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Hypoellipticity for infinitely degenerate elliptic and parabolic operators of second order

By

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§ 1. Introduction and results.

We are mainly concerned with hypoellipticity of differential operators (in $\boldsymbol{R}^3$) of the form

$$L = D_t^2+f(t)D_x^2+g(t)D_y^2 . \tag{1.1}$$

Throughout this paper, we assume that $f(t)$ and $g(t)$ are functions of class C^∞ satisfying

$$f(0) = g(0) = 0 , \quad f(t)>0 \quad \text{and} \quad g(t)>0 \qquad \text{for} \quad t\neq 0 . \tag{1.2}$$

It is well known that L is hypoelliptic if it is finitely degenerate elliptic operator. So we shall consider the case where one of $f(t)$ or $g(t)$ (or both of $f(t)$ and $g(t)$) vanishes to infinite order at $t=0$.

Before the statement of results, let us explain our motivation. On one hand, concerning the operator

$$L_1 = D_t^2+f(t)D_x^2+D_y^2$$

(here we assume $tf'(t)\geqq 0$ in addition to (1.2)), S. Kusuoka and D. Strook [4] have recently shown that it is hypoelliptic if and only if

$$\lim_{t\to 0} |t \log f(t)| = 0 . \tag{1.3}$$

(See also Y. Morimoto [7]~[10] and T. Hoshiro [2], [3].) On the other hand, by the argument of V.S. Fediĭ [1], one can see that the operator

$$L_2 = D_t^2+f(t)D_x^2+f(t)D_y^2$$

is hypoelliptic without the assumption (1.3). Concretely, L_1 with $f(t)=\exp(-1/|t|^\sigma)$ $(\sigma>0)$ is hypoelliptic if and only if $\sigma<1$, while L_2 with $f(t)=\exp(-1/|t|^\sigma)$ $(\sigma>0)$ is hypoelliptic. So one can notice that there is significant difference concerning conditions for hypoellipticity between L_1 and L_2. In the

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present paper, to understand the reason for the difference, we consider the operators of the form (1.1) generalizing L_1 and L_2.

Our main results are the followings. (In this paper, we treat partial differential operators in $\boldsymbol{R}^3$ and, since our interest is devoted to hypoellipticity, ellipticity or parabolicity of them except at $t=0$ allows us to restrict our consideration to neighborhood of $t=0$.)

Theorem 1. *Let L be a differential operator of the form* (1.1) *satisfying* (1.2). *Assume moreover that*

(1.4) $\quad f(t)$ *and* $g(t)$ *are monotone increasing for* $0<t<\delta$,

and

(1.5) *there exists a constant* $\gamma(0<\gamma<1)$ *such that*

$$\sqrt{g(\gamma t)}\,|t\log f(t)|\geqq\varepsilon>0 \qquad \textit{for} \quad 0<t<\delta\,.$$

Then L is not hypoelliptic.

Theorem 2. *Let L be a differential operator of the form* (1.1) *satisfying* (1.2). *Assume moreover that*

(1.6) $\quad f(t)$ *and* $g(t)$ *are monotone increasing for* $0<t<\delta$

and monotone decreasing for $-\delta<t<0$, *and*

$$\text{(1.7)} \qquad \begin{cases} \lim_{t\to 0} \sqrt{g(t)}\,|t\log f(t)| = 0 \\ \lim_{t\to 0} \sqrt{f(t)}\,|t\log g(t)| = 0\,. \end{cases}$$

Then L is hypoelliptic.

Note. The assumptions (1.4) and (1.6) do not play crutial roles concerning hypoellipticity (they could be replaced by more general conditions). However, they make our proofs easy.

Let us now explain the difference between L_1 and L_2. Roughly speaking, Theorem 1 asserts that large difference of "*vanishing speed*" between $f(t)$ and $g(t)$ makes L not hypoelliptic, i.e., under the condition (1.5), $f(t)$ vanishes much more rapidly than $g(t)$ ((1.5) can be written as $f(t)\leqq\exp(-\varepsilon/\sqrt{g(\gamma t)}\,t)$ because $\log f(t)<0$ for small t). On the other hand, if $f(t)\equiv g(t)$, L satisfies automatically (1.7) (because $|\log f(t)|\leqq C_\alpha|f(t)|^{-\alpha}$ for any $\alpha>0$), so L is hypoelliptic. (Notice that the condition (1.7) is not compatible with (1.5).)

To understand our results (or assumptions (1.5) and (1.7)) precisely, let us now consider the following examples.

Example 1. Let σ be a positive constant and k be a positive integer. Theorem 1 and 2 show that the operator

$$L = D_t^2 + \exp(-1/|t|^{\sigma})D_x^2 + t^{2k}D_y^2$$

is hypoelliptic if and only if $\sigma < k+1$. Also notice that, in the case where $\sigma \geqq 1$, it does not satisfy Morimoto's criterion: For any $\varepsilon > 0$ and for any compact set $K \subset \mathbf{R}^3$, there exists a constant $C_{\varepsilon,K}$ such that

$$(1.8) \qquad \|\log \langle D \rangle u\|^2 \leqq \varepsilon(Lu, u) + C_{\varepsilon,K}\|u\|^2, \quad {}^{\forall}u \in C_0^\infty(K).$$

This can be seen by taking $u_\rho(t, x, y) = u(\rho t, e^\rho x, y)$ $(\rho \to \infty)$. (See, for instance section 4 of [2].)

Example 2. Let σ_1 and σ_2 be positive numbers. Theorem 2 shows that the operator

$$L = D_t^2 + \exp(-1/|t|^{\sigma_1})D_x^2 + \exp(-1/|t|^{\sigma_2})D_y^2$$

is hypoelliptic.

Let us add here the following generalization of Theorem 2:

Theorem 3. *Let L be a differential operator of the form*

$$(1.9) \qquad L = D_t^2 + D_x(f(t, x, y)D_x) + D_y(g(t, x, y)D_y),$$

where $f(t, x, y)$ and $g(t, x, y)$ are functions of class C^∞ satisfying the following condition:

(1.10) *There exists a positive constant C such that*

$$\begin{cases} C^{-1}f(t) \leqq f(t, x, y) \leqq Cf(t) \\ C^{-1}g(t) \leqq g(t, x, y) \leqq Cg(t) \end{cases}$$

and

$$\begin{cases} \sum_{1 \leqq |k+l| \leqq 2} |D_x^k D_y^l f(t, x, y)| \leqq Cf(t) \\ \sum_{1 \leqq |k+l| \leqq 2} |D_x^k D_y^l g(t, x, y)| \leqq Cg(t) \end{cases}$$

for $-\delta < t < \delta$, $(x, y) \in \Omega$. If $f(t)$ and $g(t)$ satisfy moreover (1.2), (1.6) *and* (1.7), *then L is hypoelliptic in $(-\delta, \delta) \times \Omega$.*

Remark 1.1. Theorem 3 is of course applicable to operators with $f(t, x, y) = f(t)a(t, x, y)$ and $g(t, x, y) = g(t)\,b(t, x, y)$, where $a(t, x, y)$ and $b(t, x, y)$ are functions of class C^∞ satisfying $a(t, x, y) > 0$ and $b(t, x, y) > 0$ for $-\delta < t < \delta$, $(x, y) \in \Omega$.

By a slight modification, our arguments can also be applied to the operators of parabolic type:

Theorem 4. *Let P be a differential operator of the form*

$$(1.11) \qquad P = D_t^2 + f(t)D_x^2 + ig(t)D_y,$$

where $f(t)$ and $g(t)$ are functions of class C^∞ satisfying (1.2) *and* (1.6). *Assume moreover that*

$$\text{(1.12)} \qquad \begin{cases} \lim_{t\to 0} g(t)t^2|\log f(t)| = 0 \\ \lim_{t\to 0} \sqrt{f(t)}\,|t \log g(t)| = 0\,. \end{cases}$$

Then P is hypoelliptic.

Example 3. Theorem 4 and the argument in proof of Theorem 1 show that the operator

$$P = D_t^2 + \exp(-1/|t|^\sigma)D_x^2 + it^{2k}D_y\,,$$

where σ is a positive number and k is a positive integer, is hypoelliptic if and only if $\sigma < 2k+2$.

The outline of this article is as follows: In section 2, we prove Theorem 1. We explain and show some basic facts necessary for the proof of Theorem 2 in section 3. In section 4, we complete the proof of Theorem 2. Proofs of Theorem 3 and 4 will be given in section 5. Finally in section 6, we prove the lemma in section 4.

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§ 2. Proof of Theorem 1.

We shall begin with proving the following lemma. The idea of proof here is essentially due to Y. Morimoto [7]. We modify his argument slightly so as to apply it to the following eigen value problem (with real parameter ξ):

$$\text{(2.1)} \qquad \begin{cases} -v''(t) + f(t)\xi^2 v(t) = \lambda^2 g(t) v(t) \qquad \text{for} \quad -\delta < t < \delta\,, \\ v(\delta) = v(-\delta) = 0\,. \end{cases}$$

Here we regard $\lambda(>0)$ as an eigen value. Let us denote by $\lambda_1(\xi)$ the smallest eigen value and by $v(t;\xi)$ the corresponding eigen function normalized in such a way that $\int_{-\delta}^{\delta} |v(t;\xi)|^2 dt = 1$.

Lemma 2.1. *Suppose that $f(t)$ and $g(t)$ satisfy conditions* (1.2), (1.4) *and* (1.5). *Then*:

(2.2) *There exists a constant C_1 such that*

$$0 < \lambda_1(\xi) \leq C_1 \log|\xi|\,, \qquad \textit{for } |\xi| \textit{ large.}$$

(2.3) *For any δ' $(0 < \delta' < \delta)$ independent of ξ,*

$$\int_{-\delta'}^{\delta'} |v(t;\xi)|^2 dt \to 1 \qquad \textit{as} \quad |\xi| \to \infty\,.$$

Remark 2.1. (2.3) asserts that the mass in L^2-norm of the eigen function will concentrate to the origin as $|\xi|\to\infty$.

Proof. It is known that $v(t;\xi)$ is characterized as a function which attains the infimum of Rayleigh's ratio, i.e.,

$$\lambda_1(\xi)^2 = \inf_{\substack{v\in C_0^\infty(-\delta,\delta)\\ v\not\equiv 0}} \{\int |v'|^2 dt+\int f\xi^2|v|^2 dt\}/\int g|v|^2 dt.$$

Let us now denote by J_ξ the interval $(\gamma A(\xi), A(\xi))$, where $A=A(\xi)$ is a positive number $0<A<\delta$ such that $f(A)\xi^2=1$ ($|\xi|$ is supposed sufficiently large) and γ is the same number $0<\gamma<1$ as in the assumption (1.5). Now, notice that the assumption (1.4) implies

$$\begin{cases} f(t)\xi^2\leqq 1 \\ g(t)\geqq g(\gamma A(\xi)) \end{cases} \quad \text{for} \quad t\in J_\xi.$$

Then, with aid of Poincaré's inequality, we obtain

$$(2.4)\qquad \begin{aligned} \lambda_1(\xi)^2 &\leqq \inf_{\substack{v\in C_0^\infty(J_\xi)\\ v\not\equiv 0}} \{\int |v'|^2 dt+\int f\xi^2|v|^2 dt\}/\int g|v|^2 dt \\ &\leqq \frac{1}{g(\gamma A(\xi))}\cdot \inf_{\substack{v\in C_0^\infty(J_\xi)\\ v\not\equiv 0}} \{\int |v'|^2 dt+\int |v|^2 dt\}/\int |v|^2 dt = \\ &= \frac{1}{g(\gamma A(\xi))}\Big[\Big\{\frac{\pi}{(1-\gamma)A(\xi)}\Big\}^2+1\Big]. \end{aligned}$$

On the other hand, it follows from the assumption (1.5) that

$$(2.5)\qquad \frac{1}{g(\gamma A(\xi))}\leqq \text{const.}\,(\log|\xi|)^2 A(\xi)^2,$$

when $|\xi|$ is sufficiently large. So, combining (2.4) and (2.5), one can conclude (2.2).

To show (2.3), let us observe that

$$(2.6)\qquad \begin{aligned} &[\inf_{\delta'<|t|<\delta} f(t)]\xi^2\int_{\delta'<|t|<\delta} |v(t;\xi)|^2 dt \\ &\leqq \int_{-\delta}^{\delta} f(t)\xi^2|v(t;\xi)|^2 dt \\ &\leqq \int_{-\delta}^{\delta} |v'(t;\xi)|^2 dt+\int_{-\delta}^{\delta} f(t)\xi^2|v(t;\xi)|^2 dt \\ &= \lambda_1(\xi)^2\cdot\int_{-\delta}^{\delta} g(t)|v(t;\xi)|^2 dt\leqq \text{const.}\,\lambda_1(\xi)^2. \end{aligned}$$

Furthermore, notice that $\lambda_1(\xi)/|\xi|\to 0$ as $|\xi|\to\infty$ (recall (2.2)). Then, multiplying the both sides of (2.6) by $1/\xi^2$, we see that

$$\int_{\delta'<|t|<\delta} |v(t;\,\xi)|^2 dt \to 0 \qquad \text{as} \quad |\xi|\to\infty\,,$$

which is equivalent to (2.3). *q.e.d.*

Proof of Theorem 1. We prove it by contradiction. If L is hypoelliptic, the following inequality follows from the argument of Banach's closed graph theorem.

For any positive integer k and for any open sets ω, ω' of $\boldsymbol{R}^3$ such that $\omega' \Subset \omega$, there exist a positive integer l and a constant C such that

$$\text{(2.7)} \qquad \|D_x^k u\|^2_{L^2(\omega')} \leqq C\{\sum_{|\alpha|\leqq l} \|D^\alpha Lu\|^2_{L^2(\omega)} + \|u\|^2_{L^2(\omega)}\} \qquad \forall u \in C^\infty(\bar{\omega})\,.$$

Let us now put

$$\omega = I_\delta \times Q_1\,, \quad \omega' = I_{\delta'} \times Q_2$$

with

$$I_\delta = \{t;\ -\delta<t<\delta\}\,, \quad I_{\delta'} = \{t;\ -\delta'<t<\delta'\}$$
$$Q_1 = \{(x,\,y);\ 0<x<\delta,\ 0<y<\delta\}$$

and

$$Q_2 = \{(x,\,y);\ \delta'/2<x<\delta',\ \delta'/2<y<\delta'\}\,.$$

Furthermore, let us substitute the sequence of functions

$$\text{(2.8)} \qquad u_n = u_n(t,\,x,\,y) = \exp(inx + \lambda_1(n)y)v(t;\,n)\,, \qquad n = 1,\,2,\,\cdots\,,$$

(they are solutions of $Lu=0$ in ω) to the both sides of (2.7). Then the right hand side of (2.7) is not greater than

$$\text{(2.9)} \qquad C \times e^{2\lambda_1(n)\delta} \times \text{meas}\, Q_1 \times \int_{-\delta}^{\delta} |v(t;\,n)|^2 dt \leqq C\cdot\delta^2\cdot n^{2C_1\delta}\,.$$

On the other hand, the left hand side of (2.7) is not smaller than

$$\text{(2.10)} \qquad n^{2k} \times \text{meas}\, Q_2 \times \int_{-\delta'}^{\delta'} |v(t;\,n)|^2 dt \geqq n^{2k}\cdot\frac{\delta'^2}{4}\cdot(1-\varepsilon)\,,$$

when n is sufficiently large (recall (2.3)). Therefore, taking a positive integer k so that $k>C_1\delta$, we can see that the inequality (2.7) never holds under the assumption (1.5). This completes the proof of Theorem 1.

§ 3. Criterion for hypoellipticity.

In the present section, we are going to explain our plan of the proof of Theorem 2. The proof is divided into two steps, namely, by showing the following propositions.

Let us consider the following ordinary differential operator (with real parameters $\zeta=(\xi,\,\eta)$)

$$L_\zeta = -\frac{d^2}{dt^2} + f(t)\xi^2 + g(t)\eta^2 .$$

Proposition 3.1. *Suppose that $f(t)$ and $g(t)$ satisfy the conditions* (1.2), (1.6) *and* (1.7). *Then the following inequalities hold for L_ζ:*

Given any $\varepsilon > 0$, there exists a positive number n_0 such that

$$\int g(t)(\log |\xi|)^2 |v(t)|^2 dt \leq \varepsilon \int L_\zeta v(t) \cdot \overline{v(t)} dt \tag{3.1}$$

$$\int f(t)(\log |\eta|)^2 |v(t)|^2 dt \leq \varepsilon \int L_\zeta v(t) \cdot \overline{v(t)} dt \tag{3.2}$$

for all $v \in C_0^\infty(-\delta, \delta)$ and for all $\zeta \in \mathbf{R}^2$ satisfying $\xi^2 + \eta^2 \geqq n_0^2$.

Remark 3.1. In the right hand sides of (3.1) and (3.2), observe that

$$\int L_\zeta v(t) \cdot \overline{v(t)} dt = \int |v'(t)|^2 dt + \int f(t)\xi^2 |v(t)|^2 dt + \int g(t)\eta^2 |v(t)|^2 dt .$$

Proposition 3.2. *If L_ζ enjoys* (3.1) *and* (3.2), *then L is hypoelliptic.*

Remark 3.2. The inequalities (3.1) and (3.2) give also necessary conditions for hypoellipticity. It is because, if (3.1) does not hold, then (2.2) and (2.3) hold and this implies non-hypoellipticity of L. (Recall characterization of $\lambda_1(\xi)$ by Rayleigh's ratio.)

The proof of Proposition 3.2 will be given in the next section, using microlocal energy method. In the remaining part of this section, we shall prove Proposition 3.1. The method used there is so-called "*sew together*" argument, which has first appeared in Visik-Grušin's paper (see Fediĭ [1]).

Proof of Proposition 3.1. Inequality (3.1) is not trivial only when ζ approaches asymptotically to ξ-axis. So we prove it supposing that ζ is contained in conic neighborhood of ξ-axis. Proof of (3.2) follows from the argument here and interchanging roles of $(f(t), \xi)$ and $(g(t), \eta)$.

I) For $v(t)$ with support in $\{t \in (-\delta, \delta); f(t)|\xi|^{1/2} \geqq 1/2\}$: It is very easy to see that, if $|\xi|$ is sufficiently large,

$$\begin{aligned} \int g(t)(\log |\xi|)^2 |v(t)|^2 dt &\leq \text{const.} \int |\xi| \, |v(t)|^2 dt \\ &\leq \text{const.}\ 2 \int f(t) |\xi|^{1+1/2} |v(t)|^2 dt \\ &\leq \varepsilon \int L_\zeta v(t) \cdot \overline{v(t)} dt . \end{aligned}$$

II) For $v(t)$ with support in $\{t \in (-\delta, \delta); f(t)|\xi|^{1/2} \leqq 2\}$: Let $a = a(\xi)$ denote a positive number such that $f(a)|\xi|^{1/2} = 2$ and write $v(t)$ as

$$v(t) = -\int_t^a v'(s) ds .$$

Then, it is easy to see that

$$(3.3)\qquad \int_0^a g(t)|v(t)|^2 dt \leqq \int_0^a g(t)(a-t)dt \int |v'(s)|^2 ds$$
$$\leqq \int_0^a g(t)(a-t)dt \int L_\xi v(s)\cdot v(s)ds$$
$$\leqq \frac{1}{2} g(a)a^2 \int L_\xi v(s)\cdot v(s)ds\,.$$

On the other hand, the assumption (1.7) together with the fact that $a(\xi)\to 0$ as $|\xi|\to\infty$ yield that, if $|\xi|$ is sufficiently large,

$$(3.4)\qquad (\log|\xi|)^2 g(a)a^2 \leqq \log|\xi|)^2\cdot(\log f(a))$$
$$=\varepsilon(\log|\xi|)^2(\log 2|\xi|^{-1/2})^{-2}$$
$$\leqq \text{const. } \varepsilon\,.$$

Therefore combining (3.3) and (3.4), we obtain (3.1) for $v(t)$ with support sufficiently near to $t=0$. (The integral $\int_b^0 g(t)|v(t)|^2 dt$ can be estimated in the same way, where $b=b(\xi)$ denotes a negative number such that $f(b)|\xi|^{1/2}=2$.)

III) Now we prove (3.1) for general $v(t)\in C_0^\infty(-\delta, \delta)$. We are going to "*sew together*" the results in I) and II) which are valid in overlapping regions.

Let us first take a function $\phi=\phi(t)\in C_0^\infty$ with $0\leqq\phi(t)\leqq 1$, $\phi(t)=1$ in $|t|\geqq 1/2$ and $\phi(t)=0$ in $|t|\geqq 2$, and put

$$\begin{cases} \chi_1(t)=\phi(f(t)|\xi|^{1/2})\,, & \chi_2=1-\chi_1 \\ v_1=\chi_1 v\,, & v_2=\chi_2 v\,. \end{cases}$$

Then it follows from the results in I) and II) that

$$(3.5)\qquad \int g(t)(\log|\xi|)^2|v(t)|^2 dt \leqq$$
$$\leqq 2(\log|\xi|)^2\{\int g(t)|v_1(t)|^2 dt + \int g(t)|v_2(t)|^2 dt\}$$
$$\leqq 2\varepsilon\{\int L_\xi v_1(t)\cdot\overline{v_1(t)}dt + \int L_\xi v_2(t)\cdot\overline{v_2(t)}dt\}$$
$$\leqq 4\varepsilon\int L_\xi v(t)\cdot\overline{v(t)}dt + \text{remainder}\,.$$

The "remainder" is estimated by

$$\varepsilon 4\{2\int|\chi_1'||v'||v|dt + \int|\chi_1'|^2|v|^2 dt\}$$
$$\leqq \varepsilon 4\{2\int|v'|^2 dt + 3\int|\chi_1'|^2|v|^2 dt\}\,,$$

and furthermore, since $f(t)|\xi|^{1/2}\geqq 1/2$ in the support of χ_1',

$$|\chi_1^{\prime}|^2 \leqq \text{const.}\ |\xi| \cdot f(t)|\xi|^{1/2} \leqq \text{const.} f(t)|\xi|^2 .$$

Therefore we can see

$$\text{remainder} \leqq \varepsilon\ \text{const.} \int L_\xi v(t) \cdot \overline{v(t)}\, dt . \tag{3.6}$$

Now, Proposition 3.1 follows from (3.5) and (3.6).

§ 4. Microlocal energy method.

We start this section by preparing the Sobolev spaces and microlocal energy which are necessary for the proof of Proposition 3.2. First we define the following Sobolev spaces.

Definition. We denote by $H^{k,l}(\boldsymbol{R}^3)$ $(-\infty<k, l<\infty)$ the space of all distributions $u \in \mathcal{S}'(\boldsymbol{R}^3)$ satisfying

$$\iiint |\hat{u}(\tau, \xi, \eta)|^2 (1+\tau^2)^k (1+\xi^2+\eta^2)^l \, d\tau\, d\xi\, d\eta < \infty ,$$

where $\hat{u}$ is Fourier transform of u.

Furthermore we say that $u \in \mathcal{D}'((-\delta, \delta)\times\Omega)$ is locally of class $H^{k,l}$ at (t_0, x_0, y_0) if there exists a function $\phi \in C_0^\infty((-\delta, \delta)\times\Omega)$ with $\phi=1$ in a neighborhood of (t_0, x_0, y_0) such that $\phi u \in H^{k,l}(\boldsymbol{R}^3)$.

This definition enables us to do some reductions. At first, if $u \in \mathcal{D}'((-\delta, \delta)\times\Omega)$ and $(t_0, x_0, y_0) \in (-\delta, \delta)\times\Omega$, there exists a pair of real numbers (k, l) such that $u \in H^{k,l}$ at (t_0, x_0, y_0). The second is that, if $u \in H^{k,l}$ and $Lu \in C^\infty$ at (t_0, x_0, y_0), then we have $u \in H^{k+2,l-2}$ at (t_0, x_0, y_0). This is shown (in case of $t_0=0$) in the following way: Let $\phi_1 \in C_0^\infty((-\delta, \delta)\times\Omega)$ be a function with $\phi_1=1$ in a neighborhood of $(0, x_0, y_0)$ such that $\phi_1 u \in H^{k,l}(\boldsymbol{R}^3)$, and choose $\phi(t, x, y)=\chi(t)\psi(x, y)$ (χ and ψ are equal to 1 in a neighborhood of $t=0$ and $(x_0, y_0)=(x_0, y_0)$ respectively) so that the support of ϕ is contained in a closed set where $\phi_1=1$ (we write $\phi \Subset \phi_1$). Then the right hand side of equation

$$D_t^2(\chi\psi u) = [D_t^2, \chi](\psi u) + \chi\psi L u - \chi\psi(f D_x^2 + g D_y^2) u$$

is of class $H^{k,l-2}(\boldsymbol{R}^3)$. In fact, the second and third terms belong to C_0^∞ and $H^{k,l-2}$ respectively. The first one is of class C_0^∞ because of ellipticity of L except at $t=0$. Hence we see that $\chi\psi u \in H^{k+2,l-2}$, and furthermore by repeating this argument, that u is locally of class $\bigcap_m H^{k+m,l-m}$ at (t_0, x_0, y_0). If $t_0 \neq 0$, ellipticity of L yields $u \in \bigcap_{k,l} H^{k,l}$ $(=H^\infty)$ at (t_0, x_0, y_0).

Thus in the proof of Proposition 3.2, the partial Fourier transform of u at (t_0, x_0, y_0), i.e.,

$$\chi(t)(\psi u)^\wedge(t; \xi, \eta) = (2\pi)^{-1} \int e^{i\tau t} (\widehat{\chi\psi u})(\tau, \xi, \eta)\, d\tau$$

is smooth with respect to t for almost every (ξ, η). So we can apply (3.1) and (3.2) to it. (Lu is supposed to be smooth when we prove Proposition 3.2)

Now, we define the notion of microlocal smoothness of $u\in\mathcal{D}'((-\delta, \delta)\times\Omega)$, since our proof of Proposition 3.2 will be microlocal (it is more precise to say "semi-microlocal").

Definition. Let $(t_0, x_0, y_0)\in(-\delta, \delta)\times\Omega$ and $(\xi^0, \eta^0)\in\boldsymbol{R}^2\backslash 0$. For $u\in\mathcal{D}'((-\delta, \delta)\times\Omega)$, we say that u is microlocally of class $H^{0,\infty}$ $(=\bigcap_l H^{0,l})$ at $(t_0, x_0, y_0\colon \xi^0, \eta^0)$ if there exist a function $\phi\in C_0^\infty((-\delta, \delta)\times\Omega)$ with $\phi=1$ in a neighborhood of (t_0, x_0, y_0) and a conic neighborhood Γ_0 $(\subset\boldsymbol{R}^2)$ of (ξ^0, η^0) such that

$$\iiint_{\substack{-\infty<\tau<\infty\\(\xi,\eta)\in\Gamma_0}} |\widehat{(\phi u)}(\tau, \xi, \eta)|^2(1+\xi^2+\eta^2)^s d\tau\, d\xi\, d\eta<\infty$$

for any positive number s.

Remark 4.1. By standard argument in microlocal analysis, one can easily show that $u\in H_{\mathrm{loc}}^{0,\infty}$ at (t_0, x_0, y_0) if and only if u is microlocally of class $H^{0,\infty}$ at $(t_0, x_0, y_0;$ $\xi^0, \eta^0)$ for all (ξ^0, η^0).

Remark 4.2. In the proof of Proposition 3.2, it suffices to show that u is of class $H^{0,\infty}$ at (t_0, x_0, y_0) when Lu is of class C^∞ at (t_0, x_0, y_0). The reason is the same as in the first part of the present section, i.e., $u\in H_{\mathrm{loc}}^{0,l}$ and $Lu\in C_{\mathrm{loc}}^\infty$ imply that $u\in H_{\mathrm{loc}}^{2,l-2}$, $u\in H_{\mathrm{loc}}^{4,l-4}$, $\cdots$, thus we can see $u\in\bigcap_{k,l} H_{\mathrm{loc}}^{k,l}$ $(=H_{\mathrm{loc}}^\infty)$ when $u\in H_{\mathrm{loc}}^{0,\infty}$ and $Lu\in C_{\mathrm{loc}}^\infty$.

We end the preparation of the proof of Proposition 3.2 by recalling microlocal energy method which the author used in [3] after some refinements. The use of the method here is slightly different from that in [3], because the smoothness of u stated above does not have microlocal character but has precisely "semi-microlocal" one.

Choose first a sequence $\psi_N\in C_0^\infty(\boldsymbol{R}^2)$, $N=1, 2, \cdots$, with $\psi_N=1$ in $\{(x, y); x^2+y^2\leqq r_0^2/4\}$ and $\psi_N=0$ in $\{(x, y); x^2+y^2\geqq r_0^2\}$, satisfying:

$$|D^{p+\nu}\psi_N|\leqq C_{K_0}(CN)^{|p|}$$

for $|p|\leqq N$, $|\nu|\leqq K_0$ (here C_{K_0} and C are independent of N). Our microlocalizers $\{\alpha_n(\xi, \eta), \beta_n(x, y)\}$ are defined in such a way that

$$\alpha_n(\xi, \eta)=\psi_{N_n}\left(\frac{\xi}{n}-\xi^0, \frac{\eta}{n}-\eta^0\right), \quad \beta_n(x, y)=\psi_{N_n}(x-x, y_0-y_0)$$

where $N_n=[\log n]+1$. Our microlocal energy is

$$S_n^M(v)=\sum_{|p+q|\leqq N_n} \|c_{pq}^n\alpha_n^{(p)}(D_x, D_y)\beta_{n(q)}(x, y)v\|^2, \qquad v\in\mathcal{S}'(\boldsymbol{R}^3),$$

with

$$c_{pq}^n=M^{-|p+q|}\cdot n^{|p|}\cdot(\log n)^{-|p+q|}.$$

(Here $\alpha_n^{(p)}=\partial_\xi^{p_1}\partial_\eta^{p_2}\alpha_n$, $\beta_{n(q)}=D_x^{q_1}D_y^{q_2}\beta_n\cdot||\ ||$ stands for the normin $L^2(\boldsymbol{R}^3)$.)

Note. $S_n^M(v)$ could be called a (semi-) microlocal energy at $(x_0, y_0; n\xi^0, n\eta^0)$. Since the hypotheses (3.1) and (3.2) are very weak compared with subelliptic estimates, we are obliged to carry out carefully quantitative analysis, namely, micro-localizers $\{\alpha_n, \beta_n\}$ must be chosen as in the study of the analytic or the Gevrey wave front sets.

We have now the following lemma whose proof will be given in section 6.

Lemma 4.1. *Let $u\in\bigcup_l H^{0,l}$ locally at (t_0, x_0, y_0). Then u is microlocally of class $H^{0,\infty}$ at $(t_0, x_0, y_0; \xi^0, \eta^0)$ if and only if there exists a function $\chi(t)\in C_0^\infty(\boldsymbol{R})$ with $\chi=1$ in a neighborhood of t_0 such that microlocal energy of χu is rapidly decreasing as $n\to\infty$ (if $r_0>0$ is sufficiently small), i.e., for any positive number s, there exist constants M and C_s such that*

$$S_n^M(\chi u)\leqq C_s n^{-2s}$$

when n is large (we abbreviate as $S_n^M(\chi u)=O(n^{-2s})$.

Let us now begin the proof of Proposition 3.2. By using microlocal energy method, we show that u is microlocally of class $H^{0,\infty}$ at $(t_0, x_0, y_0; \xi^0, \eta^0)$ for every (ξ^0, η^0), when Lu is locally of class C^∞ in a neighborhood of (t_0, x_0, y_0) (receall Remark 4.1 and 4.2).

Proof of Proposition 3.2. The ellipticity of L except at $t=0$ allows us to restrict our consideration to the case of $t_0=0$. Moreover, by the same reason, the right hand side of equation

$$\psi L(\chi u) = \psi[D_t^2, \chi]u+\chi\psi Lu$$

is of class C_0^∞ if Lu is of class C^∞ in a neighborhood of $(0, x_0, y_0)$. (Here $\chi(t)\in C_0^\infty$ and $\psi(x, y)\in C_0^\infty$ have supports in small neighborhoods of $t=0$ and $(x, y)=(x_0, y_0)$ respectively.) So it suffices to show that microlocal energy of $v=\chi u$ is rapidly decreasing when ψLv is of class C_0^∞.

Assume that $|p+q|\leqq N_n$ and $r_0>0$ is chosen sufficiently small so that $\beta_n\Subset\psi$. Let us operate $\alpha_n^{(p)}\beta_{n(q)}$ to the equation $\psi Lv=h$, namely,

$$\alpha_n^{(p)}\beta_{n(q)}Lv = \alpha_n^{(p)}\beta_{n(q)}h\,.$$

Furthermore, the asymptotic expansion gives (note that $[L, \alpha_n^{(p)}]=0$)

$$Lv_{pq}+\sum_{0<|\nu|\leqq 2}(-1)^{|\nu|}\nu!^{-1}L^{(\nu)}v_{p,q+\nu} = h_{pq}\,, \tag{4.1}$$

where $v_{pq}=\alpha_n^{(p)}\beta_{n(q)}v$, $h_{pq}=\alpha_n^{(p)}\beta_{n(q)}h$ and $L^{(\nu)}$ is a differential operator with symbol $L^{(\nu)}(t; \xi, \eta)=\partial_{\xi,\eta}^\nu L(t; \tau, \xi, \eta)$. Thus we have

$$(Lv_{pq}, v_{pq})\leqq\sum_{0<|\nu|\leqq 2}\nu!^{-1}|(L^{(\nu)}v_{p,q+\nu}, v_{pq})|+\varepsilon^{-1}||h_{pq}||^2+\varepsilon||v_{pq}||^2\,. \tag{4.2}$$

Now in the following, we are going to estimate the first terms on right hand side of (4.2) (see (4.6) below), doing one by one, under the assumption: $(\xi^0, \eta^0)\in\{(\xi, \eta);\ |\eta|\leqq 2|\xi|\}$ (this implies that $c^{-1}\cdot n\leqq|\xi|\leqq c\cdot n$ for $(\xi, \eta)\in\mathrm{supp}[\alpha_n]$). In case of $(\xi^0, \eta^0)\in\{(\xi, \eta);\ |\xi|\leqq 2|\eta|\}$ (then $c^{-1}\cdot n\leqq|\eta|\leqq c\cdot n$ for $(\xi, \eta)\in\mathrm{supp}[\alpha_n]$), one can do in the parallel way, applying (3.2) and exchanging the roles of $(f(t), \xi)$ and $(g(t), \eta)$.

1) For $\nu=(k, 0)$ with $k=1$ or 2 ($L^{(\nu)}=f(t)D_x^{2-k}$): It follows from the fact that $c^{-1}\cdot n\leqq|\xi|\leqq c\cdot n$ for $(\xi, \eta)\in\mathrm{supp}[\alpha_n]$, we see

$$
\begin{aligned}
(4.3)\qquad & |(L^{(\nu)}v_{p,q+\nu}\ v_{pq})| \\
&=\Big|\iiint f(t)\xi^{2-k}\hat{v}_{p,q+\nu}(t;\ \xi,\ \eta)\overline{\hat{v}_{pq}(t;\ \xi,\ \eta)}\,dt\,d\xi\,d\eta\Big| \\
&\leqq Cn^{-k}\Big\{\iiint f\xi^2|\hat{v}_{pq}|^2 dt\,d\xi\,d\eta+\iiint f\xi^2|\hat{v}_{p,q+\nu}|^2 dt\,d\xi\,d\eta\Big\} \\
&\leqq\varepsilon(Lv_{pq},\ v_{pq})+\varepsilon(\log n)^{-2|\nu|}(Lv_{p,q+\nu},\ v_{p,q+\nu}),
\end{aligned}
$$

when n is sufficiently large.

2) For $\nu=(0, 1)$. ($L^{(\nu)}=g(t)D_y$. Also notice that $|\nu|=1$.): From the hypothesis (3.1) together with the fact that $c^{-1}\cdot n\leqq|\xi|\leqq c\cdot n$ for $(\xi, \eta)\in\mathrm{supp}[\alpha_n]$, it follows

$$
\begin{aligned}
(4.4)\qquad & |(L^{(\nu)}v_{p,q+\nu},\ v_{pq})| \\
&=\Big|\iiint g(t)\hat{v}_{p,q+\nu}(t;\ \xi,\ \eta)\eta\overline{\hat{v}_{pq}(t;\ \xi,\ \eta)}\,dt\,d\xi\,d\eta\Big| \\
&\leqq\varepsilon\iiint g\eta^2|\hat{v}_{pq}|^2 dt\,d\xi\,d\eta+\varepsilon^{-1}\iiint g|\hat{v}_{p,q+\nu}|^2 dt\,d\xi\,d\eta \\
&\leqq\varepsilon(Lv_{pq},\ v_{pp})+\varepsilon(\log n)^{-2|\nu|}(Lv_{p,q+\nu},\ v_{p,q+\nu}),
\end{aligned}
$$

when n is sufficiently large. (Recall that $\hat{v}_{pq}(t;\ \xi,\ \eta)$ is smooth with respect to t for almost all (ξ, η).)

3) For $\nu=(0, 2)$. ($L^{(\nu)}=g(t)$. Also notice that $|\nu|=2$.):

$$
\begin{aligned}
(4.5)\qquad & |(L^{(\nu)}v_{p,q+\nu},\ v_{pq})| \\
&=\Big|\iiint g(t)\hat{v}_{p,q+\nu}(t;\ \xi,\ \eta)\overline{\hat{v}_{pq}(t;\ \xi,\ \eta)}\,dt\,d\xi\,d\eta\Big| \\
&\leqq(\log n)^2\iiint g|\hat{v}_{pq}|^2 dt\,d\xi\,d\eta+(\log n)^{-2}\iiint g|\hat{v}_{p,q+\nu}|^2 dt\,d\xi\,d\eta \\
&\leqq\varepsilon(Lv_{pq},\ v_{pq})+\varepsilon(\log n)^{-2|\nu|}(Lv_{q,q+\nu},\ v_{p,q+\nu}).
\end{aligned}
$$

Thus we obtain the inequality:

$$
(4.6)\qquad (Lv_{pq},\ v_{pq})\leqq\varepsilon\sum_{0\leqq|\nu|\leqq 2}(\log n)^{-2|\nu|}(Lv_{p,q+\nu},\ v_{p,q+\nu})+\varepsilon^{-1}\|h_{pq}\|^2+\varepsilon\|v_{pq}\|^2,
$$

for any positive number ε (when n is sufficiently large).

Remaining part of the proof is quite analogous to the ones of theorem 1 and 2 in [3]. Let us now observe that

$$c_{pq}^{n}(\log n)^{-|\nu|} = M^{|\nu|}c_{p,q+\nu}^{n}.$$

Hence (4.6) implies

$$(4.7) \qquad (Lw_{pq}, w_{pp}) \leqq \varepsilon \sum_{0\leqq|\nu|\leqq 2}(Lw_{p,q+\nu}, w_{p,q+\nu}) + \varepsilon^{-1}\|c_{pq}^{n}h_{pq}\|^2 + \varepsilon\|w_{pq}\|^2,$$

where $w_{pq}=c_{pq}^{n}v_{pq}$. Next, let us sum up the both sides of (4.7) with respect to (p, q) satisfying $|p+q|\leqq N_n-2$. Then the first terms on the right hand side of (4.7) will be absorbed into the left hand side (by taking ε sufficiently small). Namely we have

$$(4.8) \qquad \sum_{|p+q|\leqq N_n}(Lw_{pq}, w_{pq}) \leqq O(n^{-2s}) + \varepsilon \sum_{|p+q|\leqq N_n}\|w_{pq}\|^2,$$

since microlocal energy of $h=\psi Lv$ is rapidly decreasing as $n\to\infty$. (To establish (4.8), notice that we may assume

$$\sum_{N_n-1\leqq|p+q|\leqq N_n}(Lw_{pq}, w_{pq}) = O(n^{-2s}),$$

by taking M sufficiently large. *Cf.* Lemma 1 of [3].)

By Poincaré's inequality,

$$(Lw_{pq}, w_{pq}) \geqq \|D_t w_{pq}\|^2 \geqq (2\delta^2)^{-1}\|w_{pq}\|^2.$$

So from (4.8), we see that, for any positive number s, there exists a constant M such that

$$S_n^M(\chi u) = \sum_{|p+q|\leqq N_n}\|w_{pq}\|^2 = O(n^{-2s}).$$

Now the proof is complete. q.e.d.

§ 5. Proofs of Theorem 3 and 4.

The proofs of Theorem 3 and 4 will be quite analogous to that of Theorem 2. So we sketch them and point out the differences.

Proof of Theorem 3. Here we shall consider the operator L whose coefficients depend also on x and y. (After modification with $f(t, x, y)$ and $g(t, x, y)$ outside of $(-\delta, \delta)\times\Omega$, we suppose that L is defined in $(-\delta, \delta)\times\boldsymbol{R}^2$, preserving the condition (1.10).) Now, the assumption (1.10) implies

$$(5.1) \qquad (Lu, u) \geqq \text{const.}\ \{\|D_t u\|^2 + (f(t)D_x u, D_x u) + (g(t)D_y u, D_y u)\},$$

where (,) and $\|\ \ \|$ denote the scalar product and the norm in $L^2(\boldsymbol{R}^3)$ respectively. So the difficulty of proof is that there are many lower order terms in the asymptotic expansion. Let us pay attention to this point. Observe now that, in proof of Theorem 3, the inequality (4.2) will become

$$(5.2) \qquad (Lv_{pq}, v_{pq}) \leqq \sum_{0<|\nu+\mu|\leqq N_0}\nu!^{-1}\mu!^{-1}|(L_{(\mu)}^{(\nu)}v_{p+\mu,q+\nu}, v_{pq})| + \varepsilon^{-1}\{\|h_{pq}\|^2 + \|r_{pq,N_0}v\|^2\} + \varepsilon\|v_{pq}\|^2,$$

where $L_{(\mu)}^{(\nu)}$ is a differential operator with symbol $L_{(\mu)}^{(\nu)}(t, x, y; \xi, \eta) = D_{xy}^{\mu}\partial_{\xi\eta}^{\nu}L(t, x, y; \tau, \xi, \eta)$ and N_0 is a positive integer which we choose below.

Let us first consider the remainder term $r_{pq,N_0}v$. Writing the symbol by oscillatory integral together with the fact that $c^{-1}\cdot n \leqq (1+|\xi|+|\eta|) \leqq c\cdot n$ for $(\xi, \eta) \in \text{supp}[\alpha_n]$, we see that

$$\begin{aligned}
&\|r_{pq,N_0}\|_{\mathcal{L}(H^{0,-k}, L^2)} \leqq \text{const. } |r_{pq,N_0}|_{l_1}^{(-k)} \\
&\leqq \text{const.} \sum_{|\nu+\mu|=N_0+1} |\alpha_n^{(p+\mu)}|_{l_2}^{(-k-2+|\nu|)} |\beta_{n(q+\nu)}|_{l_2}^{(0)} |L_{(\mu)}^{(\nu)}|_{l_2}^{(2-|\nu|)} \\
&\leqq \text{const. } n^{-|p|+k+1-N_0}(CN_n)^{|p+q|},
\end{aligned}$$

where $|\ \ |_l^{(m)}$ denotes the seminorm in $S_{1,0}^m$, i.e.,

$$|a|_l^{(m)} = \max_{|\nu+\mu|\leqq l} \sup |a_{(\mu)}^{(\nu)}(t, x, y; \xi, \eta)|/(1+|\xi|+|\eta|)^{m-|\nu|}.$$

Therefore, if u is locally of class $H^{0,-k}$ (recall that $u \in \bigcup_k H_{\text{loc}}^{0,-k}$), taking N_0 so that $N_0-k-1 \geqq s$, we have

$$\begin{aligned}
(5.3)\qquad &\sum_{|p+q|\leqq N_n} \|c_{pq}^n r_{pq,N_0}v\|^2 \\
&\leqq \text{const.} \sum_{|p+q|\leqq N_n} \|c_{pq}^n r_{pq,N_0}\|_{\mathcal{L}(H^{0,-k}, L^2)}^2 = O(n^{-2s}).
\end{aligned}$$

Next, for the first terms on the right hand side of (5.2), our purpose is to show the following inequality:

$$\begin{aligned}
(5.4)\qquad &|(L_{(\mu)}^{(\nu)} v_{p+\mu,q+\nu}, v_{pq})| \\
&\leqq \varepsilon(Lv_{pq}, v_{pq}) + \varepsilon n^{2|\nu|}(\log n)^{-2|\nu+\mu|}(Lv_{p+\mu,q+\nu}, v_{p+\mu,q+\nu}).
\end{aligned}$$

We show this as follows.

i) For (ν, μ) with $|\mu|=0$ and $1 \leqq |\nu| \leqq 2$: We can observe that the same arguments as in 1), 2) and 3) of the preceding section together with (1.10) and (5.1) yield (5.4) with these (ν, μ).

ii) For (ν, μ) such that $|\nu+\mu|>2$: Since $L_{(\mu)}^{(\nu)}$ is a differential operator of order $2-|\nu|$, we can show with aid of Poincaré's inequality in the following way:

$$\begin{aligned}
&|(L_{(\mu)}^{(\nu)} v_{p+\mu,q+\nu}, v_{pq})| \leqq \varepsilon\|v_{pq}\|^2 + \varepsilon^{-1}\|L_{(\mu)}^{(\nu)} v_{p+\mu,q+\nu}\|^2 \\
&\leqq \varepsilon \text{ const. } (Lv_{pq}, v_{pq}) + \varepsilon^{-1} \text{ const. } n^{2(2-|\nu|)}\|v_{p+\mu,q+\nu}\|^2 \\
&\leqq \varepsilon \text{ const. } (Lv_{pq}, v_{pq}) + \varepsilon n^{2|\mu|}(\log n)^{-2|\nu+\mu|}(Lv_{p+\mu,p+\nu}, v_{p+\mu,q+\kappa})
\end{aligned}$$

where *const.* *s* are constants depending only on L.

iii) For the other (ν, μ), i.e., in case of $|\mu|=|\nu|=1$, or in case of $|\nu|=0$ and $1 \leqq |\mu| \leqq 2$: $L_{(\mu)}^{(\nu)}$ is one of

$$2f_{(\mu)}D_x + f_{(\mu+j)}, \quad 2g_{(\mu)}D_y + g_{(\mu+j')}$$

where $|\mu|=1$, $f_{(\mu+j)} = D_{xy}^{\mu}D_x f$ and $g_{(\mu+j')} = D_{xy}^{\mu}D_y g$, or

$$D_x(f_{(\mu)}D_x), \qquad D_y(g_{(\mu)}D_y)$$

where $1\leqq|\mu|\leqq 2$. Here we shall consider only the first case. Taking account of the conditions (1.10), (5.1) and (3.2), we easily see that

$$
\begin{aligned}
&|(L_{(\mu)}^{(\nu)} v_{p+\mu,q+\nu}, v_{pq})|\\
&\leqq 2|(f_{(\mu)} D_x v_{p+\mu,q+\nu}, v_{pq})|+|(f_{(\mu+j)} v_{p+\mu,q+\nu}, v_{pq})|\\
&\leqq 3(fv_{pq}, v_{pq})+2(fD_x v_{p+\mu,q+\nu}, D_x v_{p+\mu,q+\nu})+(fv_{p+\mu,q+\nu}, v_{p+\mu,q+\nu})\\
&\leqq \varepsilon(Lv_{pq}, v_{pq})+\varepsilon n^{2|\mu|}(\log n)^{-2|\nu+\mu|}(Lv_{p+\mu,q+\nu}, v_{p+\mu,q+\nu})\,.
\end{aligned}
$$

Thus we arrive at the following estimate: For any positive number ε,

$$
\begin{aligned}
(Lv_{pq}, v_{pq})\leqq \varepsilon &\sum_{0\leqq|\nu+\mu|\leqq N_0} n^{2|\mu|}(\log n)^{-2|\nu+\mu|}(Lv_{p+\mu,q+\nu}, v_{p+\mu,q+\nu})\\
&+\varepsilon^{-1}\{\|h_{pq}\|^2+\|r_{pq,N_0}v\|^2\}+\varepsilon\|v_{pq}\|^2\,,
\end{aligned}
$$

when n is sufficiently large. Now, recalling (5.3), we can easily see that the same arguments as in the proof of Theorem 2 can be applied to this case.

Proof of Theorem 4. The same arguments as in the proof of Proposition 3.1 together with the assumption (1.12) yield

$$
\int g(t)(\log|\xi|)|v(t)|^2 dt\leqq\varepsilon\left|\int P_\zeta v(t)\cdot\overline{v(t)}\,dt\right| \tag{5.5}
$$

and

$$
\int f(t)(\log|\eta|)^2|v(t)|^2 dt\leqq\varepsilon\left|\int P_\zeta v(t)\cdot\overline{v(t)}\,dt\right|, \tag{5.6}
$$

where $P_\zeta=D_t^2+f(t)\xi^2+ig(t)\eta$. In order to prove Theorem 4 applying these inequalities, we have only to obtain

$$
|(Pv_{pq}, v_{pq})|\leqq\varepsilon\sum_{0\leqq|\nu|\leqq 2}(\log n)^{-2|\nu|}|(Pv_{p,q+\nu}, v_{p,q+\nu})|+\varepsilon^{-1}\|h_{pq}\|^2+\varepsilon\|v_{pq}\|^2\,. \tag{5.7}
$$

We shall show this, supposing $(\xi^0, \eta^0)\in\{(\xi, \eta);\ |\eta|\leqq 2|\xi|\}$. If $(\xi^0, \eta^0)\in\{(\xi, \eta);\ |\xi|\leqq 2|\eta|\}$, one can show (5.7), applying (5.6), by the same arguments as in the preceding section.

1′) For $\nu=(k, 0)$ with $k=1$ or $=2$ ($P^{(\nu)}=f(t)D_x^{2-k}$): We can do in the same way as 1) in the proof of Proposition 3.2.

2′) For $\nu=(0, 1)$. ($P^{(\nu)}=ig(t)$. Also notice that $|\nu|=1$): The inequality (5.5) together with the fact that $c^{-1}\cdot n\leqq|\xi|\leqq c\cdot n$ for $(\xi, \eta)\in\mathrm{supp}[\alpha_n]$ will yield

$$
\begin{aligned}
&|(P^{(\nu)} v_{p,q+\nu}, v_{pq})|\\
&=\left|\iiint g(t)\, v^\wedge_{p,q+\nu}(t;\xi,\eta)\overline{v^\wedge_{pq}(t;\xi,\eta)}\,dt\,d\xi\,d\eta\right|\\
&\leqq(\log n)\iiint g(t)|v^\wedge_{pq}|^2 dt\,d\xi\,d\eta+(\log n)^{-1}\iiint g(t)|v^\wedge_{q,p+\nu}|^2 dt\,d\xi\,d\eta\\
&\leqq\varepsilon|(Pv_{pq}, v_{pq})|+\varepsilon(\log n)^{-2|\nu|}|(Pv_{p,q+\nu}, v_{p,q+\nu})|\,.
\end{aligned}
$$

Thus we can obtain (5.7). Now by the same arguments as in section 4, we can prove Theorem 4.

§ 6. Proof of Lemma 4.1.

Here we give the proof of Lemma 4.1.

Necessity: First we suppose $\phi=\chi(t)\psi(x, y)$ plays the same role in definition of microlocal smoothness of u, where $\chi\in C_0^\infty$ with $\chi=1$ in a neighborhood of $t=t_0$ and $\psi\in C_0^\infty$ with $\psi=1$ in a neighborhood of $(x, y)=(x_0, y_0)$. Furthermore, choose $r_0>0$ sufficiently small so that $\beta_n \Subset \psi$ and $\text{supp}[\alpha_n]\subset\Gamma_0$. Now let us put $v=\chi u$ and consider the equation

$$(6.1)\qquad c_{pq}^n\alpha_n^{(p)}(D_x, D_y)\beta_{n(q)}(x, y)v = c_{pq}^u\alpha_n^{(p)}\beta_{n(q)}(\psi v)$$
$$= \sum_{|\nu|\leqq N_0}\nu!^{-1}c_{pq}^n\beta_{n(q+\nu)}\alpha_n^{(p+\nu)}(\psi v)+c_{pq}^n r_{pq,N_0}(\psi v)\,.$$

On the first terms on right hand side of (6.1), we have

$$\|\beta_{n(q+\nu)}\alpha_n^{(p+\nu)}(\psi v)\|^2\leqq(\sup|\beta_{n(q+\nu)}|\cdot\sup|\alpha_n^{(p+\nu)}|)^2\times$$
$$\times\text{const. } n^{-2s}\iiint_{\Gamma_0}|\widehat{\psi v}(\tau, \xi, \eta)|^2(1+\xi^2+\eta^2)^s d\tau\, d\xi\, d\eta\,.$$

Therefore, if $u\in H^{0,\infty}$ at $(t_0, x_0, y_0; \xi^0, \eta^0)$,

$$(6.2)\qquad \sum_{|p+q|\leqq N_n}\ \sum_{|\nu|\leqq N_0}\|c_{pq}^n\beta_{n(q+\nu)}\alpha_n^{(p+\nu)}(\psi v)\|^2 = O(n^{-2s})\,,$$

for any positive number s. (Recall that $c_{pq}^n=M^{-|p+q|}n^{-|p|}(\log n)^{-|p+q|}$ and the choice of $\{\alpha_n, \beta_n\}$.)

Concerning the last term on right hand side of (6.1), let us observe that

$$\|r_{pq,N_0}\|_{\mathcal{L}(H^{0,-k}, L^2)}\leqq\text{const. }|r_{pq,N_0}|_{l_1}^{(-k)}$$
$$\leqq\text{const.}\sum_{|\nu|=N_0+1}|\alpha_n^{(p+\nu)}|_{l_2}^{(-k)}|\beta_{n(q+\nu)}|_{l_2}^{(0)}$$
$$\leqq\text{const. } n^{k-|p|-N_0-1}(CN_n)^{|p+q|}\,.$$

Therefore, if $u\in H^{0,-k}$ at (t_0, x_0, y_0) (for some positive number k),

$$(6.3)\qquad \sum_{|p+q|\leqq N_n}\|c_{pq}^n r_{pq,N_0}(\psi v)\|^2$$
$$\leqq\text{const.}\sum_{|p+q|\leqq N_n}\|c_{pq}^n r_{pq,N_0}\|^2_{\mathcal{L}(H^{0,-k}, L^2)}$$
$$\leqq\text{const. } n^{2(k-N_0-1)}\,.$$

Moreover, the right hand side of (6.3) is estimated by const. n^{-2s} if we take N_0 so that $N_0+1-k\geqq s$.

Now, combining (6.1), (6.2) and (6.3), we can conclude that $S_n^M(\chi u)=O(n^{-2s})$ for any positive number s.

Sufficiency: First we show that, for any $\psi\in C_0^\infty(\boldsymbol{R}^2)$ with support sufficiently small such that $\psi \subset \beta_n$, $\|\alpha_n(D_x, D_y) = \psi v\| = O(n^{-s})$ if $S_n^M(v) = O(n^{-2s})$ (where $v = \chi u$).

To see this, let us consider the following inequality:

$$\begin{aligned}\|\alpha_n \psi v\| &= \|\alpha_n \psi(\beta_n v)\| \\ &\leqq \sum_{|\nu| \leqq N_0} \nu!^{-1} \sup |\psi_{(\nu)}| \|\alpha_n^{(\nu)} \beta_n v\| + \|r_{N_0} \beta_n v\| .\end{aligned}$$

First observe that,

$$\|\alpha_n^{(\nu)} \beta_n v\| = O(n^{-s-|\nu|}(\log n)^{|\nu|}) \qquad \text{if} \quad S_n^M(v) = O(n^{-2s}) .$$

Moreover, by similar argument as in the first part of the proof, we have $\|r_{N_0} \beta_n v\| = O(n^{k-N_0-1})$ when $u \in H^{0,-k}$ at (t_0, x_0, y_0). Hence, taking N_0 so that $N_0 + 1 - k \geqq s$, we see $\|\alpha_n \psi v\| = O(n^{-s})$ if $S_n^M(v) = O(n^{-2s})$ and u is locally of class $H^{0,-k}$ at (t_0, x_0, y_0).

Next, let us observe that

$$\sum_{n=1}^{\infty} \alpha_n(\xi, \eta)^2 n^{2s-1-\varepsilon} \geqq \text{const.} (1 + \xi^2 + \eta^2)^{s-\varepsilon/2} ,$$

for (ξ, η) contained in some conic neighborhood Γ_0 of (ξ^0, η^0) and $\xi^2 + \eta^2 \geqq 1$. This fact can be seen by noticing that $n^2 \geqq \text{const.} (1 + \xi^2 + \eta^2)$ for $(\xi, \eta) \in \text{supp}[\alpha_n]$ and that, for $(\xi, \eta) \in \Gamma_0$ satisfying $\xi^2 + \eta^2 \geqq 1$, the number of n such that $\alpha_n(\xi, \eta) = 1$ is estimated from below by const. $(1 + \xi^2 + \eta^2)^{1/2}$.

Thus, combining the above arguments, we see that

$$\begin{aligned}&\iint_{\Gamma_0} \int |\widehat{\psi v}(\tau, \xi, \eta)|^2 (1 + \xi^2 + \eta^2)^{s-\varepsilon/2} d\tau \, d\xi \, d\eta \\ &\leqq \sum_{n=1}^{\infty} \|\alpha_n(D_x, D_y) \psi v\|^2 \cdot n^{2s-\varepsilon-1} < \infty ,\end{aligned}$$

for every $u \in \bigcup_{k>0} H^{0,-k}$ satisfying $S_n^M(\chi u) = O(n^{-2s})$.

Now, the proof is complete. *q.e.d.*

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