lemma follows from the fact that $(J-1)W_0\subset W_1$. Part two follows from Lemma 2.3.2. Part three is a consequence of the Heisenberg construction (cf. [G1]). The last part follows from 5.4 and 5.5 in [C].

Proposition 2.4.2. *Let $\tilde{\kappa}_u$ be one of the lifts of $\tilde{\kappa}_{u,\bar{\theta}}$ to $\tilde{H}_u$. Then $\mathrm{Ind}_{\tilde{H}_u}^{Z_sK_s}(\tilde{\kappa}_u)$ is an irreducible very cuspidal representation of level N of Z_sK_s and every irreducible very cuspidal representation of level N of Z_sK_s is equivalent to some representation $\mathrm{Ind}_{\tilde{H}_u}^{Z_sK_s}(\tilde{\kappa}_u)$ with an appropriate generic element u of level $1-N$.*

Proof. This can be proved in the same way of Proposition 2.1.2 (3).

2.5. Now we shall construct the lifts of $\tilde{\kappa}_{u,\bar{\theta}}$ explicitly. We imitate the method of Moy (see Sections 3.5-3.6 in [M]). For simplicity, we shall start with the case that $r=1$ and t is a prime. We put $L=E^\times/U$, then L is a cyclic group of order t.

Lemma 2.5.1. *There are $(q^{t-1}+t-1)/t$ double cosets of $E^\times\cdot K_s^{m,1}$ in $E^\times\cdot K_s^{m,0}$.*

Proof. It suffices to see that the conjugate action of $E^\times$ on $K_s^{m,0}/K_s^{m,1}$ has no fixed point. This follows from the fact that $I-1$ induces an automorphism on W_0/W_1.

We denote by $\tilde{\kappa}_{u,\bar{\theta},i}$ $(i=1,\cdots,t)$ the lifts of $\tilde{\kappa}_{u,\bar{\theta}}$ to $E^\times\cdot K_s^{m,0}$. Let a_i be the multiplicity of $\tilde{\kappa}_{u,\bar{\theta},i}$ in $\mathrm{Ind}_{E^\times\cdot K_s^{m,1}}^{E^\times\cdot K_s^{m,0}}(\eta_{u,\theta})$.

Lemma 2.5.2. *The multiplicities a_i $(i=1,\cdots,t)$ satisfy the following equations:*

$$a_1+a_2+\cdots+a_t=q^{(t-1)/2}$$

$$a_1^2+a_2^2+\cdots+a_t^2=(q^{t-1}+t-1)/t.$$

Proof. We can prove this lemma by the same way of Lemma 3.5.30 in [M] by virtue of Lemma 2.5.1.

We use the next lemma to solve the above equations.

Lemma 2.5.3 (Lemma 3.5.33 in [M]). *If $c_1, c_2, \cdots, c_n$ are nonnegative integral solutions to the system of equations:*

$$c_1+c_2+\cdots+c_n=m$$
$$c_1^2+c_2^2+\cdots+c_n^2=(m^2+u-1)/n,$$

then either $n-1$ of the c_i's are equal to $(m+1)/n$ and one is $\{(m+1)/n\}-1$, or $n-1$ of the c_i's are equal to $(m-1)/n$ and one is $\{(m-1)/n\}+1$.

Applying Lemma 2.5.3 to the equation in Lemma 2.5.2, we obtain the next lemma.

Lemma 2.5.4. *The nonnegative solutions to the equation in Lemma 2.5.2 have $t-1$ of the a_i's equal to $\left\{q^{(t-1)/2}-\left(\frac{q}{t}\right)\right\}\Big/t$ and one of the a_i's is equal to $\left\{q^{(t-1)/2}-\left(\frac{q}{t}\right)\right\}\Big/t+\left(\frac{q}{t}\right)$. (We denote by $\left(-\right)$ the Legendre symbol.)*

We denote by $\tilde{\kappa}_{u,\theta}$ the $\tilde{\kappa}_{u,\bar{\theta},i}$ corresponding to the a_i which is different from others. By the Frobenius reciprocity and the Heisenberg construction, we have the next result on the character of $\tilde{\kappa}_{u,\theta}$.

Lemma 2.5.5. *Let $\chi_{\tilde{\kappa}_{u,\theta}}$ be the character of $\tilde{\kappa}_{u,\theta}$. If γ belongs to $E^\times\cdot K_s{}^{m,0}$, then we have:*

$$\chi_{\tilde{\kappa}_{u,\theta}}(\gamma)=\begin{cases} q^{(t-1)/2}\eta_{u,\bar{\theta}}(\gamma) & \text{for } \gamma\in H_1 \\ \left(\frac{q}{t}\right)\eta_{u,\theta}(\gamma) & \text{for } \gamma\in E^\times\cdot K_s{}^{m,1}\setminus H_1 \\ 0 \quad \text{if } \gamma \text{ is not conjugate to an element of } E^\times\cdot K_s{}^{m,1}. \end{cases}$$

We summarize the result in the next proposition.

Proposition 2.5.6. *Assume $r=1$ and t is a prime.*

(1) *Every irreducible representation of $E^\times\cdot K_s{}^{m,0}$ whose restriction on $K_s{}^{m,1}$ contains $\tilde{\psi}_u$ is written in the form $\tilde{\kappa}_{u,\theta}$ where θ is a quasi-character of $E^\times$ with the property that $\theta=\tilde{\psi}_u$ on $E^\times\cap K_s{}^{m,1}$. And the character $\chi_{\tilde{\kappa}_{u,\theta}}$ of $\tilde{\kappa}_{u,\theta}$ is given in the next formula:*

$$\chi_{\tilde{\kappa}_{u,\theta}}(\gamma)=\begin{cases} q^{(t-1)/2}\eta_{u,\bar{\theta}}(\gamma) & \text{for } \gamma\in H_1 \\ \left(\frac{q}{t}\right)\eta_{u,\theta}(\gamma) & \text{for } \gamma\in E^\times\cdot K_s{}^{m,1}\setminus H_1 \\ 0 \quad \text{if } \gamma \text{ is not conjugate to an element of } E^\times\cdot K_s{}^{m,1}. \end{cases}$$

(2) *Every irreducible very cuspidal representation of level N of Z_sK_s is equivalent to some representation $\mathrm{Ind}_{\tilde{H}_u}^{Z_sK_s}(\tilde{\kappa}_{u,\theta})$ with an appropriate generic element u and an appropriate quasi-character θ of $E^\times$.*

2.6. Now we get rid of the assumptions for r and t. Set $L=E^{\times}/U$ and $X=H_0/H_1$. We note that L is an abelian group of order relatively prime to p and the conjugate action of U on X is trivial. We denote by σ the conjugate action of L on X and regard X as an $\boldsymbol{F}_q[L]$-module where $\boldsymbol{F}_q$ is a finite field of order q. Then X is completely reducible as an $\boldsymbol{F}_q[L]$-module. For N a subgroup of L, let $\Omega_N=\{x\in X \mid \sigma(n)x=x$ for all $n\in N\}$. Ω_N is an L-invariant subspace of X. Let X_N be the L-complement in Ω_N of the $\boldsymbol{F}_q[L]$-module:

$$\sum_{N\subset M\subset L}\Omega_M$$

where the sum is over those subgroups of L which properly contain N.

Lemma 2.6.1. (1) $X=\bigoplus_{N\subset L}X_N$.

(2) *We denote by* $\langle\ ,\ \rangle_X$ *the nondegenerated alternating form on* X *defined in Lemma* 2.4.1 (2). *If* $X_N\neq\{0\}$, *the restriction of* $\langle\ ,\ \rangle_X$ *to* X_N *is also nondegenerate.*

(3) *Let* H_N *(resp.* $\tilde{N}$*) denote the subgroup of* H_0 *(resp.* $E^{\times}$*) such that* H_N/H_1 *(resp.* $\tilde{N}/U$*) is* X_N *(resp.* N*). Then* $\tilde{N}\cdot H_1$ *and* $\tilde{N}\cdot H_N$ *are normal in* $E^{\times}\cdot H_N$ *and for* $g\in E^{\times}\cdot H_N\setminus E^{\times}\cdot H_1$:

$$g^{-1}E^{\times}\cdot H_1 g\cap E^{\times}\cdot H_1=\tilde{N}\cdot H_1.$$

Proof. We set $\bar{X}=X\otimes_{\boldsymbol{F}_q}\bar{\boldsymbol{F}}_q$, $\bar{X}_N=X\otimes_{\boldsymbol{F}_q}\bar{\boldsymbol{F}}_q$ where $\bar{\boldsymbol{F}}_q$ is an algebraic closure of $\boldsymbol{F}_q$. From the definition of X_N, it is obvious that:

$$X=\sum_{N\subset L}X_N.$$

Therefore it suffices to see that:

$$\bar{X}=\sum_{N\subset L}\bar{X}_N.$$

Let $\bar{\sigma}$ be the representation of L on X defined by σ. Since L is abelian, we can show:

$$\bar{X}=\bigoplus_{\alpha\subset\bar{\sigma}}\bar{X}_\alpha$$

where the sum is over those one-dimensional representations which are contained in $\bar{\sigma}$ and $\bar{X}_\alpha=\{x\in\bar{X} \mid \bar{\sigma}(g)x=\alpha(g)x$ for all $g\in L\}$. Then from the definition of $\bar{X}_N$, we have:

$$\bar{X}_N=\bigoplus_{\substack{\alpha\subset\bar{\sigma}\\ \mathrm{Ker}\,\alpha=N}}\bar{X}_\alpha.$$

Therefore $\bar{X}=\bigoplus_{N\subset L}\bar{X}_N$.

(2) This follows from the fact that:

$$\langle\sigma(g)x,\ \sigma(g)y\rangle_X=\langle x,\ y\rangle_X$$

for x, $y\in X$ and $g\in L$. (See Remark 2.2.2).

(3) This is obvious from the definition of X_N.

We set $\eta_{u,\theta_N}=(\eta_{u,\theta})|_{\tilde{N}\cdot H_1}$. By the above lemma, the next lemma is proved by the same way of Lemma 2.4.1.

Lemma 2.6.2. (1) *The stability subgroup of* η_{u,θ_N} *in* Z_sK_s *is* $E^\times\cdot H_N$.

(2) *Let* $2D_N=\dim_{k_F}X_N$. *The induced representation* $\mathrm{Ind}_{\tilde{N}\cdot H_1}^{\tilde{N}\cdot H_N}(\eta_{u,\theta_N})$ *is a homogeneous sum of an irreducible representation* κ_{u,θ_N} *of degree* q^{D_N}.

(3) *Let* $M_N=|E^\times/\tilde{N}|$. *We can lift* κ_{u,θ_N} *to* $E^\times\cdot H_N$ *and the number of those lifts is* M_N.

We denote by $\tilde{\kappa}_{u,\theta_N,i}$ $(i=1,\cdots,M_N)$ the lifts of κ_{u,θ_N} to $E^\times\cdot H_0$. Let b_i be the multiplicity of $\tilde{\kappa}_{u,\theta_N,i}$ in $\mathrm{Ind}_{E^\times\cdot H_1}^{E^\times\cdot H_0}(\eta_{u,\theta})$. By Lemma 2.6.2 (2), we have the following analogue of Lemma 2.5.2.

Lemma 2.6.3. *The multiplicities* b_i $(i=1,\cdots,M_N)$ *satisfy the following equations:*

$$b_1+b_2+\cdots+b_{M_N}=q^{D_N}$$
$$b_1^2+b_2^2+\cdots+b_{M_N}^2=(q^{2\cdot D_N}+M_N-1)/M_N\,.$$

We can apply Lemma 2.5.3 to the equation in Lemma 2.6.3 to conclude that either

(a) M_N-1 of the b_i's are equal to $(q^{D_N}+1)/M_N$ and one is $\{(q^{D_N}+1)/M_N\}-1$

or

(b) M_N-1 of the b_i's are equal to $(q^{D_N}-1)/M_N$ and one is $\{(q^{D_N}-1)/M_N\}+1$.

We set $S(N)=-1$ (resp. $S(N)=1$) in case (a) (resp. in case (b)). In both cases, $(q^{D_N}-S(N))/M_N$ is an integer. We denote by $\tilde{\kappa}_{u,N,\theta}$ the $\tilde{\kappa}_{u,\theta_N,i}$ corresponding to the b_i which is different from others. Next lemma is the counterpart of Lemma 2.5.5.

Lemma 2.6.4. *Let* $\chi_{\tilde{\kappa}_{u,N,\theta}}$ *be the character of* $\tilde{\kappa}_{u,N,\theta}$. *If* γ *belongs to* $E^\times\cdot K_s^{m,0}$, *then we have:*

$$\chi_{\tilde{\kappa}_{u,N,\theta}}(\gamma)=\begin{cases} q^{D_N}\eta_{u,\theta}(\gamma) & \textit{for } \gamma\in\tilde{N}\cdot H_1 \\ S(N)\eta_{u,\theta_N}(\gamma) & \textit{for } \gamma\in E^\times\cdot H_1\setminus\tilde{N}\cdot H_1 \\ 0 \quad \textit{if } \gamma \textit{ is not conjugate to an element of } E^\times\cdot H_1. \end{cases}$$

Since $H_0/H_1=\bigoplus_{N\subset L}X_N$ and $(\kappa_{u,\theta_N})|_{H_1}=q^{D_N}\cdot\eta_{u,\theta}$, we can define a representation $\tilde{\kappa}_{u,\theta}$ of $E^\times\cdot K_s^{m,0}$ as follows:

$$\tilde{\kappa}_{u,\theta}(e\cdot g)=\Big(\bigotimes_N\tilde{\kappa}_{u,N,\theta}(e)\theta(e)^{-1}\Big)\theta(e)\cdot\Big(\bigotimes_N\tilde{\kappa}_{u,N,\theta}(g_N)\Big)\tilde{\kappa}_{u,N,\theta}(g_1)\,.$$

Here $e\in E^\times$, $g\in H_0$ and $g=\Big(\prod_N g_N\Big)g_1$ where $g_N\in H_N$ and $g_1\in H_1$. It is obvious that $\tilde{\kappa}_{u,\theta}$ is a lift of $\kappa_{u,\bar{\theta}}$. Since the number of lifts of $\kappa_{u,\bar{\theta}}$ to $E^\times\cdot K_s^{m,0}$ is equal to the number of lifts of $\bar{\theta}$ to $E^\times$, any lift of $\kappa_{u,\bar{\theta}}$ to $E^\times\cdot K_s^{m,0}$ is given in the form $\tilde{\kappa}_{u,\theta}$ where θ is a lift of $\bar{\theta}$ to $E^\times$. By Lemma 2.6.4, the character $\chi_{\tilde{\kappa}_{u,\theta}}$ of $\tilde{\kappa}_{u,\theta}$ is given as follows.

Lemma 2.6.5.

$$\chi_{\tilde{\kappa}_{u,\theta}}(e\cdot g)=q^{\left(\sum_{e\in N}D_N\right)}\cdot\Big(\prod_{e\notin N}S(N)\Big)\theta(e)\tilde{\psi}_u(g)$$

for $e\in E^\times$ *and* $g\in K_s^{m,1}$ *where* N *runs over the subgroups of* L *which contain (resp. do* *not contain)* e mod U *in* $\left(\sum_{e\in N} D_N\right)$ *(resp.* $\left(\prod_{e\notin N} S(N)\right)$*), and*

$$\chi_{\tilde{\kappa}_{u,\theta}}(\gamma)=0$$

if γ *is not conjugate to an element of* $E^\times\cdot K_s^{m,1}$.

Corollary 2.6.6.

$$\chi_{\tilde{\kappa}_{u,\theta}}(u\cdot g)=\left(\prod_{u\notin N} S(N)\right)\theta(u)\tilde{\psi}_u(g)$$

for $g\in K_s^{m,1}$.

Proof. Since the map $I-1$ induces an automorphism of X, $D_N=0$ if u mod $U\in N$.

We summarize the result of the odd level case. (cf. Proposition 2.4.2.)

Proposition 2.6.7. *Let* u *be a generic element of level* $2-2m$ $(m\geqq 2)$ *and* $\tilde{\psi}_u$ *be a character of* $K_s^{m,1}$ *defined by* $\tilde{\psi}_u(1+x)=\psi\left(\mathrm{tr}\, u\left(x-\frac{x^2}{2}\right)\right)$ *for* $1+x\in K_s^{m,1}$. *(cf. Lemma* 2.3.3.*) Let* θ *be a quasi-character of* $E^\times$ *with the property that* $\theta(1+x)=\psi(\mathrm{tr}\, ux)$ *for* $x\in P_E^m$ *where* $E=F(u)$.

(1) *Let* κ *be any irreducible component of* $\mathrm{Ind}_{K_s^{m,1}}^{E^\times\cdot K_s^{m,0}}(\tilde{\psi}_u)$. *(cf. Lemma* 2.3.3.*) Then* κ *is written in the form* $\tilde{\kappa}_{u,\theta}$ *which is determined by its character formula:*

$$\chi_{\tilde{\kappa}_{u,\theta}}(e\cdot g)=q^{\left(\sum_{e\in N} D_N\right)}\cdot\left(\prod_{e\notin N} S(N)\right)\theta(e)\tilde{\psi}_u(g)$$

for $e\in E^\times$ *and* $g\in K_s^{m,1}$, *and*

$$\chi_{\tilde{\kappa}_{u,\theta}}(\gamma)=0$$

if γ *is not conjugate to an element of* $E^\times\cdot K_s^{m,1}$. *(As for the definition of* $\left(\sum_{e\in N} D_N\right)$ *and* $\left(\prod_{e\notin N} S(N)\right)$, *see Lemma* 2.6.5, *Lemma* 2.6.1 *and the paragraph above Lemma* 2.6.4.*)*

(2) $\mathrm{Ind}_{E^\times\cdot K_s^{m,0}}^{Z_sK_s}(\tilde{\kappa}_{u,\theta})$ *is an irreducible very cuspidal representation of level* $2m-1$ *of* Z_sK_s *and every very cuspidal representation of level* $2m-1$ *of* Z_sK_s *is equivalent to some representation* $\mathrm{Ind}_{E^\times\cdot K_s^{m,0}}^{Z_sK_s}(\tilde{\kappa}_{u,\theta})$ *with an appropriate geneic element* u *of level* $2-2m$ *and an appropriate quasi-character* θ *of* $E^\times$.

We need determine the term $\left(\prod_{u\notin N} S(N)\right)$ to calculate the ε-factor of $\mathrm{ind}_{E^\times\cdot K_s^{m,1}}^{GL_n(F)}(\tilde{\kappa}_{u,\theta})$ in the next section.

Proposition 2.6.8. *In the above notation,*

$$\left(\prod_{u\notin N} S(N)\right)=(-1)^{r-1}\cdot\left(\frac{q}{t}\right)^r$$

where $\left(\frac{}{}\right)$ *denotes the Jacobi symbol.*

Proof. We first recall that $S(N)$ is determined by the property that:

$$\frac{q^{D_N}-S(N)}{|L/N|}\ \text{is an integer.}$$

For any element x of $E^\times$, we set $\bar{x}=x \bmod F^\times\langle u^h\rangle(1+P_E)$. Let u_0 be an element of $O_F^\times$ such that $u_0 \bmod(1+P_E)$ generates the cyclic group $k_E^\times$ and M be a subgroup of L generated by $\bar{u}_0$. We note that $L=E^\times/F^\times\langle u^h\rangle(1+P_E)$ is generated by $\bar{u}_0$ and $\bar{u}$. We shall omit the symbol '$^-$' when there is no fear of confusion. Since $(J-1)^{(p^l-1)/2}$ induces an L-module isomorphism between X and $\mathrm{Ker}(J-1)/\mathrm{Ker}(I-1)$, we can easily show that:

$$\dim_{k_F}\Omega_{\langle u_0^j\rangle}=[k_E : k_F(u_0^j)]\cdot rt-r$$

from Lemma 2.2.1 and its proof. From the definition of D_N and $S(N)$, $D_N=0$ and $S(N)=1$ if N_1 properly contains N and $\Omega_N=\Omega_{N_1}$. So if $S(\langle u_0^j\rangle)=-1$, then $j=\dfrac{q^r-1}{q^{r'}-1}$ where r' is a positive divisor of r. We set $j(r')=\dfrac{q^r-1}{q^{r'}-1}$ for any positive divisor r' of r. We shall quote the next lemma from [M].

Lemma 2.6.9 (Lemma 3.6.54 in [M]). *Suppose j, Q are integers greater than 1 and A is a nonnegative integer. If $(Q^A+1)/\{(Q^j-1)/(Q-1)\}$ is an integer, then $j=2$ and A is odd.*

Since $X_{\langle u_0^{j(r')}\rangle}$ is a $k_F(u_0^{j(r')})$-module, $D(\langle u_0^{j(r')}\rangle)$ is a multiple of r' and we can apply this lemma for $j=r/r'$, $Q=q^{r'}$ and $A=D(\langle u_0^{j(r')}\rangle)/r'$. We consider two cases according to the parity of r.

Case r odd. From the above lemma, $S(N)=1$ if N does not contain $\langle u_0\rangle$. Here:

$$\Big(\prod_{u\notin N} S(N)\Big)=\Big(\prod_{j\mid t} S(\langle u_0,\ u^j\rangle)\Big)$$

where j runs over the positive divisors of t.

Therefore we have only to determine the signature of $S(\langle u_0, u^j\rangle)$ for $j\mid t$. From the definition of $X_{\langle u_0, u^j\rangle}$, $\sum_{i\mid j} 2D_{\langle u_0, u^i\rangle}=\dim_{k_F}\Omega_{\langle u_0, u^j\rangle}$. It is easily seen that $\dim_{k_F}\Omega_{\langle u_0, u^j\rangle}=jr-r$. (See the proof of Lemma 2.2.1.) So we have:

$$2D_{\langle u_0, u^j\rangle}=\begin{cases} r\varphi(j) & \text{for } j>1 \\ 0 & \text{for } j=1 \end{cases}$$

where φ denotes Euler's φ-function, and

$$\frac{(q^r)^{\varphi(j)/2}-S(\langle u_0,\ u^j\rangle)}{j} \text{ is an integer.}$$

Then $S(\langle u_0, u^j\rangle)=-1$ if and only if j is a power of a prime, say $j=l^m$, and $\left(\frac{q^r}{l}\right)=-1$. Hence

$$\Big(\prod_{j\mid t} S(\langle u_0,\ u^j\rangle)\Big)=\Big(\frac{q^r}{t}\Big)$$
$$=\Big(\frac{q}{t}\Big)^r.$$

Case r even. From Lemma 2.6.7 and the argument of the odd case, $S(N)=-1$ if and only if $N\cap\langle u_0\rangle=\langle u_1\rangle$ and $\dim_{k_{F(u_1)}}X_N\equiv 2 \mod 4$ where $u_1=u_0^{(q^2-1)/(q-1)}$. By the argument of the odd case, $\dim_{k_{F(u_1)}}X_N\equiv 0 \mod 4$ if $N\supset\langle u_0\rangle$. Therefore in order to observe $\left(\prod_{u\notin N} S(N)\right)=-1$, it is sufficient to see that there are an odd number of subgroups $N\supset\langle u_1\rangle$ such that $\dim_{k_{F(u_1)}} X_N=2 \bmod 4$. This follows immediately from:

$$\Omega_{\langle u_1\rangle}=\bigoplus_{\langle u_1\rangle\subset N} X_N$$

and

$$\dim_{k_{F(u_1)}}\Omega_{\langle u_1\rangle}=4t-2.$$

Hence our proposition.

3. Calculation of the ε-factors

3.1. At first, we review the ε-factors of supercuspidal representations of $\mathrm{GL}_n(F)$. Godement-Jacquet [G-J] have defined the L- and ε-factors for admissible representations of $\mathrm{GL}_n(F)$. If π is an irreducible supercuspidal representation of $\mathrm{GL}_n(F)$, then $L(\pi)=1$ and the ε-factor is a scalar factor defined by:

$$\int_{\mathrm{GL}_n(F)}\hat{f}(g)\pi(g^{-1})|\det g|_F^{(n+1)/2}\mathrm{d}^\times g=\varepsilon(\pi,\psi)\int_{\mathrm{GL}_n(F)}f(g)\pi(g)|\det g|_F^{(n-1)/2}\mathrm{d}^\times g,$$

where f is a locally constant, compactly supported function on $\mathrm{M}_n(F)$, ψ is an additive character of F, $\mathrm{d}^\times g$ is a Haar measure of $\mathrm{GL}_n(F)$ defined by $\mathrm{d}^\times g=\mathrm{d}\mu(g)/|\det g|_F^n$ where μ is a self dual Haar measure on $\mathrm{M}_n(F)$ with respect to the Fourier transform:

$$\hat{f}(y)=\int_{\mathrm{M}_n(F)}f(x)\psi(xy)\mathrm{d}\mu(x).$$

The next lemma is well-known.

Lemma 3.1.1. *Let π be an irreducible supercuspidal representation of $G=\mathrm{GL}_n(F)$. If π is compactly-induced from compact modulo center subgroup H, say $\pi=\mathrm{ind}_H^G\kappa$, then*:

$$\varepsilon(\pi,\psi)=\int_H\kappa(g^{-1})|\det g|_F^{(n+1)/2}\psi(\mathrm{tr}\, g)\mathrm{d}^\times g.$$

3.2. We start with the even level case. Let u be a generic element of level $1-2m$ and $\pi=\mathrm{ind}_{E^\times\cdot K_s{}^m}^{\mathrm{GL}_n(F)}(\theta\cdot\psi_u)$. (As for the notation, see 2.1).

The next lemma is proved by the same way of Lemma 2.2.1 in [K-M].

Lemma 3.2.1. *Let μ be a self-dual Haar measure on $\mathrm{M}_n(F)$ with respect to $\psi\circ\mathrm{tr}$. Then $\mu(A_s{}^m)=q^{r^2s\cdot(1-2m/2)}$.*

From Lemma 3.1.1, we have:

$$\begin{aligned}\varepsilon(\pi,\psi)&=\int_{E^\times\cdot K_s{}^m}(\theta\cdot\psi_u)(g^{-1})|\det g|_F^{(n+1)/2}\psi(\mathrm{tr}\, g)\mathrm{d}^\times g\\&=\int_{E^\times\cdot K_s{}^m/K_s{}^m}\left(\int_{K_s{}^m}(\theta\cdot\psi_u)((hk)^{-1})|\det hk|_F^{(n+1)/2}\psi(\mathrm{tr}\, hk)\mathrm{d}^\times k\right)\mathrm{d}^\times h\\&=\int_{E^\times\cdot K_s{}^m/K_s{}^m}(\theta\cdot\psi_u)(h^{-1})|\det h|_F^{(n+1)}\left(\int_{K_s{}^m}(\theta\cdot\psi_u)(k^{-1})\psi(\mathrm{tr}\, hk)\mathrm{d}^\times k\right)\mathrm{d}^\times h.\end{aligned}$$

And in the above expression:

$$\int_{K_s{}^m}(\theta\cdot\psi_u)(k^{-1})\psi(\mathrm{tr}\ hk)\mathrm{d}^\times k$$

$$=\int_{A_s{}^m}\psi_u{}^{-1}(1+k)\psi(\mathrm{tr}\ h(1+k))\mathrm{d}\mu(k)$$

$$=\psi(\mathrm{tr}\ h)\int_{A_s{}^m}\psi(\mathrm{tr}(h-u)k))\mathrm{d}\mu(k)$$

$$=\psi(\mathrm{tr}\ h)\mu(A_s{}^m)f_{1-m}(h-u),\qquad \text{for } (A_s{}^m)^\perp=A_s{}^{1-m}.$$

Since u belongs to $A_s{}^{1-2m}$, $h-u$ belongs to $A_s{}^{1-m}$ if and only if h belongs to $u\cdot K_s{}^m$. Thus:

$$\varepsilon(\pi,\ \psi)=\mu(A_s{}^m)\psi(\mathrm{tr}\ u)(\theta\cdot\psi_u)^{-1}(u)|\det u|_F{}^{(n+1)/2}$$
$$=\mu(A_s{}^m)\psi(\mathrm{tr}\ u)\theta^{-1}(u)|u|_E{}^{(n+1)/2}.$$

From Lemma 3.1.1 and Proposition 1.2.3, $\mu(A_s{}^m)=q^{r^2s(1-2m)/2}$ and $|u|_E{}^{n/2}=(q^{-r})^{rs(1-2m)}$. So we get the next proposition.

Proposition 3.2.2. *Let* $\pi=\mathrm{ind}_{E^\times\cdot K_s{}^m}^{\mathrm{GL}_n(F)}(\theta\cdot\psi_u)$ *and* $\psi_E=\psi\circ\mathrm{tr}_{E/F}$. *Then*

$$\varepsilon(\pi,\ \psi)=\psi_E(u)\theta^{-1}(u)|u|_E{}^{1/2}.$$

3.3. We shall treat the odd level case. Let u be a generic element of level $2-2m$ and $\pi=\mathrm{ind}_{E^\times\cdot K_s{}^{m,0}}^{\mathrm{GL}_n(F)}(\tilde\kappa_{u,\theta})$. (As for the notation, see 2.3 and 2.6). By the same argument of the even level case, we have:

$$\varepsilon(\pi,\ \psi)=\int_{K_s{}^{m,0}}\tilde\kappa_{u,\theta}((uk)^{-1})|\det uk|_F{}^{(n+1)/2}\psi(\mathrm{tr}\ uk)\mathrm{d}^\times k$$
$$=|u|_E{}^{(n+1)/2}\int_{K_s{}^{m,0}}\tilde\kappa_{u,\theta}((uk)^{-1})\psi(\mathrm{tr}\ uk)\mathrm{d}^\times k.$$

The above integral is calculated as follows:

$$\int_{K_s{}^{m,0}}\tilde\kappa_{u,\theta}((uk)^{-1})\psi(\mathrm{tr}\ uk)\mathrm{d}^\times k$$

$$=\sum_{h\in K_s{}^{m,0}/K_s{}^{m,1}}\Big(\int_{K_s{}^{m,1}}\tilde\kappa_{u,\theta}((uhk)^{-1})\psi(\mathrm{tr}\ uhk)\mathrm{d}^\times k\Big)$$

$$=\sum_{y\in A_s{}^{m,0}/A_s{}^{m,1}}\Big(\int_{A_s{}^{m,1}}\tilde\kappa_{u,\theta}((1+x)^{-1})\psi(\mathrm{tr}\ u(1+y)(1+x))\mathrm{d}\mu(x)\Big)\tilde\kappa_{u,\theta}((u(1+y))^{-1})$$

$$=\sum_{y\in A_s{}^{m,0}/A_s{}^{m,1}}\Big(\int_{A_s{}^{m,1}}\psi\Big(\mathrm{tr}\ u\Big(yx+\frac{x^2}{2}\Big)\mathrm{d}\mu(x)\Big)\psi(\mathrm{tr}\ u(1+y))\tilde\kappa_{u,\theta}((u(1+y))^{-1}).$$

By taking the trace of the last term, we have:

$$\deg(\tilde\kappa_{u,\theta})\varepsilon(\pi,\ \psi)=|u|_E{}^{(n+1)/2}\psi(\mathrm{tr}\ u)\sum_{y\in A_s{}^{m,0}/A_s{}^{m,1}}\Big(\int_{A_s{}^{m,1}}\psi\Big(\mathrm{tr}\ u\Big(yx+\frac{x^2}{2}\Big)\Big)\mathrm{d}\mu(x)$$
$$\times\psi(\mathrm{tr}\ uy)\chi_{\tilde\kappa_{u,\theta}}((u(1+y))^{-1})).$$

Since $I-1$ induces a k_F-automorphism on $A_s^{m,0}/A_s^{m,1}$, there exists an element z in $A_s^{m,0}/A_s^{m,1}$ such that $uy=[u, z]$ where $[u, z]=uz-zu$ for any y in $A_s^{m,0}/A_s^{m,1}$. Set $S=u(1+y)$. Then an easy calculation shows:

$$(1+z)S(1+Z)^{-1}=u(1+u^{-1}[z, uy])(1+u^{-1}[S, z]z) \bmod K_s^{2m-1}.$$

(See (3.5.39) in [M].) The last two terms lie in K_s^{2m-2} and hence are scalars under $\tilde{\kappa}_{u,\theta}$. Therefore:

$$\begin{aligned}\chi_{\tilde{\kappa}_{u,\theta}}(S^{-1})&=\chi_{\tilde{\kappa}_{u,\theta}}((1+z)S(1+z)^{-1})^{-1})\\&=\chi_{\tilde{\kappa}_{u,\theta}}(u^{-1})\psi(\mathrm{tr}[z, uy])\psi(\mathrm{tr}[S, z]z)\\&=\chi_{\kappa_{u,\theta}}(u^{-1})\psi(\mathrm{tr}[z, uy])\psi(\mathrm{tr}([uz, z]+[uyz, z]))\\&=\chi_{\tilde{\kappa}_{u,\theta}}(u^{-1}).\end{aligned}$$

Moreover:

$$\psi(\mathrm{tr}\ uy)=\psi(\mathrm{tr}[u, z])=1$$

and

$$\begin{aligned}\psi\left(\mathrm{tr}\ u\left(yx+\frac{x^2}{2}\right)\right)&=\psi\left(\mathrm{tr}\frac{1}{2}ux^2\right)\psi(\mathrm{tr}\ u(z-u^{-1}zu)x)\\&=\psi\left(\mathrm{tr}\frac{1}{2}ux^2\right)\psi(\mathrm{tr}\ u(zx-xz))\\&=\psi\left(\mathrm{tr}\frac{1}{2}ux^2\right), \qquad \text{for } W_1^{\perp}=W_0.\end{aligned}$$

Therefore:

$$\varepsilon(\pi, \psi)=\psi(\mathrm{tr}\ u)|u|_E^{1/2}C\int_{A_s^{m,1}}\psi\left(\mathrm{tr}\frac{1}{2}ux^2\right)d\mu(x)$$

where $C=(\deg(\tilde{\kappa}_{u,\theta}))^{-1}|u|_E^{n/2}|A_s^{m,0}/A_s^{m,1}|\chi_{\tilde{\kappa}_{u,\theta}}(u^{-1})$.

Let $W_2=(J-1)^{(p^l+1)/2}W$ and $A_s^{m,2}$ be the total inverse image in A_s^{m-1} of W_2. Then:

$$\int_{A_s^{m,1}}\psi\left(\mathrm{tr}\frac{1}{2}ux^2\right)d\mu(x)=\mu(A_s^{m,2})\sum_{x\in W_1/W_2}\psi\left(\mathrm{tr}\frac{1}{2}ux^2\right)$$

since $\psi(\mathrm{tr}\ uxy)=1$ if $x\in A_s^{m,2}$ and $y\in A_s^{m,1}$. Thus:

$$\varepsilon(\pi, \psi)=\psi(\mathrm{tr}\ u)|u|_E^{1/2}C'\left(\sum_{x\in W_1/W_2}\psi\left(\mathrm{tr}\frac{1}{2}ux^2\right)\right)$$

where $C'=(\deg(\tilde{\kappa}_{u,\theta}))^{-1}|u|_E^{n/2}|A_s^{m,0}/A_s^{m,1}|\mu(A_s^{m,2})\chi_{\tilde{\kappa}_{u,\theta}}(u^{-1})$.

By Proposition 2.6.8, Lemma 2.4.1 (3), Lemma 3.2.1 and Proposition 1.2.2, we can see that $C'=q^{-r/2}\cdot\theta^{-1}(u)\cdot(-1)^{r-1}\left(\frac{q}{t}\right)^r$. Hence:

$$\varepsilon(\pi, \psi)=\psi_E(u)\theta^{-1}(u)|u|_E^{1/2}q^{-r/2}\cdot M$$

where $M=\sum_{x\in W_1/W_2}\psi\left(\mathrm{tr}\frac{1}{2}ux^2\right)$.

The rest of our work is to calculate M.

Lemma 3.3.1. $M=\sum_{x\in k_E}\psi\left(\mathrm{tr}_{k_E/k_F}\left(\frac{1}{2}\gamma x^2\right)\right)$ *where* $\gamma=(-1)^{(p^l-1)/2}t\cdot u\varpi_E^{2m-2} \bmod P_E$.

(*We note that since* $2m-2$ *is even, the right side dose not depend on the choice of* ϖ_E.)

Proof. Since W_1/W_2 is a one-dimensional vector space over k_E, $M=\sum_{x\in k_E}\psi\left(\mathrm{tr}\frac{1}{2}u(xk_0)^2\right)$ where k_0 is any element in $W_1\setminus W_2$. Let k_1 be an element in A_s^0/A_s^1 such that $k_0=\varpi_E^{m-1}(J-1)^{(p^l-1)/2}Tk_1$. Then:

$$\psi\left(\mathrm{tr}\frac{1}{2}u(xk_0)^2\right)$$
$$=\psi\left(\mathrm{tr}\frac{1}{2}u(x\varpi_E^{m-1}(J-1)^{(p^l-1)/2}Tk_1\cdot x\varpi_E^{m-1}(J-1)^{(p^l-1)/2}Tk_1)\right)$$
$$=\psi\left(\mathrm{tr}\frac{1}{2}u(x\varpi_E^{m-1}J^{-(p^l-1)/2}(-1)^{(p^l-1)/2}J^{-1}T(J-1)^{(p^l-1)}Tk_1\cdot x\varpi_E^{m-1}k_1)\right)$$
$$\text{since } \mathrm{tr}(x\cdot Iy)=\mathrm{tr}(I^{-1}x\cdot y),$$
$$=\psi\left(\mathrm{tr}\frac{1}{2}u(\varpi_E^{2m-2}x^2(-1)^{(p^l-1)/2}t(J-1)^{(p^l-1)}Tk_1\cdot k_1)\right)$$
$$\text{since } (J-1)^{(p^l-1)}Tk_1\in\mathrm{Ker}\,(I-1)=k_E.$$

Here we recall that A_s^0/A_s^1 is identified with $\mathrm{M}_r(k_F)^{Z/sZ}$ by way of the map R. Let $R(k_1)=(\mathbf{1}_r, \mathbf{0}_r, \cdots, \mathbf{0}_r)$, then $R(Ik_1)=(\mathbf{0}_r, \cdots, \mathbf{0}_r, \mathbf{1}_r, \mathbf{0}_r, \cdots, \mathbf{0}_r)$ where $\mathbf{1}_r$ lies in the $(3-2m \bmod s)$-th position. Since $(2-2m, s)=1$ and $I^sk_1=k_1$, we have:

$$R((J-1)^{(p^l-1)}Tk_1)=\frac{q^r-1}{q-1}R((I^{s-1}+\cdots+I+1)k_1)$$
$$=R((I^{s-1}+\cdots+I+1)k_1)$$
$$=(\mathbf{1}_r, \mathbf{1}_r, \cdots, \mathbf{1}_r).$$

Therefore $R((J-1)^{(p^l-1)}Tk_1\cdot k_1)=(\mathbf{1}_r, \mathbf{0}_r, \cdots, \mathbf{0}_r)$. We note that if $x\in A_s^0$ and $R(x)=(\gamma_0, \cdots, \gamma_{s-1})$, then $(\mathrm{tr}\,x) \bmod P_F=\sum_{i=0}^{s-1}\mathrm{tr}\,\gamma_i$. Hence to prove the lemma it suffices to see that $\mathrm{tr}_{k_E/k_F}(e)=\mathrm{tr}\,e_0$ if $R(e)=(e_0, \cdots, e_{s-1})$. This follows from the fact that $k_E=k_F(\varpi_F^{(2-2m)}u^s \bmod P_E)$ and $\mathrm{tr}_{k_E/k_F}(\varpi_F^{(2-2m)}u^s \bmod P_E)=\mathrm{tr}\,\beta_i (i=0, \cdots, s-1)$ if $R(\varpi_F^{(2-2m)}u^s)=(\beta_0, \cdots, \beta_{s-1})$.

Theorem 3.3.2. *Let* u *be a generic element of level* $1-N$ $(N\geqq 2)$ *and* $E=F(u)$, *which is an extension of* F *whose ramification degree is* s *and residual degree is* r. *Let* $m=\left[\frac{N+1}{2}\right]$, $s=p^l\cdot t$ *where* $(t, p)=1$, ψ_u *be as in* 1.3 *and* θ *be a quasi-character of* $E^\times$ *which coincides with* ψ_u *on* $1+P_E{}^m$. *When* N *is even, let* $\pi_{u,\theta}=\mathrm{ind}_{E^\times\cdot K_s{}^m}^{\mathrm{GL}_n(F)}(\theta\cdot\psi_u)$. *When* N *is odd, let* $\tilde{\kappa}_{u,\theta}$ *as in* 2.6 *and* $\pi_{u,\theta}=\mathrm{ind}_{E^\times\cdot K_s{}^{m,0}}^{\mathrm{GL}_n(F)}(\tilde{\kappa}_{u,\theta})$. (*As for the definitions of* $K_s{}^m$ *and* $K_s{}^{m,0}$, *see* 1.1.2 *and* 2.3.3). *Let* $\psi_E=\psi\circ\mathrm{tr}_{E/F}$. *Then:*

(1) $\varepsilon(\pi_{u,\theta}, \psi)=\psi_E(u)\theta^{-1}(u)|u|_E^{1/2}$ *when* N *is even,*

(2) $\varepsilon(\pi_{u,\theta}, \psi)=\psi_E(u)\theta^{-1}(u)|u|_E^{1/2}(-1)^{r-1}\left(\frac{q}{t}\right)^r\frac{1}{\sqrt{q_E}}\sum_{x\in k_E}\psi\left(\mathrm{tr}_{k_E/k_F}\left(\frac{1}{2}\gamma x^2\right)\right)$ *when* N *is odd*

where $\gamma=(-1)^{(p^l-1)/2}tu\varpi_E{}^{N-1} \bmod P_E$.

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