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On the boundary behavior of holomorphic mappings of plane domains to Riemann surfaces

Dedicated to Professor Kōtarou Oikawa on his 60th birthday

By

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§ 1. Introduction

The classical function theory has some theorems about the removability of singularities of holomorphic functions, e.g., Riemann's theorem, big Picard's theorem. The purpose of this paper is to extend such theorems for holomorphic mappings to Riemann surfaces. As for big Picard's theorem for holomorphic mappings to Riemann surfaces, Ohtsuka [5] showed the following;

Theorem 1. *Let f be a holomorphic mapping of the punctured disk $D^*=\{z\in \boldsymbol{C}; 0<|z|<1\}$ to a Riemann surface R of genus g. Theu, the followings are valid;*

(i) *If R is the Riemann sphere $\hat{\boldsymbol{C}}$ and $\lim_{z\to 0} f(z)$ does not exist, then f takes all points except at most two points of $\hat{\boldsymbol{C}}$ infinitely many times. If R is a torus T and $\lim_{z\to 0} f(z)$ does not exist, then f takes all points of T infinitely many times. In particular, if R is contained in $\hat{\boldsymbol{C}}-\{a_1, a_2, a_3\}$ or $T-\{b\}$ $(a_j>\hat{\boldsymbol{C}}, j=1, 2, 3,$ and $b\in T)$, then f has a limit in $\bar{R}$ as $z\to 0$ and is extended to a holomorphic mapping of the unit disk D to $\bar{R}$, where $\bar{R}$ is a Riemann surface filled up the punctures of R.*

(ii) *If $2\leqq g\leqq +\infty$, then $\lim_{z\to 0} f(z)$ exists in R or there exists a boundary neighbourhood V of R such that V is conformally equivalent to D^*, and $f(z)$ converges to a boundary point of V corresponding to the origin of D^* as $z\to 0$.*

First, we shall give an elementary proof of Theorem 1 by using Fuchsian groups. Such a method was taken by Marden, Richard and Rodin [4] and Royden [6], but our proof is different from theirs.

Secondly, we shall extend the exceptional set in Theorem 1 and consider the removability for holomorphic mappings to Riemann surfaces satisfying certain conditions (Proposition and Theorem 2).

Finally, we shall constuct some examples related to the above results.

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§ 2. An elementary proof of Theorem 1

Lemma 1. *Let R be a Riemann surface whose universal coveriug surface is conformally equivalent to the upper half plane U. Then, for a holomorphic mapping f of D^* to R, the same result of* (ii) *in Theorem* 1 *holds.*

Proof. Let Γ be a Fuchsian group on U such that $U/\Gamma \cong R$. The punctured disk D^* is also represented by $U/\langle A\rangle$, where $A(z)=z+a$ $(a>0)$. There exist a holomorphic mapping $\tilde{f}$ of U to U and $\gamma_A \in \Gamma$ such that the diagram

$$\begin{array}{ccc} U & \xrightarrow{\ \tilde{f}\ } & U \\ {\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle \pi} \\ D^*=U/\langle A\rangle & \xrightarrow{\ f\ } & U/\Gamma=R \end{array}$$

is commutative and

$$\tilde{f}\circ A^n=\gamma_A{}^n\circ\tilde{f} \qquad (n\in \boldsymbol{Z}), \tag{1}$$

where π_0 and π are the canonical projections of U onto D^* and R, respectively.

If γ_A is not the identity, then it is parabolic. Indeed, for the Poincaré distance ρ on U, we have

$$\rho(\tilde{f}(z),\ \tilde{f}(w))\leqq\rho(z,\ w) \qquad (z,\ w\in U),$$

since $\tilde{f}$ is holomorphic (Schwarz's lemma). Thus,

$$\rho(\gamma_A(\tilde{f}(z)),\ \tilde{f}(z))=\rho(\tilde{f}(A(z)),\ \tilde{f}(z))\leqq\rho(A(z),\ z).$$

Hence, we verify that

$$\inf\{\rho(\gamma_A(z),\ z)\,;\ z\in U\}=0$$

because of $\inf\{\rho(A(z),\ z)\,;\ z\in U\}=0$. This implies that γ_A is parabolic. We may assume that $\gamma_A(z)=z+1$. Shimizu's lemma (cf. Kra [3] p. 60) says that there exists a constant $c>0$ such that $U_c=\{z\in U\,;\ \mathrm{Im}\, z>c\}$ is precisely invariant by $\langle\gamma_A\rangle$ in Γ. Consider the circles $C_t=\{|z|=t\}$ in D_* $(0<t<1)$ and their lifts $\tilde{C}_t$ on U with the end points z_t and $z_t+a(=\gamma_A(z_t))$. From (1) we have $\tilde{f}(z_t+a)=\tilde{f}(z_t)+1$. Because of Schwarz's lemma again, we have

$$(\text{the Poincaré length of } \tilde{f}(\tilde{C}_t))\leqq(\text{the Poincaré length of } \tilde{C}_t)\longrightarrow 0 \qquad (t\to 0)$$

and

$$\rho(\tilde{f}(z_t),\ \tilde{f}(z_t)+1)=\rho(\tilde{f}(z_t),\ \tilde{f}(z_t+a))\leqq\rho(z_t,\ z_t+a)\longrightarrow 0 \qquad (t\to 0).$$

Therefore, $\mathrm{Im}\,\tilde{f}(z_t)\to+\infty$ as $t\to 0$ and $\tilde{f}(\tilde{C}_t)\subset U_c$ for sufficiently small $t(>0)$. Set $V=\pi(U_c)$, then V is conformally equivalent to D^* and

$$f(C_t)=f(\pi_0(\tilde{C}_t))=\pi(\tilde{f}(\tilde{C}_t))\subset\pi(U_c)=V,$$

for sufficiently small $t(>0)$. Hence, f is regarded as a bounded holomorphic function on a neighbourhood of $z=0$ in D^*, and can be extended holomorphically to $z=0$ by Riemann's theorem.

If γ_A is the identity, then $\pi^{-1}\circ f$ is regarded as a holomorphic function of D^* to U. Thus, $z=0$ is a removable singularity of f.

Proof of Theorem 1. If $g\geqq 2$, then the universal covering surface of R is the upper half plane. Hence, the statement of (ii) follows from Lemma 1. Next, we consider the case $R=T$. Suppose that $\lim_{z\to 0} f(z)$ does not exist and there exists a point b of T such that such that $f(z)=b$ for only finitely many z in D^*. Consider the restriction $f\,|\,D_r^*$ for a sufficiently small $r>0$, where $D_r^*=\{0<|z|<r\}$. Then, we may assume that $f\,|\,D_r^*$ is a holomorphic mapping of D_r^* to $T-\{b\}$. Since the universal covering of $T-\{b\}$ is U, $\lim_{z\to 0} f(z)$ exists in T from Lemma 1. This contradicts the assumption. Thus, we have shown the statement of (i) for $R=T$. Similarly, we can show it for $R=\hat{C}$.

§ 3. The Carathéodory metric and C-nondegenerate Riemann surfaces

In this section, we prepare some definitions for § 4.

Definition 1. Let M be a complex manifold and $H_p(M; D)$ the family of all holomorphic functions f from M to the unit disk D with $f(p)=0$ $(p\in M)$. For each tangent vector v at p, we set

$$C_M(p, v)=\sup\{|\langle\partial f(p), v\rangle|\,;\, f\in H_p(M; D)\},$$

where $\langle\alpha, v\rangle$ is the natural pairing between a cotangent vector α and a tangent vector v. We call C_M *the Carathéodory metric for* M.

Usual normal-family argument guarantees that $C_M(p, v)$ is continuous in (p, v). Furthermore, it is easily seen that C_M has the distance decreasing property, that is, if $f\,;\, M\to M'$ is a holomorphic mapping, then

$$C_{M'}(f(p), f^*(v))\leqq C_M(p, v).$$

The Carathéodory metric for D is equal to the Poincaré metric, i.e.,

$$C_D(z, v)=|v|/(1-|z|^2).$$

For each smooth curve $w\,;\,[a, b]\to M$ on M, we define the Carathéodory length $C_M L(w)$ of w by

$$C_M L(w)=\int_a^b C_M\Big(w(t),\, w^*\Big(\frac{\partial}{\partial t}\Big)\Big)dt.$$

Definition 2. Let R be a Riemann surface with $C_R(p, \cdot)\neq 0$ for each $p\in R$. Then R is called C-nondegenerate if there exists $\varepsilon>0$ such that for every non-trivial smooth closed curve w on R, the inequality

$$C_M L(w)>\varepsilon \qquad (2)$$

holds.

Clearly, a compact bordered Riemann surface (with no puncture) is a C-nondegenerate Riemann surface. Examples of C-nondegenerate Riemann surfaces of infinite genus will be given in § 5 (Examples 2 and 3).

Next, we consider a compact subset E of $\boldsymbol{C}$.

Definition 3. A compact subset E of $\boldsymbol{C}$ is called *AB-removable* if every bounded holomorphic function f defined on $W-E$ is extended to a holomorphic function on W, where W is a neighbourhood of E.

§ 4. Removability of holomorphic mappings to C-nondegenerate Rieman surfaces

Theorem 2. *Let R be a C-nondegenerate Riemann surface and $\tilde{R}$ a Riemaun surface which is a (possibly brauched) covering surface over R with the projection $\tilde{\pi}$; $\tilde{R}\to R$. Suppose that for each p in R there exists a neighbourhood V of p such that every component of $\tilde{\pi}^{-1}(V)$ is also a C-nondegenerate Riemann surface. Then, for every AB-removable compact subset E of D, every holomorphic mapping f of $D-E$ to $\tilde{R}$ is extended to a holomorphic mapping of D to $\tilde{R}$.*

Remark. The conclusion of the theorem is not valid if $\tilde{R}$ is a compact Riemann surface. In fact, there exist $\tilde{R}$, E, and f such that E is AB-removable but f; $D-E\to\tilde{R}$ can not be extended to a holomorphic mapping of D to $\tilde{R}$ (§ 5 Example 1).

Proof. First, we show the statement when $\tilde{R}=R$, i.e., $\tilde{\pi}=id.$.

The universal covering surfaces of R and $D-E$ are conformally equivalent to the upper half plane U. Hence, R and $D-E$ are represented by torsion free Fuchsian groups Γ and Γ_0 as U/Γ and U/Γ_0, respectively. As in the proof of Theorem 1, there exists a holomorphic mapping $\tilde{f}$ of U to U such that the following diagram is commutative;

$$\begin{array}{ccc} U & \xrightarrow{\ \tilde{f}\ } & U \\ \pi_0\big\downarrow & & \big\downarrow\pi \\ D-G=U/\Gamma_0 & \xrightarrow{\ f\ } & U/\Gamma=R \end{array}$$

where π_0 and π are the canonical projections of U onto $D-E$ and R, respectively. Furthermore, there exists a group homomorphism θ of Γ_0 to Γ with

$$\tilde{f}\circ\gamma=\theta(\gamma)\circ\tilde{f} \qquad \text{for all } \gamma\in\Gamma_0.$$

We show that $\theta(\gamma)=id.$, namely, $\tilde{f}\circ\gamma=\tilde{f}$ for all γ in Γ_0. To do this, it suffices to show that for any simple closed curve γ, $f(\gamma)$ is homotopic to a trivial curve in R. We may assume that γ consists of a finite number of horizontal segments and vertical ones. Denote by π the inside of γ.

First, we draw vertical and horizontal lines in π such that they divide π into small rectangles π_{ij} $(i, j=1, 2, \cdots, k)$ and the radii are less than a sufficiently small number $\delta>0$.

Next, we consider small circles c_{ij}^{α} $(\alpha=1, 2, 3, 4)$ with the radii $\delta\times(\operatorname{diam}\pi_{ij})$ centered at the vertices of π_{ij}, where diam π_{ij} is the diameter of π_{ij}.

Since E is an AB-removable subset of D, it is of class $N_{\mathfrak{B}}$ and of class $N_{\mathfrak{D}}$ in the sence of Ahlfors-Beurling [1]. Thus, when we take arbitrary points p_{ij}^{α} and p_{ij}^{β} on $c_{ij}^{\alpha}\cap\pi_{ij}$ and on $c_{ij}^{\beta}\cap\pi_{ij}$ $(1\leqq\alpha,\ \beta\leqq 4)$, respectively, p_{ij}^{α} and p_{ij}^{β} can be joined by a curve in $\pi_{ij}-E$ whose length differs arbitrarily little from $|p_{ij}^{\alpha}-p_{ij}^{\beta}|$. The similar is valid for p_{ij}^{α} and p_{lk}^{β} $((i, j)\neq(l, k))$ ([1] Theorem 10). So, we can easily construct an open covering $\{\mathfrak{D}_{mn}\}$ of π such that

(a) each $\mathfrak{D}_{mn}$ is a Jordan region,

(b) the boundary $\partial\mathfrak{D}_{mn}$ is in $\pi-E$ and the length is less than δ,

(c) $\mathfrak{D}_{mn}\cap\mathfrak{D}_{kl}\neq\emptyset$ $((m, n)\neq(k, l))$ and $\bigcup_{m,n}\mathfrak{D}_{mn}=\pi$.

The AB-removability of E implies that the Carathéodory metric C_{D-E} is equal to the one on D and the Poincaré metric on D. Hence, if we take δ sufficiently small, then we have from (b)

$$C_{D-E}L(\partial\mathfrak{D}_{mn})<\varepsilon \quad \text{for each } \mathfrak{D}_{mn} \tag{3}$$

where ε is the number given at Definition 2 for a C-nondegenerate Riemann surface R, because the ratio between the Poincaré metric and the Euclidean metric is bounded on a compact subset of D. From the distance decreasing property of the Carathéodory metric, we have

$$C_R L(f(\partial\mathfrak{D}_{mn}))<\varepsilon .$$

Hence, each $f(\partial\mathfrak{D}_{mn})$ is a trivial curve in R. Since $\gamma=\partial\pi$ is freely homotopic to the sum $\sum\partial\mathfrak{D}_{mn}$, so is $f(\gamma)$ to $\sum f(\partial\mathfrak{D}_{mn})$. Thus, $f(\gamma)$ is a trivial curve and we have shown the statement when $\tilde{R}=R$.

As for the general case, consider a holomorphic mapping $\tilde{\pi}\circ f$ of $D-E$ to R. Since $\tilde{\pi}\circ f$ has a holomorphic extension Φ on D as above, we can take a neighbourhood V' of $\Phi(z)$ and U_z of z for each point z in E such that $\Phi(U_z)$ is contained in V' and every component V of $\tilde{\pi}^{-1}(V')$ in R satisfies the condition of the theorem for $\tilde{\pi}$; $\tilde{R}\to R$. Then, $\tilde{\pi}\circ f(U_z\cap(D-E))$ is contained in some component of $\tilde{\pi}^{-1}(V)$. Hence, $f|(U_z\cap(D-E))$ is a holomorphic mapping to a C-nondegenerate Riemann surface by the assumption of the theorem, and it has a holomorphic extension from the above argument. So, f; $D-E\to\tilde{R}$ is extended to a holomorphic mapping of D to $\tilde{R}$.

§ 5. Examples

In this section, we shall construct some examples about the preceding sections.

Example 1. We give a compact Riemann surface R, an AB-removable compact subset E of D and a holomorphic mapping f of $D-E$ to R which has not a holomorphic extension of D to R.

Let R_0 be a compact bordered Riemann surface whose double is of genus $g\geqq 2$. Then, R_0 is represented as U/Γ_0 by a finitely generated Fuchsian group Γ_0 of the second kind on U. It is easily seen that the limit set Λ $(\subset\boldsymbol{R}\cup\{\infty\})$ of Γ_0 is a closed set of linear measure zero. Indeed, let $\hat{u}$ be the harmonic measure of Λ in U. Since

Λ is Γ_0-invariant, $\hat{u}$ is automorphic for Γ_0. Thus, $\hat{u}$ is projected to a bounded harmonic function u on R_0. Obviously, u vanishes on ∂R_0, so $u\equiv 0\equiv \hat{u}$. This implies that Λ is of linear measure zero and AB-removable.

Since Γ_0 is of the second kind, the region of discontinuity $\Omega=\hat{C}-\Lambda$ is connected and Ω/Γ_0 is a compact Riemann surface R of genus $g\geqq 2$. Take an open disk B in C such that $B\cap\Lambda\neq\emptyset$ and $\partial B\subset\Omega$. Then, $f=\pi|B\cap\Omega$ is a holomorphic mapping of $B-(B\cap\Lambda)$ to R, where π is the natural projection of Ω onto R. $B\cap\Lambda$ is a compact subset of B and AB-removable, but f can not be extended to a holomorphic mapping of B because Λ is the limit set of Γ_0. Thus, we have constructed a desired example.

Example 2. (A C-nondegenerated Riemann surface of infinite genus) Let $\{I_n\}_{n=1}^{\infty}$ be a set of closed line segments on the unit disk D satisfying the conditions;

(*) (the Poincaré distance between I_n and I_m)$>\varepsilon\cdot(n\neq m)$ and (the Poincaré length of I_n)$>\varepsilon$ $(n=1, 2, \cdots)$ for some $\varepsilon>0$.

We denote by D_1, D_2 two copies of D with the slits $\{I_n\}_{n1}^{\infty}$, and connect them to each other by identifying two edges of I_n $(n=1, 2, \cdots)$ crosswisely. A two-sheeted Riemann surface R over D with the branched points over the endpoints of I_n $(n=1, 2, \cdots)$ is obtained. Obviously, R is of infinite genus. So, it suffices to show that R is a C-nondegenerate Riemann surface.

Let w be a non-trivial smooth closed curve on R. Then, from the distance decreasing property of the Carathéodory metric, we have

$$C_R L(w)\geqq C_D L(p(w)), \tag{4}$$

where p is the natural projection of R onto D. Since w is non-trivial on R, it is easily seen that $p(w)$ joins two distinct segments of $\{I\}_{n=1}^{\infty}$ or contains a subarc of $p(w)$ rounds some I_n. In any case, the Poincaré length of $p(w)$ is more than ε by the assumption. Since the Carathéodory metric for D is the Poincaré metric for D, we have

$$C_D L(p(w))>\varepsilon .$$

Thus, from (4) $C_R L(w)>\varepsilon$, which implies that R is a C-nondegenerate Riemann surface.

Next, we note the following;

Proposition. *Let R be a compact bordered Riemann surface and $\tilde{R}$ a Riemann surface which is a (possibly branched) covering snrface over R with the projection $\tilde{\pi}$; $\tilde{R}\to R$. Suppose that for each $p\in R$ there exists a neighbourhood V of p such that every component of $\tilde{\pi}^{-1}(V)$ is a compact bordered Riemanu surface. Then for an AB-removable compact subset E of D, every holomorphic mapping f of $D-E$ to $\tilde{R}$ is extended to a holomorphic mapping of D to $\tilde{R}$.*

Proof. By the same argument as in the proof of Theorem 2, it suffices to show that the statement of Proposition is valid when $R=\tilde{R}$. We may assume that the boundary ∂R of R consists of a finite number of analytic Jordan curves. It is known (cf. Heins [2]) that there exists a holomorphic function h of $R\cup\partial R$ such that $|h|<1$ in R and $|h|=1$ on ∂R. Since $h\circ f$ is a bounded holomorphic function of $D-E$, it has a holomorphic extension Ψ on D. Noting that R is regarded as a finite branched covering over D via h, we verify that for each z in E, there exists a neighbourhood of V' of $\Psi(z)$ such that each component of $h^{-1}(V')$ in R is simply connected. We can take a neighbourhood U_z of z in D so small that $h\circ f(U_z\cap(D-E))$ is contained in V'. Then, $f(U_z\cap(D-E))$ is in V for some component V of $h^{-1}(V')$. Since V is conformally equivalent to the unit disk, $f|U_z\cap(D-E)$ is regarded as a bounded holomorphic function and has a holomorphic extension F_z on U_z. Since $z\in E$ is arbitrary, there exists a holomorphic mapping F of D such that $F=F_z$ on U_z, and we have shown the statement when $\tilde{R}=R$.

The proof of the above proposition is simpler than that of Theorem 2. So, we need construct a Riemann surface $\tilde{R}$ which is C-nondegenerate but does not satisfy the condition of Proposition for any compact bordered Riemann surface.

Example 3. We give a C-nondegenerate Riemann surface which does not satisfy the condition of Proposition.

To do it, we note the following;

Lemma 2. *We can take an annulus* $A=\{z\in \boldsymbol{C}\,;\,0<r<|z|<1\}$ *and segments* I_n *in* A *with the endpoints* a_n *and* b_n $(n=1, 2, \cdots)$ *satisfying the followings;*

(i) $\{I_n\}_{n=1}^{\infty}$ *satisfies the condition* (*) *in Example* 2 *with respect to the Poincaré metric for the uuit disk* D.

(ii) *a bounded holomorphic function on* A *vanishing at* $\cup_{n=1}^{\infty}(a_n\cup b_n)$ *is zero on* A.

Proof. We take A and $\{I_n\}_{n=1}^{\infty}$ by the following way. Let Γ be a Fuchsian group acting on the upper half plane U such that U/Γ is a compact Riemann surface. Furthermore, we assume that U/Γ is symmetric with respect to a certain simple geodesic α on U/Γ, and that the imaginary axis on U is mapped to α by the canonical projection π of U onto U/Γ. Let $g(z)=kz(k>1)$ be in Γ corresponding to α. Then, there exists a fundamental region ω for Γ such that ω is contained in $U(k)=U\cap\{1<|z|\leqq k\}$ and symmetric with respect to the imaginary axis. We take a closed segment s in ω with the endpoints z_1, z_2 and set $S=\{\gamma\in\Gamma\,;\,\gamma(s)\subset U(k)\cap\{\mathrm{Re}\,z>0\}\}$. Since $U(k)$ is a fundamental region for a cyclic group $\langle g\rangle$ and $A=U/\langle g\rangle$ is an annulus, $\cup\{\gamma(s)\,;\,\gamma\in S\}$ is corresponding to a set of segments $\{I_n\}_{n=1}^{\infty}$ in A accumulating to one of the boundary component of A. We set $A=\{r<|z|<1\}$ such that $\{I_n\}_{n=1}^{\infty}$ accumulates to the unit circle.

We must show that $\{I_n\}_{n=1}^{\infty}$ satisfies the above conditions. Let h be a bounded holomorphic function on A vanishing at $\cup_{n=1}^{\infty}(a_n\cup b_n)$. Then, $h\circ\pi'$ is a bounded holomorphic function on U, where π' is the canonical projection of U onto $A=U/\langle g\rangle$. Furthermore, it vanishes at $\cup_{n=1}^{\infty}\{g^n\circ\gamma(z_1)\cup g^n\circ\gamma(z_2):\gamma\in S\}$. On the other hand, U/Γ is a compact Riemann surface. Hence, Γ is a Fuchsian group of divergence type, namely, for each z in U,

$$\sum_{\gamma\in\Gamma}(1-|F\circ\gamma(z)|)=+\infty,$$

where F is a Möbius transformation of U onto D. Since each $\gamma(s)$ in $U(k)$ corresponds to a coset of $\Gamma\diagdown\langle g\rangle$, we have for a certain F

$$\sum_{\substack{\gamma\in S\\ n\in Z}}(1-|F\circ g^n(\gamma(z_j))|)=2^{-1}\sum_{\gamma\in\Gamma}(1-|F\circ\gamma(z_j)|)$$

$$=+\infty \qquad (j=1, 2).$$

Hence, the zeros $\{F\circ g^n(\gamma(z_j)), j=1, 2: \gamma\in S\}$ of a bounded holomorphic function $h\circ\pi'\circ F^{-1}$ on D does not satisfy the Blaschke condition (cf. Tsuji [7]), and we conclude that $h\circ\pi'\circ F^{-1}\equiv 0$ and $h\equiv 0$. We have shown that $\{I_n\}_{n=1}^{\infty}$ satisfies the condition (ii).

Let $\rho_D(z)|dz|$ and $\rho_A(z)|dz|$ be the Poincaré metrics on D and A, respectively. If $z\in A$ is in $\{(1+r)/2<|z|<1\}$, an inequality

$$\rho_A(z)\leqq M\rho_D(z) \tag{5}$$

holds for some constant $M>0$ because a disk $\{w\in A: |w-z|<1-|z|\}$ is contained in A. It is easily seen that $\{I_n\}_{n=1}^{\infty}$ satisfies the condition (*) in Example 2 with respect to ρ_A. Therefore, from (5) we verify that $\{I_n\}_{n=1}^{\infty}$ satisfies it with respect to ρ_D.

We denote by D_1 (resp. D_2) a copy of D (resp. $D-\bigcup_{n=1}^{\infty}L'_n$) with the slits $\{I_n\}_{n=1}^{\infty}\subset A$ (resp. $\{I_n\}_{n=1}^{\infty}$ and $\{L_n\}_{n=1}^{\infty}$), where I_n and A are taken as in Lemma 2, $L'_n=[d_{n+1}, c_n]$ and $L_n=(c_n, d_n)$ with $0<c_n<d_n<c_{n-1}<d_{n-1}<r$ and $\lim_{n\to\infty} c_n=\lim_{n\to\infty} d_n=0$. First, we connect D_1 and D_2 along $\{I_n\}_{n=1}^{\infty}$ as in Example 2 and construct a two sheeted Riemann surface R_0 over D. Next, we set $D(n)=D-[0, c_n]\cup[d_n, 1)$ and take $k(n)$ copies $D(n)_1, \cdots, D(n)_{k(n)}$ of $D(n)$, where $k(n)$ is a natural number with

$$k(n)\int_0^{c_n}\rho_D(z)|dz|>\varepsilon>0.$$

Cutting $D(n)_j$ along (c_n, d_n) $(j=1, \cdots, k(n), n=1, 2, \cdots)$, we construct a Riemann surface R by identifying the upper edge of (c_n, d_n) in $D(n)_j$ with the lower edge of (c_n, d_n) in $D(n)_{j+1}$ $(j=0, 1, \cdots, k(n), n=1, 2, \cdots)$, where $D(n)_0=D(n)_{k(n)+1}=R_0$. R is is our desired surface, i.e., R is a C-nondegenenerate Riemann surface but does not satisfy the condition of Proposition. It is easily seen that R is a C-nondegenerate surface. Indeed, let w be a non-trivial simple closed curve and p the natural projection of R to D. Then, $p(w)$ has a subarc which rounds some $(0, c_n]$ $k(n)$-times, rounds some $I_n\subset A$, rounds $(0, d_1)$ or connects two distinct segments of $\{I_n\}_{n=1}^{\infty}$. Thus, we have

$$\inf\{C_D L(p(w)): w \text{ is a non-trivial simple closed curve in } R\}>0.$$

From the distance decreasing property of the Carathéodory metric as before, we conclude that R is a C-nondegeuerate Riemann surface. Next, suppose that R is a covering surface over some compact bordered Riemann surface R_0. Let π be the projection of R to R_0 and let h be a bounded holomorphic function on R_0 such that $|h|<1$ in R_0 and $|h|=1$ on ∂R_0. Then, $F=h\circ\pi$ is a bounded holomorphic function on R, in particular, F is a bounded holomorphic function on $\pi^{-1}(A)$, where A is the annulus given at first. We denote by V_A the component of $\pi^{-1}(A)$ which is contained in $D_1 \cup D_2$ $(\subset R)$ jointed at $\{I_n\}_{n=1}^{\infty}$. Consider a function $(F(z)-F(z'))^2=H(\zeta)$ on A, where z and z' are two points on V_A with $\pi(z)=\pi(z')=\zeta \in A$. H is well-defined and vanishes at $\bigcup_{n=1}^{\infty}(a_n \cup b_n)$. Therefore, $H \equiv 0$ by the construction of $\{I_n\}_{n=1}^{\infty}$. This implies that $F|V_A$ is a lift of holomorphic function on the annulus A via π. On the other hand, D_1 has no slit in $\{|z| \leqq r\}$ where F is holomorphic. Hence, F is regarded as a lift of a holomorphic function on the unit disk D via π. Thus, for every sequence $\{z_n\}_{n=1}^{\infty}$ on D_2 with $\lim \pi(z_n)=0$, $\lim F(z_n)=\alpha$ exists and $|\alpha|<1$. For a neighbourhood V of α in D, we verify that there exists a component V' of $F^{-1}(V)$ such that V' is not a compact bordered Riemann surface. Since $h^{-1}(\alpha)$ consists of finite points of R_0, for any neighbourhood $\hat{V}$ of $h^{-1}(\alpha)$ $\pi^{-1}(\hat{V})$ contains a Riemann surface which is not a compact bordered Riemann surface.

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