J. Math. Kyoto Univ. (JMKYAZ)
29-3 (1989) 463–487

Almost periodic solutions of one dimensional wave equations with periodic coefficients

By

Masaru YAMAGUCHI

1. Introduction

Let Ω be an open interval $(0, \pi)$. Consider the following one dimensional wave equation:

$$\Box u+\varepsilon(p(t)u+q(t)\partial_t u)=f(x, t) \qquad \text{in} \quad (x, t)\in\Omega\times R^1, \tag{1.1}$$

where $\Box$ is the D'Alembertian $\partial_t^2-\partial_x^2$, $p(t)$ and $q(t)$ are periodic with period $2\pi/\omega$ and f is quasiperiodic in t (For definition see §2), and ε is a small parameter. We impose initial and boundary conditions:

$$u(x, 0) = \phi(x)$$
$$\partial_t u(x, 0) = \psi(x), \qquad x\in\Omega \tag{1.2}$$
$$u(0, t) = u(\pi, t) = 0, \qquad t\in R. \tag{1.3}$$

The aim of this paper is to show that *all the solutions of the initial boundary value problem* (*IBVP*) (1.1)–(1.3) *are almost periodic under some incommensurability conditions between the autonomous almost periodic oscillation and the forcing term* f (See Theorem 3.1 and Theorem 3.4).

We remark that this paper mainly deals with *the case where* (*i*) *the ratio of the period* $2\pi/\omega$ *and the length* π *of* Ω *has an irrational ratio, and* (*ii*) $\int_0^{2\pi/\omega} q(t)dt$ *is equal to zero.* In the rational ratio case or in the damped case there are very many works on qualitative behavior of solutions of the wave equations with periodic terms (even with semilinear terms) (see Vejvoda [7]). However it seems to the author that both in the irrational ratio case and in the undamped case they are scarecely investigated in detail. Especially (ii) contains the case $q(t)\equiv 0$ (i.e. the undamped case).

As is known from Hill's operator theory (Magnus and Winkler [4]) even a scalar ordinary differential equation $\ddot{u}+a^2u+p(t)u=0$ has unbounded solutions in $t\in R^1$ if a^2 belongs to one of instability zones. Thus the IBVP (1.1)–(1.3) is not necessarily expected to have time-bounded solutions, much less almost periodic

---

Communiacted by Prof. S. Mizohata December 7, 1987

solutions. As for this direction we know only the results of Vaghi [6]. Vaghi showed that the IBVP (1.1)—(1.3) has a unique almost periodic solution for almost periodic f, under the assumption that the solution is a-priori bounded. But generally it is difficult to show the boundedness of the solutions in the undamped case. As we show, the problem is of very subtle character. To clarify this structure, our starting point is the work owe to [5,11]. Secondly, it needs number-theoretic relations of the period of p and q, the basic frequencies of f and the approximate eigenvalues of $-\partial_x^2$.

Finally we note that different from other works, the coefficient $q(t)$ of the damping term can take both positive and negative values, more precisely the damping effect is determined by the signature of the mean $\int_0^{2\pi/\omega} q(t)dt$ but not the signature of q itself.

In §2 notation and definitions are introduced. §3 deals with the IBVP (1.1)—(1.3) and main theorems are shown. In §4 some examples are given in which the incommensurability conditions are not satisfied. In such cases the solutions are generally unbounded in time $t \in R^1$. In §5 we mention some results on semilinear wave equations with periodic coefficients which are the extension of Yamaguchi [9,10]. In Appendix we explain some results on second order ordinary differential equations with periodic coefficients, mainly owe to [5,11].

I would like to express my sincere gratitude to Professors S. Mizohata and S. Miyatake for their valuable advices and to Professor T. Kakita for his constant encouragement and advices.

2. Notation and Definitions

Let B be one of the following sets: the real numbers R^1, the nonnegative numbers R^+, an interval $[0, T]$, an m-dimensional torus $T^m = [0, 2\pi]^m$ and any products of them. Let X be a Banach space. Let $C^k(B, X)$ be the class of k-times continuously differentiable mappings of B into X. We set $C(B, X) = C^0(B, X)$. Let $L^2(\Omega)$, $H^p(\Omega)$ and $H_0^p(\Omega)$ are the usual Lebesgue space and the Sobolev spaces (resp.) with norms $|\cdot|_{L^2}$ and $|\cdot|_{H^p}$. $K^p(\Omega)$ $(p \geq 1)$ is a subspace of $H^p(\Omega)$ whose elements u satisfy $\partial_x^{2\alpha} u \in H_0^1(\Omega)$, $0 \leq \alpha \leq [(p-1)/2]$. $K^p(\Omega)$ is a Banach space with norm $|\cdot|_{H^p}$. We set, for $s \geq 1$,

$$|u(t)|_{E,s} = |u(t)|_{H^s} + |\partial_t u(t)|_{H^{s-1}}.$$

Definision 2.1. A function $f(t, a) \in C(R^1 \times R^b, X)$ is called a ξ-quasiperiodic (ξ-$q.p.$) function in t if there exist a function $\mathring{f}(\theta, a) \in C(T^m \times R^b, X)$ and a vector $\xi = (\xi_1, \cdots, \xi_m) \in R^m$ such that $f(t, a) = \mathring{f}(\xi t, a)$. The function $\mathring{f}(\theta, a)$ is called the corresponding function ($c.p.$ function) to $f(t, a)$ and is always denoted by a $\circ$ over the $q.p.$ function $f(t, a)$. Without loss of generality we assume that the numbers $\xi_1, \cdots, \xi_m$ are rationally independent. The numbers $\xi_1, \cdots, \xi_m$ are called basic frequencies ($b.f.$) of $f(t, a)$.

Definition 2.2. A matrix function $M(t)$ on R^1 is called ξ-*q.p.* if every component $m_{ij}(t)$ of $M(t)$ is ξ-*q.p.* in t. We define a norm of ξ-*q.p.* matrix $M(t)$ by

$$|M|_0=\max_i \sum_j \max_{\theta\in T^m} |\mathring{m}_{ij}(\theta)|\,.$$

Definition 2.3. A function $f\in C(R^1, X)$ is almost periodic in X if for any $\varepsilon>0$ there exists a relatively dense set $\{\tau\}$ such that

$$|f(t+\tau)-f(t)|_X<\varepsilon$$

for all $t\in R^1$.

Remark 2.4. (i) It is well-known [1] that for any almost periodic function $f(t)$ in X there exists a real sequence $\{\mu_k\}$ such that $\hat{f}(\alpha)=\lim_{T\to\infty}(1/T)\int_0^T f(t)\exp(-i\alpha t)dt$ is equal to zero if $\alpha\neq\mu_k$. We call $\sum_{k=1}^{\infty} f_k \exp(i\mu_k t)$ the *Fourier series* of f, where $f_k=\hat{f}(\mu_k)$. The above $\{\mu_k\}$ is called the *Fourier exponents of f. If the Fourier series is uniformly convergent, then it is identically equal to* $f(t)$. More generally, given an almost periodic function

$$f(t)\sim\sum_{k=1}^{\infty} f_k \exp(i\mu_k t)\,,$$

there exists a sequence of trigonometric polynomials $\{\sum_{k=1}^{n} r_{k,m} f_k \exp(i\mu_k t)\}_{m=1}^{\infty}$, $n=n(m)$, which converges uniformly to $f(t)$. The numbers $r_{k,m}$ are rational and depend on μ_k and m, but not on f_k.
(ii) Two distinct almost periodic functions have distinct Fourier series.
(iii) If a sequence of almost periodic functions in X uniformly converges to a function, then the limit function is almost periodic in X.

3. Almost periodic solutions and quasiperiodic solutions of IBVP (1.1)—(1.3)

In this section we deal with the IBVP:

$$\Box u+\varepsilon(p(t)u+q(t)\partial_t u)=f(x, t) \quad \text{in} \quad (x, t)\in\Omega\times R^1\,, \tag{1.1}$$

$$u(x, 0)=\phi(x)$$

$$\partial_t u(x, 0)=\psi(x)\,, \qquad x\in\Omega \tag{1.2}$$

$$u(0, t)=u(\pi, t)=0\,, \qquad t\in R^1\,. \tag{1.3}$$

We assume:
(A1) $p(t)$ and $q(t)$ are $(2\pi/\omega)$-periodic and C^∞-functions.
(A2) The frequency ω satisfies the following incommensurabilty condition:

$$|n\omega-m|\geqq C/n$$

for some $C>0$, all $n\in N$ and all $m\in Z$.
(A3) ϕ and ψ belong to $K^s(\Omega)$ and $K^{s-1}(\Omega)$ (resp.), where $s\geqq 3$ is an integer.

From now on through this paper s is a fixed integer ≥ 3.

Remark 3.1. (A2) is satisfied for a dense set in R^1. This set contains all irrational solutions of algebraic equations with rational coefficients of degree 2 (see [2]).

Existence and uniqueness of the global solution is known for the IBVP (1.1)—(1.3) (see for example [3]):

Proposition. *Assume* ($A1$) *and* ($A3$). *Let* f *be an element of* $C(R^1, K^{s-1}(\Omega))$. *Then the IBVP* (1.1)—(1.3) *has a unique solution in* $\bigcap_{i=0}^{2} C^i(R^1, K^{s-i}(\Omega))$.

Now we show the representation theorem of solutions of IBVP (1.1)—(1.3) which plays an essential role in investigating various properties of behavior of the solutions. To this end we use the modification of the result in [5]. *See Appendix A, Part II for more detailed version,* but we very shortly state the difference for explanation.

Remark 3.2. Condiser a second-order ODE:

$$(*) \qquad \ddot{u}+(\lambda^2+r(t))u = 0\,,$$

where $r(t)$ is $q.p.$ in t with basic frequencies $\omega=(\omega_1, \cdots, \omega_m)$ and $\mathring{r}(\theta)$ is l-times differentiable in $\theta\in T^m$ and λ^2 is a positive parameter. One can construct a linear transformation $V(\theta)$ and a constant $\lambda_0>0$ so that Eq. (*) is reduced to

$$\ddot{v}+\mu^2 v = 0 \qquad (\mu = \mu(\lambda)\in R^1) \quad \text{by} \quad \begin{bmatrix} v \\ \dot{v} \end{bmatrix} = V(\omega t)\begin{bmatrix} u \\ \dot{u} \end{bmatrix},$$

provided that $\lambda\geq\lambda_0$ and λ does not belong to a set $G\subset \bigcup_{n\in Z^m}\{\lambda\colon |\lambda-(n, \omega)/2|\leq C/(1+|(n, \omega)|)$ for some constant $C>0$.

In the above result λ_0 is given by $c|\mathring{r}|_l/M^0$ for a given small M^0, where c is a constant depending only on l and m (see Lemma 1, (5), (6) in [5]). In our case $r(t)$ is replaced by $\varepsilon p(t)$, m is equal to 1 and λ^2 is equal to k^2. In this case also the argument of [5] holds. In fact, we have εP_k instead of P_k in (5), (6) in [5]. If $\lambda^2\geq 1$, then the above reduction holds for any ε in $[-\hat{\varepsilon}_0, \hat{\varepsilon}_0]$, where $\hat{\varepsilon}_0=M^0/(c|\mathring{r}|_l)$. Here C, $\mu(\lambda)$ and λ_0 in the above are replaced by C_ε, $\mu_\varepsilon(\lambda)$ and 1 (resp.), It holds that $\lim_{\varepsilon\to 0} C_\varepsilon=0$ and $\lim_{\varepsilon\to 0}\mu_\varepsilon(\lambda)=\lambda$. We note: $\lim_{\varepsilon\to 0} C_\varepsilon=0$ implies that one can take $C_\varepsilon<C/2$ for small ε, where C is the constant given in the assumption (A2). That is, (A2) is the condition that all k, the roots of the eigenvalues k^2, do not belong to the exceptional intervals.

Thus we have obtained the Green function of (*).

Using this function we respresent the solution of nonhomogeneous ODE:

$$\ddot{v}+\lambda^2 v+\varepsilon(p(t)v+q(t)\dot{v}) = f(t)$$

(See Theorem A.1 and A.2).

Expanding the solution u of (1.1)—(1.3) into Fourier series formally, we have an infinite number of second order ODEs of the form (*). Applying Theorem A.1 and A.2 and the energy estimates, we have the representation theorem.

Before stating the theorem, we note the following example concerning the condition (A2). By Hill theory it is clear that the above G with $m=1$ contains all instability zones of the Hill equation $\ddot{z}+(\lambda^2+\varepsilon p(t))z=0$.

Example Consider IBVP (1.1)—(1.3) with $f(x, t)\equiv 0$ and $q(t)\equiv 0$:

$$\begin{aligned} &\Box u+\varepsilon p_0(t)u = 0\,, \qquad (x, t)\in\Omega\times R^1\,,\\ &u|_{\partial\Omega} = 0\,, \qquad t\in R^1\,,\\ &u(0) = \phi\,, \quad \partial_t u(0) = \psi\,, \qquad x\in\Omega\,. \end{aligned}$$

Using the continued fractions ([2]), one can take ω so as to satisfy

$$1/(2k_j^{n+3})\leq m_j-k_j\omega/2\leq 1/k_j^{n+3}\,.$$

for two suitable sequences of natural numbers $\{k_j\}$ and $\{m_j\}$, where k_j, $m_j\to\infty$ as $j\to\infty$. Then (A2) does not hold. One may assume without loss of generality that $\{k_j\}$ consists of odd numbers ([2]). Let $p_0(t)$ be a negative-valued $2\pi/\omega$-periodic function whose n-th derivative is as follows:

$$p_0^{(n)}(t) = \begin{cases} -1 & 0\leq t<\pi/\omega\\ 1 & -\pi/\omega\leq t<0\,. \end{cases}$$

Let $\varepsilon>0$. Then it holds (cf. Theorem 4.1 and 4.2 in [12]) that the endpoints α_k and β_k of the k-th instability interval $J_k=[\alpha_k, \beta_k]$ have the property:

$$\alpha_k = k\omega/2+\sum_{j=0}^{[(n+2)/2]} a_{2j+1}^{\varepsilon}/(2k)^{2j+1}-\varepsilon\,|e_k^{(n)}|/(2k)^{n+1}+\delta_k^{\varepsilon}/k^{n+2}$$

$$\beta_k = k\omega/2+\sum_{j=0}^{[(n+2)/2]} a_{2j+1}^{\varepsilon}/(2k)^{2j+1}+\varepsilon\,|e_k^{(n)}|/(2k)^{n+1}+\delta_k'^{\varepsilon}/k^{n+2}\,,$$

where a_{2j+1}^{ε} are real constants depending on ε and j, but not on k and $a_{2j+1}^{\varepsilon}=O(\varepsilon)$, ($\varepsilon\to 0$), the real sequences $\{\delta_k^{\varepsilon}\}$ and $\{\delta_k'^{\varepsilon}\}$ belong to l^2 and δ_k^{ε}, $\delta_k'^{\varepsilon}=O(\varepsilon)$ ($\varepsilon\to 0$), and $e_k^{(n)}=(\omega/2\pi)\int_0^{2\pi/\omega} e^{-ik\omega t}p_0^{(n)}(t)dt$. It holds

$$e_k^{(n)} = (i/k)\{1+(-1)^{k-1}\}\,.$$

Since

$$a_1^{\varepsilon} = (\omega\varepsilon/2\pi)\int_0^{2\pi/\omega}(-p_0(t))\,dt \quad ([12])$$

and $p_0(t)<0$, we have $a_1^{\varepsilon}>0$. Hence for sufficiently large k_j one can take $\varepsilon>0$ suitably small so that m_j belongs to J_{k_j}. Thus if $p_{m_j}=(\phi, \sin m_jx)_{L^2}\neq 0$ or $q_{m_j}=(\psi, \sin m_jx)_{L^2}\neq 0$, then the solution of the above IBVP is unbounded in $t\in R^1$, for the solution of $\ddot{z}_j+(m_j^2+\varepsilon p_0(t))z_j=0$, $z_j(0)=p_j$ or $\dot{z}_j(0)=q_j$ is unbounded in t.

Theorem 3.1. *Assume (A1), (A2) and (A3). Let f be an element of $C(R^1, K^{s-1}(\Omega))$. Then there exists $\varepsilon_0>0$ such that for any $\varepsilon\in[-\varepsilon_0, \varepsilon_0]$ the solution of (1.1)–(1.3) is of the form*

$$\begin{bmatrix} u(x, t) \\ \partial_t u(x, t) \end{bmatrix} = \sum_{k=1}^{\infty} \begin{bmatrix} u_k(t) \\ \dot{u}_k(t) \end{bmatrix} \sin kx, \tag{3.1}$$

$$\begin{bmatrix} u_k(t) \\ \dot{u}_k(t) \end{bmatrix} = \exp\left(-\varepsilon/2 \int_0^t q(\tau)d\tau\right) \begin{bmatrix} 1 & 0 \\ (-\varepsilon/2)q(t) & 1 \end{bmatrix}$$

$$\times J_k(t)B_k(t)\left[\int_0^t (J_k(\tau)B_k(\tau))^{-1}g_k(\tau)d\tau + J_k(0)^{-1}z_k\right], \tag{3.2}$$

$$J_k(t) = \begin{bmatrix} 1 & 1 \\ ik & -ik \end{bmatrix} V_k(t), \tag{3.3}$$

where

(i) each $V_k(t)$ is a regular $2\pi/\omega$-periodic 2×2 matrix depending on k, ε, p and q but not on ϕ, ψ and f,

(ii) $B_k(t)$ is equal to $\begin{bmatrix} \exp(ia_k t) & 0 \\ 0 & \exp(-ia_k t) \end{bmatrix}$. Here each a_k is a real constant depending on ε, p and q, satisfying $\lim_{\varepsilon\to 0} a_k = k$. It also satisfies an asymptotic property

$$|a_k - k| \leq \hat{C}/k \tag{3.4}$$

for some constant $\hat{C}$ and any $k\in N$,

(iii) $g_k(t)$ is equal to $\begin{bmatrix} 0 \\ (f, \sin kx)_{L^2} \exp\left(\varepsilon/2 \int_0^t q(\tau)d\tau\right) \end{bmatrix}$, and z_k is equal to $\begin{bmatrix} 1 & 0 \\ -(\varepsilon/2)q(0) & 1 \end{bmatrix} \begin{bmatrix} (\phi, \sin kx)_{L^2} \\ (\psi, \sin kx)_{L^2} \end{bmatrix}$.

Proof. The functions f, ϕ and ψ have the Fourier expansions:

$$f(\cdot, t) = \sum_{k=1}^{\infty} f_k(t) \sin kx \quad \text{in} \quad K^{s-1}(\Omega), \tag{3.5}$$

$$\phi = \sum_{k=1}^{\infty} p_k \sin kx, \quad \psi = \sum_{k=1}^{\infty} q_k \sin kx \quad \text{in } K^s(\Omega) \text{ and } K^{s-1}(\Omega) \text{ (resp.)}, \tag{3.6}$$

where $f_k(t)=(f(t), \sin kx)_{L^2}$, $p_k=(\phi, \sin kx)_{L^2}$ and $q_k=(\psi, \sin kx)_{L^2}$. We look for a solution of the form:

$$u(\cdot, t) = \sum_{k=1}^{\infty} u_k(t) \sin kx. \tag{3.7}$$

Substituting (3.5)–(3.7) into (1.1) and (1.2), we obtain

$$\ddot{u}_k + k^2 u_k + \varepsilon(p(t)u_k + q(t)\dot{u}_k) = f_k(t), \tag{3.8}$$

$$u_k(0) = p_k, \quad \dot{u}_k(0) = q_k, \qquad k = 1, 2, \cdots \tag{3.9}$$

Applying Theorem A.1 to the problem (3.8) and (3.9), there exists $\varepsilon_0>0$ such that for any $\varepsilon\in[-\varepsilon_0, \varepsilon_0]$ (3.8) and (3.9) has a unique solution of the form (3.2) and (3.3). We show the convergence of the series (3.7) in $\bigcap_{i=0}^{2} C^i(R^1, K^{s-i}(\Omega))$. First we show the following estimate:

$$\begin{aligned}
&k^2|u_k(t)|^2+|\dot u_k(t)|^2\\
&\quad\le C\exp\Big(-\varepsilon\int_0^t q(\tau)d\tau\Big)\Big[\Big(\int_0^t \exp\Big(\varepsilon/2\int_0^T q(\nu)d\nu\Big)|f_k(\tau)|d\tau\Big)^2\\
&\qquad +k^2p_k^2+q_k^2\Big],
\end{aligned}\tag{3.10}$$

where C is a positive constant not depending on k, f and (p_k, q_k). Set $|z|_2^2=|z_1|^2+|z_2|^2$ for $z=(z_1, z_2)$. From (A.11) in Theorem A.2 in Appendix A we observe

$$\begin{aligned}
&k^2|u_k(t)|^2+|\dot u_k(t)|^2\\
&=\left|\begin{bmatrix}k&0\\0&1\end{bmatrix}\begin{bmatrix}u_k(t)\\ \dot u_k(t)\end{bmatrix}\right|_2^2\\
&=\Big|\begin{bmatrix}k&0\\0&1\end{bmatrix}\exp\Big(-\varepsilon/2\int_0^t q(\tau)d\tau\Big)\begin{bmatrix}1&0\\(-\varepsilon/2)q(t)&1\end{bmatrix}J_k(t)B_k(t)\\
&\quad\times\Big[\int_0^t (J_k(\tau)B_k(\tau))^{-1}\exp\Big(\varepsilon/2\int_0^T q(\nu)d\nu\Big)g_k(\tau)d\tau+J_k(0)^{-1}z_k\Big]\Big|_2^2\\
&\le 4\exp\Big(-\varepsilon\int_0^t q(\tau)d\tau\Big)\left|\begin{bmatrix}k&0\\0&1\end{bmatrix}\begin{bmatrix}1&0\\(-\varepsilon/2)q(t)&1\end{bmatrix}J_k(t)B_k(t)\right|_0^2\\
&\quad\times\Big|\int_0^t (J_k(\tau)B_k(\tau))^{-1}\exp\Big(\varepsilon/2\int_0^T q(\nu)d\nu\Big)g_k(\tau)d\tau+J_k(0)^{-1}z_k\Big|_2^2\\
&\le 8\exp\Big(-\varepsilon\int_0^t q(\tau)d\tau\Big)\left|\begin{bmatrix}k&0\\0&1\end{bmatrix}\right|_0^2\left|\begin{bmatrix}1&0\\(-\varepsilon/2)q(t)&1\end{bmatrix}\right|_0^2|V_k(t)|_0^2|B_k(t)|_0^2\\
&\quad\times\Big(\Big|\int_0^t (J_k(\tau)B_k(\tau))^{-1}\exp\Big(\varepsilon/2\int_0^T q(\nu)d\nu\Big)g_k(\tau)d\tau\Big|_2^2+|J_k(0)^{-1}z_k|_2^2\Big)\\
&\le c_0^2c_4k^2\exp\Big(-\varepsilon\int_0^t q(\tau)d\tau\Big)\Big[\Big(\int_0^t \exp\Big(\varepsilon/2\int_0^T q(\nu)d\nu\Big)|(J_k(\tau)B_k(\tau))^{-1}g_k(\tau)|_2d\tau\Big)^2\\
&\quad+|J_k(0)^{-1}z_k|_2^2\Big],
\end{aligned}$$

where we used (A.9) and (A.10) in Theorem A.1 in Appendix, and $c_0, \cdots, c_3$ are the same constants as in Theorem A.1 and c_4 is a positive constant depending on p, q and ε_0. We have

$$|(J_k(t)B_k(t))^{-1}g_k(t)|_2\le c_1c_5(1/k)|f_k(t)|$$

and

$$|(J_k(0)^{-1}z_k|_2^2\le c_1^2c_6(1/k^2)(k^2p_k^2+q_k^2)$$

for suitable positive constants c_5 and c_6 depending on q and ε_0, whence (3.10) is proved. Now we show the convergence of (3.7). Let u_{pq} equal to $\sum_{k=p}^{q} u_k(t)\sin kx$. Using (3.10) we have

$$|\partial_x^s u_{pq}(\cdot, t)|_{L^2}^2 + |\partial_x^{(s-1)}\partial_t u_{pq}(\cdot, t)|_{L^2}^2$$
$$= \sum_{k=p}^{q} (k^{2s}|u_k(t)|^2 + k^{2(k-1)}|\dot{u}_k(t)|^2)$$
$$\leq \text{Const.} \exp(-\varepsilon \int_0^t q(\tau)d\tau)$$
$$\times [\sum_{k=p}^{q} k^{2(s-1)}(\int_0^t \exp(\varepsilon/2 \int_0^{\tau} q(\nu)d\nu)|f_k(\tau)|d\tau)^2 + \sum_{k=p}^{q}(k^{2s}p_k^2 + k^{2(s-1)}q_k^2] .$$

Applying the Minkowski inequality

$$\sum_{r=1}^{a} |\int_B h_r(x)dx|^2 \leq [\int_B (\sum_{r=1}^{a} |h_r(x)|^2)^{1/2} dx]^2 ,$$

we have

$$|\partial_x^s u_{pq}(\cdot, t)|_{L^2}^2 + |\partial_x^{(s-1)}\partial_t u_{pq}(\cdot, t)|_{L^2}^2$$
$$\leq \text{Const.} \exp(-\varepsilon \int_0^t q(\tau)d\tau)$$
$$\times [(\int_0^t \exp(\varepsilon/2 \int_0^{\tau} q(\nu)d\nu)(\sum_{k=p}^{q} k^{2(s-1)}|f_k(\tau)|^2)^{1/2} d\tau)^2$$
$$+ \sum_{k=p}^{q}(k^{2s}p_k^2 + k^{2(s-1)}q_k^2)] .$$

Since f belongs to $C(I, K^{s-1}(\Omega))$,

$$\int_0^t \exp(\varepsilon/2 \int_0^{\tau} q(\nu)d\nu)(\sum_{k=p}^{q} k^{2(s-1)}|f_k(\tau)|^2)^{1/2} d\tau$$
$$\leq T \max_{0\leq t\leq T} \exp(\varepsilon/2 \int_0^t q(\tau)d\tau) \max_{0\leq t\leq T} (\sum_{k=p}^{q} k^{2(s-1)}|f_k(t)|^2)^{1/2}$$
$$\to 0 \quad \text{as } p, q \to \infty .$$

Hence it follows

$$|\partial_x^s u_{pq}(\cdot, t)|_{L^2} + |\partial_x^{(s-1)} u_{pq}(\cdot, t)|_{L^2}$$
$$\to 0 \quad \text{uniformly in } t \in [0, T] \quad \text{as} \quad p, q \to \infty .$$

Also it follows, by (3.8),

$$|\partial_x^{(s-2)}\partial_t^2 u_{pq}(\cdot, t)|_{L^2}^2 = \sum_{k=p}^{q} k^{2(s-2)}|\ddot{u}_k(t)|^2$$
$$= \sum_{k=p}^{q} k^{2(s-2)}|k^2 u_k + \varepsilon(p(t)u_k + q(t)\dot{u}_k) - f_k(t)|^2$$
$$\leq 4\sum_{k=p}^{q}(k^{2s}|u_k|^2 + \varepsilon^2 p(t)^2 k^{2(s-1)}|u_k|^2 + \varepsilon^2 q(t)^2 k^{2(s-1)}|\dot{u}_k|^2 + k^{2(s-1)}|f_k(t)|^2)$$
$$\leq \text{Const.} \sum_{k=p}^{q}(k^{2s}|u_k|^2 + k^{2(s-1)}|\dot{u}_k|^2 + k^{2(s-1)}|f_k|^2)$$
$$\to 0 \quad \text{uniformly in } t \in [0, T] \quad \text{as} \quad p, q \to \infty ,$$

where the constant does not depend on k, u and f. Thus $u = \sum_{k=1}^{\infty} u_k(\cdot) \sin kx$

converges in $\bigcap_{i=0}^{2} C^i([0, T], K^{s-i}(\Omega))$. By the similar argument to the above that this solution is unique in $\bigcap_{i=0}^{2} C^i([0, T], K^{s-i}(\Omega))$. Since T is an arbitrary positive number and f belongs to $C(R^1, K^{s-1}(\Omega))$ we have $u \in \bigcap_{i=0}^{2} C^i(R^+, K^{s-i}(\Omega))$. We also have $u \in \bigcap_{i=0}^{2} C^i((-\infty, 0], K^{s-i}(\Omega))$. Hence u belongs to $\bigcap_{i=0}^{2} C^i(R^1, K^{s-i}(\Omega))$. The asymptotic property of $\{a_k\}$ immediately follows from Theorem A.1. Q.E.D.

Theorem 3.2. *Assume* (A1), (A2) *and* (A3). *Let* $f(x, t)$ *identically vanish. Let* ε_0 *and* ε *be the same as in Theorem* 3.1 *and* b *equal to* $\varepsilon \int_0^{2\pi/\omega} q(t)dt$. *Then it holds:*

(i) *If b is positive, then the solution u in Theorem* 3.1 *satisfies*

$$|u(\cdot, t)|_{E,s} \leq C \exp((-b/2)t) .$$

(ii) *If b is equal to zero, then u is almost periodic in $K^s(\Omega)$ and $\partial_t u$ is almost periodic in $K^{s-1}(\Omega)$.*

(iii) *If b is negative, then*

$$\overline{\lim_{t\to\infty}} |u(\cdot, t)|_{E,s} = \infty .$$

Remark 3.3. As is seen from Thoerem 3.1, the almost periodic solution in Theorem 3.2 (ii) has the Fourier exponents contained in a set $\{a_k\}_{k=1}^{\infty} \cup \{\omega l\}_{l=1}^{\infty}$ (see Remark 2.4).

The proof of Theorem 3.2 is clear from Theorem 3.1.

We investigate the behavior of solutions in the case where $f(\cdot, t)$ is *q.p.* In this case we have to take into account the interaction between the autonomous almost periodic oscillation and the forcing term f (i.e. between the Fourier exponents in Remark 3.3 and the b.f. of f).

Assume that

(f1) The function $f(\cdot, t)$ is ξ-*q.p.* ($\xi \in R^m$) and its corresponding function $\mathring{f}(\cdot, \theta)$ belongs to $C^\infty(T^m, K^{s-1}(\Omega))$.

(f2) The $(m+1)$-dimensional vector $\eta = (\omega, \xi)$ and the sequence $\{a_k\}$ satisfy, for $r \geq m+2$ and some $K > 0$,

$$(*) \qquad |a_k - (\eta, l)| \geq K|l|^{-r}$$

for all $l \in Z^{m+1} \setminus \{0\}$ and any k which satisfies $(f, \sin kx)_{L^2} \neq 0$.

Remark 3.4. (i) The assumption (f2) is satisfied in many applications (see Proposition B.1).

(ii) In (f2), the restriction "$r \geq m+2$" is not necessary in order that Theorem 3.3 and 3.4 hold. However if $r < m+2$, then the set of $\eta \in R^{m+1}$ satisfying (f2) may be the empty set.

(iii) In (f1) we assumed that f is of C^∞-class, *for simplicity*. However as will be seen from the following proof of Thoerem 3.3, it is enough to assume that $f \in C^{\rho}(T^m, K^{s-1}(\Omega))$, where $\rho = (m+2)([r+2]+1)$ and r is in (f2). Thus weaker incommensurability needs more differentiability.

Theorem 3.3. *Assume* ($A1$), ($A2$), ($f1$) *and* ($f2$). *Let* ε_0 *be the same constant as in Theorem* 3.1. *Let* $\int_0^{2\pi/\omega} q(t)dt$ *be equal to zero. Then for any* $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ *the boundary value problem* (1.1) *and* (1.3) *has only one* η*-q.p. solution in* $\bigcap_{i=0}^{2} C^i(R^1, K^{s-i}(\Omega))$.

Theorem 3.4. *Assume* ($A1$), ($A2$), ($A3$), ($f1$) *and* ($f2$). *Let* $\int_0^{2\pi/\omega} q(t)dt$ *be equal to zero. Then the solution* $u(\cdot, t)$ *in Theorem* 3.1 *of the IBVP* (1.1)—(1.3) *is almost periodic in* $K^s(\Omega)$ *and* $\partial_t u(\cdot, t)$ *is almost periodic in* $K^{s-1}(\Omega)$. *Their Fourier exponents are contained in a set* $\{(\eta, l): l \in Z^{m+1}\} \cup \{a_k\}$.

Example 3.1. Consider the IBVP. Let the ω of p and q be any irrational solution of an algebraic equation with rational coefficients of degree 2. Then ω satisfies (A2). By Theorem 3.1 the sqeuence $\{a_k\}$ is set up and satisfies $\lim_{\varepsilon \to 0} a_k = k$ and $a_k = k + O(1/k)$ $(k \to \infty)$. Let f equal to $\sin \mu t \sin jx$, $\mu \in R^1, j \in N$. We can take μ so as to satisfy, for some $K > 0$,

$$|a_j - (l_0\omega + l_1\mu)| \geq K/(|l_0| + |l_1|)^3$$

for any $l_0 \in Z$ and any $l_1 \in Z$, $|l_0| + |l_1| \neq 0$. This is possible for a suitably large j, for a_k satisfies the above asymptotic relation. Thus all assumptions (A1), (A2), (A3), (f1) and (f2) are satisfied. Hence by Theorem 3.4 the solution is almost periodic.

Remark 3.5. More generally than Example 3.1, even in case where f is of the form of the finite Fourier series

$$f = \sum_{k \in X} f_k(t) \sin kx \quad (X \text{ is a suitable finite set of } Z),$$

where $f_k(t)$ is ξ-*q.p.* for any $k \in X$, the assumptions in Theorem 3.4 are all satisfied for a suitable choice of ξ by the similar arguments to the above.

Remark 3.6. In Theorem 3.3 and 3.4, if $\int_0^{2\pi/\omega} q(t)dt \neq 0$ is assumed, then we need not take into account the resonances between autonomous oscillation and the forcing term (See Yamaguchi [8]). In this case the high differentiability of $\mathring{f}$ is not necessary. The solution u of (1.1)—(1.3) is of the form $u_1 + \exp(-bt)u_2$, where b is equal to $\varepsilon/2 \int_0^{2\pi/\omega} q(t)dt$, u_1 is η-*q.p.* and u_2 is almost periodic in $K^s(\Omega)$.

We prove Theorem 3.3 and 3.4. We prepare the following lemma:

Lemma 3.1. *Assume* ($A1$), ($A2$), ($f1$) *and* ($f2$). *Then for any* $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ *Eq.* (3.8) *has a unique* η*-q.p. solution* $u_k(t)$ *satisfying*

$$k^2|u_k(t)|^2+|\dot{u}_k(t)|^2\leq C\|\mathring{f}_k\|_{(m+1)P}\,,$$

where P is any positive integer, $C>0$ is a constant depending on P, p and q, and $\|\mathring{f}\|_j$ is equal to

$$[\sum_{|\sigma|\leq j}(\int_{T^m}|D_\theta^\sigma\mathring{f}(\theta)|\,d\theta)^2]^{1/2}.$$

Proof of Lemma 3.1. From Theorem 3.1 we have

$$\begin{bmatrix}u_k(t)\\ \dot{u}_k(t)\end{bmatrix}=\begin{bmatrix}1 & 0\\ (-\varepsilon/2)q(t) & 1\end{bmatrix}J_k(t)\exp(-\varepsilon/2\int_0^t q(\tau)d\tau)$$
$$\times[(1/2ik)\int_0^t B_k(\tau)^{-1}\zeta_k(\tau)d\tau+B_k(t)J_k(0)^{-1}z]\,, \tag{3.11}$$

where $\zeta_k(t)=\begin{bmatrix}\zeta_k^1(t)\\ \zeta_k^2(t)\end{bmatrix}$ and $\zeta_k^j(t)=\exp(\varepsilon/2\int_0^t q(\tau)d\tau)f_k(t)[(V_k(t)^{-1})_{j1}-(V_k(t)^{-1})_{j2}]$, $j=1, 2$. The functions $\zeta_k^j(t)$ are clearly η-$q.p.$ and their corresponding functions are in $C^{(m+1)P}$-class on T^{m+1}. Expanding $\zeta_k^j(t)$ into Fourier series

$$\zeta_k^j(t)=\sum_{l\in Z^{m+1}}\chi_l^j\exp(i(\eta, l)t)$$

and substituting these into the above integral in (3.11), we have

$$\begin{aligned}&\int_0^t\zeta_k^j(\tau)\exp((-1)^j ia_k\tau)d\tau\\ &=\int_0^t\exp((-1)^j ia_k\tau)\sum_l\chi_l^j\exp(i(\eta, l)\tau)d\tau\\ &=\sum_l\chi_l^j\int_0^t\exp(i((\eta, l)+(-1)^j a_k)\tau)d\tau\\ &=\sum_l\chi_l^j/\{i((\eta, l)+(-1)^j a_k)\}[\exp(i((\eta, l)+(-1)^j a_k)t)-1].\end{aligned}$$

Hence we have

$$[\quad]\text{ in (3.11)}=(1/2ik)[\alpha_k(t)-B_k(t)(\alpha_k(0)-2ikJ_k(0)^{-1}z)]\,,$$

where $\alpha_k(t)$ is equal to $\sum_l\rho_l\exp(i(\eta, l)t)$, $\rho_l=\begin{bmatrix}\rho_l^1\\ \rho_l^2\end{bmatrix}$ and $\rho_l^j=\chi_l^j/\{i((\eta, l)+(-1)^j a_k)\}$.

Hence from (3.11) we have the formal solution of the form:

$$\begin{bmatrix}u_k(t)\\ \dot{u}_k(t)\end{bmatrix}=\begin{bmatrix}1 & 0\\ (-\varepsilon/2)q(t) & 1\end{bmatrix}J_k(t)\exp(-\varepsilon/2\int_0^t q(\tau)d\tau)$$
$$\times(1/2ik)[\alpha_k(t)+B_k(t)(-\alpha_k(0)+2ikJ_k(0)^{-1}z)]\,. \tag{3.12}$$

We have to show the convergence of the formal series $\alpha_k(t)$. Plainly

$$|\alpha_k(t)|_2^2\leq\sum_{j=1}^2(\sum_l|\chi_l^j|/|(\eta, l)+(-1)^j a_k|)^2\,.$$

Using (*) and $|\chi_l^j|\leq\text{Const.}\,\|\zeta_k^j\|_{(m+1)P}/\prod_{i=1}^m(1+|l_i|)^P$, we obtain

$$|\alpha_k(t)|_2^2 \leq \text{Const.}\,(\sum_l \prod_{i=1}^{m+1}(1+|l_i|)^{-P+r})^2 \sum_{j=1}^{2} \|\zeta_k^j\|_{(m+1)P}^2$$
$$\leq \text{Const.} \sum_{j=1}^{2} \|\zeta_k^j\|_{(m+1)P}^2$$

for $P \geq r+2$. Thus we have shown that $\alpha_k(t)$ is η-$q.p.$ and

$$|\alpha_k(t)|_2^2 \leq \text{Const.} \sum_{j=1}^{2} \|\zeta_k^j\|_{(m+1)P}^2 ,$$

where Const. s depend on m, p, q, K and P. Now we choose initial values a and b so as to satisfy

$$-\alpha_k(0)+2ikJ_k(0)^{-1}z = 0$$

i.e.

$$\begin{bmatrix} a \\ b \end{bmatrix} = (1/2ik)\begin{bmatrix} 1 & 0 \\ (-\varepsilon/2)q(0) & 1 \end{bmatrix} J_k(0)\alpha_k(0) .$$

Then from (3.12) the solution is η-$q.p.$ It is clear that η-$q.p.$ solution is unique. Q.E.D.

Proof of Theorem 3.3. Again using the Fourier expansion method, we have Eq. (3.8). From the hypotheses in Theorem 3.3 all assumptions in Lemma 3.1 are satisfied. Hence Lemma 3.1 implies that each Eq. (3.8) has a unique η-$q.p.$ solution $u_k(t)$ satisfying

$$k^2|u_k(t)|^2+|\dot{u}_k(t)|^2 \leq \text{Const.}\,\|\mathring{f}_k\|_{(m+2)\tilde{p}} , \qquad t\in R^1 \tag{3.13}$$

where $\tilde{p}$ is any integer $\geq r+2$. Setting $\mathring{u}_{pq}(\theta)=\sum_{k=p}^{q}\mathring{u}_k(\theta)\sin kx$ and using the Minkowski inequality and (3.13), we have

$$\begin{aligned}
&|\partial_x^s u_{pq}(\cdot, t)|^2+|\partial_x^{s-1}u_{pq}(\cdot, t)|^2 \\
&\leq \text{Const.} \sum_{k=p}^{q} k^{2(s-1)}\|\mathring{f}_k\|_{(m+2)\tilde{p}}^2 \\
&\leq \text{Const.}\,\{\int_{T^m}(\sum_{k=p}^{q}\sum_{|\sigma|\leq(m+2)\tilde{p}} k^{2(s-1)}|D_\theta^\sigma \mathring{f}_k(\theta)|^2)^{1/2}d\theta\}^2 \\
&\leq \text{Const.}\sup_{\theta\in T^m}\sum_{k=p}^{q}(\sum_{|\sigma|\leq(m+2)\tilde{p}} k^{2(s-1)}|D_\theta^\sigma \mathring{f}_k(\theta)|^2) \\
&\to 0 \quad \text{as} \quad p, q \to \infty ,
\end{aligned}$$

for (f1) holds. Hence $\sum_{k=1}^{\infty} u_k(t)\sin kx$ uniformly converges with respect to the norm $|\partial_x^s u(\cdot, t)|+|\partial_x^{s-1}\dot{u}(\cdot, t)|$ so that $\sum_{k=1}^{\infty} u_k(t)\sin kx$ converges in $\bigcap_{i=0}^{2} C^i(R^1, K^{s-i}(\Omega))$. The uniqueness of η-$q.p.$ solution is shown in the similar way. Q.E.D.

Proof of Theorem 3.4. From all assumptions in Theorem 3.4 all the hypotheses in Theorem 3.1 follow. Hence the IBVP (1.1)—(1.3) has a unique global solution in $\bigcap_{i=0}^{2} C^i(R^1, K^{s-i}(\Omega))$ of the form (3.1)—(3.3). Let u_1 be the η-$q.p.$ solution in Theorem 3.3 and v be the solution in $\bigcap_{i=0}^{2} C^i(R^1, K^{s-i}(\Omega))$ of (1.1) with $f(x, t)\equiv 0$ and initial data $v(\cdot, 0)=\phi-u_1(\cdot, 0)$ and $\partial_t v(\cdot, 0)=\psi-\partial_t u_1(\cdot, 0)$. By Theorem 3.1 and Theorem 3.2 the solution is almost periodic in $K^s(\Omega)$, and its Fourier exponents are contained in the set $\{(\eta, l): l\in Z^{m+1}\} \cup \{a_k\}$. Then $u=u_1+v$ is the solution in $\bigcap_{i=0}^{2} C^i(R^1, K^{s-i}(\Omega))$ of the IBVP (1.1)—(1.3). Thus the theorem is proved. Q.E.D.

4. Remark

We have assumed (f1) and (f2). The essential assumption is (f2). Improving Theorem 3.4, in this section we consider the necessity of (f2). Hereafter we restricted ourselves to the case $\varepsilon=0$. In this case $a_k=k$. This makes our consideration easier. As will be seen in the following example, if $|a_k-(\eta, l)|=0$ holds for some k and l, then there exists an unbounded solution of IBVP (1.1)—(1.3).

Example 4.1. Consider IBVP for (1.1) with $\varepsilon=0$:

$$\begin{aligned} &\Box u = f(x, t)\,, \qquad (x, t)\in\Omega\times R^1 \\ &u(x, 0) = 0\,, \quad \partial_t u(x, 0) = 0\,, \qquad x\in\Omega \\ &u(0, t) = u(\pi, t) = 0\,, \qquad t\in R^1\,, \end{aligned}$$

where

$$f(x, t) = 2\sum_{j=1}^{\infty} (\cos jt \sin jx)/(j-1)!\,.$$

Clearly $f(x, t)$ is 2π-periodic in t. Then $m=1$, $l\in Z^2$, ω is arbitrary and $\xi_1=1$. Hence $a_k-(\eta, l)=0$ with $k=1$, $l_0=0$ and $l_1=1$. It is easy to show that

$$u(x, t) = \sum_{j=1}^{\infty} (t \sin jt \sin jx)/j!$$

is a unique solution of the above IBVP which is unbounded in $t\in R^1$.

As is mentioned in Remark 3.4 (iii), it is enough to assume only finite order of differentiability of f for the almost-periodicity of the solutions of (1.1)—(1.3). However if the differentiability of f is smaller or the irrationality of η in (f2) is less, then the solution may become unbounded in t for suitable choice of f. Indeed, the following example gives such an unbounded solution.

Example 4.2. Consider IBVP for (1.1) with $\varepsilon=0$:

$$\begin{aligned} &\Box u = f(x, t)\,, \qquad (x, t)\in\Omega\times R^1 \\ &u(x, 0) = \phi(x)\,, \quad \partial_t u(x, 0) = \psi(x)\,, \qquad x\in\Omega \\ &u(0, t) = u(\pi, t) = 0\,, \qquad t\in R^1\,. \end{aligned}$$

Since $\varepsilon=0$, a_k is equal to k. Without loss of generality we can assume $\phi=\psi=0$. We deal with the case where (f2) is broken but $a_k-(\eta, l)\neq 0$ for all k and l. Using the continued fractions (Khinchin [2]) we can take $\xi(>0)$ and the sequences $\{k_n\}$ and $\{m_n\}$ of natural numbers such that

$$C/k_n^N \leq |k_n\xi - m_n| \leq C_0/k_n^N$$

for some positive constants C and C_0, and a natural number $N>r$. By this (f2) is violated. We set $\alpha_n = k_n\xi - m_n$. Also without loss of generality (if necessary, take the subsequences of $\{k_n\}$ and $\{m_n\}$) we assume that the sequence $\{\alpha_n\}$ is positive ([2]) and monotonically decreasing. Note that $\{m_n\}$ contains an infinite number of odd numbers ([2]). Choose a pair of the subsequences of $\{k_n\}$ and $\{m_n\}$ such that (i) for a fixed $x_0 \in \Omega$ it holds $\sin m_n x_0 \geq \delta > 0$ (or $\sin m_n x_0) \leq -\delta < 0$) for all n and some constant $\delta>0$, and (ii) $k_n \leq M k_{n+1}$ for all n, where $M=(C/2C_0)^{1/2}$. Here we again have written the subsequences as $\{k_n\}$ and $\{m_n\}$, and without loss of generality we assume $\sin m_n x_0 \geq \delta$ for all n. We define, for some positive integer p,

$$f(x, t) = \sum_{n=1}^{\infty} (1/k_n^p) \sin (k_n \xi t) \sin m_n x .$$

The function f is of $C^{p-1}(\Omega \times R^1)$ and is $2\pi/\xi$-periodic in t. We set

$$u = \sum_{n=1}^{\infty} \frac{1}{m_n} [\int_0^t f_{k_n}(\tau) \sin m_n(t-\tau) d\tau] \sin m_n x ,$$

where $f_{k_n}(t) = (1/k_n^p) \sin k_n \xi t$. Then it is easy to show that the above u is a unique solution of IBVP. We have

$$u(x, t) = \operatorname{Im} \sum_{n=1}^{\infty} 1/(2ik_n^p m_n) \left\{ \frac{\exp (i(k_n\xi - m_n)t) - 1}{i(k_n\xi - m_n)} \exp (im_n t) \right.$$
$$\left. - \frac{\exp (i(k_n\xi + m_n)t) - 1}{i(k_n\xi + m_n)} \exp (-im_n t) \right\} \sin m_n x .$$

The second term of the above bracket { } contributes to the almost periodic part of u. Hence we consider

$$v = \operatorname{Im} \sum_{n=1}^{\infty} 1/(ik_n^p m_n) \frac{\exp (i\alpha_n t) - 1}{i\alpha_n} \exp (im_n t) \sin m_m x .$$

Decompose v into $v_1 + v_2$, where

$$v_1 = \operatorname{Im} \sum_{n \leq n_0 - 1} 1/(ik_n^p m_n) \frac{\exp (i\alpha_n t) - 1}{i\alpha_n} \exp (im_n t) \sin m_n x$$

and

$$v_2 = \operatorname{Im} \sum_{n > n_0 - 1} 1/(ik_n^p m_n) \frac{\exp (i\alpha_n t) - 1}{i\alpha_n} \exp (im_n t) \sin m_n x .$$

Here $n_0 = n_0(t)$ is the minimum number n which satisfies

$$\alpha_n t \leq \pi/2$$

for any given $t>0$. Setting $r_\nu = [\alpha_\nu^{-1}/4]$ and $t_\nu = 2\pi r_\nu$, $\nu = 1, 2, \cdots$, we have

$$v(x, t_\nu) = \sum_{n=1}^{\infty} 1/(k_n^p m_n) \frac{\sin \alpha_n t_\nu}{\alpha_n} \sin m_n x\,, \qquad \nu = 1, 2, \cdots$$

Also

$$\begin{aligned} t_\nu \alpha_\nu &= 2\pi r_\nu \alpha_\nu \\ &= 2\pi[\alpha_\nu^{-1}/4]\alpha_\nu \leq \pi/2 \end{aligned}$$

and

$$\begin{aligned} t_\nu \alpha_{\nu-1} &= 2\pi[\alpha_\nu^{-1}/4]\alpha_{\nu-1} \\ &> 2\pi(\alpha_\nu^{-1}/4-1)\alpha_{\nu-1} \\ &\geq 2\pi(\alpha_\nu^{-1}/8)\alpha_{\nu-1} \\ &\geq (\pi/4)(C/C_0)(k_\nu^N/k_{\nu-1}^N) \\ &\geq (\pi/4)(C/C_0)(1/M)^N \\ &\geq (\pi/4)(C/C_0)(2C_0/C) \\ &= \pi/2\,, \end{aligned}$$

whence $n_0(t_\nu)=\nu$.

Now we estimate v_1 and v_2. We have

$$\begin{aligned} |v_1| &\leqq \sum_{n\leq\nu-1} 1/(k_n^p m_n)(1/\alpha_n) \\ &\leq (1/C)(2/\xi) \sum_{n\leq\nu-1} k_n^{N-p-1} \\ &\leq (2/C\xi) \sum_{n=1}^{\nu-1} (M^{N-p-1})^n k_\nu^{N-p-1} \\ &\leq (2/C\xi)(M^{N-p-1}/(1-M^{N-p-1}))k_\nu^{N-p-1} \end{aligned}$$

Next noting that

$$\sin \alpha_n t/(\alpha_n t) \geq \min_{0<x\leq\pi/2} (\sin x/x)(\equiv L)$$

for $0<\alpha_n t\leq\pi/2$, and $\sin m_n x\geq\delta$, $x=x_0$, we have

$$\begin{aligned} v_2 &= \sum_{n=\nu}^{\infty} 1/(k_n^p m_n) \frac{\sin \alpha_n t_\nu}{\alpha_n t_\nu} t_\nu \sin m_n x \\ &\geq \sum_{n=\nu}^{\infty} (1/k_n^{p+1})(L\delta t_\nu/\xi) \\ &= (2\pi L\delta/\xi) \sum_{n=\nu}^{\infty} (1/k_n^{p+1})[\alpha_\nu^{-1}/4] \\ &\geq (2\pi L\delta/\xi) \sum_{n=\nu}^{\infty} (1/k_n^{p+1})(\alpha_\nu^{-1}/4-1) \\ &\geq (\pi L\delta/4\xi) \sum_{n=\nu}^{\infty} (1/k_n^{p+1})(k_\nu^N/C_0) \\ &\geq (\pi L\delta/4C_0\xi)k_\nu^{N-p-1}\,. \end{aligned}$$

Hence taking N suitably large, M^{N-p} is sufficiently small so that

$$\nu = \nu_1+\nu_2 \geq \text{Const. } k_\nu^{N-p-1} ,$$

where Const. is positive and independent of ν. Since

$$2\pi(k_\nu^N/4C_0-1) \leq t_\nu < 2\pi k_\nu^N/(4C) ,$$

it holds $k_\nu \sim t_\nu^{1/N}$. Hence

$$\nu \geq \text{Const. } t_\nu^{1-(p+1)/N} .$$

This implies that $u(x_0, t_\nu)$ tends to infinity as $\nu \to \infty$; i.e. u is unbounded in t.

The case where ϕ and ψ are not identically zero is quite clear from the above result, for ϕ and ψ only contribute to the periodic part of solutions.

5. Semilinear wave equations with periodic potential and periodic damping

In this section we deal with the following problem:

$$\Box u+\varepsilon(p(t)u+q(t)\partial_t u) = \mu F(x, t, u, \partial_x u, \partial_t u), \qquad (x, t) \in \Omega \times R^+ \tag{1.1$_1$}$$

$$u(x, 0) = \phi(x) , \quad \partial_t u(x, 0) = \psi(x) , \qquad x \in \Omega \tag{1.2}$$

$$u(x, t)|_{\partial\Omega} = 0 , \qquad t \in R^+ \tag{1.3}$$

and

$$\Box u+\varepsilon(p(t)u+q(t)\partial_t u)+G(x, u, \partial_x u, \partial_t u) = \mu F(x, t, u, \partial_x u, \partial_t u) , \quad (x, t) \in \Omega \times R^+ \tag{1.1$_2$}$$

$$u(x, t)|_{\partial\Omega} = 0 , \qquad t \in R^+ , \tag{1.3}$$

where μ is a small parameter. For brevity we set $s \geq 3, f(t, u(t)) = f(\cdot, t, u(\cdot, t), \partial_x u(\cdot, t), \partial_t u(\cdot, t))$ and $u(\cdot, t) = u(t)$ etc.

We assume the following:

(A) The conditions (A1), (A2) and (A3) in §3 are satisfied. For ε_0 given in Theorem 3.1 we fix ε so as to satisfy $|\varepsilon| \leq \varepsilon_0$.

(F) The function $F(x, t, a)$, $a=(a_1, a_2, a_3)$, is defined on $\bar{\Omega} \times R^+ \times R^3$.

(i) The F is in C^{s+1}-class in (x, a). The F and its derivatives $\partial_x^k \partial_a^l F(x, t, a)$, $0 \leq k+l \leq s+1$, are continuous in $t \in R^+$.

(ii) For any $u \in \bigcap_{i=0}^{1} C^i(R^+, K^{s-i}(\Omega))$ $F(t, u(t))$ belongs to $C(R^+, K^{s-1}(\Omega))$.

(iii) $\int_0^\infty \max_{(x,a) \in \bar{\Omega} \times B(Q)} |\partial_x^k \partial_a^l F(x, t, a)| dt$ converges for every pair (k, l), $0 \leq k+l \leq s+1$, and $Q>0$, where $B(Q)$ is a bounded domain in R^3 of the form $\{a \in R^3 : |a_i| \leq Q, i=1, 2, 3\}$. We set $y(t; M) = \max_{0 \leq k+l \leq s+1} \max_{(x,a) \in \bar{\Omega} \times B(M)} |\partial_x^k \partial_a^l F(x, t, a)|$.

(G) The function $G(x, a)$ is defined on $\bar{\Omega} \times R^3$.

(i) The G is in C^{s+2}-class in (x, a).

(ii) $G(x, 0) = \partial_a G(x, 0) = \partial_a^2 G(x, 0) = 0$.

(iii) For any $u\in\bigcap_{i=0}^{1}C^{i}(R^{+}, K^{s-i}(\Omega))$ $G(u(t))$ belongs to $C(R^{+}, K^{s-1}(\Omega))$.

Theorem 5.1. *Assume (A), (G) and (F). Let $\int_{0}^{2\pi/\omega}q(t)dt$ be equal to zero. Let $y(t; M)$ satisfy*

$$y(t;\ M)\leq H(M)/(t+1)^{1+\alpha}\qquad \text{for any}\quad M>0\,,$$

where $H(M)$ is a constant depending on M and α is a constant ≥ 1. Then there exists $\mu_{0}>0$ such that for any $\mu\in[-\mu_{0}, \mu_{0}]$ BVP $(1.1)_{2}$ and (1.3) has a time-decaying solution satisfying

$$|u(t)|_{H^{s}}+|\dot{u}(t)|_{H^{s-1}}\leq c_{0}/(t+1)^{\alpha}\,,$$

where c_{0} is a constant.

Theorem 5.2. *Assume (A) and (F). Let $\int_{0}^{2\pi/\omega}q(t)dt$ be equal to zero. Let $M_{0}>0$ be given and let (ϕ, ψ) satisfy $|\phi|_{H^{s}}+|\psi|_{H^{s-1}}\leq M_{0}$. Then for any constant $M>M_{0}$ there exists a positive constant $\mu_{0}=\mu_{0}(F, M, M_{0})$ such that for any $\mu\in[-\mu_{0}, \mu_{0}]$, IBVP $(1.1)_{1}$–(1.3) has a unique classical bounded solution in $\bigcap_{i=0}^{2}C^{i}(R^{+}, K^{s-i}(\Omega))$ satisfying*

$$|u(t)|_{H^{s}}+|\dot{u}(t)|_{H^{s-1}}\leq CM\qquad \text{for}\quad t\in R^{+}\,,$$

where $C>1$ is a constant depending on p and q. The solution has a form $u_{0}+u_{1}$. Here u_{0} and u_{1} satisfy the following:

(i) *u_{0} and u_{1} belong to $\bigcap_{i=0}^{2}C^{i}(R^{+}, K^{s-i}(\Omega))$.*

(ii) *$u_{0}(t)$ is almost periodic in $K^{s}(\Omega)$ and its Fourier exponents are contained in the set $\{a_{k}\}\cup\{\omega l\colon l\in Z\}$.*

(iii) *$u_{1}(t)$ satisfies*

$$|u_{1}(t)|_{H^{s}}+|\dot{u}_{1}(t)|_{H^{s-1}}\leq \text{Const.}\int_{t}^{\infty}y(\tau;\ CM)d\tau\,.$$

The proofs of Theorem 5.1 and Theorem 5.2 are done in the similar way to Theorem 4.1 and Theorem 4.5 in [10].

Proof of Theorem 5.2. We have only to show the energy estimate of solutions of the linear problem:

$$\square u+\varepsilon(p(t)u+q(t)\partial_{t}u)=f(x,\ t)\quad \text{with (1.2) and (1.3)}\,.$$

Let f belong to $C(R^{+}, K^{s-1}(\Omega))$. Then the problem becomes (3.8) and (3.9) in §3. The solution of (3.8) and (3.9) satisfies the estimate (3.10) in §3. So we have

$$k^{2}u_{k}(t)^{2}+\dot{u}_{k}(t)^{2}\leq \text{Const.}\,[(\int_{0}^{t}|f_{k}(\tau)|d\tau)^{2}+(k^{2}p_{k}^{2}+q_{k}^{2})]\,,$$

where Const. does not depend on k, f_k and (p_k, q_k). Hence using the Minkowski inequality, we have

$$\sum_{k=1}^{\infty}(k^{2s}u_k(t)^2+k^{2(s-1)}\dot{u}_k(t)^2)$$
$$\leq \text{Const.}\,[(\int_0^t(\sum_{k=1}^{\infty}k^{2(s-1)}f_k(\tau)^2)^{1/2}d\tau)^2+\sum_{k=1}^{\infty}(k^{2s}p_k^2+k^{2(s-1)}q_k^2)]\,.$$

Hence it follows

$$|u(t)|_{H^s}+|\dot{u}(t)|_{H^{s-1}}\leq \text{Const.}\,[\int_0^t|f(\tau)|_{H^{s-1}}d\tau+|\phi|_{H^s}+|\psi|_{H^{s-1}}]\,.$$

Let

$$\begin{bmatrix} v_k(t) \\ \dot{v}_k(t) \end{bmatrix}=\exp(-\varepsilon/2\int_0^t q(\tau)d\tau)\begin{bmatrix} 1 & 0 \\ (-\varepsilon/2)q(t)) & 1 \end{bmatrix}J_k(t)B_k(t)$$
$$\times[\int_0^{\infty}(J_k(\tau)B_k(\tau))^{-1}g_k(\tau)d\tau+J_k(0)^{-1}z_k]\,.$$

and

$$\begin{bmatrix} w_k(t) \\ \dot{w}_k(t) \end{bmatrix}=\exp(-\varepsilon/2\int_0^t q(\tau)d\tau)\begin{bmatrix} 1 & 0 \\ (-\varepsilon/2)q(t) & 1 \end{bmatrix}J_k(t)B_k(t)$$
$$\times\int_t^{\infty}(J_k(\tau)B_k(\tau))^{-1}g_k(\tau)d\tau\,.$$

Then $v_k(t)$ and $w_k(t)$ are the solutions of Eq.s.

$$\ddot{u}_k+k^2u_k+\varepsilon(p(t)u_k+q(t)\dot{u}_k)=0$$

and

$$\ddot{u}_k+k^2u_k+\varepsilon(p(t)u_k+q(t)\dot{u}_k)=f_k(t)$$

(resp.). Similar calculations to (3.10) imply:

$$k^2v_k(t)^2+\dot{v}_k(t)^2$$
$$\leq \text{Const.}\,[(\int_0^{\infty}|f_k(t)|dt)^2+k^2p_k^2+q_k^2]$$

and

$$k^2w_k(t)^2+\dot{w}_k(t)^2$$
$$\leq \text{Const.}\,(\int_t^{\infty}|f_k(\tau)|d\tau)^2$$

where Const.s do not depend on k. Hence we have

$$\sum_{k=1}^{\infty}(k^{2s}v_k(t)^2+k^{2(s-1)}v_k(t)^2)$$
$$\leq \text{Const.}\,[(\int_0^{\infty}(\sum_{k=1}^{\infty}k^{2(s-1)}f_k(\tau)^2)^{1/2}d\tau)^2+\sum_{k=1}^{\infty}(k^{2s}p_k^2+k^{2(s-1)}q_k^2)]$$

and

$$\sum_{k=1}^{\infty}(k^{2s}w_k(t)^2+k^{2(s-1)}\dot{w}_k(t)^2)$$

$$\leq \text{Const.}\left[\int_t^{\infty}(\sum_{k=1}^{\infty}k^{2(s-1)}f_k(\tau)^2)^{1/2}d\tau\right]^2$$

We set $v=\sum_{k=1}^{\infty}v_k\phi_k$ and $w=\sum_{k=1}^{\infty}w_k\phi_k$. Then v and w satisfy

$$|v(t)|_{H^s}+|\dot{v}(t)|_{H^{s-1}}$$

$$\leq \text{Const.}\left[\int_0^{\infty}|f(t)|_{H^{s-1}}dt+|\phi|_{H^s}+|\psi|_{H^{s-1}}\right]$$

$$|w(t)|_{H^s}+|\dot{w}(t)|_{H^{s-1}}$$

$$\leq \text{Const.}\int_t^{\infty}|f(\tau)|_{H^{s-1}}d\tau\,. \tag{5.1}$$

These show that v and w belong to $\bigcap_{i=0}^{2}C^i(R^1, K^{s-i}(\Omega))$. It is clear that $v(t)$ is almost periodic in $K^s(\Omega)$ and $w(t)$ decays to zero as $t\to\infty$ if $|f(t)|_{H^{s-1}}$ is integrable in $[0, \infty)$. $u=v+w$ is the solution of IBVP.

Now consider the nonlinear problem. Setting $f(t)=\mu F(t, u(t))$ and applying the above estimates of v and w, it follows that the solution of IBVP $(1.1)_1$–(1.3) has the form u_0+u_1 and

$$|u_1(t)|_{H^s}+|\dot{u}_1(t)|_{H^{s-1}}$$

$$\leq \text{Const.}\int_t^{\infty}|\mu F(\tau, u(\tau))|_{H^{s-1}}d\tau$$

$$\leq \text{Const.}\int_t^{\infty}y(\tau; Q)(1+|u(\tau)|_{H^{s-1}})d\tau$$

$$\leq \text{Const.}\int_t^{\infty}y(\tau; Q)d\tau\,.$$

This proves the theorem. Q.E.D.

Proof of Theorem 5.1. We consider the linear problem:

$$\square u+\varepsilon(p(t)u+q(t)\partial_t u)=f(t) \tag{5.2}$$

with (1.3).

We determine p_k and q_k so as to satisfy

$$\int_0^{\infty}(J_k(t)B_k(t))^{-1}g_k(t)dt+J_k(0)^{-1}z_k=0\,. \tag{5.3}$$

Setting $\phi=\sum_{k=1}^{\infty}p_k\phi_k$ and $\psi=\sum_{k=1}^{\infty}q_k\psi_k$, ϕ and ψ belong to $K^s(\Omega)$ and $K^{s-1}(\Omega)$ (resp.). As we showed in the proof of Theorem 5.2, IBVP (5.2), (1.2) and (1.3) has a solution of the form u_0+u_1. However by (5.3) $u_0(t)$ vanishes identically. $u_1(t)$ satisfies (5.1). So we established the existence of time-decaying solution of the linear problem and derived the decay estimate (5.1). Employing Picard's iteration method, we obtain a time-decaying solution of the nonlinear problem $(1.1)_2$–(1.3), provided ε is small. From (5.1) the solution satisfies

$$\begin{aligned}|u(t)|_{H^s}+|\dot{u}(t)|_{H^{s-1}} &\le \text{Const.}\int_t^\infty |\mu F(\tau, u(\tau))-G(u(\tau))|_{H^{s-1}}d\tau \\ &\le \text{Const.}\int_t^\infty 1/(t+1)^{1+\alpha}d\tau\,.\end{aligned}$$

So the theorem is proved. Q.E.D.

Appendix A

In this appendix, first we shall mention some results on second-order ODEs which are used to obtain the representation theorem of the solutions of IBVP (1.1)–(1.3) (Theorem 3.1). These results are based on the method of reduction by Parashuk [5] and Dinaburg-Sinai [11]. (Their methods are originated from Kolmogorov-Arnold-Moser's iteration method.) In the second part we shall briefly explain the method of [5] in our case for the convinience for readers who are further interested in the construction and the structure of the sequence $\{a_k\}$, the exceptional intervals and the transformation matrices V_k.

I. Consider second order ordinary differential equations with real Cauchy data:

$$\text{(A.1)}\qquad \ddot{u}_k+k^2u_k+\varepsilon(p(t)u_k+q(t)\dot{u}_k)=f(t)$$

$$\text{(A.2)}\qquad u(0)=a\,,\quad \dot{u}(0)=b\,,$$

where $p(t)$ and $q(t)$ satisfy the conditions (A1) and (A2) in §3, k is a positive integer and ε is a small parameter.

First we show that (A.1) can be written in another form by using the Liouville transformation and a diagonalization. We write u instead of u_k. Put

$$\text{(A.3)}\qquad u=\exp(-\varepsilon/2\int_0^t q(\tau)d\tau)v\,.$$

Then (A.1) is transformed to

$$\text{(A.4)}\qquad \ddot{v}+k^2v+r(t)v=\exp(\varepsilon/2\int_0^t q(\tau)d\tau)f(t)\,,$$

where $r(t)=\varepsilon(p(t)-(\varepsilon/4)q(t)^2-(1/2)\dot{q}(t))$. Write (A.4) in matrix form

$$\text{(A.5)}\qquad \frac{d}{dt}\begin{bmatrix} v \\ \dot{v}\end{bmatrix}=\begin{bmatrix} 0 & 1 \\ -k^2-r(t) & 0\end{bmatrix}\begin{bmatrix} v \\ \dot{v}\end{bmatrix}+\exp(\varepsilon/2\int_0^t q(\tau)d\tau)f(t)\begin{bmatrix} 0 \\ 1\end{bmatrix}.$$

Ignoring $r(t)$ we diagonalize the above (A.5). Observe that $N=\frac{1}{2}\begin{bmatrix} 1 & (ik)^{-1} \\ 1 & -(ik)^{-1}\end{bmatrix}$ is a diagonalizer of $\begin{bmatrix} 0 & 1 \\ -k^2 & 0\end{bmatrix}$ and $N^{-1}=\begin{bmatrix} 1 & 1 \\ ik & -ik\end{bmatrix}$. Denoting $N\begin{bmatrix} v \\ \dot{v}\end{bmatrix}=\phi$, we obtain

(A.6) $$\dot{\phi} = i\begin{bmatrix} k & 0 \\ 0 & -k \end{bmatrix}\phi + \frac{ir(t)}{2k}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\phi + G_k(t),$$

where $G_k(t) = \exp(\varepsilon/2\int_0^t q(\tau)d\tau)f(t)/2k\begin{bmatrix} -i \\ i \end{bmatrix}$.

The proof of the following theorem is done in the similar way to that of [5]. Note that $r(t)$ contains not only p but $\dot{q}$, q^2 and small parameter ε. However, if we take ε sufficiently small instead of large λ, the iteration is well-done and the argument of Remark 3.2 also holds (See the next part of this appendix).

Theorem A.1. *Assume that $f(t)$ is in C^0-class. Let σ be any natural number. Then there exist $\hat{\varepsilon}_0 = \hat{\varepsilon}_0(p, q, \sigma)$ and $C = C(p, q, \sigma)$ independent of k such that for any $\varepsilon \in [-\hat{\varepsilon}_0, \hat{\varepsilon}_0]$ one can construct an invertible linear transformation*

$$V_k(t): \psi \to \phi, \quad V_k(t+2\pi/\omega) = V_k(t)$$

such that Eq. (A.6) *is reduced to the system*

(A.7) $$\dot{\psi} = i\Gamma_k\psi + H_k(t), \quad \psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}.$$

Here the components of V_k are C^σ-class, and $\Gamma_k = \begin{bmatrix} a_k & 0 \\ 0 & -a_k \end{bmatrix}$, $H_k(t) = V_k(t)^{-1}G_k(t)$, and $a_k = a_k(p, q, \varepsilon)$ is a real constant satisfying

(A.8) $$|a_k - k| \le C/k, \quad \lim_{\varepsilon\to 0} a_k = k.$$

Moreover the matrix V_k satisfies:

(A.9) $$c_0^{-1} \le |V_k|_0 \le c_0, \quad c_1^{-1} \le |V_k^{-1}|_0 \le c_1, \quad c_0, c_1 > 1,$$

and

(A.10) $$|V_k|_\sigma \le c_2, \quad |V_k^{-1}|_\sigma \le c_3,$$

where c_j, $j = 0, \cdots, 3$ are positive constants depending on ε_0, σ, p and q but not on k, (a, b) and f.

Using this theorem, we obtain the representation formula of the solutions of (A.1) and (A.2) as follows:

Theorem A.2. *Under the same assumptions as in Theorem* A.1, *for any $\varepsilon \in [-\hat{\varepsilon}_0, \hat{\varepsilon}_0]$ the Cauchy problem* (A.1) *and* (A.2) *has a unique solution of the form*

(A.11) $$\begin{bmatrix} u_k(t) \\ \dot{u}_k(t) \end{bmatrix} = \exp(-\varepsilon/2\int_0^t q(\tau)d\tau)\begin{bmatrix} 1 & 0 \\ (-\varepsilon/2)q(t) & 1 \end{bmatrix} J_k(t)B_k(t)$$
$$\times[\int_0^t (J_k(\tau)B_k(\tau))^{-1}\exp(\varepsilon/2\int_0^\tau q(\nu)d\nu)F(\tau)d\tau + J_k(0)^{-1}z].$$

Here $J_k(t) = \begin{bmatrix} 1 & 1 \\ ik & -ik \end{bmatrix} V_k(t)$,

$$B_k(t)=\begin{bmatrix}\exp(ia_kt) & 0\\ 0 & \exp(-ia_kt)\end{bmatrix},$$

$$z=\begin{bmatrix}1 & 0\\ (-\varepsilon/2)q(0) & 1\end{bmatrix}^{-1}\begin{bmatrix}a\\ b\end{bmatrix} \quad and \quad F=\begin{bmatrix}0\\ f\end{bmatrix},$$

where V_k and a_k satisfy the same estimates (A.8)–(A.10) *as in Theorem* A.1.

Proof of Theorem A.2. By the transformations (A.3) and V_k we have

$$\text{(A12)} \quad \begin{bmatrix}u_k(t)\\ \dot{u}_k(t)\end{bmatrix}=\exp\left(-\varepsilon/2\int_0^t q(\tau)d\tau\right)\begin{bmatrix}1 & 0\\ (-\varepsilon/2)q(t) & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ ik & -ik\end{bmatrix}V_k(t)\psi .$$

Solving Eq. (A.7) with Cauchy data

$$\psi(0)=V_k(0)^{-1}\begin{bmatrix}1 & 1\\ ik & -ik\end{bmatrix}\begin{bmatrix}1 & 0\\ (-\varepsilon/2)q(0) & 1\end{bmatrix}^{-1}\begin{bmatrix}a\\ b\end{bmatrix},$$

we obtain

$$\psi(t)=\frac{1}{2k}B_k(t)\left[\int_0^t (V_k(\tau)B_k(\tau))^{-1}G_k(\tau)d\tau+\psi(0)\right],$$

whereas

$$\psi(t)=\frac{1}{2k}B_k(t)\Big[\int_0^t (V_k(\tau)B_k(\tau))^{-1}\exp\left(\varepsilon/2\int_0^\tau q(\nu)d\nu\right)f(\tau)\begin{bmatrix}0\\ 1\end{bmatrix}d\tau$$
$$+J_k(0)^{-1}\begin{bmatrix}1 & 0\\ (-\varepsilon/2)q(0) & 1\end{bmatrix}^{-1}\begin{bmatrix}a\\ b\end{bmatrix}\Big].$$

Hence by (A.12) the formula (A.11) follows. Q.E.D.

II. In this part we briefly explain the method of reduction in [5] in our case. From now on we assume that all functions of θ are defined for θ mod. 2π, so 2π-periodic in θ.

Consider a system:

$$\text{(A.13)} \qquad \dot{\phi}=i\lambda J\phi+R(\theta;\varepsilon,\lambda)\phi, \quad \dot{\theta}=\omega,$$

where $\lambda\geq 1$, $J=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$, $R=\frac{i\mathring{r}(\theta)}{2\lambda}\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}$ and $\mathring{r}(\theta)$ is a corresponding function of $r(t)$; i.e. $\mathring{r}(\theta)=\varepsilon(\mathring{p}(\theta)-(\varepsilon/4)\mathring{q}(\theta)^2-(1/2)(\dot{\mathring{q}})(\theta))$. By constructing a linear transformation

$$\text{(A.14)} \qquad \phi=V(\theta;\varepsilon,\lambda)\psi,$$

we will reduce (A.13) to a system with constant coefficients:

$$\dot{\psi}=iaJ\psi, \quad a=a(\varepsilon;\lambda)\in R^1 .$$

By step-by-step reduction, we inductively establish a sequence of linear transformations $\{I+W_j(\theta;\varepsilon,\lambda)\}$ which tends to $V(\theta;\varepsilon,\lambda)$ as $j\to\infty$, where I is an identity matrix. It is important to note that in order to complete this, at each step of iteration we will have to remove intervals $[n\omega/2-\varepsilon_j, n\omega/2+\varepsilon_j']$ ($n=1, 2, \cdots, N_j$) in the λ-axis (ε_j, ε_j': small constants; N_j: a suitable integer). These are the exceptional intervals. The procedure is as follows. Put $|\mathring{r}|_l=\max_{0\le k\le l}\max_{\theta\in[0,2\pi]}|D_\theta^k\mathring{r}(\theta)|$, $\|P\|_h=\max_i\sum_j\sup_{|\mathrm{Im}\,\theta|<h}|P_{ij}(\theta)|$ for matrix $P=(P_{ij})$, $h_j=1/2^j$, $M_j=c|\mathring{r}|_l h_j^l/\lambda$ and $N_j=4h_j^{-1}\log(1/M_j)$.

First an approximate sequence $R_j(\theta;\varepsilon,\lambda)$ of $R(\theta;\varepsilon,\lambda)$ will be constructed. By Moser's lemma we can approximate $\mathring{r}(\theta)$ by $\{r_j(\theta)\}$ analytic in $|\mathrm{Im}\,\theta|<h_j$. Then put

$R_j(\theta;\varepsilon,\lambda)=\dfrac{ir_j}{2\lambda}\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}$. The sequence $\{R_j\}$ approximates R.

Second, by induction we can construct $\{W_j\}$ with the following properties (i)–(iii).

(i) By $\phi=(I+W_j)\psi$, the system

$$\dot{\phi}=i\lambda J\phi+R_j\phi \tag{A.15}$$

is reduced to

$$\dot{\psi}=i\lambda_j J\psi+S_j(\theta;\varepsilon,\lambda)\psi\,, \tag{A.16}$$

where $\lambda_j=\lambda_j(\lambda)>0$ is a constant and $J=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$.

(ii) Each matrix W_j and S_j are analytic in $|\mathrm{Im}\,\theta|<h_j$.

(iii) The W_j, S_j and λ_j have the following estimates:

$$\|W_j\|_{h_j}\le\sum_{s=0}^{j-1}M_s\,, \tag{A.17}$$

$$\|S_j\|_{h_j}\le M_j\,, \tag{A.18}$$

$$1-\sum_{s=0}^{j-1}M_s\le\frac{d\lambda_j}{d\lambda}\le 1+\sum_{s=0}^{j-1}M_s\,, \tag{A.19}$$

$$|\lambda_j-\lambda_{j+1}|\le M_j \tag{A.20}$$

and

$$\|W_j-W_{j+1}\|_{h_{j+1}}\le Ch_{j+1}^{\sigma}\,, \tag{A.21}$$

where $\nu<\dfrac{1}{2}$ is a positive constant depending on l, and $\sigma=l(1-\nu)-2$. From (i)–(iii) we can deduce: as $j\to\infty$, then $R_j\to R$, $S_j\to 0$ and there exist V and $a\ge 1$ such that $I+W_j\to V$ and $\lambda_j\to a$. Thus the reduction of (A.13) to (A.14) is done by the above V.

We can construct W_{j+1} from W_j, S_j and λ_j. To this end, for given S_j and λ_j, w_j is obtained by solving a simple first-order differential equation and we put $W_{j+1}=(I+W_j)(I+w_j)$. At j-th step in the induction we have to remove the following exceptional intervals $\Delta_n^j=\{\lambda;\ |n\omega/2-\lambda_j(\lambda)|\le M_j^{\nu}/n\}$, $n=1,\cdots,[N_j]$. It follows that for any fixed n Δ_n^j is monotonically decreasing if $j\ge j(n)$, where $j(n)$ is the least j satisfying $n\le N_j$. Hence for any fixed n, $\Delta_n^j\subset\Delta_n^{j(n)}$. So the union of $\{\Delta_n^j\}$ are contained in $\bigcup_{n=1}^{\infty} \Delta_n^{j(n)}$. Also $\Delta_n^{j(n)}$ is contained in $\Delta_n = \{\lambda; |\frac{1}{2}n\omega - \lambda| \leq c|\mathring{r}|_l^\nu/n\}$, where c is a constant depending only on l. Note that if (A2) in §3 holds, then every k, root of the eigenvalue of $-\partial_x^2$, does not belong to $\bigcup_{n=1}^{\infty} \Delta_n$, provided that $|\mathring{r}|_l$ is small. Hence (A.1), $k=1, 2, \cdots$, have the uniform reduction.

The asymptotic property of $\{a(\lambda)\}$ follows from (A.20):

$$|a(\lambda)-\lambda| \leq \sum_{s=0}^{\infty} M_s = c|\mathring{r}|_l/\lambda\,,$$

where c is a constant depending only on l. When we apply this result to Theorem A.1, we first set σ suitably large and then put $l=[(\sigma+2)/(1-\nu)]+1$.

Appendix B

Proposition B.1. *Almost all $(m+1)$-dimensional vectors satisfy the inequalities* (*) in (f2).

Proof. Let K be a fixed positive number and B be a bounded domain in R^{m+1}. We denote by $D_{l,k,K}$ a set of $\eta \in B$ which satisfy

$$|a_k + (\eta, l)| \leq K|l|^{-(m+2)}$$

for some $k \geq 1$ and $l \in Z^{m+1} \setminus \{0\}$. Then we have, for $\eta \in D_{l,k,K}$,

$$\begin{aligned}|k+(\eta, l)| &\leq |k-a_k| + |a_k+(\eta, l)| \\ &\leq C_3/k + K|l|^{-(m+2)}\end{aligned}$$

by (3.4). Hence it follows

$$k \leq K|l|^{-(m+2)} + |\eta|\,|l| + c\,,$$

where $c=\sqrt{C_3}$. Thus for any fixed $l \in Z^{m+1} \setminus \{0\}$ the number $N(l) = \#\{k: k \leq K|l|^{-(m+2)} + |\eta|\,|l| + c\}$ is less than Const. $(|l|+1)$, where Const. depends on C_3 and B, but not on $K \leq 1$. Since $m(D_{l,k,K})$, the Lebesgue measure of the set $D_{l,k,K}$, is less than $M(B)K|l|^{-(m+2)}$, where $M(B)$ is a constant depending only on B, $m(\bigcup_{l \in Z^{m+1}\setminus\{0\}} \bigcup_{k=1}^{N(l)} D_{l,k,K})$ is estimated as follows:

$$\begin{aligned} m(\bigcup_l \bigcup_k D_{l,k,K}) &\leq \sum_{l \in Z^{m+1}\setminus\{0\}} \sum_{k=1}^{N(l)} m(D_{l,k,K}) \\ &\leq M(B)K \sum_l \sum_{k=1}^{N(l)} |l|^{-(m+2)} \\ &\leq \text{Const. } M(B)K \sum_l (|l|+1)|l|^{-(m+2)} \\ &\leq \text{Const. } K \sum_l (1/|l|^{(m+1)}) \\ &\leq \text{Const. } K\,, \end{aligned}$$

where Const.s depend on B, m and C_3. Taking $K\to 0$, we have

$$m(\bigcap_{K>0}(\bigcup_{l}\bigcup_{k}D_{l,k,K}))=0\,.$$

Since B is any bounded domain, the proposition is proved. Q.E.D.

Department of Mathematical Sciences
Tokai University

References

[1] L. Amerio and G. Prouse, Almost Periodic Functions and Functional Equations, Van Nostrand, 1971.

[2] A. Khinchin, Continued Fractions, The University of Chicago Press, 1964.

[3] O. Ladyzenskaja, The Mixed Problem for Hyperbolic Equations (in Russian), Moscow, 1953.

[4] W. Magnus and S. Winkler, Hill's Equation, Interscience, 1966.

[5] I. Parashuk, On the instability zones of Schrödinger equations with smooth quasiperiodic potential, Ukrain. Math. J., 30–1 (1978), 70–78.

[6] C. Vaghi, Su un'equazione iperbolica con coefficienti periodici e termine noto quasiperiodico, Rend. Ist. Lombardo, **100** (1966), 155–180.

[7] O. Vejvoda, Partial Differential Equations, Sijthoff Nordhoff, 1981.

[8] M. Yamaguchi, Almost periodic solutions of some wave equations with Dirichlet boundary values, Proc. of Colloquium on Qualitative Theory of Differential Equations, Colloquia Matematica Societatis Janos Bolyai, **30** (1981), 1061–1074.

[9] M. Yamaguchi, Existence and stability of global bounded classical solutions of initial boundary value problem for semilinear wave equations, Funkc. Ekvac., **23** (1980), 289–308.

[10] M. Yamaguchi, Time-decaying solutions and asymptotically almost periodic solutions of nonlinear evolution equations, Funkc. Ekvac., **24** (1981), 281–306.

[11] E. Dinaburg and Ya. G. Sinai, On the one-dimensional Schrödinger equation with quasi-periodic potential, Funct. Anal. Appl., **9**–4, 279–289 (translation (English)).

[12] V. Marcenko and I. Ostrovskii, A charaterization of the spectrum of Hill's operator, Mat. Sbornik, **97** (1975), 493–554.