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Hypoellipticity for infinitely degenerate elliptic and parabolic operators II, operators of higher order

By

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§1. Introduction and results

The present article is a continuation of the previous work [3] concerning the hypoellipticity of second order operators. Here we are mainly concerned with hypoellipticity of differential operators of the form

$$L = D_t^{2m} + f(t) D_x^{2m} + g(t) D_y^{2m} . \tag{1.1}$$

Throughout this paper, we assume that $f(t)$ and $g(t)$ are functions of class C^∞ satisying the following condition:

(A1) i) *$f(0)=g(0)=0$, $f(t)>0$ and $g(t)>0$ for $t \neq 0$.*
ii) *Both of $f(t)$ and $g(t)$ are monotone increasing for $0<t<\delta$, and monotone decreasing for $-\delta<t<0$.*

Then, the same argument as in the proof of theorem 1 of [3] will give the following (we omit its proof):

Theorem 1. *Let L be a differential operator of the form* (1.1) *with $f(t)$ and $g(t)$ satisfying* (A.1). *Assume moreover:*

(A.2) *there exists a constant α with $0<\alpha<1$ such that*

$$\sqrt[2m]{g(\alpha t)}\,|t \log f(t)| \geqq \varepsilon > 0 \qquad \text{for} \quad 0<t<\delta .$$

Then L is not hypoelliptic.

This result gives a necessary condition of hypoellipticity for operator L. Our main purpose of the present paper is to prove that a certain condition (see (A.4) below) almost complimentary to (A.2) is also sufficient for hypoellipticity of L, under the assumption (see (A.3) below) concerning magnitudes of derivatives of $f(t)$ and $g(t)$.

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Theorem 2. *Let L be a differential operator of the form* (1.1) *with $f(t)$ and $g(t)$ satisfying* (A.1). *Assume moreover the followings:*

(A.3) *Functions $a(t)=\{f(t)\}^{1/2m}$ and $b(t)=\{g(t)\}^{1/2m}$ satisfy*

$$\left|\left(\frac{d}{dt}\right)^j a(t)\right| \leqq Ca(t)^{1-\sigma j} \quad \text{and} \quad \left|\left(\frac{d}{dt}\right)^j b(t)\right| \leqq Cb(t)^{1-\sigma j}$$

for $-\delta<t<\delta$, $1\leqq j\leqq m$, with some positive constants σ $(0<\sigma<1/m)$ and C.

$$\text{(A.4)} \qquad \begin{cases} \lim\limits_{t\to 0} \sqrt[2m]{g(t)}\,|t \log f(t)| = 0 \\ \lim\limits_{t\to 0} \sqrt[2m]{f(t)}\,|t \log g(t)| = 0\,. \end{cases}$$

Then L is hypoelliptic.

Remark 1.1. Of course, the assumption (A.3) is equivalent to

$$\text{(A.3)}' \qquad \left|\left(\frac{d}{dt}\right)^j f(t)\right| \leqq C' f(t)^{1-\sigma' j}, \quad \left|\left(\frac{d}{dt}\right)^j g(t)\right| \leqq C' g(t)^{1-\sigma' j},$$

$j=1, \cdots, m$ with another positive constant C' and $\sigma'=\sigma/(2m)$.

Remark 1.2. If $g(t)=O(t^{\alpha})$ at $t=0$, then the assumption (A.3) implies $\alpha\geqq 2m/\sigma$. So in the case where both of $f(t)$ and $g(t)$ vanish finitely at $t=0$, the assumption (A.3) is very restirictive. But the hypoellipticity of L in that case is by now well known, so we are interested in the case where one of (or both of) f(t) or $g(t)$ vanishes infinitely at $t=0$. Moreover, if $f(t)\leqq$Const $g(t)$ for $-\delta<t<\delta$, the assumption for $g(t)$ (or $b(t)$) in (A.3) can be removed. This will be seen at the end of §2.

To understand our results, let us pay attention to the following examples.

Example 1. We put $f(t)=\exp(-1/|t|^{\gamma})$ with some positive constant γ, and $g(t)=t^{2k}$ with some positive integer k. Then $f(t)$ satisfies the assumption (A.3) for any $\sigma>0$, and $f(t)\leqq g(t)$ in a neighborhood of $t=0$. Hence, Theorem 1 and 2 show that the operator

$$L = D_t^{2m}+\exp(-1/|t|^{\gamma})D_x^{2m}+t^{2k}D_y^{2m}$$

is hypoelliptic if and only if $\gamma<k/m+1$.

Example 2. Let γ_1 and γ_2 be positive numbers. Theorem 2 shows that the poerator

$$L = D_t^{2m}+\exp(-1/|t|^{\gamma_1})D_x^{2m}+\exp(-1/|t|^{\gamma_2})D_y^{2m}$$

is hypoelliptic.

Remark 1.3. Let $f_j(t)$ $(j=1, \cdots, n)$ be functions of class C^{∞} satisfying the conditions (A.1), (A.3) and

$$\lim_{t\to 0} {}^{2m}\sqrt{f_j(t)}\,|t \log f_k(t)| = 0 \qquad \text{for} \quad j,\, k = 1, \cdots, n\,.$$

Then our arguments in the proof of Theorem 2 can also be applied to show that the operator

$$L = D_t^{2m} + \sum_{j=1}^{n} f_j(t) D_{x_j}^{2m}$$

is hypoelliptic in $\boldsymbol{R}^{n+1}$.

Remark 1.4. Theorem 2 has a generalization to the operators of the form

$$L = D_t^{2m} + D_x^m(f(t, x, y)D_x^m) + D_y^m(g(t, x, y)D_y^m)\,,$$

where $f(t, x, y)$ and $g(t, x, y)$ are functions of class C^∞ satisfying the following condition: There exist a pair of functions $f(t)$, $g(t)$ satisfying the conditions (A.1), (A.3), (A.4), and a positive constant C such that

$$\begin{cases} C^{-1}f(t) \leqq f(t, x, y) \leqq Cf(t) \\ C^{-1}g(t) \leqq g(t, x, y) \leqq Cg(t) \end{cases}$$

and

$$\begin{cases} \displaystyle\sum_{1\leqq|k+l|\leqq 2m} |D_x^k D_y^l f(t, x, y)| \leqq Cf(t) \\ \displaystyle\sum_{1\leqq|k+l|\leqq 2m} |D_x^k D_y^l g(t, x, y)| \leqq Cg(t)\,. \end{cases}$$

For detail, c.f. theorem 3 of [3].

By a slight modification, our arguments can also be applied to the operators of parabolic type:

Theorem 3. Let P be a differential operator of the form

$$P = D_t^{2m} + f(t) D_x^{2m} + ig(t) D_y\,, \tag{1.2}$$

where $f(t)$ and $g(t)$ are functions of class C^∞ satisfying the conditions (A.1) and (A.3). Assume moreover that

$$\begin{cases} \lim_{t\to 0} g(t) t^{2m} |\log f(t)| = 0 \\ \lim_{t\to 0} {}^{2m}\sqrt{f(t)}\,|t \log g(t)| = 0\,. \end{cases} \tag{A.5}$$

Then P is hypoelliptic.

Example 3. Let γ be a positive constant, and let k be a positive integer. Then, the operator

$$P = D_t^{2m} + \exp(-1/|t|^\gamma) D_x^{2m} + it^{2k} D_y$$

is hypoelliptic if and only if $\gamma < 2k + 2m$. The proof of necessity is due to the same argument as in proof of Theorem 1 (c.f. section 2 of [3]). Also recall Remark 1.2 for the proof of the sufficiency.

Our plan of the present paper, which is quite analogous to that of [3], is as follows: In §2, we explain some basic facts necessary for the proof of Theorem 2. The presence of Lemma 1 there is the most significant difference from the previous article [3]. In §3, we complete the proof of Theorem 2. Finally, §4 will be devoted to the proof of Theorem 3.

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§2. A priori estimates

We start this section by explaining our plan of the proof of Theorem 2. At the beginning, let us introduce the ordinary differential opeartor with real parameters $\zeta=(\xi, \eta)$ as follows:

$$L_\zeta = (-1)^m \frac{d^{2m}}{dt^{2m}}+f(t)\xi^{2m}+g(t)\eta^{2m}\,. \tag{2.1}$$

The proof of Theorem 2 is devided into the following two propositions.

Proposition 1. *Suppose that $f(t)$ and $g(t)$ satisfy the assumptions of Theorem 2. Then, we have the following inequality:*

(2.2) *Given any $\varepsilon>0$, there exists a positive number n_ε such that*

$$\int f(t)(\log \langle\eta\rangle)^{2m}|v(t)|^2dt+\int g(t)(\log\langle\xi\rangle)^{2m}|v(t)|^2dt$$
$$\leqq \varepsilon\int L_\zeta v(t)\cdot\overline{v(t)}dt\,,$$

for all $v\in C_0^{2m}(-\delta, \delta)$ and for all $\zeta\in\boldsymbol{R}^2$ satisfying $\xi^2+\eta^2\geqq n_\varepsilon^2$. (Here we denote $\langle\xi\rangle=(1+|\xi|^2)^{1/2}$).

Remark 2.1. In the right hand side of (2.2) ,notice that

$$\int L_\zeta v(t)\cdot\overline{v(t)}dt$$
$$=\int|v^{(m)}|^2dt+\int f\xi^{2m}|v|^2dt+\int g\eta^{2m}|v|^2dt\,.$$

Proposition 2. *If L_ζ enjoyes* (2.2), *then L is hypoelliptic.*

The proof of Proposition 2 will be given in the next section, using microlocal energy method. We shall devote the remaining part of this section to the proof of Proposition 1. The technique of its proof here is quite analogous to the one in Višik-Grušin's paper [10].

At first, let us prove the following lemma, which will be necessary in the proof of Proposition 1.

Lemma 1. *If* $f(t)=a(t)^{2m}$ *satisfies the conditions* (A.1) *and* (A.3), *then it holds the following inequality:*

For any γ *satisfying* $0<\gamma<(1-\sigma m)/(1+\sigma)$, *there exists a positive constant* C_γ *such that*

$$\sum_{1\leq k\leq m-1}\int|(a(t)|\xi|^{\gamma})^k D_t^{m-k}v(t)|^2 dt \tag{2.3}$$
$$\leq C_\gamma\{\int|D_t^m v(t)|^2 dt+\int f(t)\xi^{2m}|v(t)|^2 dt\},$$

for all $v\in C_0^m(-\delta, \delta)$ *and for all* $\xi\in\boldsymbol{R}$.

Proof of Lemma 1. Choose γ' so that

$$\gamma/(1-\sigma m)<\gamma'<(1+\sigma)^{-1}. \tag{2.4}$$

I. First we prove (2.3) for $v(t)$ supported in $\{t\in(-\delta, \delta);\ a(t)|\xi|^{\gamma'}\geq 1/2\}$. Let us make a change of vairable as

$$h(\xi, t)dt = dy,$$

with

$$h(\xi, t) = a(t)|\xi|.$$

Furthermore we put

$$w(y) = h^{m-(1/2)}v(t).$$

Then

$$\begin{aligned}\int\{|D_t^m v|^2+f(t)\xi^{2m}|v|^2\}dt \\ =\int\{|D_t^m v|^2+|h^m v|^2\}dt \\ =\int\{|(hD_y)^m(h^{-m+(1/2)}w)|^2+|w|^2h\}h^{-1}dy \\ =\int\{|D_y^m w|^2+|w|^2\}dy+\text{remainder},\end{aligned} \tag{2.5}$$

where "remainder" is of the form

$$\mathrm{Re}\int\{|\sum_{1\leq k\leq m}\text{coefficient}\times D_y^{m-k}w|^2+$$
$$+2D_y^m w\cdot\overline{\sum_{1\leq k\leq m}\text{coefficient}\times D_y^{m-k}w}\}dy.$$

We can prove recursively that "coefficient" is the sum of terms of the form

$$\text{Const.}\prod_j\{h^{(j)}\cdot h^{j-1}\}^{p_j}$$

satisfying

$$\sum_j j\cdot p_j = k.$$

(Here we denote $h^{(j)}=\left(\frac{d}{dt}\right)^j h$.)

Now we can see that $|h^{(j)}\cdot h^{-j-1}|$ is small in the support of $w(y)$ (or $v(t)$) if $|\xi|$ is sufficiently large, because the assumption (A.3) implies

$$\begin{aligned}|h^{(j)}\cdot h^{-j-1}| &= |a^{(j)}|\cdot a^{-j-1}\cdot |\xi|^{-j} \\ &\leqq \text{Const.}\ (a^{-1-\sigma}\cdot|\xi|^{-1})^j \\ &\leqq \text{Const.}\ \{|\xi|^{-1+\gamma'(1+\sigma)}\}^j .\end{aligned}$$

It is clear that

$$\sum_{1\leqq k\leqq m-1}\int |D_y^k w|^2 dy \leqq \text{Const.}\int \{|D_y^m w|^2+|w|^2\} dy .$$

So from (2.5) and the above observation, it follows that

$$\begin{aligned}&\sum_{0\leqq k\leqq m}\int |(h^{-1}D_t)^k(h^{m-(1/2)}v)|^2 h dt \\ &=\sum_{0\leqq k\leqq m}\int |D_y^k w|^2 dy \\ &\leqq \text{Const.}\int \{|D_t^m v|^2+|h^m v|^2\} dt .\end{aligned}$$

Moreover, using the above argument again, we can see

$$\begin{aligned}&\sum_{1\leqq k\leqq m-1}\int |h^{m-k}D_t^k v|^2 dt \\ &\leqq \text{Const.}\int \{|D_t^m v|^2+|h^m v|^2\} dt ,\end{aligned}\tag{2.6}$$

which is a stronger inequality compared with (2.3). (Notice that

$$h^{1/2}(h^{-1}D_t)^k(h^{m-(1/2)}v) = h^{m-k}D_t^k v+\sum_{l<k}\text{coefficient}\times h^{m-l}D_t^l v$$

and that "coefficient" is small in the suppost of $v(t)$ if $|\xi|$ is large.)

II. For $v(t)$ supported in $\{t\in(-\delta, \delta);\ a(t)|\xi|^{\gamma'}\leqq 2\}$, it is very easy to see that

$$\begin{aligned}&\int |(a(t)|\xi|^{\gamma})^k D_t^{m-k}v|^2 dt \\ &\leqq |\xi|^{-2(\gamma'-\gamma)k}\times \text{Const.}\int |D_t^{m-k}v|^2 dt \\ &\leqq \text{Const.}\ |\xi|^{-2(\gamma'-\gamma)k}\int |D_t^m v|^2 dt ,\end{aligned}\tag{2.7}$$

which is a stronger inequality compared with (2.3).

III. Let us prove (2.3) for general $v\in C_0^m(-\delta, \delta)$. Choose a function $\phi\in C_0^\infty(\boldsymbol{R})$ so that $\phi(t)=1$ for $|t|\leqq 1/2$ and $\phi(t)=0$ for $|t|\geqq 2$. Furthermore let us put

$$\chi_1(t)=\phi(a(t)|\xi|^{\gamma'}), \quad \chi_2(t)=1-\chi_1(t),$$

and

$$v_1(t)=\chi_1 v, \qquad v_2(t)=\chi_2 v.$$

Then, (2.6) and (2.7) will yield

$$\begin{aligned}
&\int|(a(t)|\xi|^{\gamma})^k D_t^{m-k}v(t)|^2dt\\
&\leqq 2\{\int|(a|\xi|^{\gamma})^k D_t^{m-k}v_1|^2dt+\int|(a|\xi|^{\gamma})^k D_t^{m-k}v_1|^2dt\}\\
&\leqq \text{Const. }\{|\xi|^{-2(\gamma'-\gamma)k}\int|D_t^m v_1|^2dt+|\xi|^{-2(1-\gamma)k}\int(|D_t^m v_2|^2+|h^m v_2|^2)dt\}\\
&\leqq \text{Const.}\times|\xi|^{-2(\gamma'-\gamma)k}\int(|D_t^m v|^2+|h^m v|^2)dt+\text{remainder}.
\end{aligned}$$

(To see the last estimate, notice that $\gamma'>\gamma$.)

Observe now that "remainder" is estimated by

$$\text{Const.}|\xi|^{-2(\gamma'-\gamma)k}\int|[D_t^m, \chi^1]v|^2dt,$$

and that the coefficient of D_t^{m-k} $(k=1, \cdots, m)$ in $[D_t^m, \chi_1]$ is a sum of terms of the form

$$\text{Const. }\phi^{(k')}(a|\xi|^{\gamma'})\prod_j\{a\{^{(j)}|\xi|^{\gamma'}\}^{p_j}$$

with

$$\sum_j j\cdot p_j=k \quad \text{and} \quad k'=\sum_j p_i.$$

Furthermore, observe that the assumption (A.3) yields, in the support of $\phi^{(k')}(a|\xi|^{\gamma'})$,

$$\begin{aligned}
\prod_j\{a^{(j)}|\xi|^{\gamma'}\}^{p_j}&\leqq\text{Const. }\prod_j\{a^{1-\sigma j}|\xi|^{\gamma'}\}^{p_j}\\
&\leqq\text{Const. }\prod_j|\xi|^{\sigma\cdot\gamma'\cdot j\cdot p_j}\\
&\leqq\text{Const. }|\xi|^{\sigma\gamma' m}.
\end{aligned}$$

So we can see that "remainder" is estimated by

$$\text{Const. }|\xi|^{-2((\gamma'-\gamma)-k-\sigma m\gamma')}\int|D_t^m v|^2dt.$$

Since we have chosen γ and γ' so that $(\gamma'-\gamma)-\sigma m\gamma'>0$, we can now see that the inequality (2.3) holds for all $v\in C_0^m(-\delta, \delta)$. $\square$

Now, we come to the proof of Proposition 1.

Proof of Proposition 1. We are going to prove the inequality:

$$\int g(t)(\log\langle\xi\rangle)^{2m}|v(t)|^2dt \leqq \varepsilon\int L_\xi v(t)\cdot\overline{v(t)}dt\,, \tag{2.8}$$

$$\forall v\in C_0^{2m}(-\delta,\delta),\ |\xi|\,;\text{ sufficiently large}\,,$$

in the region $\Lambda_1=\{(\xi,\eta);\ |\eta|\leqq 2|\xi|\}$. In Λ_1, it is clear the that

$$\int f(t)(\log\langle\eta\rangle)^{2m}|v(t)|^2dt \leqq \varepsilon\int f(t)\xi^{2m}|v(t)|^2dt\,,$$

if $|\xi|$ is sufficiently large. So we can see that (2.3) holds in Λ_1, once (2.8) is established. The proof of (2.2) in $\Lambda_2=\{(\xi,\eta);\ |\xi|\leqq 2|\eta|\}$ is parallel, interchanging the roles of $(f(t),\xi)$ and $(g(t),\eta)$.

I. First we shall prove (2.8) for $v(t)$ supported in $\{t\in(-\delta,\delta);\ a(t)|\xi|^{\gamma/2}\geqq 1/2\}$, where γ is the same number as in Lemma 1. It will be easily seen that, if $|\xi|$ is sufficiently large,

$$\begin{aligned}
&\int g(t)(\log\langle\xi\rangle)^{2m}|v(t)|^2dt \\
&\leqq \varepsilon\int |\xi|^{\gamma/2}|v(t)|^2dt \\
&\leqq \varepsilon\int (a(t)|\xi|)^{2m}|v(t)|^2dt \\
&\leqq \varepsilon\int L_\xi v(t)\cdot\overline{v(t)}dt\,.
\end{aligned} \tag{2.9}$$

II. To prove (2.8) for $v(t)$ supported in $\{t\in(-\delta,\delta);\ a(t)|\xi|^{\gamma/2}\leqq 2\}$, we must do the following consideration.

Let $z=z(\xi)$ denote a positive number such that $a(z)|\xi|^{\gamma/2}=2$ and write $v(t)$ as

$$v(t) = -\int_t^z \frac{(z-s)^{m-1}}{(m-1)!}\cdot v^{(m)}(s)ds\,.$$

Then it holds

$$\begin{aligned}
&\int_0^z g(t)|v(t)|^2dt \\
&\leqq \int_0^z g(t)\frac{(z-t)^{2m-1}}{(2m-1)(m-1)!^2}dt\int |v^{(m)}(s)|^2ds \\
&\leqq \frac{g(z)\cdot z^{2m}}{2m(2m-1)(m-1)!^2}\int |v^{(m)}(s)|^2ds\,.
\end{aligned} \tag{2.10}$$

On the other hand, the assumption (A.4) together with the fact that $z(\xi)\to 0$ as $|\xi|\to\infty$ yield, if $|\xi|$ is sufficiently large,

$$\begin{aligned}
&(\log\langle\xi\rangle)^{2m}g(z)z^{2m} \\
&\leqq (\log\langle\xi\rangle^{2m})\,\varepsilon(\log f(z))^{-2m} \\
&= \varepsilon(\log\langle\xi\rangle)^{2m}(\log 2^{2m}|\xi|^{-m\gamma})^{-2m} \\
&\leqq \text{Const. }\varepsilon\,.
\end{aligned} \tag{2.11}$$

Therefore, combining (2.10) and (2.11), we obtain (2.8) for $v(t)$ with support sufficiently near to $t=0$. (The integral $\int_{z'}^{0} g(t)|v(t)|^2 dt$ can be estimated in the same way, where $z'=z'(\xi)$ denotes a negative number such that $a(z')|\xi|^{\gamma/2}=2$.)

III. Now, we shall prove (2.8) for general $v(t)\in C_0^{2m}(-\delta, \delta)$. Choose a function $\phi\in C_0^\infty$ so that $\phi(t)=1$ for $|t|\leqq 1/2$ and $\phi(t)=0$ for $|t|\geqq 2$. Put

$$\tilde{\chi}_1(t) = \phi(a(t)|\xi|^{\gamma/2}), \quad \tilde{\chi}_2(t) = 1-\tilde{\chi}(t)$$

and

$$v_1 = \tilde{\chi}_1 v_1, \quad v_2 = \tilde{\chi}_2 v .$$

Then, the results in I and II above yield

$$\begin{aligned}
&\int g(t)(\log\langle\xi\rangle)^{2m}|v(t)|^2 dt \\
&\leqq 2(\log\langle\xi\rangle)^{2m}\left\{\int g|v_1|^2 dt+\int g|v_2|^2 dt\right\} \\
&\leqq 2\varepsilon\left\{\int L_\xi v_1\cdot\bar{v}_1 dt+\int L_\xi v_2\cdot\bar{v}_2 dt\right\} \\
&\leqq 8\varepsilon\int L_\xi v\cdot\bar{v}\,dt+\text{remainder} .
\end{aligned}$$

Observe now that the remainder is estimated by

$$8\varepsilon\int|[D_t^m, \tilde{\chi}_1]v|^2 dt ,$$

and that the coefficient of D_t^{m-k} in $[D_t^m, \tilde{\chi}_y]$ is the sum of terms of the form

$$\text{Const.}\ \phi^{(k')}(a|\xi|^{\gamma/2})\prod_j\{a^{(j)}|\xi|^{\gamma/2}\}^{p_j} \tag{2.12}$$

with

$$\sum_j j\cdot p_j = k \quad \text{and} \quad k' = \sum_j p_j .$$

Moreover, since $a|\xi|^{\gamma/2}\geqq 1/2$ in the support of $\phi^{(k')}(a|\xi|^{\gamma/2})$, (2,.12) is not larger than

$$\text{Const.}\ |\xi|^{k\gamma/2}\times\{a(t)|\xi|^{\gamma/2}\}^k .$$

Now, recalling Lemma 1, we can see that (2.8) holds for all $v\in C_0^{2m}(-\delta, \delta)$ and $|\xi|$; sufficiently large. □

Remark 2.2. If $f(t)\leqq$ Const. $g(t)$, then it is easily seen that

$$\begin{aligned}
&\int f(t)(\log\langle\eta\rangle)^{2m}|v(t)|^2 dt \\
&\leqq \varepsilon\int g(t)\eta^{2m}|v(t)|^2 dt \\
&\leqq \varepsilon\int L_\xi v(t)\cdot\overline{v(t)}dt ,
\end{aligned}$$

if $|\eta|$ is sufficidntly large. Moreover, in the region Λ_2, it is clear that

$$\begin{aligned}&\int g(t)(\log\langle\xi\rangle)^{2m}|v(t)|^2dt\\&\quad\leqq\varepsilon\int g(t)\eta^{2m}|v(t)|^2dt\\&\quad\leqq\varepsilon\int L_\zeta v(t)\cdot\overline{v(t)}dt\,,\end{aligned}$$

if $|\eta|$ is sufficiently large. So in this case, we can prove (2.2) in the region Λ_2, without using the condition for $g(t)$ in (A.3). On the other hand, notice that the condition for $g(t)$ in (A.3) is not in need for the proof of (2.2) in the region Λ_1. Thus we can remove the condition for $g(t)$ in (A.3), in the case where $f(t)\leqq$ Const. $g(t)$ for $-\delta<t<\delta$.

§3. Microlocal energy method

To prove Proposition 2 in the preceding section, let us recall some reductions for the proof of hypoellipticity in [3]. At first, we define the following Sobolev space.

Definition. We denote by $H^{k,l}(\boldsymbol{R}^3)$ $(-\infty<k,\ l<\infty)$ the space of all distributions $u\in\mathcal{S}'(\boldsymbol{R}^3)$ satisfying

$$\iiint|\hat{u}(\tau,\ \xi,\ \eta)|^2(1+\tau^2)^k(1+\xi^2+\eta^2)^l d\tau\,d\xi\,d\eta<\infty\,,$$

where $\hat{u}$ is Fourier transform of u.

Furthermore we say that $u\in\mathcal{D}'$ is locally of class $H^{k,l}$ at $(t_0,\ x_0,\ y_0)$ if there exists a function $\phi\in C_0^\infty(\boldsymbol{R}^3)$ with $\phi=1$ in a neighborhood of $(t_0,\ x_0,\ y_0)$ such that $\phi u\in H^{k,l}(\boldsymbol{R}^3)$.

Now we can see tha folloeing following facts. The first is that, if $u\in\mathcal{D}'((-\delta,\ \delta)\times\Omega)$ and $(t_0,\ x_0,\ y_0)\in(-\delta,\ \delta)\times\Omega$, then there exists a pair of real numbers $(k,\ l)$ such that $u\in H^{k,l}$ at $(t_0,\ x_0,\ y_0)$. The second is that, if $u\in H^{k,l}$ and $Lu\in C^\infty$ at $(t_0,\ x_0,\ y_0)$, then we have $u\in H^{k+2m,l-2m}$ at $(t_0,\ x_0,\ y_0)$. Moreover repeating this fact, we can see that u is locally of class $\bigcap_j H^{k+j,k-j}$ at $(t_0,\ x_0,\ y_0)$. Thus in the proof of Proposition 2, the partial Fourier transform of u at $(t_0,\ x_0,\ y_0)$, i.e.,

$$\chi(t)(\psi u)^\wedge(t;\ \xi,\ \eta)=(2\pi)^{-1}\int e^{i\tau t}\widehat{(\chi\psi u)}(\tau,\ \xi,\ \eta)d\tau\,,$$

where $\chi(t)\in C_0^\infty(\boldsymbol{R})$ with $\chi(t)=1$ in a neighborhood of $t=t_0$ and $\psi(x,\ y)\in C_0^\infty(\boldsymbol{R}^2)$ with $\psi(x,\ y)=1$ in a neighborhood of $(x,\ y)=(x_0,\ y_0)$, is smooth with respect to t for almost every $(\xi,\ \eta)$.

Next, we define the notion of microlical smoothness of $u\in\mathcal{D}'((-\delta,\ \delta)\times\Omega)$, since our proof of Proposition 2 will be microlocal. (It is more precise to say "semi-microlocal".)

Definition. Let $(t_0, x_0, y_0) \in (-\delta, \delta) \times \Omega$ and $(\xi^0, \eta^0) \in \boldsymbol{R}^2 \backslash 0$. For $u \in \mathcal{D}'((-\delta, \delta) \times \Omega)$ we say that u is microlocally of class $H^{0,\infty}$ $(=\bigcap_l H^{0,l})$ at $(t_0, x_0, y_0; \xi^0, \eta^0)$ if there exist a function $\phi \in C_0^\infty((-\delta, \delta) \times \Omega)$ with $\phi = 1$ in a neighborhood of (t_0, x_0, y_0) and a conic neighborhood Γ_0 $(\subset \boldsymbol{R}^2)$ of (ξ^0, η^0) such that

$$\iiint_{\substack{-\infty<\tau<\infty \\ (\xi,\eta)\in\Gamma_0}} |\widehat{(\phi u)}(\tau, \xi, \eta)|^2 (1+\xi^2+\eta^2)^s d\tau d\xi\, d\eta < \infty$$

for any positive number s.

Remark 3.1. By standard argument in microlocal analysis, one can easily show that $u \in H^{0,\infty}$ at (t_0, x_0, y_0) if and only if u is microlocally of class $H^{0,\infty}$ at $(t_0, x_0, y_0; \xi^0, \eta^0)$ for all (ξ^0, η^0).

Remark 3.2. In the proof of Proposition 2, it suffices to show that u is of class $H^{0,\infty}$ at (t_0, x_0, y_0) if Lu is of class C^∞ at (t_0, x_0, y_0). The reason is that the smoothness of u in t-direction follows from the smoothness of u in (x, y)-direction, since L is non-characteristic in t-direction.

We end the preparation of the proof of Proposition 2 by recalling microlical energy method in [3].

Choose first a sequence $\psi_N \in C_0^\infty(\boldsymbol{R}^2)$, $N=1, 2, \cdots$ with $\psi_N = 1$ in $\{(x, y); x^2+y^2 \leqq r_0^2/4\}$ and $\psi_N = 0$ in $\{(x, y); x^2+y^2 \geqq r_0^2\}$ satisfying:

$$|D^{p+\nu}\psi_N| \leqq C_{K_0}(CN)^{|p|}$$

for $|p| \leqq N$, $|\nu| \leqq K_0$. (Here C and C_{K_0} are independent of N.) We define the microlocalizers $\{\alpha_n(\xi, \eta), \beta_n(x, y)\}$ by

$$\alpha_n(\xi, \eta) = \psi_{N_n}\left(\frac{\xi}{n} - \xi^0, \frac{\eta}{n} - \eta^0\right), \quad \beta_n(x, y) = \psi_{N_n}(x - x_0, y - y_0)$$

with

$$N_n = [\log n] + 1 .$$

Our microlocal energy of $v \in \mathcal{D}'$ is

$$S_n^M(v) = \sum_{|p+q| \leqq N_n} \|c_{pq}^n \alpha_n^{(p)}(D_x, D_y)\beta_{n_{(q)}}(x, y)v\|^2 ,$$

with

$$c_{pq}^n = M^{-|p+q|} \cdot n^{|p|} \cdot (\log n)^{-|p+q|} .$$

(Here $\alpha_n^{(p)} = \partial_\xi^{p_1}\partial_\eta^{p_2}\alpha_n(\xi, \eta)$, $\beta_{n_{(q)}} = D_x^{q_1}D_y^{q_2}\beta_n(x, y)$. $\|\ \|$ stands for the norm in $L^2(\boldsymbol{R}^3)$.)

Then, we have the following lemma whose proof is given in section 6 of [3].

Lemma 2. *Let $u \in \bigcup_l H^{0,l}$ locally at (t_0, x_0, y_0). Then u is mocrolocally of class $H^{0,\infty}$ at $(t_0, x_0, y_0; \xi^0, \eta^0)$ if and only if there exists a function $\chi(t) \in C_0^\infty(\boldsymbol{R})$ with $\chi = 1$*

in a neighborhood of $t=t_0$ *such that the microlocal energy of* χu *is rapidly decreasing as* $n\to\infty$ *(if* $r_0>0$ *is small), i.e., for any positive number* s*, there exists constants* M *and* C_s *such that*

$$S_n^M(\chi u)\leqq C_s n^{-2s}$$

when n *is large.* (*We abbreviate as* $S_n^M(\chi u)=O(n^{-2s})$.)

Now we come to the proof of Proposition 2. We will show that u is microlocally of class $H^{0,\infty}$ at $(0, x_0, y_0; \xi^0, \eta^0)$ for every (ξ^0, η^0) if Lu is locally of class C^∞ at $(0, x_0, y_0)$. (Notice that the ellipticity of L except at $t=0$ allows us to restrict our consideration to the case $t_0=0$.)

Proof of Proposition 2. Let $\chi(t)$ and $\psi(x, y)$ be smooth functions whose suports are contained in small neighborhoods of $t=0$ and $(x, y)=(x_0, y_0)$ respectively. Observe now that the right hand side of the equation

$$\psi L(\chi u) = \psi[D_t^{2m}, \chi]u+\chi\psi Lu$$

is of class C_0^∞ if Lu is class C^∞ at $(0, x_0, y_0)$. Therefore it suffices to show that the microlocal energy of $v=\chi u$ is rapidly decreasing if so is that of ψLv.

Assume that $|p+q|\leqq N_n-2m$ and $r_0>0$ is chosen sufficiently small so that $\beta_n\subset\subset\psi$. Let us first operate $\alpha_n^{(p)}(D_x, D_y)\beta_{n_{(q)}}(x, y)$ to the equation $\psi Lv=h$, namely,

$$\alpha_n^{(p)}\beta_{n_{(q)}}Lv = \alpha_n^{(p)}\beta_{n_{(q)}}h\,.$$

The asymptotic expansion gives (notice that $[L, \alpha_n^{(p)}]=0$)

$$Lv_{pq}+\sum_{0<|\nu|\leqq 2m}(-1)^{|\nu|}\nu!^{-1}L^{(\nu)}v_{p,q+\nu} = h_{pq}\,, \tag{3.1}$$

where $v_{pq}=\alpha_n^{(p)}\beta_{n_{(q)}}v$, $h_{pq}=\alpha_n^{(p)}\beta_{n_{(q)}}h$ and $L^{(\nu)}$ is a differential operator with symbol $L^{(\nu)}(t; \xi, \eta)=\partial_{\xi,\eta}^{\nu}L(t; \tau, \xi, \eta)$. Thus we have

$$(Lv_{pq}, v_{pq})\leqq\sum_{0<|\nu|2m}|(L^{(\nu)}v_{p,q+\nu}, v_{pq})|+ \varepsilon^{-1}||h_{pq}||^2+\varepsilon||v_{pq}||^2\,. \tag{3.2}$$

Moreover, multiplying the both sides of (3.2) by $(c_{pq}^n)^2$, we obtain

$$\begin{aligned}(Lw_{pq}, w_{pq})\leqq&\sum_{0<|\nu|\leqq 2m}(\log n)^{|\nu|}M^{|\nu|}|(L^{(\nu)}w_{p,q+\nu}, w_{pq})|\\ &+\varepsilon^{-1}||c_{pq}^n h_{pq}||^2+\varepsilon||w_{pq}||^2\,,\end{aligned} \tag{3.3}$$

where $w_{pq}=c_{pq}^n v_{pq}$. (Notice that $c_{pq}^n=M^{|\nu|}(\log n)^{|\nu|}c_{p,q+\nu}^n$.)

Now in the following, we are going to estimate the first terms of right hand side of (3.3) (see (3.4) below), under the assumption $(\xi^0, \eta^0)\in\Lambda_1=\{(\xi, \eta);\ |\eta|\leqq 2|\xi|\}$ (this implies $c^{-1}\cdot n\leqq|\xi|\leqq c\cdot n$ for $(\xi, \eta)\in\mathrm{supp}\,[\alpha_n]$). In the case of $(\xi^0, \eta^0)\in\Lambda_2=\{(\xi, \eta);\ |\xi|\leqq 2|\eta|$, one can do in the parl a parallel way, interchanging the roles of $(f(t), \xi)$ and $(g(t), \eta)$.

1) For $\nu=(k, 0)$ (i.e. $L^{(\nu)}=\mathrm{const.}\,f(t)D_x^{2m-k}$), we see the following: Since $c^{-1}\cdot n\leqq|\xi|\leqq c\cdot n$ for $(\xi, \eta)\in\mathrm{supp}\,[\alpha_n]$, we can see

$$
\begin{aligned}
&|(L^{(\nu)}w_{p,q+\nu}, w_{pq})| \\
&= \text{Const.}\ \left|\iiint f(t)\xi^{2m-k} w^{\wedge}_{p,q+\nu}(t;\xi,\eta)\cdot\overline{w^{\wedge}_{pq}(t;\xi,\eta)}dt\,d\xi\,d\eta\right| \\
&\leqq \text{Const.}\ n^{-k}\left\{\iiint f\cdot\xi^{2m}(|w^{\wedge}_{p,q+\nu}|^2+|w^{\wedge}_{pq}|^2)dt d\xi d\eta\right\}.
\end{aligned}
$$

Therefore it follows

$$
\begin{aligned}
&(\log n)^{|\nu|}M^{|\nu|}|(L^{(\nu)}w_{p,q+\nu}, w_{pq})| \\
&\quad\leqq\varepsilon\{(Lw_{p,q+\nu}, w_{p,q-\nu})+(Lw_{pq}, w_{pq})\},
\end{aligned}
$$

if n is sufficiently large.

2) For $\nu=(0, k)$ (i.e. $L^{(\nu)}=\text{const.}\,g(t)D_y^{2m-k}$), we see the following: Observe that

$$
\begin{aligned}
&(\log n)^{|\nu|}M^{|\nu|}|(L^{(\nu)}w_{p,q+\nu}, w_{pq})| \\
&= \text{Const.}\ (\log n)^k M^k\left|\iiint g(t)\eta^{2m-k}w^{\wedge}_{p,q+\nu}(t;\xi,\eta)\cdot\overline{w^{\wedge}_{qp}(t;\xi,\eta)}dt\,d\xi\,d\eta\right|
\end{aligned}
$$

and

$$
(M\log n)^k|\eta|^{2m-k}\leqq\varepsilon\cdot p^{-1}\cdot\eta^{2m}+\varepsilon^{-r}\cdot q^{-1}\cdot(M\log n)^{2m}
$$

where $p=2m/(2m-k)$, $q=2m/k$ and $r=(2m-k)/k^2$. Therefore it holds

$$
\begin{aligned}
&(\log n)^{|\nu|}M^{|\nu|}|(L^{(\nu)}w_{p,q-\nu}, w_{pq})| \\
&\leqq \text{Const.}\iiint\{\varepsilon\cdot p^{-1}\cdot\eta^{2m}+\varepsilon^{-r}\cdot q^{-1}\cdot(M\log n)^{2m}\}\times \\
&\qquad\qquad\times g\{|w^{\wedge}_{p,q+\nu}|^2+|w^{\wedge}_{pq}|^2\}dt\,d\xi\,d\eta \\
&\leqq \text{Const.}\ \varepsilon\{(Lw_{p,q+\nu}, w_{p,q+\nu})+(Lw_{pq}, w_{pq})\},
\end{aligned}
$$

if n is sufficienly large (notice that $\log n\sim\log\langle\xi\rangle$ in the support of $w^{\wedge}_{pq}(t;\xi,\eta)$ with $|p+q|\leqq N_n$).

Thus we obtain, for any $\varepsilon>0$,

$$
(3.4)\qquad (Lw_{pq}, w_{pq})\leqq\varepsilon\sum_{0\leqq|\nu|\leqq 2m}(Lw_{p,q+\nu}, w_{p,q+\nu})+\varepsilon^{-1}\|c^n_{pq}h_{pp}\|^2+\varepsilon\|w_{pq}\|^2,
$$

if n is sufficiently large. Now let us sum up (3.4) with respect to (p, q) satisfying $|p+q|\leqq N_n-2m$. Then the first terms on right hand side of (3.4) will be absorbed into the left hand side (by taking $\varepsilon>0$ sufficiently small). Namely we have

$$
(3.5)\qquad \sum_{|p+q|\leqq N_n}(Lw_{pq}, w_{pq}) = O(n^{-2s})+\varepsilon\sum_{|p+q|\leqq N_n}\|w_{pq}\|^2,
$$

for any positive number s. (To see (3.5), notice that we may assume

$$
\sum_{N_n-2m\leqq|p+q|\leqq N_n}(Lw_{pp}, w_{pp}) = O(n^{-2s}),
$$

by taking M sufficiently large. C.f. lemma 1 of [2].)

On the other hand, it follows from Poincare 's inequality

$$(Lw_{pp}, w_{pq}) \geqq ||D_t^m w_{pq}||^2 \geqq \text{Const.}\ ||w_{pq}||^2 .$$

So it holds: for any $s>0$, there exists a constant M such that

$$S_n^M(\chi u) = \sum_{|p+q|\leqq N_n} ||w_{pq}||^2 = O(n^{-2s}) .$$

The proof of Proposition 2 is now complete. □

§4. Proof of Theorem 3

The proof of Theorem 3 will be quite analogous to that of Theorem 2. So we sketch it and point out the difference.

Proof of Theorem 3. The same arguments as in the proof of Proposition 1 together with the astumptions (A.1), (A.3) and (A.5) yield

$$\int g(t)(\log\langle\xi\rangle)|v(t)|^2 dt \leqq \varepsilon \left| \int P_\zeta v(t)\cdot\overline{v(t)}dt \right| \tag{4.1}$$

and

$$\int f(t)(\log\langle\eta\rangle)^{2m}|v(t)|^2 dt \leqq \varepsilon \left| \int P_\zeta v(t)\cdot\overline{v(t)}dt \right| , \tag{4.2}$$

where we denote $P_\zeta = D_t^{2m} + f(t)\xi^{2m} + ig(t)\eta$. In order to prove Theorem 3, we have only to obtain, fro any $\varepsilon>0$,

$$\begin{aligned}|(Pw_{pq}, w_{pq})| \leqq \varepsilon \sum_{0\leqq|\nu|\leqq 2m} &|(Pw_{p,q+\nu}, w_{p,q+\nu})| \\ &+\varepsilon^{-1}||c_{pq}^n h_{pq}||^2 + \varepsilon||w_{pq}||^2 ,\end{aligned} \tag{4.3}$$

if n is sufficiently large. (For the above notations, recall §3.) We shall show this, in the region $\Lambda_1 = \{(\xi, \eta); |\eta| \leqq 2|\xi|\}$, applying (4.1). One can show (4.3) in the region $\Lambda_2 = \{(\xi, \eta)\}; |\xi| \leqq 2|\eta|\}$, applying (4.2) and the same arguments as in the preceding section.

First observe that the argument in the first part of §3 yields

$$\begin{aligned}|(Pw_{pq}, w_{pq})| \leqq \sum_{0<|\nu|\leqq 2m} &(M\log n)^{|\nu|}|(P^{(\nu)}w_{p,q+\nu}, w_{pq})| \\ &+\varepsilon^{-1}||c_{pq}^n h_{pq}||^2 + \varepsilon||w_{pq}||^2 .\end{aligned} \tag{4.4}$$

We are going to estimate the first terms on right Sand side of the inequality (4.4).

1′) For $\nu=(k, 0)$ with $1\leqq k\leqq 2m$ (i.e. $P^{(\nu)}=\text{const.} f(t)D_x^{2m-k}$), we can do in the same way as 1) in the preceding section.

2′) For $\nu=(0, 1)$ (i.e. $P^{(\nu)}=ig(t)$), we can see the following: The inequality (4.1) together with the fact that $c^{-1}\cdot n \leqq |\xi| \leqq c\cdot n$ for $(\xi, \eta)\in \text{supp}\,[\alpha_n]$ will yield

$$\begin{aligned}&(M\log n)^{|\nu|}|(P^{(\nu)}w_{p,q+\nu}, w_{pq})| \\ &= (M\log n)^{|\nu|}\left|\iiint g(t)\hat{w}_{p,q+\nu}(t;\xi,\eta)\cdot\overline{\hat{w}_{pq}(t;\xi,\eta)}dt\,d\xi\,d\eta\right|\end{aligned}$$

$$\leqq (M \log n)^{|\nu|} \{ \iiint g(|w^{\wedge}_{p,q+\nu}|^2 + |w^{\wedge}_{pq}|^2) dt\, d\xi\, d\eta \}$$

$$\leqq \varepsilon \{ |(Pw_{p,q+\nu}, w_{p,q+\nu})| + |(Pw_{pq}, w_{pq})| \}$$

if n is sufficiently large (notice that $|\nu|=1$).

Thus we obtain (4.3) and with this the proof of Theorem 3 is complete. □

§5. Appendix

Here we present an extension of theorem 2 in the preceding paper [3]. We consider the operator (in $\boldsymbol{R}^{l+1}$) of the form

$$L = D_t^2 + \sum_{j,k=1}^{l} a_{jk}(t) D_{x_j} D_{x_k} \tag{5.1}$$

with real-valued C^∞ functions $a_{jk}(t)$ satisfying $a_{jk}(t) = a_{kj}(t)$ for $j, k = 1, \cdots, l$. Now let us denote $A(t) = (a_{jk}(t))$, $\zeta = {}^t(\xi_1, \cdots, \xi_l)$ and introduce

$$\begin{cases} \lambda_1(t) = \inf\limits_{|\zeta|=1} (A(t)\zeta, \zeta) \\ \lambda_2(t) = \sup\limits_{|\zeta|=1} (A(t)\zeta, \zeta) . \end{cases} \tag{5.2}$$

Then we have:

Theorem 4. *Let L be a differential operator of the form* (5.1). *Assume:*

(i) $\lambda_1(0)=0$. $\lambda_1(t)>0$ *for* $t \neq 0$.

(ii) $\lambda_1(t) \in C^1(\boldsymbol{R})$. *Both of $\lambda_1(t)$ and $\lambda_2(t)$ are monotone increasing for* $0<t<\delta$, *and monotone decreasing for* $-\delta<t<\delta$.

(iii) $$\lim_{t\to 0} \sqrt{\lambda_2(t)}\, |t \log \lambda_1(t)| = 0 .$$

Then L is hypoelliptic in $\boldsymbol{R}^{l+1}$.

Proof. At first, let us observe some basic inequalities (see (5.4), (5.5) and (5.6) below) necessary for the proof. Set

$$\begin{aligned} L_\zeta &= -\frac{d^2}{dt^2} + \sum_{j,k=1}^{l} a_{jk}(t) \xi_j \xi_k \\ &= -\frac{d^2}{dt^2} + (A(t)\zeta, \zeta) . \end{aligned} \tag{5.3}$$

Then the definition (5.2) implies

$$\begin{aligned} \int L_\zeta v(t) \cdot \overline{v(t)} dt &= \int \{ |v'(t)|^2 + (A(t)\zeta, \zeta) |v(t)|^2 \} dt \\ &\geqq \int \{ |v'(t)|^2 + \lambda_1(t) |\zeta|^2 |v(t)|^2 \} dt . \end{aligned}$$

We can see by the argument in section 3 of [3] that the assumptions (i), (ii) and (iii) yield

$$\begin{aligned}(5.4)\qquad &\int \lambda_2(t)(\log\langle\zeta\rangle^2)|v(t)|^2dt\\ &\leqq \varepsilon\int\{|v'(t)|^2+\lambda_1(t)|\zeta|^2|v(t)|^2\}dt\\ &\leqq \varepsilon\int L_\zeta v(t)\cdot\overline{v(t)}dt\,.\end{aligned}$$

On the other hand, from the fact that $A(t)$ is positive swmi-definite, it follows

$$(5.5)\qquad \begin{cases} 0\leqq a_{ij}(t)\leqq\lambda_2(t)\,, & j=1,\cdots,l\\ |a_{jk}(t)|\leqq\sqrt{a_{jj}(t)\cdot a_{kk}(t)}\leqq\lambda_2(t)\,, & \text{for}\quad j\neq k\end{cases}$$

and

$$\begin{aligned}(5.6)\qquad &\left|\frac{\partial}{\partial\xi_j}(A(t)\zeta,\zeta)\right|\leqq 2\sqrt{a_{jj}(t)}\sqrt{(A(t)\zeta,\zeta)}\\ &\qquad\leqq 2\sqrt{\lambda_2(t)}\sqrt{(A(t)\zeta,\zeta)}\,,\qquad j=1,\cdots,l\,.\end{aligned}$$

Now, let us explain our proof. To prove Theorem 4, we have only to obtain, for any $\varepsilon>0$,

$$\begin{aligned}(5.7)\qquad (Lw_{pq},w_{pq})\leqq\varepsilon\sum_{0\leqq|\nu|\leqq2}(Lw_{p,q+\nu},w_{p,q+\nu})\\ +\varepsilon^{-1}\|c^n_{pq}h_{pq}\|^2+\varepsilon\|w_{pq}\|^2\,,\end{aligned}$$

if n is sufficiently large. (For the above notations, recall §3.) Observe now that the argument in the first part of §3 yields

$$\begin{aligned}(5.8)\qquad (Lw_{pq},w_{pq})\leqq\sum_{0<|\nu|\leqq2}(M\log n)^{|\nu|}|(L^{(\nu)}w_{p,q+\nu},w_{pq})|\\ +\varepsilon^{-1}\|c^n_{pq}h\|^2+\varepsilon\|w_{pq}\|^2\,.\end{aligned}$$

In order to show the inequality (5.7), let us estimate the first terms on right hand side of (5.8) in the following way:

1″) For $|\nu|=2$, the inequalities (5.4) and (5.5) together with the fact that $c^{-1}\cdot n\leqq|\zeta|\leqq c\cdot n$ for $\zeta\in\operatorname{supp}[\alpha_n]$ will yield

$$\begin{aligned}&\sum_{|\nu|=2}(M\log n)^{|\nu|}|(L^{(\nu)}w_{p,q+\nu},w_{pq})|\\ &\leqq 2(M\log n)^2\sum_{|\nu|=2}\iint\sum_{j,k=1}^{l}|a_{jk}(t)|\,|\hat{w}_{p,q+\nu}(t;\zeta)\cdot\overline{\hat{w}_{pq}(t;\zeta)}|\,dtd\zeta\\ &\leqq\text{Const.}\sum_{|\nu|=2}\iint\lambda_2(t)(\log\langle\zeta\rangle)^2(|\hat{w}_{p,q+\nu}|^2+|\hat{w}_{pq}|^2)dtd\zeta\\ &\leqq\varepsilon\sum_{|\nu|=2}\{(Lw_{pq},w_{pq})+(Lw_{p,q+\nu},w_{p,q+\nu})\}\end{aligned}$$

if n is sufficiently large.

2″) For $|\nu|=1$, the inequalities (5.4) and (5.6) together with the fact that $c^{-1}\cdot n\leqq|\zeta|\leqq c\cdot n$ for $\zeta\in\operatorname{supp}[\alpha_n]$ will yield

$$\sum_{|\nu|=1}(M\log n)^{|\nu|}|(L^{(\nu)}w_{p,q+\nu},\ w_{pq})|$$

$$\leqq (M\log n)\sum_{|\nu|=1}\iint\sum_{j=1}^{l}\left|\frac{\partial}{\partial\xi_j}(A(t)\zeta,\ \zeta)\right||\hat{w}_{p,q+\nu}(t;\ \zeta)\cdot\overline{\hat{w}_{pq}(t;\ \zeta)}|\,dt\,d\zeta$$

$$\leqq \text{Const.}\sum_{|\nu|=1}\iint\{\varepsilon^{-1}\lambda_2(t)(\log\langle\zeta\rangle)^2+\varepsilon(A(t)\zeta,\ \zeta)\}(|\hat{w}_{p,q+\nu}|^2+|\hat{w}_{pq}|^2)\,dt\,d\zeta$$

$$\leqq \text{Const.}\ \varepsilon\sum_{|\nu|=1}\{(Lw_{pq},\ w_{pq})+(Lw_{p,q+\nu},\ w_{p,q+\nu})\}$$

if n is sufficiently large.

Thus we obtain (5.7) and this proves Theorem 4. □

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