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The decomposition of the spaces of cusp forms of half-integral weight and trace formula of Hecke operators

By

Masaru UEDA

Introduction.

Let k be a positive integer and N a positive integer divisible by 4. For an even character χ modulo N, we denote by $S(k+1/2, N, \chi)$ the space of cusp forms with weight $k+1/2$, level N and character χ. Suppose $k \geqq 2$. For a primitive form F of $S(2k, N/2, \chi^2)$, we define a subspace $S(k+1/2, N, \chi; F)$ by:

$$S\left(k+\frac{1}{2}, N, \chi;\ F\right)=\left\{\begin{array}{c} S\left(k+\frac{1}{2}, N, \chi\right) \ni f;\ f \mid \tilde{T}(p^2)=\lambda_F(p) f \\ \text{for all prime numbers } p \nmid N \end{array}\right\}.$$

Here, we denote by $\tilde{T}(p^2)$ the Hecke operator on $S(k+1/2, N, \chi)$ and $\{\lambda_F(p)\}$ is the system of eigen values of F with respect to the Hecke operator $T(p)$ on $S(2k, N/2, \chi^2)$. Then, the following decomposition is well-known:

$$(1) \qquad S\left(k+\frac{1}{2}, N, \chi\right)=\bigoplus_F S\left(k+\frac{1}{2}, N, \chi;\ F\right),$$

where the direct sum is extended over all primitive forms of $S(2k, N/2, \chi^2)$(cf. [Sh 1] Lemma 7). Note that we can also obtain a similar decomposition for the case $k=1$ after slight modifications. Then, from the decomposition (1), we can expect that there exist some relations between traces of $\tilde{T}(p^2)$ and traces of $T(p)$.

Our main purpose in this paper is to study relations between these two traces for several cases. In [N], S. Niwa already took up this problem for the case of a cubic-free level N and the trivial character χ_0. He calculated the trace of the Hecke operator $\tilde{T}(n^2)$ on $S(k+1/2, N, \chi_0)$ for all natural numbers n with $(n, N)=1$ and compared them with the traces of the Hecke operator $T(n)$ on $S(2k, N/2, \chi_0)$. Then, he found that these two traces have a simple relation. For example, if $N/4$ is square-free, these two traces coincide.

We shall generalize these results in § 1 and § 3. In § 1, we shall explicitly calculate the trace of the Hecke operator on $S(k+1/2, N, \chi)$ under the assumption: $\chi^2=1$, and in § 3, we shall prove a relation between these traces.

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Next, suppose that $N=4M$ with $(M, 2)=1$ and $\chi^2=1$. Then, in [K], W. Kohnen defined a canonical subspace $S(k+1/2, N, \chi)_K$ of $S(k+1/2, N, \chi)$ and some Hecke operators on that subspace (cf. § 0 (**d**)). Moreover, when M is square-free, he calculated the traces of those operators and found that those traces coincide with the traces of the Hecke operators on $S(2k, M, \chi_0)$, where χ_0 is the trivial character.

We shall also generalize these results in § 2 and § 3. In § 2, we shall explicitly calculate those traces for any odd integer M and prove a relation between traces in § 3. Moreover, in § 4, we shall give some examples of the explicit decomposition of $S(k+1/2, N, \chi)_K$, which is the same type as the decomposition (1).

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§ 0. Preliminaries.

(a) General notations.

Let k denote a positive integer. If $z\in \boldsymbol{C}$ and $x\in \boldsymbol{C}$, we put $z^x=\exp(x\cdot\log(z))$ with $\log(z)=\log(|z|)+\sqrt{-1}\arg(z)$, $\arg(z)$ being determined by $-\pi<\arg(z)\leqq\pi$. Also, we put $\boldsymbol{e}(z)=\exp(2\pi\sqrt{-1}z)$.

Let $\mathfrak{H}$ be the complex upper half plane. For a complex-valued function $f(z)$ on $\mathfrak{H}$, $\alpha=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in GL_2^+(\boldsymbol{R})$, $\gamma=\begin{pmatrix} u & v \\ w & x\end{pmatrix}\in\Gamma_0(4)$ and $z\in\mathfrak{H}$, we define functions $J(\alpha, z)$, $j(\gamma, z)$ and $f|[\alpha]_k(z)$ on $\mathfrak{H}$ by: $J(\alpha, z)=cz+d$, $j(\gamma, z)=\left(\frac{-1}{x}\right)^{-1/2}\left(\frac{w}{x}\right)(wz+x)^{1/2}$ and $f|[\alpha]_k(z)=(\det\alpha)^{k/2}J(\alpha, z)^{-k}f(\alpha z)$.

For a natural number n, we denote by $\varphi(n)$ the cardinality of $(\boldsymbol{Z}/n\boldsymbol{Z})^\times$. Put $h(-n)=$ the class number of proper ideal classes of the order with discriminant $-n$ in the imaginary quadratic number field $\boldsymbol{Q}(\sqrt{-n})$, $w(-n)=$a half of the cardinality of the unit group of the above order and $h'(-n)=h(-n)/w(-n)$.

For a real number x, $[x]$ means the greatest integer m with $x\geqq m$. When $n=\prod_{q|n} q^\nu$ is the decomposition to prime numbers q of a natural number n, we put

$$\alpha_u(n)=\prod_{q|n}\left\{(q^{\nu+1}-1)-\left(\frac{u}{q}\right)(q^\nu-1)\right\}/(q-1).$$

For a finite-dimensional vector space V over $\boldsymbol{C}$ and a linear operator T on V, trace$(T|V)$ denotes the trace of T on V.

(b) Modular forms of integral weight.

Let N be a positive integer. By $S(2k, N)$, we denote the space of all holomorphic cusp forms of weight $2k$ with the trivial character on the group $\Gamma=\Gamma_0(N)$.

Let $\alpha\in GL_2^+(\boldsymbol{R})$. If Γ and $\alpha^{-1}\Gamma\alpha$ are commensurable, we define a linear operator $[\Gamma\alpha\Gamma]_{2k}$ on $S(2k, N)$ by: $f|[\Gamma\alpha\Gamma]_{2k}=(\det\alpha)^{k-1}\sum_{\alpha_i} f|[\alpha_i]_{2k}$, where α_i runs over a system of representatives for $\Gamma\backslash\Gamma\alpha\Gamma$.

For a natural number n with $(n, N)=1$, we put $T_{2k,N}(n)=\sum_{ad=n}\left[\Gamma\begin{pmatrix} a & 0 \\ 0 & d\end{pmatrix}\Gamma\right]_{2k}$, where the sum is extended over all pairs of integers (a, d) such that $a, d>0$, $a|d$, $ad=n$.

(c) Modular forms of half-integral weight.

Let N be a positive integer divisible by 4 and χ an even character modulo N such that $\chi^2=1$. Put $\mu=\mathrm{ord}_2(N)$, $M=2^{-\mu}N$ and $\Gamma=\Gamma_0(N)$. Then, there is a square-free odd positive divisor M_0 of M such that $\chi=\left(\frac{M_0}{}\right)$ or $\left(\frac{2M_0}{}\right)$ (the Kronecker symbol).

Let $G(k+1/2)$ be the group consisting of pairs (α, φ), where $\alpha=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in GL_2^+(\boldsymbol{R})$ and φ is a holomorphic function on $\mathfrak{H}$ satisfying $\varphi(z)=t(\det\alpha)^{-k/2-1/4}J(\alpha, z)^{k+1/2}$ with $t\in\boldsymbol{C}$ and $|t|=1$. The group law is defined by: $(\alpha, \varphi(z))\cdot(\beta, \psi(z))=(\alpha\beta, \varphi(\beta z)\psi(z))$. For a complex-valued function f on $\mathfrak{H}$ and $(\alpha, \varphi)\in G(k+1/2)$, we define a function $f|(\alpha, \varphi)$ on $\mathfrak{H}$ by: $f|(\alpha, \varphi)(z)=\varphi(z)^{-1}f(\alpha z)$.

By $\Delta=\Delta_0(N, \chi)=\Delta_0(N, \chi)_{k+1/2}$, we denote the subgroup of $G(k+1/2)$ consisting of all pairs (γ, φ), where $\begin{pmatrix} a & b \\ c & d\end{pmatrix}=\gamma\in\Gamma$ and $\varphi(z)=\chi(d)j(\gamma, z)^{2k+1}$. We denote by $G(k+1/2, N, \chi)$ (resp. $S(k+1/2, N, \chi)$) the space of integral (resp. cusp) forms of weight $k+1/2$ with the character χ on the group Γ, namely, the space of all the complex-valued holomorphic function f on $\mathfrak{H}$ which satisfies $f|\xi=f$ for all $\xi\in\Delta$ and which is holomorphic (resp. is holomorphic and vanish) at all cusps of Γ. In particular, we write $S(k+1/2, N)=S(k+1/2, N, \chi)$ if χ is the trivial character.

Now, for $\nu=0$ or 1, we denote by $\Omega^\nu(N, \chi)$ the set of all pairs (ψ, t), where ψ is a primitive character modulo r with $\psi(-1)=(-1)^\nu$ and t is a positive integer, which satisfy the following two conditions:

(0.1). $4tr^2|N$.

(0.2). $\chi=\left(\frac{\psi(-1)t}{}\right)\psi$ *as a character modulo* N.

Then, we consider the theta series of the following type: $h^\nu(\psi; z)=(1/2)\sum_{m\in\boldsymbol{Z}}\psi(m)m^\nu e(m^2z)$, where $z\in\mathfrak{H}$ and $\nu=0$ or 1.

For the case $\nu=0$, we know that $\{h^0(\psi; tz)|(\psi, t)\in\Omega^0(N, \chi)\}$ is a $\boldsymbol{C}$-basis of the space $G(1/2, N, \chi)$ (cf. [S-S]). For the case $\nu=1$, let $U(N, \chi)$ be the subspace of $S(3/2, N, \chi)$ generated by $\{h^1(\psi; tz)|(\psi, t)\in\Omega^1(N, \chi)\}$ over $\boldsymbol{C}$. By $V(N, \chi)$, we denote the orthogonal complement of $U(N, \chi)$ in $S(3/2, N, \chi)$ with respect to the Petersson inner product.

Let $\xi\in G(k+1/2)$. If Δ and $\xi^{-1}\Delta\xi$ are commensurable, we define a linear operator $[\Delta\xi\Delta]_{k+1/2}$ on $G(k+1/2, N, \chi)$ and $S(k+1/2, N, \chi)$ by: $f|[\Delta\xi\Delta]_{k+1/2}=\sum_{\eta} f|\eta$, where η runs over a system of representatives for $\Delta\backslash\Delta\xi\Delta$. Then, for a natural number n with $(n, N)=1$, we put

$$\tilde{T}_{k+1/2,N,\chi}(n^2)$$
$$=n^{k-3/2}\sum_{ad=n} a\left[\Delta\left(\begin{pmatrix} a^2 & 0 \\ 0 & d^2 \end{pmatrix}, (d/a)^{k+1/2}\right)\Delta\right]_{k+1/2},$$

where the sum is extended over all pairs of integers (a, d) such that $a, d>0$, $a|d$ and $ad=n$.

For $k=1$, from [Sh 2] Theorem 1.7, it follows that $h^1(\psi; tz)$ with $(\psi, t)\in \Omega^1(N, \chi)$ is an eigen function of the Hecke operators $\tilde{T}_{3/2,N,\chi}(p^2)$ for all prime numbers $p\nmid N$. Hence, we see that $U(N, \chi)$ and $V(N, \chi)$ are invariant under the action of the Hecke operators $\tilde{T}_{3/2,N,\chi}(n^2)$ for all natural numbers n with $(n, N)=1$ (cf. [Sh 1] Lemma 5).

$U(N, \chi)$ corresponds to the space of the Eisenstein series through the Shimura correspondence and only the elements of $V(N, \chi)$ correspond to the cusp forms (cf. [St]). Hence, when $k=1$, we shall be dealing with $V(N, \chi)$ in place of $S(3/2, N, \chi)$ and consider only the traces of $\tilde{T}_{3/2,N,\chi}(n^2)$ on $V(N, \chi)$.

(d) The Kohnen subspace.

Suppose that $N=4M$ and M is an odd natural number. Then, $\chi=\left(\frac{M_0}{\cdot}\right)$ for some positive divisor M_0 of M (cf. §0 (**c**)). Put $\varepsilon=\left(\frac{-1}{M_0}\right)$. Then, the Kohnen subspace $S(k+1/2, N, \chi)_K$ is defined as follows:

$$S\left(k+\frac{1}{2}, N, \chi\right)_K=\left\{\begin{array}{l} S\left(k+\frac{1}{2}, N, \chi\right)\ni f(z)=\sum_{n=1}^{\infty} a(n)e(nz)\,; \\ a(n)=0 \qquad \text{for} \quad \varepsilon(-1)^k n\equiv 2, 3 (\mathrm{mod}\, 4) \end{array}\right\}.$$

In particular, we write $S(k+1/2, N)_K=S(k+1/2, N, \chi)_K$ if χ is the trivial character.

Put $\xi=\xi_{k+1/2,\varepsilon}=\left(\begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}, \varepsilon^{k+1/2}e((2k+1)/8)\right)\in G\left(k+\frac{1}{2}\right)$ and $Q=Q_{k+1/2,N,\chi}=[\Delta\xi\Delta]_{k+1/2}$. Then, Q becomes a hermitian operator on $S(k+1/2, N, \chi)$. Moreover, from [K] Proposition 1, we know that $S(k+1/2, N, \chi)_K$ is the α-eigen subspace of $S(k+1/2, N, \chi)$ with respect to the operator Q, where $\alpha=(-1)^{[(k+1)/2]}2\sqrt{2}\,\varepsilon$.

For $k=1$, from the definitions of $S(3/2, N, \chi)_K$ and $U(N, \chi)$, it is easily shown that $S(3/2, N, \chi)_K$ contains $U(N, \chi)$. Then, we denote by $V(N, \chi)_K$ the orthogonal complement of $U(N, \chi)$ in $S(3/2, N, \chi)_K$ with respect to the Petersson inner product.

From [K] §3 and §4, we know that $S(k+1/2, N, \chi)_K$ (resp. $V(N, \chi)_K$) is invariant under the action of the Hecke operators $\tilde{T}_{k+1/2,N,\chi}(n^2)$ (resp. $\tilde{T}_{3/2,N,\chi}(n^2)$) for all natural numbers n with $(n, N)=1$. Hence, we can consider the traces of those Hecke operators on $S(k+1/2, N, \chi)_K$ and $V(N, \chi)_K$.

§ 1. The trace formula for the Hecke operator of half-integral weight.

Throughout this section, we shall use the same notations and assumptions as in § 0 (**a**) and (**c**).

Now, we shall give an explanation of the Shimura's trace formula (cf. [Sh 3]).

We take $\tau=(\alpha, h)\in G(k+1/2)$ with $\alpha\in SL_2(\boldsymbol{R})$ which satisfies the following conditions:

(1.1) *$\Gamma=\Gamma_0(N)$ and $\alpha^{-1}\Gamma\alpha$ are commensurable.*

(1.2) *We define a proper lifting L by:*

$L(\gamma)=(\gamma, \chi(d)j(\gamma, z)^{2k+1})\in G\left(k+\frac{1}{2}\right)$, $\gamma=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\Gamma$ and $z\in\mathfrak{H}$. Then, $L(\alpha\gamma\alpha^{-1})=\tau L(\gamma)\tau^{-1}$ for all $\gamma\in\Gamma\cap\alpha^{-1}\Gamma\alpha$.

For this τ, from [Sh 2] Proposition 1.1, we have the bijection: $\Gamma\alpha\Gamma\ni\gamma_1\alpha\gamma_2\to L(\gamma_1)\tau L(\gamma_2)\in\Delta\tau\Delta$, where $\Delta=\Delta_0(N, \chi)_{k+1/2}$ and $\gamma_1, \gamma_2\in\Gamma$. Moreover, Δ and $\tau^{-1}\Delta\tau$ are commensurable. In the following, we denote by $\beta^*=(\beta, h(\beta; z))$ the image of $\beta\in\Gamma\alpha\Gamma$ with respect to the above bijection.

Next, we put $\tau'=(\alpha^{-1}, h(\alpha^{-1}z)J(\alpha^{-1}, z)^2)$. Then, τ' also satisfies the conditions (1.1) and (1.2) with respect to a proper lifting L': $\Gamma\ni\gamma=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\to(\gamma, \chi(d)j(\gamma, z)^{3-2k})$. Hence, $\Delta'=\Delta_0(N, \chi)_{3/2-k}$ and $\tau'^{-1}\Delta'\tau'$ are also commensurable.

From [Sh 3] Theorem 4.5 and the assumption: $\chi^2=1$, we have the following trace formula:

$$\begin{aligned}(1.3)\qquad &\mathrm{trace}\left([\Delta\tau\Delta]_{k+1/2}\,|\,S\left(k+\frac{1}{2}, N, \chi\right)\right)\\ &-\mathrm{trace}\left([\Delta'\tau'\Delta']_{3/2-k}\,|\,G\left(\frac{3}{2}-k, N, \chi\right)\right)\\ &=\sum_{C\in\Phi(\Gamma\alpha\Gamma/\Gamma)} J(C).\end{aligned}$$

Here, the notations are as follows:

Let $\Phi(\Gamma\alpha\Gamma)$ denote the subset of $\Gamma\alpha\Gamma$ consisting of: all scalar elements, all elliptic elements, all hyperbolic elements whose upper fixed points (cf. [Sh 3] § 3.6) are cusps of Γ and all parabolic elements whose fixed points are cusps of Γ. We call two elements β and β' in $\Phi(\Gamma\alpha\Gamma)$ equivalent if: When β and β' are scalars or elliptic or hyperbolic, $\gamma\beta\gamma^{-1}=\beta'$ for some $\gamma\in\Gamma$; When β and β' are parabolic, $\gamma\beta'\gamma^{-1}\in Z_\Gamma(\beta)\beta$ for some $\gamma\in\Gamma$, where $Z_\Gamma(\beta)=\{\gamma\in\Gamma\,|\,\gamma\beta=\beta\gamma\}$. We denote by $\Phi(\Gamma\alpha\Gamma/\Gamma)$ the set of all equivalence classes in $\Phi(\Gamma\alpha\Gamma)$ with respect to the above equivalence relation. For each $C\in\Phi(\Gamma\alpha\Gamma/\Gamma)$, we pick any β from C. Then, the complex number $J(C)$ is given as follows:

(i) If $\beta^*=(\pm1, \eta)$, $J(C)=(1/8)(2k-1)\eta^{-1}|\Gamma_0(4):\Gamma_0(N)|$.

(ii) When β is elliptic, let $z_0\in\mathfrak{H}$ be the fixed point of β,

$$\alpha=\begin{pmatrix}\bar{z}_0 & z_0\\ 1 & 1\end{pmatrix},\ \alpha^{-1}\beta\alpha=\begin{pmatrix}\bar{\lambda} & 0\\ 0 & \lambda\end{pmatrix},\ \eta=h(\beta;\ z_0)$$

and

$$\sigma(\beta)=\#\{\gamma\in\Gamma\mid\gamma z_0=z_0\}.$$

Then, $J(C)=\{\sigma(\beta)\eta(1-\lambda^{-2})\}^{-1}$.

(iii) When β is hyperbolic, let $z_0\in\boldsymbol{Q}\cup\{\infty\}$ be the upper fixed point of β. Take an element $\rho^*=(\rho,\varphi)\in G(k+1/2)$ such that $\rho\in SL_2(\boldsymbol{R})$ and that $\rho(\infty)=z_0$. Then, we put

$$\rho^{*-1}\beta^*\rho^*=\left(\begin{pmatrix}\lambda^{-1} & x\\ 0 & \lambda\end{pmatrix},\ \eta\right)\quad\text{and}\quad J(C)=-(1/2)\{\eta(1-\lambda^{-2})\}^{-1}.$$

(iv) When β is parabolic, let $z_0\in\boldsymbol{Q}\cup\{\infty\}$ be the fixed point of β and σ an element of Γ which generates $\{\gamma\in\Gamma\mid\gamma z_0=z_0\}/\{\pm1\}$. Take an element $\rho^*=(\rho,\varphi)\in G(k+1/2)$ such that $\rho\in SL_2(\boldsymbol{R})$ and that $\rho(\infty)=z_0$ and that $\rho^{-1}\sigma\rho=\pm\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$. We write $\rho^{*-1}L(\sigma)\rho^*=\left(\pm\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix},\ \boldsymbol{e}(\delta)\right)$ with $0<\delta\leqq1$ and $\rho^{*-1}\beta^*\rho^*=\left(\pm\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix},\ \eta\right)$ with $x\in\boldsymbol{R}$. Then, for $\beta\in\Gamma$ (resp. $\beta\notin\Gamma$), we put $J(C)=\eta^{-1}\boldsymbol{e}(\delta x)(1/2-\delta)$ (resp. $\eta^{-1}\boldsymbol{e}(\delta x)(1-\boldsymbol{e}(x))^{-1}$).

Our purpose in this section is to prove the following proposition.

Proposition 1. (1) *Suppose* $k\geqq2$. *For all natural numbers* n *with* $(n,N)=1$, *we have*:

$$\operatorname{trace}\left(\tilde{T}_{k+1/2,N,\chi}(n^2)\mid S\left(k+\frac{1}{2},\ N,\ \chi\right)\right)=T(s)+T(p)+T(e)+T(h).$$

(2) *Let* n *be the same as in* (1). *Then, we have*:

$$\operatorname{trace}(\tilde{T}_{3/2,N,\chi}(n^2)\mid V(N,\chi))=T(s)+T(p)+T(e)+T(h)+T(d).$$

Here, the terms $T(s)$, $T(p)$, $T(e)$, $T(h)$ *and* $T(d)$ *are given by the formulas* (1.6), (1.9), (1.10), (1.11) *and* (1.12) *in the following calculations.*

Proof. For a simplicity, we use the following notations:
We put

$$\Gamma=\Gamma_0(N),\ \Delta=\Delta_0(N,\chi)_{k+1/2},\ \alpha(n)=\begin{pmatrix}n^{-1} & 0\\ 0 & n\end{pmatrix}$$

and

$$\tau(k;\ n)=(\alpha(n),\ n^{k+1/2})\in G\left(k+\frac{1}{2}\right).$$

For a prime number p,

$$\operatorname{ord}_p(N)=\tilde{\nu}_p=\tilde{\nu}=\begin{cases}\nu_p=\nu, & \text{if } p \text{ is odd};\\ \mu, & \text{if } p=2.\end{cases}$$

By χ_p, we denote the p-component of the character χ for any prime number $p\mid N$ and, by $f(\chi_p)$, the conducter of χ_p. Put

$$\delta_0(\sqrt{n})=\begin{cases}1, & \text{if } n \text{ is square;}\\ 0, & \text{otherwise;}\end{cases}$$

and

$$\delta_1=\begin{cases}1, & \text{if } \mu=2;\\ 0, & \text{otherwise.}\end{cases}$$

For a natural number n, let $n=n_0^2 n_1$, where n_0 is a positive integer and n_1 is a square-free positive integer.

Now, we shall firstly calculate the trace of the operator $[\Delta\tau(k;\ n)\Delta]$ by using the Shimura's trace formula (1.3). Secondly, by summing up them, we shall obtain the trace of $\tilde{T}_{k+1/2,N,\chi}(n^2)$ on $S(k+1/2, N, \chi)$ and $V(N, \chi)$.

Let $\tilde{s}(n)$, $\tilde{p}(n)$, $\tilde{e}(n)$ and $\tilde{h}(n)$ be the contribution from the scalar, parabolic, elliptic and hyperbolic equivalence classes in $\Phi=\Phi(\Gamma\alpha(n)\Gamma/\Gamma)$ respectively. Moreover, if $k\geqq 2$, $G(3/2-k, N, \chi)=\{0\}$. Hence, the contribution from the trace on $G(3/2-k, N, \chi)$ occurs only when $k=1$. Then, we put

$$\tilde{d}(n)=\text{trace}([\Delta'\tau'(n)\Delta']_{1/2}|G(1/2, N, \chi))$$

$$-\text{trace}([\Delta\tau(1;\ n)\Delta]_{3/2}|U(N, \chi)),$$

where

$$\tau'(n)=(\alpha(n)^{-1}, n^{-1/2}) \quad\text{and}\quad \Delta'=\Delta_0(N, \chi)_{1/2}.$$

By using these notation, we can write

$$\text{trace}\Big([\Delta\tau(k;\ n)\Delta]|S\Big(k+\frac{1}{2}, N, \chi\Big)\Big)$$

$$=\tilde{s}(n)+\tilde{p}(n)+\tilde{e}(n)+\tilde{h}(n) \qquad \text{if} \quad k\geqq 2,$$

and

$$\text{trace}([\Delta\tau(1;\ n)\Delta]|V(N, \chi))$$

$$=\tilde{s}(n)+\tilde{p}(n)+\tilde{e}(n)+\tilde{h}(n)+\tilde{d}(n).$$

Now, before calculating the each term, we give some remarks.

Remark (1.4). If an equivalence class $C\in\Phi$ is not scalar, we can choose an element $\beta=n^{-1}\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ from C such that $(a, c)=1$ and that $c\neq 0$ (cf. [N] Remark 1). Put $\beta'=n\beta=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma\begin{pmatrix}1 & 0\\ 0 & n^2\end{pmatrix}\Gamma$, then, there exist u, v and $w\in\boldsymbol{Z}$ such that $\beta'=\sigma_1\begin{pmatrix}1 & 0\\ 0 & n^2\end{pmatrix}\sigma_2$ wlth $\sigma_1=\begin{pmatrix}a & -v\\ c & u\end{pmatrix}\in\Gamma$ and $\sigma_2=\begin{pmatrix}1 & w\\ 0 & 1\end{pmatrix}\in\Gamma$. Therefore, from $c\equiv 0 \pmod{N}$,

$$\beta^*=L(\sigma_1)\tau(k;\ n)L(\sigma_2)$$

$$=(\sigma_1, \chi(u)j(\sigma_1, z)^{2k+1})\tau(k;\ n)(\sigma_2, j(\sigma_2, z)^{2k+1})$$

$$=\Big(\beta, \chi(a)\Big(\frac{-1}{a}\Big)^{-k-1/2}\Big(\frac{c}{a}\Big)J(\beta, z)^{k+1/2}\Big).$$

Remark (1.5). Suppose that β is parabolic or hyperbolic. Then, β has a fixed point κ which is the cusp of Γ. Since $c\neq 0$, we know $\kappa\neq\infty$. Let $\rho=\begin{pmatrix}\kappa & \kappa-1\\ 1 & 1\end{pmatrix}\in SL_2(\boldsymbol{R})$ and $\rho^*=(\rho, J(\rho, z)^{k+1/2})\in G(k+1/2)$. Then, by calculating with attention to the signature of the branch, we have

$$\rho^{*-1}\beta^*\rho^*=(\rho^{-1}, J(\rho^{-1}, z)^{k+1/2})\left(\beta, \chi(a)\left(\frac{-1}{a}\right)^{-k-1/2}\left(\frac{c}{a}\right)J(\beta, z)^{k+1/2}\right)$$
$$\times(\rho, J(\rho, z)^{k+1/2})$$
$$=\left(\begin{pmatrix}\lambda^{-1} & y\\ 0 & \lambda\end{pmatrix}, \left(\frac{\operatorname{sgn}(\lambda)}{\operatorname{sgn}(c)}\right)\chi(a)\left(\frac{-1}{a}\right)^{-k-1/2}\left(\frac{c}{a}\right)\lambda^{k+1/2}\right),$$

where $\lambda=(a-c\kappa)/n$, $y=(-a+d-c+2c\kappa)/n$ and $\operatorname{sgn}(x)=1, -1$ according to $x\geqq 0$, $x<0$.

Now, we shall begin the calculation of the each term.

1. The calculation of $\tilde{s}(n)$.

Obviously, $\Gamma\alpha(n)\Gamma$ contains a scalar element if and only if $n=1$. In that case, since $(\pm 1)^*=(\pm 1, 1)$, we have

$$\tilde{s}(n)=\begin{cases} 2^{\mu-4}(2k-1)M\prod_{p\mid M}(p+1)/p, & \text{if } n=1;\\ 0, & \text{otherwise}.\end{cases}$$

Therefore, the contribution to the trace of $\tilde{T}_{k+1/2, N, \chi}(n^2)$ on $S(k+1/2, N, \chi)$ and $V(N, \chi)$ is:

$$(1.6)\qquad T(s)=n^{k-3/2}\sum_{0<a\mid n_0} a\tilde{s}(n/a^2)$$
$$=\delta_0(\sqrt{n})n^{k-1}2^{\mu-4}(2k-1)M\prod_{p\mid M}(p+1)/p.$$

2. The calculation of $\tilde{p}(n)$.

Now, all Γ-equivalence classes for the cusps of Γ are represented by the number t^{-1}, where t runs over the set:

$$S=\left\{\begin{array}{l} t=\zeta\prod_{p\mid N}p^e>0;\ 0\leqq e=e_p\leqq\nu \text{ and } \zeta \text{ runs over a system of repre-}\\ \text{sentatives, which is prime to } N, \text{ for } (\boldsymbol{Z}/\prod_{p\mid N}p^{\min(e,\nu-e)}\boldsymbol{Z})^\times.\end{array}\right\}$$

Let $A_t=\begin{pmatrix}1 & 0\\ t & 1\end{pmatrix}$. Then, the stabilizer of t^{-1} in $\Gamma/\{\pm 1\}$ is genrated by

$$\sigma=\begin{pmatrix}1-ut & u\\ -ut^2 & 1+ut\end{pmatrix}=A_t\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}A_t^{-1},$$

where u is the least natural number such that $ut^2\equiv 0 \pmod{N}$, namely, $u=\prod'_{p\mid N}p^{\nu-2e}$. Here, the meaning of the symbol $\prod'_{p\mid N}$ is as follows:

For any complex number $a(p)$, we put

$$a'(p)=\begin{cases} a(p), & \text{if the prime number } p \text{ satisfies the condition } 2e<\tilde{\nu}\,; \\ 1, & \text{otherwise.} \end{cases}$$

Then, we define $\prod_{p|N}' a(p)=\prod_{p|N} a'(p)$.

Let us write out all parabolic equivalence classes in $\Phi=\Phi(\Gamma\alpha(n)\Gamma/\Gamma)$. Let β, β_1 be two parabolic elements in $\Phi(\Gamma\alpha(n)\Gamma)$. Then, from the definition of the equivalence relation, it is easily seen that, if β and β_1 are equivalent, the fixed point of β must be Γ-equivalent to the fixed point of β_1. Hence, we may assume that the unique fixed point of β is t^{-1} with $t\in S$. Then, we have $A_t^{-1}\beta A_t=\pm n^{-1}\begin{pmatrix} n & \tau \\ 0 & n\end{pmatrix}$ for some non-zero real number τ. Hence,

$$\beta'=n\beta=\pm A_t\begin{pmatrix} n & \tau \\ 0 & n\end{pmatrix}A_t^{-1}=\pm\begin{pmatrix} n-t\tau & \tau \\ -t^2\tau & n+t\tau\end{pmatrix}.$$

Since $\Gamma\begin{pmatrix} 1 & 0 \\ 0 & n^2\end{pmatrix}\Gamma=\left\{\begin{matrix} M_2(\boldsymbol{Z})\ni\begin{pmatrix} a & b \\ c & d\end{pmatrix};\ c\equiv 0 \ (\mathrm{mod}\ N),\ (a,\ N)=1 \\ (a,\ b,\ c,\ d)=1,\quad ad-bc=n^2.\end{matrix}\right\}$,

we see that $\beta'\in\Gamma\begin{pmatrix} 1 & 0 \\ 0 & n^2\end{pmatrix}\Gamma$ if and only if $t^2\tau\equiv 0\ (\mathrm{mod}\ N)$ and $(n,\ \tau)=1$. In that case, τ becomes a multiple of u. Put $x=\tau/u$ and $\beta'(t,\ x)=\begin{pmatrix} n-txu & xu \\ -t^2xu & n+txu\end{pmatrix}$.

Suppose that such elements $\beta'(t,\ x_1)$ and $\beta'(t,\ x_2)$ are equivalent. Then, from the definition of the equivalence relation, there exist γ_1 and γ_2 in the stabilizer of t^{-1} in $\Gamma/\{\pm 1\}$ such that $\gamma_1^{-1}\gamma_2\beta'(t,\ x_1)\gamma_1=\beta'(t,\ x_2)$. Since that stabilizer is generated by σ, we can write $\gamma_1=\pm\sigma^a$ and $\gamma_2=\pm\sigma^b$ with $a,\ b\in\boldsymbol{Z}$. Hence,

$$\begin{aligned}\beta'(t,\ x_2)&=A_t\begin{pmatrix} n & x_2u \\ 0 & n\end{pmatrix}A_t^{-1}=\gamma_1^{-1}\gamma_2\beta'(t,\ x_1)\gamma_1 \\ &=\pm\sigma^{b-a}\beta'(t,\ x_1)\sigma^a=\pm A_t\begin{pmatrix} n & x_1u+bnu \\ 0 & n\end{pmatrix}A_t^{-1}.\end{aligned}$$

Therefore, we have $x_2=x_1+bn$.

From the above results, a system of representatives of all parabolic equivalence classes in Φ is formed by the matrices $\beta(t,\ x)=n^{-1}\beta'(t,\ x)$, where $t=\zeta\prod_{p|N} p^e$ runs over the set S and x runs over a system of representativatives for $(\boldsymbol{Z}/n\boldsymbol{Z})^\times$ which satisfies the condition $x\neq 0$. Here, by the suitable choice of the representative, we may assume that $4|x$ and that $(\zeta,\ n)=1$.

Now, we shall determine the number $J(\beta)=J(C)$ for the equivalence class C containing the matrix $\beta=\beta(t,\ x)$. Let $\rho=A_t\begin{pmatrix} u^{1/2} & 0 \\ 0 & u^{-1/2}\end{pmatrix}\in SL_2(\boldsymbol{R})$, then, we have $\rho^{-1}\sigma\rho=\begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix}$. Since $\beta=1+(x/n)(\sigma-1)$, we have $\rho^{-1}\beta\rho=\begin{pmatrix} 1 & x/n \\ 0 & 1\end{pmatrix}$.

Next, we take a lift $\rho^*\in G(k+1/2)$ of ρ, then, we must determine the numbers δ and η such that $\rho^{*-1}L(\sigma)\rho^{*}=\left(\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, e(\delta)\right)$ with $0<\delta\leqq 1$ and that $\rho^{*-1}\beta^{*}\rho^{*}=\left(\begin{pmatrix}1 & x/n\\ 0 & 1\end{pmatrix}, \eta\right)$. From an elementary calculation, it is easily shown that δ and η are invariant when we replace ρ^{*} by $\left(\begin{pmatrix}t^{-1} & t^{-1}-1\\ 1 & 1\end{pmatrix}, (z+1)^{k+1/2}\right)$. Moreover, if we write $\beta=n^{-1}\begin{pmatrix}a & b\\ c & d\end{pmatrix}$, from the definitions of the letters, we have $c\neq 0$ and $(a, c)=1$, and then, $t^{-1}=(a-d)/2c$ and $a-ct^{-1}=(a+d)/2=n$. Therefore, by using the Remarks (1.4) and (1.5), we obtain:

$$\eta=\chi(a)\left(\frac{-1}{a}\right)^{-k-1/2}\left(\frac{c}{a}\right)=\chi(n-txu)\left(\frac{-1}{n-txu}\right)^{-k-1/2}\left(\frac{-t^{2}xu}{n-txu}\right).$$

From $tu=\zeta\prod_{p|N}p^{\max(e,\tilde{\nu}-e)}$ and $4|x$, we have $txu\equiv 0 \pmod{8\prod_{p|M}p}$. Hence,

$$\left(\frac{-1}{n-txu}\right)=\left(\frac{-1}{n}\right) \quad\text{and}\quad \left(\frac{-t^{2}xu}{n-txu}\right)=\left(\frac{-1}{n}\right)\left(\frac{t^{2}xu}{n-txu}\right)=\left(\frac{-1}{n}\right)\left(\frac{xu}{n}\right).$$

Moreover, from §0 (**c**), we have a square-free positive divisor M_0 of M such that $\chi=\left(\frac{M_0}{}\right)$ or $\left(\frac{2M_0}{}\right)$. Hence, we have $\chi(n-txu)=\chi(n)$. Thus, we obtain

$$\eta=\chi(n)\left(\frac{-1}{n}\right)^{-k+1/2}\left(\frac{xu}{n}\right).$$

In a similar way, we have

$$e(\delta)=\chi(1+ut)\left(\frac{-1}{1+ut}\right)^{-k-1/2}\left(\frac{-ut^{2}}{1+ut}\right)=\chi(1+ut)\left(\frac{-1}{1+ut}\right)^{-k+1/2}\left(\frac{u}{1+ut}\right)$$

$$=\chi(1+ut)\left(\frac{-1}{1+ut}\right)^{-k+1/2}\prod_{p|N}{}'\left(\frac{p}{1+ut}\right)^{\tilde{\nu}}.$$

From these results, the value of δ is calculated as follows:

Case (i) ($\mu=2$).

$$\delta=\begin{cases}1, & \text{if } e_2\neq 1;\\ (1/2)-((-1)^{k}/4)\left(\frac{-1}{M_0}\right)\prod_{p|M}{}'\left(\frac{-1}{p}\right)^{\nu}, & \text{if } e_2=1.\end{cases}$$

Case (ii) ($\mu=3$).

$$\delta=\begin{cases}1, & \text{if } f(\chi_2)|4 \text{ and } e_2\neq 1;\\ 1/2, & \text{if } f(\chi_2)|4 \text{ and } e_2=1;\\ 1, & \text{if } f(\chi_2)=8 \text{ and } e_2\neq 2;\\ 1/2, & \text{if } f(\chi_2)=8 \text{ and } e_2=2.\end{cases}$$

Case (iii) ($\mu=4$).

$$\delta=\begin{cases}1, & \text{if } f(\chi_2)|4;\\ 1, & \text{if } f(\chi_2)=8 \text{ and } e_2\neq 2;\\ 1/2, & \text{if } f(\chi_2)=8 \text{ and } e_2=2.\end{cases}$$

Case (iv) ($\mu\geqq 5$).

$\delta=1$.

Now, we can determine $J(\beta)$. Obviously, Γ contains $\beta=\beta(t, x)$ if and only if $n=1$. In that case, we have $\eta=1$. Also, from the assumption $4|x$, we have $e(\delta x)=1$. Therefore, we obtain:

$$J(\beta)=\begin{cases}(1/2)-\delta, & \text{if } n=1;\\ \chi(n)\left(\frac{-1}{n}\right)^{k-1/2}\left(\frac{xu}{n}\right)e(\delta x/n)(1-e(x/n))^{-1}, & \text{if } n>1.\end{cases}$$

Before calculating $\tilde{p}(n)$, we prepair the following two lemmas.

Lemma (1.7). *We have the following equalities.*

$$(1)\quad \sum_{\substack{0\leq e_p\leq \nu_p\\ p|M}} \prod_{p|M}{}'\left(\frac{p}{n}\right)^{\nu}\prod_{p|M}\varphi(p^{\min(e,\nu-e)})=\prod_{p|M}\left(p^{[\nu/2]}+\left(\frac{p}{n}\right)^{\nu}p^{[(\nu-1)/2]}\right).$$

$$(2)\quad \sum_{\substack{0\leq e_p\leq \nu_p\\ p|M}} \prod_{p|M}{}'\left(\frac{-n}{p}\right)^{\nu}\prod_{p|M}\varphi(p^{\min(e,\nu-e)})=\prod_{p|M}\left(p^{[\nu/2]}+\left(\frac{-n}{p}\right)^{\nu}p^{[(\nu-1)/2]}\right).$$

Proof. It follows from some elementary calculations.

Lemma (1.8). *We have the following equalities.*

$$(1)\quad \sum_{a\in(\mathbf{Z}/n\mathbf{Z})^{\times}}\left(\frac{4a}{n}\right)e(4a/n)(1-e(4a/n))^{-1}$$
$$=\begin{cases}-(1/2)\delta_0(\sqrt{n})\varphi(n), & \textit{if}\quad n\equiv 1 \pmod 4;\\ \sqrt{-n}\,h'(-n), & \textit{if}\quad n\equiv 3 \pmod 4.\end{cases}$$

$$(2)\quad \sum_{a\in(\mathbf{Z}/n\mathbf{Z})^{\times}}\left(\frac{4a}{n}\right)e(2a/n)(1-e(4a/n))^{-1}$$
$$=\begin{cases}0, & \textit{if}\quad n\equiv 1 \pmod 4;\\ \left(\left(\frac{2}{n}\right)-1\right)\sqrt{-n}\,h'(-n), & \textit{if}\quad n\equiv 3 \pmod 4.\end{cases}$$

$$(3)\quad \sum_{a\in(\mathbf{Z}/n\mathbf{Z})^{\times}}\left(\frac{4a}{n}\right)e(a/n)(1-e(4a/n))^{-1}$$
$$=\begin{cases}(1/2)\sqrt{n}\,h'(-4n), & \textit{if}\quad n\equiv 1 \pmod 4;\\ (1/2)\left(1-\left(\frac{2}{n}\right)\right)\sqrt{-n}\,h'(-n), & \textit{if}\quad n\equiv 3 \pmod 4.\end{cases}$$

Here, the above sum $\sum_{a\in(\mathbf{Z}/n\mathbf{Z})^{\times}}$ *is extended over a system of representatives for* $(\mathbf{Z}/n\mathbf{Z})^{\times}$. *We shortly write* $\sum_a$ *for this sum.*

Proof. Since $e(a/n)(1-e(a/n))^{-1}=-(1/2)+(\sqrt{-1}/2)\cot(\pi a/n)$, we have

$$\sum_a\left(\frac{4a}{n}\right)e(4a/n)(1-e(4a/n))^{-1}=\sum_a\left(\frac{a}{n}\right)e(a/n)(1-e(a/n))^{-1}$$

$$=-(1/2)\sum_a\left(\frac{a}{n}\right)+(\sqrt{-1}/2)\sum_a\left(\frac{a}{n}\right)\cot(\pi a/n).$$

Obviously, the first term is equal to $-(1/2)\delta_0(\sqrt{n})\varphi(n)$. If $n\equiv 1 \pmod 4$, the second term is equal to zero. If $n\equiv 3 \pmod 4$, we have

$$\sum_a\left(\frac{a}{n}\right)\cot(\pi a/n)=(2n/\pi)L\left(1, \left(\frac{}{n}\right)\right)=2\sqrt{n}\,h'(-n).$$

Thus, we obtain the equality (1). We can also apply the similar procedure for the cases of the equalities (2) and (3).

Now, we return to the calculation of $\tilde{p}(n)$.

First, we suppose $n=1$ and shall calculate $\tilde{p}(1)$ for the Cases (i)-(iv).

Case (i) ($\mu=2$).

In this case, we have:

$$J(\beta)=\begin{cases} -1/2, & \text{if } e_2\neq 1; \\ ((-1)^k/4)\left(\frac{-1}{M_0}\right)\prod'_{p\mid M}\left(\frac{-1}{p}\right)^{\nu}, & \text{if } e_2=1. \end{cases}$$

Hence, $\tilde{p}(1)=\tilde{p}_1+\tilde{p}_2$ with $\tilde{p}_1=-(1/2)\times\#\{\beta(t, x)\mid S\ni t$ such that $e_2\neq 1\}$ and $\tilde{p}_2=((-1)^k/4)\left(\frac{-1}{M_0}\right)\sum_1\prod'_{p\mid M}\left(\frac{-1}{p}\right)^{\nu}$, where the sum $\sum_1$ is extended over all the matrices $\beta(t, x)$ such that $e_2=1$. By using the Lemma (1.7), we have $\tilde{p}_1=-\prod_{p\mid M}(p^{[\nu/2]}+p^{[(\nu-1)/2]})$ and

$$\begin{aligned}\tilde{p}_2&=((-1)^k/4)\left(\frac{-1}{M_0}\right)\varphi(2^{\min(1,2-1)})\sum_{\substack{0\leq e_p\leq\nu_p\\ p\mid M}}\prod'_{p\mid M}\left(\frac{-1}{p}\right)^{\nu}\varphi\left(\prod_{p\mid M}p^{\min(e,\nu-e)}\right)\\ &=((-1)^k/4)\left(\frac{-1}{M_0}\right)\prod_{p\mid M}\left(p^{[\nu/2]}+\left(\frac{-1}{p}\right)^{\nu}p^{[(\nu-1)/2]}\right).\end{aligned}$$

Case (ii) ($\mu=3$).

In a similar way as in the calculation of $\tilde{p}_1$ of the case (i), we have:

$$\tilde{p}(1)=-(3/2)\prod_{p\mid M}(p^{[\nu/2]}+p^{[(\nu-1)/2]}).$$

Case (iii) ($\mu=4$).

In a similar way as in the case (ii), we have:

$$\tilde{p}(1)=\begin{cases} -(1/2)\prod_{p\mid N}(p^{[\tilde{\nu}/2]}+p^{[(\tilde{\nu}-1)/2]}), & \text{if } f(\chi_2)\mid 4; \\ -2\prod_{p\mid M}(p^{[\nu/2]}+p^{[(\nu-1)/2]}), & \text{if } f(\chi_2)=8. \end{cases}$$

Case (iv) ($\mu\geqq 5$).

In a similar way as in the case (ii), we have:

$$\tilde{p}(1)=-(1/2)\prod_{p\mid N}(p^{[\tilde{\nu}/2]}+p^{[(\tilde{\nu}-1)/2]}).$$

Next, we suppose $n>1$. In the following, since $4\mid x$, we write $x=4x_0$. Then,

$$J(\beta)=\chi(n)\Big(\frac{-1}{n}\Big)^{k-1/2}\Big(\frac{4x_0u}{n}\Big)\boldsymbol{e}(4\delta x_0/n)(1-\boldsymbol{e}(4x_0/n))^{-1}.$$

Now, we shall calculate $\tilde{p}(n)$ for the Cases (i)-(iv).

Case (i) ($\mu=2$).

In this case, we have $\tilde{p}(n)=\tilde{p}_1+\tilde{p}_2$ with

$$\tilde{p}_1=\chi(n)\Big(\frac{-1}{n}\Big)^{k-1/2}\sum_{x_0\in(\mathbf{Z}/n\mathbf{Z})^\times}\sum_{\substack{0\le e_p\le\tilde{\nu}_p\\ p\mid N,\, e_2\neq 1}}\sum_{\zeta}\Big(\frac{4x_0u}{n}\Big)\boldsymbol{e}(4x_0/n)(1-\boldsymbol{e}(4x_0/n))^{-1}$$

and

$$\tilde{p}_2=\chi(n)\Big(\frac{-1}{n}\Big)^{k-1/2}\sum_{x_0\in(\mathbf{Z}/n\mathbf{Z})^\times}\sum_{\substack{0\le e_p\le\tilde{\nu}_p\\ p\mid N,\, e_2=1}}\sum_{\zeta}\Big(\frac{4x_0u}{n}\Big)$$

$$\times\boldsymbol{e}((2-(-1)^a)x_0/n)(1-\boldsymbol{e}(4x_0/n))^{-1},$$

where $(-1)^a=(-1)^k\Big(\frac{-1}{M_0}\Big)\prod_{p\mid M}'\Big(\frac{-1}{p}\Big)^{\nu}$.

From the Lemmas (1.7) and (1.8), It follows that:

$$\tilde{p}_1=\begin{cases} -\delta_0(\sqrt{n})\varphi(n)\prod_{p\mid M}(p^{[\nu/2]}+p^{[(\nu-1)/2]}), & \text{if } n\equiv 1 \pmod 4;\\ 2(-1)^k\chi(n)\sqrt{n}\,h'(-n)\prod_{p\mid M}\Big(p^{[\nu/2]}+\Big(\frac{p}{n}\Big)^{\nu}p^{[(\nu-1)/2]}\Big), & \text{if } n\equiv 3 \pmod 4.\end{cases}$$

Similarly, we have:

$$\tilde{p}_2=\begin{cases} ((-1)^k/2)\Big(\frac{-1}{M_0}\Big)\chi(n)\sqrt{n}\,h'(-4n)\prod_{p\mid M}\Big(p^{[\nu/2]}+\Big(\frac{-n}{p}\Big)^{\nu}p^{[(\nu-1)/2]}\Big),\\ \qquad\qquad \text{if } n\equiv 1 \pmod 4;\\ ((-1)^k/2)\Big(1-\Big(\frac{2}{n}\Big)\Big)\chi(n)\sqrt{n}\,h'(-n)\prod_{p\mid M}\Big(p^{[\nu/2]}+\Big(\frac{p}{n}\Big)^{\nu}p^{[(\nu-1)/2]}\Big),\\ \qquad\qquad \text{if } n\equiv 3 \pmod 4.\end{cases}$$

Here, we used the identity:

$$\boldsymbol{e}(3x_0/n)(1-\boldsymbol{e}(4x_0/n))^{-1}=-\boldsymbol{e}(-x_0/n)(1-\boldsymbol{e}(-4x_0/n))^{-1}.$$

We note that these expressions also fit for the case $n=1$.

Case (ii) ($\mu=3$).

In a similar way as in the Case (i), from the Lemmas (1.7) and (1.8), we obtain the following results:

$$\tilde{p}(n)=\begin{cases} -(3/2)\delta_0(\sqrt{n})\varphi(n)\prod_{p\mid M}(p^{[\nu/2]}+p^{[(\nu-1)/2]}),\\ \qquad\qquad \text{if } f(\chi_2)\mid 4 \text{ and } n\equiv 1 \pmod 4;\\ 3(-1)^k\chi(n)\sqrt{n}\,h'(-n)\prod_{p\mid M}\Big(p^{[\nu/2]}+\Big(\frac{p}{n}\Big)^{\nu}p^{[(\nu-1)/2]}\Big),\\ \qquad\qquad \text{if } f(\chi_2)\mid 4 \text{ and } n\equiv 3 \pmod 4;\end{cases}$$

$$\begin{cases} -(3/2)\delta_0(\sqrt{n})\varphi(n)\prod_{p\mid M}(p^{[\nu/2]}+p^{[(\nu-1)/2]}), \\ \qquad\qquad \text{if } f(\chi_2)=8 \text{ and } n\equiv 1 \pmod 4; \\ 3(-1)^k\left(\frac{2}{n}\right)\chi(n)\sqrt{n}\,h'(-n)\prod_{p\mid M}\left(p^{[\nu/2]}+\left(\frac{p}{n}\right)^{\nu}p^{[(\nu-1)/2]}\right), \\ \qquad\qquad \text{if } f(\chi_2)=8 \text{ and } n\equiv 3 \pmod 4. \end{cases}$$

We note that these expressions also fit for the case $n=1$.

Case (iii) ($\mu=4$).

In a similar way as in the Case (i), from the Lemmas (1.7) and (1.8), we obtain the following results:

$$\tilde{p}(n)=\begin{cases} -(1/2)\delta_0(\sqrt{n})\varphi(n)\prod_{p\mid N}(p^{[\tilde{\nu}/2]}+p^{[(\tilde{\nu}-1)/2]}), \\ \qquad\qquad \text{if } f(\chi_2)\mid 4 \text{ and } n\equiv 1 \pmod 4; \\ (-1)^k\chi(n)\sqrt{n}\,h'(-n)\prod_{p\mid N}\left(p^{[\tilde{\nu}/2]}+\left(\frac{p}{n}\right)^{\tilde{\nu}}p^{[(\tilde{\nu}-1)/2]}\right), \\ \qquad\qquad \text{if } f(\chi_2)\mid 4 \text{ and } n\equiv 3 \pmod 4; \\ -2\delta_0(\sqrt{n})\varphi(n)\prod_{p\mid M}(p^{[\nu/2]}+p^{[(\nu-1)/2]}), \\ \qquad\qquad \text{if } f(\chi_2)=8 \text{ and } n\equiv 1 \pmod 4; \\ 2(-1)^k\left(1+\left(\frac{2}{n}\right)\right)\chi(n)\sqrt{n}\,h'(-n)\prod_{p\mid M}\left(p^{[\nu/2]}+\left(\frac{p}{n}\right)^{\nu}p^{[(\nu-1)/2]}\right), \\ \qquad\qquad \text{if } f(\chi_2)=8 \text{ and } n\equiv 3 \pmod 4. \end{cases}$$

We note that these expressions also fit for the case $n=1$.

Case (iv) ($\mu\geqq 5$).

In a similar way as in the Case (i), from the Lemmas (1.7) and (1.8), we obtain the following results:

$$\tilde{p}(n)=\begin{cases} -(1/2)\delta_0(\sqrt{n})\varphi(n)\prod_{p\mid N}(p^{[\tilde{\nu}/2]}+p^{[(\tilde{\nu}-1)/2]}), \\ \qquad\qquad \text{if } n\equiv 1 \pmod 4; \\ (-1)^k\chi(n)\sqrt{n}\,h'(-n)\prod_{p\mid N}\left(p^{[\tilde{\nu}/2]}+\left(\frac{p}{n}\right)^{\tilde{\nu}}p^{[(\tilde{\nu}-1)/2]}\right), \\ \qquad\qquad \text{if } n\equiv 3 \pmod 4. \end{cases}$$

We note that these expressions also fit for the case $n=1$.

Finally, we must calculate the contribution to the trace of $\tilde{T}_{k+1/2,N,\chi}(n^2)$ on $S(k+1/2, N, \chi)$ and $V(N, \chi)$. But, that is easy calculation if we use the identity: $\sum_{0<a\mid n_0} a\varphi(n/a^2)=n$.

The results are as follows:

(1.9) *For* $n\equiv 1 \pmod 4$,

$T(p)=p_1+p_2$ *with*

$$p_1=-(1/2)\delta_0(\sqrt{n})n^{k-1/2}\prod_{p|M}(p^{[\nu/2]}+p^{[(\nu-1)/2]})$$

$$\times\begin{cases}2, & \textit{if}\quad \mu=2;\\ 3, & \textit{if}\quad \mu=3;\\ 4, & \textit{if}\quad \mu=4\quad \textit{and}\quad f(\chi_2)=8;\\ 2^{[\mu/2]}+2^{[(\mu-1)/2]}, & \textit{if}\quad \mu\geqq 5,\ \textit{or}\ \mu=4\quad \textit{and}\quad f(\chi_2)|4;\end{cases}$$

and

$$p_2=\delta_1((-1)^k/2)\left(\frac{-1}{M_0}\right)\chi(n)n^{k-1}\prod_{p|M}\left(p^{[\nu/2]}+\left(\frac{-n}{p}\right)^{\nu}p^{[(\nu-1)/2]}\right)$$
$$\times\sum_{0<a|n_0}h'(-4n/a^2).$$

For $n\equiv 3 \pmod 4$,

$$T(p)=(-1)^k\chi(n)n^{k-1}\prod_{p|M}\left(p^{[\nu/2]}+\left(\frac{p}{n}\right)^{\nu}p^{[(\nu-1)/2]}\right)\sum_{0<a|n_0}h'(-n/a^2)$$

$$\times\begin{cases}(1/2)\left(5-\left(\frac{2}{n}\right)\right), & \textit{if}\quad \mu=2;\\ 3, & \textit{if}\quad \mu=3\quad \textit{and}\quad f(\chi_2)|4;\\ 3\left(\frac{2}{n}\right), & \textit{if}\quad \mu=3\quad \textit{and}\quad f(\chi_2)=8;\\ 2\left(1+\left(\frac{2}{n}\right)\right), & \textit{if}\quad \mu=4\quad \textit{and}\quad f(\chi_2)=8;\\ 2^{[\mu/2]}+\left(\frac{2}{n}\right)^{\mu}2^{[(\mu-1)/2]}, & \textit{if}\quad \mu\geqq 5,\ \textit{or}\ \mu=4\quad \textit{and}\quad f(\chi_2)|4.\end{cases}$$

3. The calculation of $\tilde{e}(n)$.

Let C be an elliptic equivalence class in Φ. Take $\beta=n^{-1}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in C$ such that $(a, c)=1$ and that $c\neq 0$.

Since N is divisible by 4, Γ has no elliptic point. Therefore, by using the Remarks (1.4) and (1.5), we have

$$J(\beta)=J(C)=(1/2)\chi(\beta)\zeta(\beta)^{-2k-1}(1-\zeta(\beta)^{-4})^{-1}\quad\text{with}\quad \chi(\beta)=\chi(a)\left(\frac{c}{a}\right)$$

and

$$\zeta(\beta)=\left(\frac{-1}{a}\right)^{-1/2}J(\beta, z_0)^{1/2}$$
$$=\left(\frac{-1}{a}\right)^{-1/2}(2\sqrt{n})^{-1}(\sqrt{2n+a+d}+\operatorname{sgn}(c)\sqrt{a+d-2n}).$$

Here, $z_0\in\mathfrak{H}$ is the fixed point of β.

We put $w=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$ and $\Gamma^*=\Gamma\cup\Gamma w$ and $W(\beta)=|Z_{\Gamma^*}(\beta): Z_\Gamma(\beta)|$, where $Z_{\Gamma^*}(\beta)$ (resp. $Z_\Gamma(\beta)$) is the centralizer of β in Γ^* (resp. Γ). We note that, in the elliptic case, the equivalence relation is the usual Γ-conjugacy relation and that any element of Γ^* acts on $\Gamma\alpha(n)\Gamma$ by means of the inner automorphism. Then, it is easy to see $J(w\beta w)=\overline{J(\beta)}$ and, from the definition of $J(\beta)$, we have $J(-\beta)=J(\beta)$. Therefore, we have $\tilde{e}(n)=\sum_1 J(\beta)=\sum_1 J(w\beta w)=(1/2)\sum_1(J(\beta)+J(w\beta w))=\sum_2(J(\beta)+J(w\beta w))W(\beta)^{-1}$, where β in the sum $\sum_1$ (resp. $\sum_2$) runs over all representatives for the elliptic Γ (resp. Γ^*)-conjugacy classes in Φ. Moreover, since $n\beta=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma\begin{pmatrix} 1 & 0 \\ 0 & n^2 \end{pmatrix}\Gamma$, we have $c\equiv 0 \pmod 4$ and $a\equiv d\equiv \pm 1 \pmod 4$. Hence, β is not Γ^*-conjugate to $-\beta$. Therefore, we have $\tilde{e}(n)=2\sum_3(J(\beta)+J(w\beta w))W(\beta)^{-1}$, where β in the sum $\sum_3$ runs over all representatives for the elliptic Γ^*-conjugacy classes in Φ which is congruent to $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ modulo 4.

Thus, by using: $|\zeta(\beta)|=1$, we obtain:

$$\tilde{e}(n)=\sum_3 \chi(\beta)\Xi(\beta)W(\beta)^{-1}.$$

Here,

$$\Xi(\beta)=(\zeta(\beta)^{-2k+1}-\zeta(\beta)^{2k-1})(\zeta(\beta)^2-\zeta(\beta)^{-2})^{-1}.$$

Now, we shall give all representatives for the elliptic Γ^*-conjugacy classes in Φ which is congruent to $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ modulo 4, by using the method in [H] (or [D-M] chapter 6).

Let t be an integer such that $|t|<2n$ and that $t\equiv 2 \pmod 4$. We write $t^2-4n^2=m^2u$, where m is a positive integer and u is a fundamental discriminant, namely, the discriminant of some imaginary quadratic field. Then, let f be a positive integer such that $f|m$ and that $(f, n)=1$, and let ξ be a representative for $(\boldsymbol{Z}/\prod_{p|N} p^{\nu+\rho}\boldsymbol{Z})$ which satisfies the conditions:

$$(\xi, Nn)=1,\ \xi\equiv 1 \pmod 4 \quad \text{and} \quad F_t(\xi)\equiv 0 \pmod{Nf^2}$$

with

$$\rho=\rho_p=\mathrm{ord}_p(f) \quad \text{and} \quad F_t(X)=X^2-tX+n^2.$$

We put

$$S(\xi)=\left\{\begin{array}{l}\text{the prime divisor } p \text{ of } N \text{ such that } t^2-4n^2\equiv 0 \pmod{p^{2\rho+1}} \\ \text{and that } F_t(\xi)\equiv 0 \pmod{p^{\nu+2\rho+1}}\end{array}\right\}$$

and let S be a subset of $S(\xi)$.

For these t, f, ξ and S, we define the matrix $\varphi=\varphi(t, f, \xi, S)$ by:

$$\varphi=\begin{pmatrix} \xi & -f\prod_{p\in S} p^{-\nu}|F_t(\xi)f^{-2}|_p^{-1} \\ f^{-1}F_t(\xi)\prod_{p\in S} p^{\nu}|F_t(\xi)f^{-2}|_p & t-\xi \end{pmatrix}.$$

Moreover, we put $R=\begin{pmatrix} \boldsymbol{Z} & \boldsymbol{Z} \\ N\boldsymbol{Z} & \boldsymbol{Z} \end{pmatrix}$, $U=\prod_q (R\otimes_{\boldsymbol{Z}}\boldsymbol{Z}_q)^\times \times GL_2^+(\boldsymbol{R})$, $\boldsymbol{Q}_A[\varphi]=\boldsymbol{Q}[\varphi]\otimes_{\boldsymbol{Q}}\boldsymbol{Q}_A$ and $\Lambda=R\cap\boldsymbol{Q}[\varphi]$, where q runs over all prime numbers. Then, we have the bijections:

$$\Gamma\backslash U\boldsymbol{Q}_A[\varphi]^\times\cap GL_2(\boldsymbol{Q})/\boldsymbol{Q}[\varphi]^\times\cong\boldsymbol{Q}_A[\varphi]^\times/\{(\boldsymbol{Q}_A[\varphi]^\times\cap U)\boldsymbol{Q}[\varphi]^\times\}$$

$$\cong \textit{the proper ideal class group of the order } \Lambda$$

(cf. [H] or [D-M] Chapter 6). Hence, we can choose a system $\{\delta\}$ of representatives for $\Gamma\backslash U\boldsymbol{Q}_A[\varphi]^\times\cap GL_2(\boldsymbol{Q})/\boldsymbol{Q}[\varphi]^\times$ as follows:

For the double coset corresponding to the principal proper ideal class with the above bijection, we take $\delta=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$. In the other case, the corresponding proper ideal class contains a prime ideal P of Λ such that $(P, nN(t^2-4n^2))=1$ and that $\#\Lambda/P$ is a prime number p. Then, there exist the elements $v\in\boldsymbol{Q}_A[\varphi]^\times$ and $u\in U$ such that $P=\Lambda v$ and that $uv\in GL_2(\boldsymbol{Q})$. If necessary, by multipling some element of Γ from the left, we can see that $uv=\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}$ or $\begin{pmatrix}1 & j\\ 0 & p\end{pmatrix}$ with $0\leqq j<p$. Then, we take $\delta=uv$.

Now, when t, f, ξ, S and δ vary under the above conditions, the matrix $\beta=n^{-1}\delta\varphi\delta^{-1}$ forms a complete system of representatives for the elliptic Γ^*-conjugacy classes in Φ which is congruent to $\begin{pmatrix}1 & *\\ 0 & 1\end{pmatrix}$ modulo 4.

We write the above $\beta=n^{-1}\begin{pmatrix}a & b\\ c & d\end{pmatrix}$. Then, by using tne above conditions and the inequality: $F_t(\xi)>0$, we have $(a, c)=1$ and $c>0$. Hence, we have $\zeta(\beta)=(2\sqrt{n})^{-1}(\sqrt{t+2n}+\sqrt{t-2n})$. Therefore, $\zeta(\beta)$ and $\Xi(\beta)$ depend only on t, and so we write $\zeta(\beta)=\zeta_t$ and $\Xi(\beta)=\Xi_t$.

Next, since $Z_{\Gamma^*}(\beta)=\delta\Lambda^\times\delta^{-1}$, we have $W(\beta)=w((t^2-4n^2)f^{-2})$ and, by using the same method as in [N] p. 196, we have

$$\chi(n^{-1}\delta\varphi\delta^{-1})=\chi(\varphi)\left(\frac{t+2n}{p}\right) \quad \text{for } \delta=\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix} \text{ or } \begin{pmatrix}1 & j\\ 0 & p\end{pmatrix}.$$

Therefore, we obtain:

$$\tilde{e}(n)=\sum_t\Xi_t\sum_f w((t^2-4n^2)f^{-2})^{-1}\sum_\xi\sum_S\chi(\varphi)$$

$$\times\sum_\delta\begin{cases}1, & \text{if } \delta=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix};\\ \left(\frac{t+2n}{p}\right), & \text{if } \delta=\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix} \text{ or } \begin{pmatrix}1 & j\\ 0 & p\end{pmatrix}.\end{cases}$$

Since $\left(\frac{t+2n}{\quad}\right)$ can be considered as the genus character on the proper ideal class group of Λ(cf. [N] p. 196-197), the last sum is equal to the class number $h((t^2-4n^2)f^{-2})$ or zero, according as $t+2n$ is square or not.

Thus, we may assume that $t+2n=s^2$ with $s>0$. Then, $t^2-4n^2=s^2(s^2-4n)$ and $\zeta_t=(2\sqrt{n})^{-1}(s+(s^2-4n)^{1/2})$. By using: $|\zeta_t|=1$, we have $\Xi_t=-n^{-k+3/2}s^{-1}\pi_k(s, n)$, where $\pi_k(s, n)$ is defined as follows: Let x and y are the solutions of $X^2-sX+n=0$. Then, we define

$$\pi_k(s, n)=(x^{2k-1}-y^{2k-1})(x-y)^{-1}.$$

Finally, we discuss

$$\sum_{\xi}\sum_{S}\chi(\varphi)=\sum_{\xi}\chi(\xi)\sum_{S}\left(\frac{f}{\xi}\right)\prod_{p\in S}\left(\frac{p^{\tilde{\nu}}|F_t(\xi)|_p}{\xi}\right).$$

Since $(f, t-2n, t+2n)$ divides some power of 2, we have the following decomposition: $f=2^{\rho_2}f_1f_2$, $(f_1f_2, 2)=1$, $f_1>0$, $f_2>0$, $f_1|s$, $(f_1, s^2-4n)=1$, $f_2^2|(s^2-4n)$ and $(f_2, s)=1$. From $\xi\equiv 1 \pmod 4$ and the reciprocity law, $\left(\frac{f_i}{\xi}\right)=\left(\frac{\xi}{f_i}\right)$, $(i=1, 2)$. Put $t'=t/2$. Since $0\equiv F_t(\xi)\equiv(\xi-t')^2 \pmod{f_1^2f_2^2}$, we have $\xi\equiv t' \pmod{f_1f_2}$. Hence,

$$\left(\frac{\xi}{f_1}\right)=\left(\frac{t'}{f_1}\right)=\left(\frac{2t}{f_1}\right)=\left(\frac{s^2-4n}{f_1}\right) \text{ and also } \left(\frac{\xi}{f_2}\right)=\left(\frac{t'}{f_2}\right)=\left(\frac{s^2}{f_2}\right)=1.$$

Therefore, $\left(\frac{f}{\xi}\right)=\left(\frac{2}{\xi}\right)^{\rho_2}\left(\frac{s^2-4n}{f_1}\right)$.

Next, by using the same argument in [N] p. 198, we can prove that $\left(\frac{|F_t(\xi)|_p}{\xi}\right)=1$ for $p\in S$ and that $\left(\frac{p}{\xi}\right)=\left(\frac{\xi}{p}\right)=\left(\frac{2s^2-4n}{p}\right)$ for any odd prime $p\in S$. Therefore,

$$\begin{aligned}\sum_{\xi}\sum_{S}\chi(\varphi)&=\left(\frac{s^2-4n}{f_1}\right)\sum_{\xi}\chi(\xi)\left(\frac{2}{\xi}\right)^{\rho_2}\sum_{S}\prod_{p\in S}\left(\frac{p}{\xi}\right)^{\tilde{\nu}}\\&=\left(\frac{s^2-4n}{f_1}\right)\sum_{\xi}\chi(\xi)\left(\frac{2}{\xi}\right)^{\rho_2}\prod_{p\in S(\xi)}\left(1+\left(\frac{p}{\xi}\right)^{\tilde{\nu}}\right)\\&=\left(\frac{s^2-4n}{f_1}\right)\sum_{\xi}\chi(\xi)\left(\frac{2}{\xi}\right)^{\rho_2}\prod_{p|N}\left(1+\delta_p(\xi)\left(\frac{p}{\xi}\right)^{\tilde{\nu}}\right)\\&=\left(\frac{s^2-4n}{f_1}\right)\sum_{\xi}\chi_2(\xi)\left(\frac{2}{\xi}\right)^{\rho_2}\left(1+\delta_2(\xi)\left(\frac{2}{\xi}\right)^{\mu}\right)\\&\quad\times\prod_{p|M}\chi_p(\xi)\left(1+\delta_p(\xi)\left(\frac{2s^2-4n}{p}\right)^{\nu}\right).\end{aligned}$$

Here, for any prime number $p|N$ and a representative η of $(\boldsymbol{Z}/p^{\tilde{\nu}+\rho}\boldsymbol{Z})$ such that $F_t(\eta)\equiv 0 \pmod{p^{\tilde{\nu}+2\rho}}$, we define

$$\delta_p(\eta)=\begin{cases}1, & \text{if } s^2(s^2-4n)\equiv 0 \pmod{p^{2\rho+1}} \text{ and } F_t(\eta)\equiv 0 \pmod{p^{\tilde{\nu}+2\rho+1}};\\ 0, & \text{otherwise.}\end{cases}$$

The quantity $\chi_p(\xi)\left(1+\delta_p(\xi)\left(\frac{2s^2-4n}{p}\right)^{\nu}\right)$ for any prime number $p|M$ (resp. $\chi_2(\xi)\left(\frac{2}{\xi}\right)^{\rho_2}\left(1+\delta_2(\xi)\left(\frac{2}{\xi}\right)^{\mu}\right)$) depends only on ξ modulo $p^{\nu+\rho}$ (resp. ξ modulo $2^{\mu+\rho}$). Therefore, we have:

$$\sum_{\xi}\sum_{S}\chi(\varphi)=\left(\frac{s^2-4n}{f_1}\right)\prod_{p|N}c_p(s, f).$$

Here, we define

$$c_p(s, f)=\begin{cases}\sum_{\eta}\chi_p(\eta)\left(1+\delta_p(\eta)\left(\frac{2s^2-4n}{p}\right)^{\nu}\right), & \text{if } p|M;\\ \sum_{\eta}\chi_2(\eta)\left(\frac{2}{\eta}\right)^{\rho}\left(1+\delta_2(\eta)\left(\frac{2}{\eta}\right)^{\mu}\right), & \text{if } p=2;\end{cases}$$

where the above sum $\sum_\eta$ runs over all representatives η of $(\boldsymbol{Z}/p^{\tilde{\nu}+\rho}\boldsymbol{Z})$ such that $F_t(\eta)\equiv 0 \pmod{p^{\tilde{\nu}+2\rho}}$ and besides that $\eta\equiv 1 \pmod 4$ in the case $p=2$.

Combining all the above results, we obtain:

$$\tilde{e}(n)=-n^{-k+3/2}\sum_s s^{-1}\pi_k(s, n)\sum_f h'(s^2(s^2-4n)f^{-2})\Big(\frac{s^2-4n}{f_1}\Big)\prod_{p\mid N} c_p(s, f).$$

Here, the integer s runs over all even integers such that $0<s<2\sqrt{n}$, then, we can write $s^2-4n=m_1^2u$ with $m_1>0$. Put $\theta_p=\mathrm{ord}_p(sm_1)$ for any prime number $p\mid N$ and $s'=s\prod_{p\mid N}p^{-\mathrm{ord}_p(s)}$. Similarly, for f_1, f_2 and m_1, we define the number f_1', f_2' and m_1'. Moreover, we decompose $s'=r(s')u(s')$ and $m_1'=r(m_1')u(m_1')$ such that $(r(s'), s', m_1')=(r(m_1'), s', m_1')=1$ and that $u(s')$ and $u(m_1')$ divide some power of (s', m_1').

Under these notations, we easily see that f_1' runs over the set $A_1=\{\boldsymbol{Z}\ni f_1';\ 0<f_1'\mid r(s')$ and $(f_1', u)=1\}$ and that f_2' runs over the set $A_2=\{\boldsymbol{Z}\ni f_2';\ 0<f_2'\mid r(m_1')\}$. Moreover, we introduce the following notations:

$$d_p(\theta_p, \rho_p)=\begin{cases} 1-\Big(\dfrac{u}{p}\Big)p^{-1}, & \text{if } \theta_p\neq\rho_p;\\ 1, & \text{if } \theta_p=\rho_p.\end{cases}$$

$$n_p(\theta_p)=\sum_{\rho_p=0}^{\theta_p} p^{\theta_p-\rho_p}c_p(s, f)d_p(\theta_p, \rho_p)\times\begin{cases}\Big(\dfrac{u}{p}\Big)^{\rho_p}, & \text{if } p\mid M \text{ and } p\mid s;\\ 1, & \text{otherwise}.\end{cases}$$

Here, we remark that the last sum depends only on θ_p.

Now, observing that $c_p(s, f)$ depends only on $\rho_p=\mathrm{ord}_p(f)$ when we fix s, we have

$$\begin{aligned}
&\sum_f h'(s^2(s^2-4n)f^{-2})\Big(\frac{s^2-4n}{f_1}\Big)\prod_{p\mid N}c_p(s, f)\\
&=\sum_{\substack{0\le\rho_p\le\theta_p\\ p\mid N}}\sum{}^{(1)}\sum{}^{(2)} h'\Big(\prod_{p\mid N}p^{2(\theta_p-\rho_p)}(s'm_1'/f_1'f_2')^2u\Big)\\
&\qquad\times\Big(\frac{s^2-4n}{f_1}\Big)\prod_{p\mid N}c_p(s, f)\\
&=h'(u)u(s')u(m_1')\prod_{q\mid u(s')u(m_1')}\Big(1-\Big(\frac{u}{q}\Big)q^{-1}\Big)\prod_{p\mid N}n_p(\theta_p)\\
&\qquad\times\sum{}^{(1)}\Big(\frac{u}{f_1'}\Big)(r(s')/f_1')\prod_{q\mid(r(s')/f_1')}\Big(1-\Big(\frac{u}{q}\Big)q^{-1}\Big)\\
&\qquad\times\sum{}^{(2)}(r(m_1')/f_2')\prod_{q\mid(r(m_1')/f_2')}\Big(1-\Big(\frac{u}{q}\Big)q^{-1}\Big).
\end{aligned}$$

Here, the notations are as follows: f_1' (resp. f_2') in the sum $\sum^{(1)}$ (resp. $\sum^{(2)}$) runs over the set A_1 (resp. A_2) and q denotes a prime number.

From an elementary calculation, we see that the part of the sum $\sum^{(1)}$ is equal to $r(s')$ and the part of the sum $\sum^{(2)}$ is equal to $\alpha_u(r(m_1'))$(cf. §0 (**a**)). Moreover, observing that $q\mid u(s')u(m_1')$ if and only if $q\mid(s', m_1')$ and that

$s^{-1}u(s')r(s')=\prod_{p|N} p^{-\mathrm{ord}_p(s)}$, we have:

$$\tilde{e}(n)=-n^{-k+3/2}\sum_s \pi_k(s,\ n)h'(u)u(m_1')\alpha_u(r(m_1'))$$

$$\times\prod_{q|(s',m_1')}\left(1-\left(\frac{u}{q}\right)q^{-1}\right)\prod_{p|N}\{p^{-\mathrm{ord}_p(s)}n_p(\theta_p)\}.$$

Therefore, the contribution to the trace of $\tilde{T}_{k+1/2,N,\chi}(n^2)$ on $S(k+1/2,\ N,\ \chi)$ and $V(N,\ \chi)$ is as follows:

$$T(e)=n^{k-3/2}\sum_{0<a|n_0} a\tilde{e}(n/a^2)$$

$$=-\sum_{0<a|n_0}\sum_s a^{2k-2}\pi_k(s,\ n/a^2)h'(u)u(m_1')\alpha_u(r(m_1'))$$

$$\times\prod_{q|(s',m_1')}\left(1-\left(\frac{u}{q}\right)q^{-1}\right)\prod_{p|N}\{p^{-\mathrm{ord}_p(s)}n_p(\theta_p)\}.$$

Here, s, u, m_1, etc. are defined in the same way as above, when we replace n with n/a^2.

Now, we put $\tilde{s}=as$, $\tilde{m}_1=am_1$, $\tilde{x}=ax$ and $\tilde{y}=ay$. Then, we have that $\tilde{x}$ and $\tilde{y}$ are the solutions of $X^2-\tilde{s}X+n=0$ and that $\tilde{s}^2-4n=\tilde{m}_1^2u$. Since $(a,\ N)=1$, $\mathrm{ord}_p(\tilde{s}\tilde{m}_1)=\theta_p$. Hence,

$$T(e)=-\sum_{0<a|n_0}\sum{}^{(3)}\pi_k(\tilde{s},\ n)h'(u)\prod_{p|N}\{p^{-\mathrm{ord}_p(\tilde{s})}n_p(\theta_p)\}u(m_1')\alpha_u(r(m_1'))$$

$$\times\prod_{q|(s',m_1')}\left(1-\left(\frac{u}{q}\right)q^{-1}\right)$$

$$=-\sum{}^{(4)}\pi_k(\tilde{s},\ n)h'(u)\prod_{p|N}\{p^{-\mathrm{ord}_p(\tilde{s})}n_p(\theta_p)\}$$

$$\times\sum_{0<a|(\tilde{s},n_0)} u(m_1')\alpha_u(r(m_1'))\prod_{q|(s',m_1')}\left(1-\left(\frac{u}{q}\right)q^{-1}\right).$$

Here, $\tilde{s}$ in the sum $\sum^{(3)}$ (resp. $\sum^{(4)}$) runs over all even integers such that $2\sqrt{n}>\tilde{s}>0$ and that $a|\tilde{s}$ (resp. $2\sqrt{n}>\tilde{s}>0$).

We observe that an integer a divides the odd part of $(\tilde{s},\ \tilde{m}_1)$ if and only if $a|(\tilde{s},\ n_0)$. Hence, from an elementary calculation, we have

$$\sum_{0<a|(\tilde{s},n_0)} u(m_1')\alpha_u(r(m_1'))\prod_{q|(s',m_1')}\left(1-\left(\frac{u}{q}\right)q^{-1}\right)=\alpha_u(\tilde{m}_1\prod_{p|N}p^{-\mathrm{ord}_p(\tilde{m}_1)}).$$

Therefore, rewriting the notations: $\tilde{s}\to s$, $\tilde{m}_1\to t$, etc., we obtain the following formula:

(1.10) $$T(e)=-\sum{}^{(e)}\pi_k(s,\ n)h'(u)\alpha_u(t_0)\prod_{p|N}\{p^{-\mathrm{ord}_p(s)}n_p(\theta_p)\}.$$

Here, s in the sum $\sum^{(e)}$ runs over all even integers such that $2\sqrt{n}>s>0$. The other notations are as follows: Let x and y be the solutions of $X^2-sX+n=0$. Then, $\pi_k(s,\ n)=(x^{2k-1}-y^{2k-1})(x-y)^{-1}$. Put $s^2-4n=t^2u$ with a positive integer t and a fundamental discriminant u. Put $t_0=t\prod_{p|N}p^{-\mathrm{ord}_p(t)}$ and $\theta=\theta_p=\mathrm{ord}_p(st)$ for *any prime divisor p of N. The constant $n_p(\theta_p)$ is defined in the same way as in the above calculation. Finally, we remark that the explicit determination of the constant $n_p(\theta_p)$ needs an elementary but very long calculation. So, we omit it. For the explicit value of the constant $n_p(\theta_p)$, see the Appendix 1.*

4. The calculation of $\tilde{h}(n)$.

Let C be a hyperbolic equivalence class in Φ. Take $\beta=n^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in C$ such that $(a, c)=1$ and that $c\neq 0$. Let κ be the upper fixed point of β which is a cusp of Γ and $\lambda(\beta)=(a-c\kappa)/n$. Since $\kappa=(2c)^{-1}\{a-d-\mathrm{sgn}(a+d)((a+d)^2-4n^2)^{1/2}\}$, we have: $\mathrm{sgn}(\lambda(\beta))=\mathrm{sgn}(a+d)$ and

$$\lambda(\beta)=(2n)^{-1}\{a+d+\mathrm{sgn}(a+d)((a+d)^2-4n^2)^{1/2}\}.$$

Moreover, by using the Remarks (1.4) and (1.5), we have

$$J(\beta)=J(C)=-(1/2)\left(\frac{\mathrm{sgn}(a+d)}{\mathrm{sgn}(c)}\right)\chi(a)\left(\frac{-1}{a}\right)^{k+1/2}\left(\frac{c}{a}\right)\lambda(\beta)^{-k-1/2}(1-\lambda(\beta)^{-2})^{-1}.$$

Hence, we have $J(-\beta)=J(\beta)$ and $J(w\beta w)=\left(\frac{-1}{a}\right)\mathrm{sgn}(a+d)J(\beta)$, where $w=\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

Let Γ^* and $W(\beta)$ be the same as in the elliptic case. Then, we have $\tilde{h}(n)=\sum_1 J(\beta)=\sum_1 J(w\beta w)=(1/2)\sum_1(J(\beta)+J(w\beta w))=(1/2)\sum_2(J(\beta)+J(w\beta w))=(1/2)\sum_3(J(\beta)+J(-\beta)+J(w\beta w)+J(-w\beta w))=\sum_3(J(\beta)+J(w\beta w))=2\sum_4(J(\beta)+J(w\beta w))W(\beta)^{-1}=4\sum_4 J(\beta)W(\beta)^{-1}$, where β in $\sum_1$ runs over all representatives for the hyperbolic Γ-conjugacy classes in Φ, β in $\sum_2$ runs over those such that $\left(\frac{-1}{a}\right)\mathrm{sgn}(a+d)=1$, β in $\sum_3$ runs over those such that $a+d>0$ and that $a\equiv 1 \pmod 4$, and β in the $\sum_4$ runs over those for the hyperbolic Γ^*-conjugacy classes in Φ such that $a+d>0$ and that $a\equiv 1 \pmod 4$.

Thus, we obtain:

$$\tilde{h}(n)=-2\sum_4\chi(\beta)W(\beta)^{-1}\lambda(\beta)^{-k-1/2}(1-\lambda(\beta)^{-2})^{-1} \quad \text{with} \quad \chi(\beta)=\chi(a)\left(\frac{c}{a}\right).$$

Let t be an integer such that $t\equiv 2 \pmod 4$ and that $t>2n$ and that t^2-4n^2 is square. Then, we write $t^2-4n^2=m^2$ with $m>0$. For these t and m, let f, ξ, S and φ be the same as in the elliptic case. But, for simplifying the calculation, we assume the additional condition: $\xi\neq 1$.

Next, let $\Lambda=\boldsymbol{Q}[\varphi]\cap R$ with $R=\begin{pmatrix} \boldsymbol{Z} & \boldsymbol{Z} \\ N\boldsymbol{Z} & \boldsymbol{Z} \end{pmatrix}$. Then, we know that $\boldsymbol{Q}_A[\varphi]^\times/\{(\prod_p(\Lambda\otimes_Z \boldsymbol{Z}_p)^\times\times\boldsymbol{R}[\varphi]^\times)\boldsymbol{Q}[\varphi]^\times\}$ is isomorphic to the proper ideal class group of the order Λ. Then, for the principal proper ideal class, we set $\delta=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. In the other case, any proper ideal class contains a prime ideal P such that $\#\Lambda/P$ is a prime number p and that $(p, nN(t^2-4n^2))=1$. So, there exist $v\in\boldsymbol{Q}_A[\varphi]^\times$ and $u\in U$ such that $\Lambda v=P$ and that $GL_2(\boldsymbol{Q})\ni uv=\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}$ with $0\leq j$

$<p$. Then, we set $\delta=uv$. Here, U is the same as in the elliptic case.

When these t, f, ξ, S and $\{\delta\}$ vary under the above conditions, the matrix $n^{-1}\delta\varphi\delta^{-1}=n^{-1}\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ forms a complete system of all representatives for the hyperbolic Γ^*-conjugacy classes in Φ, such that $a+d>0$ and that $a\equiv 1 \pmod 4$.

Therefore, we have

$$\tilde{h}(n)=-2\sum_t\sum_f\sum_\xi\sum_S\sum_\delta \chi(n^{-1}\delta\varphi\delta^{-1})W(n^{-1}\delta\varphi\delta^{-1})^{-1}\lambda(\varphi)^{-k-1/2}(1-\lambda(\varphi)^{-2})^{-1}.$$

Moreover, in the same way as in the elliptic case, we may assume that $t+2n$ is square. Then, we write $t+2n=s^2$ with $s>0$. Hence, we have:

$$\begin{aligned}\tilde{h}(n)&=-\sum_t\sum_f\sum_\xi\sum_S \lambda(\varphi)^{-k-1/2}(1-\lambda(\varphi)^{-2})^{-1}\varphi(m/f)\chi(\varphi)\\ &=-n^{-k+3/2}\sum_s s^{-1}y^{2k-1}(x-y)^{-1}\sum_f \varphi(s(s^2-4n)^{1/2}f^{-1})\prod_{p\mid N} c_p(s,f),\end{aligned}$$

where x and y are the solutions of $X^2-sX+n=0$ such that $x>y$, and the constant $c_p(s, f)$ are the same as in the elliptic case.

We can deduce the contribution to the trace of $\tilde{T}_{k+1/2,N,\chi}(n^2)$ on $S(k+1/2, N, \chi)$ and $V(N, \chi)$ by using the same method as in the elliptic case. The results is as follows:

$$\begin{aligned}(1.11)\quad T(h)&=n^{k-3/2}\sum_{0<a\mid n_0} a\tilde{h}(n/a^2)\\ &=-\sum\nolimits^{(h)}((s-t)/2)^{2k-1}\prod_{p\mid N} m_p(\theta_p),\end{aligned}$$

where s in the sum $\sum^{(h)}$ runs over all even integers such that $s>2\sqrt{n}$ and that s^2-4n is square. The other notations are as follows: Let $t=(s^2-4n)^{1/2}$ and $\theta=\theta_p=\mathrm{ord}_p(st)$ for any prime divisor p of N. The explicit value of the constant $m_p(\theta_p)$ is given by the Appendix 2.

5. The calculation of $\tilde{d}(n)$.

Put $\beta_N=\begin{pmatrix} 0 & -1 \\ N & 0\end{pmatrix}$, $\tau_N=(\beta_N, N^{1/4}(-\sqrt{-1}z)^{1/2})\in G(1/2)$ and $\Delta''=\Delta_0\left(N, \chi\left(\frac{N}{}\right)\right)_{1/2}$. Then, from [Sh 2] Proposition 1.4, we know that τ_N induces an isomorphism between $G(1/2, N, \chi)$ and $G\left(1/2, N, \chi\left(\frac{N}{}\right)\right)$, and $\tau_N^{-1}\Delta''\tau_N=\Delta'$. Hence,

$$\tau_N^{-1}\Delta''\tau(0;\, n)\Delta''\tau_N=\Delta'\tau_N^{-1}\tau(0;\, n)\tau_N\Delta'=\Delta'\tau'(n)\Delta'.$$

If we write $\Delta''\tau(0;\, n)\Delta''=\bigcup_i \Delta''\xi_i$ (disjoint union), we have $\Delta'\tau'(n)\Delta'=\bigcup_i \Delta'\tau_N^{-1}\xi_i\tau_N$. Therefore, as an operator on $G(1/2, N, \chi)$, we have:

$$\tau_N^{-1}[\Delta''\tau(0;\, n)\Delta'']_{1/2}\tau_N=[\Delta'\tau'(n)\Delta']_{1/2}$$

and

$$\begin{aligned}&\mathrm{trace}([\Delta'\tau'(n)\Delta']\mid G(1/2, N, \chi))\\ &=\mathrm{trace}\left([\Delta''\tau(0;\, n)\Delta'']\mid G\left(1/2, N, \chi\left(\frac{N}{}\right)\right)\right).\end{aligned}$$

From [Sh 2] Theorem 1.7, it follows that, for $(\psi, t)\in\Omega^0\left(N, \chi\left(\frac{N}{-}\right)\right)$, $h^0(\psi; tz)$ is an eigen function of $\tilde{T}_{1/2,N,\chi\left(\frac{N}{-}\right)}(p^2)$ for any prime number $p\nmid N$ and that the eigen value is $\psi(p)(1+p^{-1})$.

Let $n=\prod_{p\mid n} p^{\tau}$. Then, we know the following equalities as operators:

$$[\Delta''\tau(0; n)\Delta'']=\left[\Delta''\left(\begin{pmatrix}1 & 0\\ 0 & n^2\end{pmatrix}, n^{1/2}\right)\Delta''\right]$$
$$=\prod_{p\mid n}\left[\Delta''\left(\begin{pmatrix}1 & 0\\ 0 & p^{2\tau}\end{pmatrix}, p^{\tau/2}\right)\Delta''\right], \text{ and for } \tau\geqq 2,$$
$$\left[\Delta''\left(\begin{pmatrix}1 & 0\\ 0 & p^{2\tau}\end{pmatrix}, p^{\tau/2}\right)\Delta''\right]$$
$$=\left[\Delta''\left(\begin{pmatrix}1 & 0\\ 0 & p^{2}\end{pmatrix}, p^{1/2}\right)\Delta''\right]\left[\Delta''\left(\begin{pmatrix}1 & 0\\ 0 & p^{2\tau-2}\end{pmatrix}, p^{(\tau-1)/2}\right)\Delta''\right]$$
$$-\begin{cases} p^2+p, & \text{if } \tau=2;\\ p^2\left[\Delta''\left(\begin{pmatrix}1 & 0\\ 0 & p^{2\tau-4}\end{pmatrix}, p^{(\tau-2)/2}\right)\Delta''\right], & \text{if } \tau\geqq 3;\end{cases}$$

(cf. [N] Introduction).

From these results, we obtain inductively that, for $(\psi, t)\in\Omega^0\left(N, \chi\left(\frac{N}{-}\right)\right)$, $h^0(\psi; tz)$ has the eigen value $\psi(n)\sqrt{n}\prod_{p\mid n}(p^{\tau}+p^{\tau-1})$ with respect to the operator $[\Delta''\tau(0; n)\Delta'']$.

We can also discuss the case of $U(N, \chi)$ in a similar way and obtain that, for $(\psi, t)\in\Omega^1(N, \chi)$, $h^1(\psi; tz)$ has the eigen value $\psi(n)\sqrt{n}\prod_{p\mid n}(p^{\tau}+p^{\tau-1})$ with respect to the operator $[\Delta\tau(1; n)\Delta]$, where $n=\prod_{p\mid n} p^{\tau}$. Moreover, $\left\{h^0(\psi; tz)\mid(\psi, t)\in\Omega^0\left(N, \chi\left(\frac{N}{-}\right)\right)\right\}$ and $\{h^1(\psi; tz)\mid(\psi, t)\in\Omega^1(N, \chi)\}$ are $\boldsymbol{C}$-basis of $G\left(1/2, N, \chi\left(\frac{N}{-}\right)\right)$ and $U(N, \chi)$ respectively (cf. § 0, (**c**)).

From the above results, we have

$$\tilde{d}(n)=\left\{\sum\nolimits_0 a_0\left(\chi\left(\frac{N}{-}\right); \psi\right)\psi(n)-\sum\nolimits_1 a_1(\chi; \psi)\psi(n)\right\}\sqrt{n}\prod_{p\mid n}(p^{\tau}+p^{\tau-1}).$$

Here, $n=\prod_{p\mid n} p^{\tau}$. For $\nu=0$ or 1, ψ in the sum $\sum_{\nu}$ runs over all primitive characters with $\psi(-1)=(-1)^{\nu}$ and $a_{\nu}(\chi; \psi)$ is the number of all positive integers t which satisfy the conditions (0.1) and (0.2) (cf. § 0 (**c**)) with respect to χ and ψ.

From the equality: $a_0(\chi; \psi)=a_0\left(\chi\left(\frac{N}{-}\right); \psi\right)$, it follows that $\tilde{d}(n)=\sum^* a^*(\chi; \psi)\psi(-n)\sqrt{n}\prod_{p\mid n}(p^{\tau}+p^{\tau-1})$, where ψ in the sum $\sum^*$ runs over all primitive characters and $a^*(\chi; \psi)$ is the number of all positive integers t which satisfy the conditions (0.1) and (0.2) (cf. § 0 (**c**)) with respect to χ and ψ.

The contribution to the trace of $\tilde{T}_{3/2,N,\chi}(n^2)$ on $V(N, \chi)$ is as follows:

$$(1.12)\quad T(d)=n^{-1/2}\sum_{0<a\mid n_0} a\tilde{d}(n/a^2)$$
$$=n^{-1/2}\sum_{0<a\mid n_0} a\sum^* a^*(\chi; \psi)\psi(-n/a^2)(n/a^2)^{1/2}\prod_{p\mid(n/a^2)}(p^{\tau}+p^{\tau-1})$$

$$=\sum^* a^*(\chi;\psi)\psi(-n)\sum_{0<a|n_0}\prod_{p|(n/a^2)}(p^{\tau}+p^{\tau-1}\cdots$$

$$=\sum^* a^*(\chi;\psi)\psi(-n)\prod_{p|n}(p^{\tau+1}-1)/(p-1).$$

Here, $n=\prod_{p|n}p^{\tau}$ *and let* $\sum^*$ *and* $a^*(\chi;\psi)$ *be the same as above.*

Now, the proof of the Proposition 1 is completed.

§2. The trace formula for the Hecke operator on the Kohnen subspace.

Throughout this section, we shall use the same notations and assumptions as in §0 (**a**) and (**d**), and moreover, we suppose that n is any positive integer such that $(n, N)=1$.

Our purpose in this section is to calculate the trace of the Hecke operator $\tilde{T}_{k+1/2,N,\chi}(n^2)$ on $S(k+1/2, N, \chi)_K$ and $V(N, \chi)_K$.

Let $\boldsymbol{Pr}=\boldsymbol{Pr}_{k+1/2,N,\chi}=(\alpha-\beta)^{-1}(Q_{k+1/2,N,\chi}-\beta)$ be the orthogonal projection from $S(k+1/2, N, \chi)$ onto $S(k+1/2, N, \chi)_K$. (cf. [K]) Then, we have:

$$\text{(2.1)}\qquad \operatorname{trace}\Big(\tilde{T}_{k+1/2,N,\chi}(n^2)\Big|S\Big(k+\frac{1}{2}, N, \chi\Big)_K\Big)$$

$$=\operatorname{trace}\Big(\tilde{T}_{k+1/2,N,\chi}(n^2)\boldsymbol{Pr}\Big|S\Big(k+\frac{1}{2}, N, \chi\Big)\Big)$$

$$=(\sqrt{2}/6)(-1)^{[(k+1)/2]}\varepsilon\operatorname{trace}\Big(\tilde{T}_{k+1/2,N,\chi}(n^2)Q\Big|S\Big(k+\frac{1}{2}, N, \chi\Big)\Big)$$

$$+(1/3)\operatorname{trace}\Big(\tilde{T}_{k+1/2,N,\chi}(n^2)\Big|S\Big(k+\frac{1}{2}, N, \chi\Big)\Big),$$

where $Q=Q_{k+1/2,N,\chi}$. (cf. [K])

For a simplicity, we write $\Gamma=\Gamma_0(N)$, $\Delta=\Delta_0(N,\chi)_{k+1/2}$ and $n=n_0^2n_1$ with a positive integer n_0 and a positive square-free integer n_1. Then, we have:

$$\text{(2.2)}\qquad \tilde{T}_{k+1/2,N,\chi}(n^2)Q=n^{k-3/2}\sum_{0<a|n_0}a[\Delta\tau(k;n/a^2)\Delta]Q,$$

where $\tau(k;n)$ is the same as in §1. Hence, from the results of §1, it is sufficient to calculate the trace of the operator $[\Delta\tau(k;n)\Delta]Q$ for our purpose.

We can prove the following lemma by modifying the proof of the Lemma 1 in [K] §4.

Lemma (2.3). *We have, as elements of the abstract Hecke algebra,*

$$\Delta\Big(\begin{pmatrix}1&0\\0&n^2\end{pmatrix}, n^{k+1/2}\Big)\Delta\cdot\Delta\xi_{k+1/2,\varepsilon}\Delta=\Delta\Big(\begin{pmatrix}4&1\\0&4n^2\end{pmatrix}, \varepsilon^{k+1/2}e((2k+1)/8)n^{k+1/2}\Big)\Delta,$$

for any positive integer n *with* $(n, N)=1$.

From this lemma, we have:

$$\text{(2.4)}\qquad [\Delta\tau(k;n)\Delta]Q=\varepsilon^{-k-1/2}e(-(2k+1)/8)[\Delta\tau_0(k;n)\Delta],$$

where $G(k+1/2)\ni\tau_0(k;n)=(\alpha_0(n), n^{k+1/2})$ with $\alpha_0(n)=\begin{pmatrix} n^{-1} & (4n)^{-1} \\ 0 & n \end{pmatrix}$.

Let L be the same proper lifting as in the condition (1.2) of § 1. Then, it is easily shown that $\tau_0(k;n)$ satisfies the conditions (1.1) and (1.2) with respect to L. Hence, we have the bijection: $\Gamma\alpha_0(n)\Gamma\ni\sigma_1\alpha_0(n)\sigma_2\rightarrow L(\sigma_1)\tau_0(k;n)L(\sigma_2)\in\Delta\tau_0(k;n)\Delta$, and denote by β^* the image of $\beta\in\Gamma\alpha_0(n)\Gamma$.

Now, we shall calculate the trace of the operator $[\Delta\tau_0(k;n)\Delta]$ by using the Shimura's trace formula (1.3) in § 1. Since $\Gamma\alpha_0(n)\Gamma$ has no scalar element, the contribution to the trace from the scalar elements is zero. Let $\tilde{p}_0(n)$, $\tilde{e}_0(n)$ and $\tilde{h}_0(n)$ be the contribution to the trace from the parabolic, elliptic and hyperbolic equivalence classes in $\Phi=\Phi(\Gamma\alpha_0(n)\Gamma/\Gamma)$ respectively. Moreover, when $k=1$, we put

$$\tilde{d}_0(n)=\mathrm{trace}([\Delta'\tau_0'(n)\Delta']_{1/2}|G(1/2, N, \chi))-\mathrm{trace}([\Delta\tau_0(1;n)\Delta]_{3/2}|U(N, \chi)),$$

where $\tau_0'(n)=(\alpha_0(n)^{-1}, n^{-1/2})$ and $\Delta'=\Delta_0(N, \chi)_{1/2}$.

By using these notations, we can write

$$\mathrm{trace}\Big([\Delta\tau_0(k;n)\Delta]\Big|S\Big(k+\frac{1}{2}, N, \chi\Big)\Big)=\tilde{p}_0(n)+\tilde{e}_0(n)+\tilde{h}_0(n),$$

if $k\geqq 2$, and

$$\mathrm{trace}\Big([\Delta\tau_0(1;n)\Delta]\Big|V(N, \chi)\Big)=\tilde{p}_0(n)+\tilde{e}_0(n)+\tilde{h}_0(n)+\tilde{d}_0(n),$$

Before calculating the each term, we give some remarks.

Remark (2.5). We have

$$\Gamma\begin{pmatrix} 4 & 1 \\ 0 & 4n^2 \end{pmatrix}\Gamma=\left\{\begin{array}{l} M_2(\boldsymbol{Z})\ni\begin{pmatrix} a & b \\ c & d \end{pmatrix};\ ad-bc=16n^2,\ c\equiv 0 \pmod{16M}, \\ a\equiv d\equiv 0 \pmod 4,\ (a, M)=1 \text{ and } (a, b, c, d)=1. \end{array}\right\}$$

(cf. [K] § 4 Lemma 2).

Remark (2.6). Let $\beta=(4n)^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma\alpha_0(n)\Gamma$. If $(b, d)=1$, then, we have: $\beta^*=\Big(\beta, \Big(\frac{\mathrm{sgn}(d)}{-\mathrm{sgn}(c)}\Big)\Big(\frac{d}{b}\Big)\Big(\frac{-1}{b}\Big)^{-k-1/2}\Big(\frac{\varepsilon}{b}\Big)\Big(\frac{a}{M_0}\Big)((cz+d)/4n)^{k+1/2}\Big)$, where M_0 is the square-free positive divisor of M such that $\chi=\Big(\frac{M_0}{\ }\Big)$, and $\mathrm{sgn}(x)=1$ or -1, according as $x\geqq 0$ or $x<0$. We can prove this assertion by slightly modifying the proof of [K] § 4 Lemma 3.

Remark (2.7). For an elliptic or hyperbolic matrix $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in M_2(\boldsymbol{Z})$. Put $t=a+d$ and $f=(a-d, b, c)$. Then, t and f (and also the signature of c if A is elliptic) are invariant under the $SL_2(\boldsymbol{Z})$-cojugation. Moreover, every elliptic or hyperbolic Γ-conjugacy class in $\Gamma\alpha_0(n)\Gamma$ contains an element $(4n)^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $d>0$, $(b, d)=1$ and $(b/f, (t^2-64n^2)/f^2)=1$, where $t=a+d$ and $f=(a-d, b, c)$

(cf. [K] § 4 Lemma 4).

Remark (2.8). Let $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in M_2(\boldsymbol{Z})$ be an elliptic or hyperbolic matrix such that $ad-bc=16n^2$, $t=a+d\equiv 0 \pmod 4$ and $(a, b, c, d)=1$ and that $f=(a-d, b, c)$ is an odd integer. For this A, we define a set $D(A)$ by: $D(A)=\{SL_2(\boldsymbol{Z})\ni B \mid (4n)^{-1}B^{-1}AB\in\Gamma\alpha_0(n)\Gamma\}$. Then, Γ operates on $D(A)$ by multiplication from the right and we have: $\#D(A)/\Gamma=\prod_{p\mid M}\tilde{c}_p(t, f)$, where

$$\tilde{c}_p(t, f)=\begin{cases} p^{\nu-1}(p+1), & \text{if } \rho\geqq\nu; \\ p^\rho\times\#\{(\boldsymbol{Z}/p^\nu\boldsymbol{Z})\ni x \mid x^2-(t/4)x+n^2\equiv 0 \pmod{p^{\nu+\rho}}\} & \text{if } \rho<\nu; \end{cases}$$

with $\nu=\nu_p=\mathrm{ord}_p(M)$ and $\rho=\rho_p=\mathrm{ord}_p(f)$. The proof of this assertion will be given at the Appendix 3.

Now, we return to the calculation of the each term.

1. The calculation of $\tilde{p}_0(n)$.

By using the Remark (2.5) and the same argument as in § 1, we can write out all parabolic equivalence classes in Φ. The result is as follows: Put

$$\tilde{S}=\left\{\begin{array}{l} t=4\zeta\prod_{p\mid M}p^e>0;\ 0\leqq e=e_p\leqq\mathrm{ord}_p(M)=\nu=\nu_p \text{ and} \\ \zeta \text{ runs over a system of representatives, such} \\ \text{that } (\zeta, N)=1, \quad \text{for } (\boldsymbol{Z}/\prod_{p\mid M}p^{\min(e,\nu-e)}\boldsymbol{Z})^\times \end{array}\right\}.$$

For $t=4\zeta\prod_{p\mid M}p^e\in\tilde{S}$, we write $u=\prod'_{p\mid M}p^{\nu-2e}$ and $\tilde{\beta}(t, x)=(4n)^{-1}\begin{pmatrix} 4n-txu & xu \\ -t^2xu & 4n+txu \end{pmatrix}$, where the symbol $\prod'_{p\mid M}$ means the same as in § 1. Then, a system of representatives of all parabolic equivalence classes in Φ is formed by the matrices $\tilde{\beta}(t, x)$, where $t=4\zeta\prod_{p\mid M}p^e$ runs over the set $\tilde{S}$ and x runs over a system of representatives for $(\boldsymbol{Z}/4n\boldsymbol{Z})^\times$. Here, by the suitable choice of the representative, we may assume that $x>0$.

Now, we shall determine the number $J(\beta)=J(C)$ for the equivalence class C containing the matrix $\beta=\tilde{\beta}(t, x)$. The stabilizer of t^{-1}, which is the fixed point of β, in $\Gamma/\{\pm 1\}$ is generated by $\sigma=\begin{pmatrix} 1-ut & u \\ -ut^2 & 1+ut \end{pmatrix}$. Put $A_t=\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$ and $\rho=A_t\begin{pmatrix} u^{1/2} & 0 \\ 0 & u^{-1/2} \end{pmatrix}$. Then, we have $\rho^{-1}\sigma\rho=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and, since $\beta=1+(x/4n)(\sigma-1)$, $\rho^{-1}\beta\rho=\begin{pmatrix} 1 & x/4n \\ 0 & 1 \end{pmatrix}$.

Let $\rho^*=(A_t, j(A_t, z)^{2k+1})\left(\begin{pmatrix} u^{1/2} & 0 \\ 0 & u^{-1/2} \end{pmatrix}, u^{-k/2-1/4}\right)$, then, since $t\equiv 0 \pmod 4$ and $tu\equiv 0 \pmod{\prod_{p\mid M}p}$, we have:

$$\begin{aligned}\rho^{*-1}L(\sigma)\rho^* &=\rho^{*-1}(A_t, j(A_t, z)^{2k+1})\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, 1\right)(A_t^{-1}, j(A_t^{-1}, z)^{2k+1})\rho^* \\ &=\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1\right).\end{aligned}$$

Moreover, we note that $(xu, 4n+txu)=(xu, 4n)=1$. Hence, by using the Remark (2.6), we have

$$\beta^*=\left(\beta, \left(\frac{\varepsilon n}{xu}\right)\left(\frac{n}{M_0}\right)\left(\frac{-1}{xu}\right)^{-k-1/2}((cz+d)/4n)^{k+1/2}\right) \quad \text{with } \beta=(4n)^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Here, we used the assumption $x>0$. Therefore, we have

$$\rho^{*-1}\beta^*\rho^*=\left(\begin{pmatrix} 1 & x/4n \\ 0 & 1 \end{pmatrix}, \left(\frac{\varepsilon n}{xu}\right)\left(\frac{n}{M_0}\right)\left(\frac{-1}{xu}\right)^{-k-1/2}\right).$$

Finally, since $\alpha_0(n)\notin\Gamma$, we have always $\beta\notin\Gamma$. Thus, we obtain

$$J(\beta)=\left(\frac{\varepsilon n}{xu}\right)\left(\frac{n}{M_0}\right)\left(\frac{-1}{xu}\right)^{k+1/2}e(x/4n)(1-e(x/4n))^{-1}.$$

Since $e(x/4n)(1-e(x/4n))^{-1}=-(1/2)+(\sqrt{-1}/2)\cot(\pi x/4n)$, we have $\tilde{p}_0(n)=\tilde{p}_1(n)+\tilde{p}_2(n)$ with

$$\tilde{p}_1(n)=-(1/2)\left(\frac{n}{M_0}\right)\sum_{\substack{0\leq e_p\leq\nu_p \\ p\mid M}}\sum_{\zeta}\left(\frac{\varepsilon n}{u}\right)\sum_x\left(\frac{\varepsilon n}{x}\right)\left(\frac{-1}{xu}\right)^{k+1/2}$$

and

$$\tilde{p}_2(n)=(\sqrt{-1}/2)\left(\frac{n}{M_0}\right)\sum_{\substack{0\leq e_p\leq\nu_p \\ p\mid M}}\sum_{\zeta}\left(\frac{\varepsilon n}{u}\right)\sum_x\left(\frac{\varepsilon n}{x}\right)\left(\frac{-1}{xu}\right)^{k+1/2}\cot(\pi x/4n).$$

Put

$$A_0=\sum_x\left(\frac{\varepsilon n}{x}\right)\left(\frac{-1}{xu}\right)^{k+1/2} \quad \text{and} \quad B_0=\sum_x\left(\frac{\varepsilon n}{x}\right)\left(\frac{-1}{xu}\right)^{k+1/2}\cot(\pi x/4n),$$

where x in the sum $\sum_x$ runs over any system of all representatives for $(\boldsymbol{Z}/4n\boldsymbol{Z})^\times$. Then, we have:

$$\begin{aligned} A_0&=(1/2)\sum_x\left\{\left(\frac{\varepsilon n}{x}\right)\left(\frac{-1}{xu}\right)^{k+1/2}+\left(\frac{\varepsilon n}{-x}\right)\left(\frac{-1}{-xu}\right)^{k+1/2}\right\} \\ &=(1/2)\sum_x\left(\frac{\varepsilon n}{x}\right)\left\{\left(\frac{-1}{xu}\right)^{k+1/2}+\varepsilon\left(\frac{-1}{-xu}\right)^{k+1/2}\right\} \\ &=(1/2)\sum_x\left(\frac{\varepsilon n}{x}\right)\left(\frac{\varepsilon}{xu}\right)(1+\varepsilon(-1)^k\sqrt{-1})=((1+\varepsilon(-1)^k\sqrt{-1})/2)\left(\frac{\varepsilon}{u}\right)\sum_x\left(\frac{n}{x}\right) \\ &=(1+\varepsilon(-1)^k\sqrt{-1})\left(\frac{\varepsilon}{u}\right)\delta_0(\sqrt{n})\varphi(n), \end{aligned}$$

where $\delta_0(\sqrt{n})$ is the same as in §1. Similarly, by using the Dirichlet's class number formula, we have:

$$\begin{aligned} B_0&=(1/2)\sum_x\left\{\left(\frac{\varepsilon n}{x}\right)\left(\frac{-1}{xu}\right)^{k+1/2}\cot(\pi x/4n)+\left(\frac{\varepsilon n}{-x}\right)\left(\frac{-1}{-xu}\right)^{k+1/2}\cot(-\pi x/4n)\right\} \\ &=(1/2)\sum_x\left(\frac{\varepsilon n}{x}\right)\cot(\pi x/4n)\left\{\left(\frac{-1}{xu}\right)^{k+1/2}-\varepsilon\left(\frac{-1}{-xu}\right)^{k+1/2}\right\} \end{aligned}$$

$$=(1/2)\sum_x\left(\frac{\varepsilon n}{x}\right)\cot(\pi x/4n)\left(\frac{-\varepsilon}{xu}\right)(1-\varepsilon(-1)^k\sqrt{-1})$$

$$=2(1-\varepsilon(-1)^k\sqrt{-1})\left(\frac{-\varepsilon}{u}\right)\sqrt{n}\,h'(-4n).$$

Hence, by using the Lemma (1.7) in § 1,

$$\tilde{p}_1(n)=-((1+\varepsilon(-1)^k\sqrt{-1})/2)\delta_0(\sqrt{n})\varphi(n)\prod_{p\mid M}(p^{[\nu/2]}+p^{[(\nu-1)/2]})$$

and

$$\tilde{p}_2(n)=(\sqrt{-1}+\varepsilon(-1)^k)\left(\frac{n}{M_0}\right)\sqrt{n}\,h'(-4n)\prod_{p\mid M}\left(p^{[\nu/2]}+\left(\frac{-n}{p}\right)^\nu p^{[(\nu-1)/2]}\right).$$

Therefore, in the same way as in § 1, we have:

(2.9)
$$n^{k-3/2}\sum_{0<a\mid n_0}a\tilde{p}_0(n/a^2)$$
$$=-((1+\varepsilon(-1)^k\sqrt{-1})/2)\delta_0(\sqrt{n})n^{k-1/2}\prod_{p\mid M}(p^{[\nu/2]}+p^{[(\nu-1)/2]})$$
$$+(\sqrt{-1}+\varepsilon(-1)^k)\left(\frac{n}{M_0}\right)n^{k-1}\prod_{p\mid M}\left(p^{[\nu/2]}+\left(\frac{-n}{p}\right)^\nu p^{[(\nu-1)/2]}\right)$$
$$\times\sum_{0<a\mid n_0}h'(-4n/a^2).$$

2. The calculation of $\tilde{e}_0(n)$.

Let C be an elliptic equivalence class in Φ. By using the Remark (2.7), we can take $\beta=(4n)^{-1}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in C$ with $d\neq 0$, $(b, d)=1$ and $(b/f, (t^2-64n^2)/f^2)=1$, where $t=a+d$ and $f=(a-d, b, c)$. We note that $|t|<8n$ because C is elliptic. From the Remark (2.6), we have: $J(\beta)=J(C)=(1/2)\chi(\beta)\zeta(\beta)^{-2k-1}(1-\zeta(\beta)^{-4})^{-1}$ with $\chi(\beta)=\left(\frac{\mathrm{sgn}(d)}{-\mathrm{sgn}(c)}\right)\left(\frac{a}{M_0}\right)\left(\frac{d}{b}\right)\left(\frac{\varepsilon}{b}\right)$ and $\zeta(\beta)=\left(\frac{-1}{b}\right)^{-1/2}((cz_0+d)/4n)^{1/2}=\left(\frac{-1}{b}\right)^{-1/2}(4\sqrt{n})^{-1}(\sqrt{t+8n}+\mathrm{sgn}(c)\sqrt{t-8n})$. Here, $z_0\in\mathfrak{H}$ is the fixed point of β.

Put $w=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$. Then, since $w\beta w=(4n)^{-1}\begin{pmatrix}a & -b\\ -c & d\end{pmatrix}$, we have $\chi(w\beta w)=\varepsilon\chi(\beta)$, $\zeta(w\beta w)=-\sqrt{-1}\zeta(\beta)^{-1}$ and $J(w\beta w)=((\varepsilon(-1)^k\sqrt{-1})/2)\chi(\beta)\zeta(\beta)^{2k+1}(1-\zeta(\beta)^4)^{-1}$. Hence,

$$\tilde{e}_0(n)=\textstyle\sum_1 J(\beta)=\sum_2(J(\beta)+J(w\beta w))$$

$$=(1/2)\textstyle\sum_2\chi(\beta)(\zeta(\beta)^{-2k+1}-\varepsilon(-1)^k\sqrt{-1}\zeta(\beta)^{2k-1})(\zeta(\beta)^2-\zeta(\beta)^{-2})^{-1},$$

where β in $\sum_1$ runs over all representatives for the elliptic Γ-conjugacy classes in Φ and β in $\sum_2$ runs over those such that $c>0$.

Put $\lambda(t)=(\sqrt{t+8n}+\sqrt{t-8n})/2$ and $p(t)=(\lambda(t)^{-2k+1}-\overline{\lambda(t)}^{-2k+1})(\lambda(t)-\overline{\lambda(t)})^{-1}$, then $\lambda(-t)=\sqrt{-1}\,\overline{\lambda(t)}$ and $p(-t)=(-1)^k(\lambda(t)^{-2k+1}+\overline{\lambda(t)}^{-2k+1})(\lambda(t)+\overline{\lambda(t)})^{-1}$. Then, for β in $\sum_2$, we have $\zeta(\beta)=\left(\frac{-1}{b}\right)^{-1/2}(2\sqrt{n})^{-1}\lambda(t)$. Hence, from $|\zeta(\beta)|=1$, we can rewrite $\tilde{e}_0(n)$ as follows:

$$\tilde{e}_0(n)=2^{2k}n^{k+1/2}\sum_2\chi(\beta)\Big(\frac{-1}{b}\Big)^{k+1/2}(\lambda(t)^2-\overline{\lambda(t)}^2)^{-1}$$

$$\times(\lambda(t)^{-2k+1}-\varepsilon(-1)^k\sqrt{-1}\Big(\frac{-1}{b}\Big)\overline{\lambda(t)}^{-2k+1})$$

$$=2^{2k-1}n^{k+1/2}\sum_2\chi(\beta)\Big\{\Big(\frac{-1}{b}\Big)^{k+1/2}\Big(1+\varepsilon(-1)^k\sqrt{-1}\Big(\frac{-1}{b}\Big)\Big)p(t)(t+8n)^{-1/2}$$

$$+(-1)^k\Big(\frac{-1}{b}\Big)^{k+1/2}\Big(1-\varepsilon(-1)^k\sqrt{-1}\Big(\frac{-1}{b}\Big)\Big)p(-t)(t-8n)^{-1/2}\Big\}$$

$$=2^{2k-1}n^{k+1/2}\sum_2\chi(\beta)\Big\{\Big(\frac{\varepsilon}{b}\Big)(1+\varepsilon(-1)^k\sqrt{-1})p(t)(t+8n)^{-1/2}$$

$$+(-1)^k\Big(\frac{-\varepsilon}{b}\Big)(1-\varepsilon(-1)^k\sqrt{-1})p(-t)(t-8n)^{-1/2}\Big\}.$$

Moreover, by using the correspondence: $\beta\rightarrow-w\beta w$, we have

$$\sum_2\chi(\beta)\Big(\frac{-\varepsilon}{b}\Big)p(-t)(t-8n)^{-1/2}=\sqrt{-1}\varepsilon\sum_2\chi(\beta)\Big(\frac{\varepsilon}{b}\Big)p(t)(t+8n)^{-1/2}.$$

Therefore,

$$\tilde{e}_0(n)=2^{2k}n^{k+1/2}(1+\varepsilon(-1)^k\sqrt{-1})\sum_2\mathrm{sgn}\,(d)\Big(\frac{a}{M_0}\Big)\Big(\frac{d}{b}\Big)p(t)(t+8n)^{-1/2}.$$

Now, we shall give a system of all representatives for the elliptic ${}^{\prime}\Gamma$-conjugacy classes in Φ, such that $c>0$.

Let t be an integer such that $|t|<8n$ and that $t\equiv0\pmod 4$. Then, we write $t^2-64n^2=m^2u$ with a fundamental discriminant u and a positive integer m. Let f be a positive integer such that $f^2|(t^2-64n^2)$ and that $(f, 2n)=1$. We put

$$B(t, f)=\left\{\begin{array}{l} M_2(\boldsymbol{Z})\ni\begin{pmatrix} a & b\\ c & d\end{pmatrix};\ a+d=t,\ (a-d,\ b,\ c)=f\,, \\ (a,\ b,\ c,\ d)=1,\ ad-bc=16n^2,\ c>0 \end{array}\right\}.$$

and, for $A\in B(t, f)$, we define a set $D(A)$ in the same way as in the Remark (2.8). Then, $SL_2(\boldsymbol{Z})$ operates on $B(t, f)$ by means of the inner automorphism in $GL_2(\boldsymbol{R})$. By $Z(A)=Z_{SL_2(\boldsymbol{Z})}(A)$, we denote the centralizer of A in $SL_2(\boldsymbol{Z})$. Then, $Z(A)$ operates on $D(A)$ by multiplication from the left.

Take a representative A of a $SL_2(\boldsymbol{Z})$-conjugacy class in $B(t, f)$, and, for such A, take a representative B of $Z(A)\backslash D(A)/\Gamma$. Then, when t, f, A and B vary under the above conditions, the matrix $(4n)^{-1}B^{-1}AB=(4n)^{-1}\begin{pmatrix} a & b\\ c & d\end{pmatrix}$ forms a complete system of all representatives for the elliptic Γ-conjugacy classes in Φ, such that $c>0$. Here, by the suitable choice of the representative, we may assume that $d>0$, $(b, d)=1$ and $(b/f, (t^2-64n^2)/f^2)=1$ (cf. Remark (2.7)).

Thus, we have:

$$\tilde{e}_0(n)=2^{2k}n^{k+1/2}(1+\varepsilon(-1)^k\sqrt{-1})\sum_t p(t)(t+8n)^{-1/2}\sum_f\sum_A\sum_B\Big(\frac{a}{M_0}\Big)\Big(\frac{d}{b}\Big).$$

Since $(f, t+8n, t-8n)=(M_0, t+8n, t-8n)=1$, we can decompose $f=f_1f_2$ and $M_0=M_1M_2$ with $0<f_1$, $0<f_2$, $f_1^2|(t+8n)$, $f_2^2|(t-8n)$, $(f_1, t-8n)=(f_2, t+8n)=1$, $0<M_1$, $0<M_2$, $M_1|(t+8n)$ and $(M_2, t+8n)=1$. In the same way as in [K] p. 52, we have

$$\left(\frac{d}{b}\right)=\left(\frac{(t+8n)/f_1^2}{b/f}\right)\left(\frac{t-8n}{f_1}\right)\left(\frac{t+8n}{f_2}\right)$$

and

$$\left(\frac{a}{M_0}\right)=\left(\frac{a}{M_1}\right)\left(\frac{a}{M_2}\right)=\left(\frac{t-8n}{M_1}\right)\left(\frac{t+8n}{M_2}\right).$$

Hence,

$$\tilde{e}_0(n)=2^{2k}n^{k+1/2}(1+\varepsilon(-1)^k\sqrt{-1})\sum_t p(t)(t+8n)^{-1/2}\left(\frac{t-8n}{M_1}\right)\left(\frac{t+8n}{M_2}\right)$$
$$\times\sum_f\left(\frac{t-8n}{f_1}\right)\left(\frac{t+8n}{f_2}\right)\sum_A\sum_B\left(\frac{(t+8n)/f_1^2}{b/f}\right).$$

Therefore, from the same arguement in [K] p. 53, we may assume that $t+8n$ is a square integer, and then, by using the Remark (2.8), we have:

$$\sum_A\sum_B\left(\frac{(t+8n)/f_1^2}{b/f}\right)=h((t^2-64n^2)/f^2)\sharp(Z(A)\backslash D(A)/\Gamma)$$
$$=h'((t^2-64n^2)/f^2)\prod_{p|M}\tilde{c}_p(t, f).$$

We can write $t+8n=4s^2$ with $s>0$. Then, we have $t^2-64n^2=16s^2(s^2-4n)$, $\lambda(t)=s+(s^2-4n)^{1/2}$ and $p(t)=-4^{-k}n^{-2k+1}\pi_k(s, n)$, where $\pi_k(s, n)$ is the same as in § 1.

Thus, we obtain:

$$\tilde{e}_0(n)=-((1+\varepsilon(-1)^k\sqrt{-1})/2)n^{-k+3/2}\sum_s s^{-1}\pi_k(s, n)\left(\frac{s^2-4n}{M_1}\right)$$
$$\times\sum_{f_1, f_2}\left(\frac{s^2-4n}{f_1}\right)h'(16s^2(s^2-4n)(f_1f_2)^{-2})\prod_{p|M}\tilde{c}_p(4s^2-8n, f_1f_2),$$

where s runs over all integers such that $0<s<2\sqrt{n}$, f_1 runs over a set $\{\boldsymbol{Z}\ni f_1>0; (f_1, 2n)=1, f_1|s\}$ and f_2 runs over a set $\{\boldsymbol{Z}\ni f_2>0; (f_2, 2n)=1, f_2^2|(s^2-4n)\}$.

In the same way as in § 1 (elliptic case), we rewrite the above formula as follows:

$$\tilde{e}_0(n)=-((1+\varepsilon(-1)^k\sqrt{-1})/2)n^{-k+3/2}\sum_s\pi_k(s, n)h'(u)\left(4-2\left(\frac{u}{2}\right)\right)$$
$$\times 2^{\mathrm{ord}_2(m_1)}u(m_1')\alpha_u(r(m_1')) \prod_{q|(s', m_1')}\left(1-\left(\frac{u}{q}\right)q^{-1}\right)$$
$$\times\prod_{p|M}\{p^{-\mathrm{ord}_p(s)}n_p(\theta_p)\},$$

where $s^2-4n=m_1^2u$ with $m_1>0$, $\theta_p=\mathrm{ord}_p(sm_1)$, $s'=s\prod_{p|N}p^{-\mathrm{ord}_p(s)}$ and $m_1'=$

$m_1\prod_{p|N} p^{-\mathrm{ord}_p(m_1)}$, $u(m_1')$ and $r(m_1')$ are the same as in §1 (elliptic case), the constant $n_p(\theta_p)$ is given by the table (case (1)-(3)) in the Appendix 1.

Moreover, in the same way as in §1, we obtain:

$$(2.10)\qquad n^{k-3/2}\sum_{0<a|n_0} a\tilde{e}_0(n/a^2)$$
$$=-((1+\varepsilon(-1)^k\sqrt{-1})/2)\sum\nolimits^{(e)}\pi_k(\tilde{s},\ n)h'(u)\Big(4-2\Big(\frac{u}{2}\Big)\Big)2^{\mathrm{ord}_2(\tilde{m}_1)}$$
$$\times\alpha_u(\tilde{m}_1\prod_{p|N}p^{-\mathrm{ord}_p(\tilde{m}_1)})\prod_{p|M}\{p^{-\mathrm{ord}_p(\tilde{s})}n_p(\theta_p)\},$$

where $\tilde{s}$ in the sum $\sum^{(e)}$ runs over all integers such that $0<\tilde{s}<2\sqrt{n}$, $\tilde{s}^2-4n=\tilde{m}_1^2u$ with a fundamental discriminant u and a positive integer $\tilde{m}_1$, $\theta_p=\mathrm{ord}_p(\tilde{s}\tilde{m}_1)$. Also $\pi_k(\tilde{s},\ n)$ and $n_p(\theta_p)$ are the same as above.

3. The calculation of $\tilde{h}_0(n)$.

Let C be a hyperbolic equivalence class in Φ and $w=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$. From the definition of $J(C)$, we have $J(C)=J(-C)$. Hence, $\tilde{h}_0(n)=\sum_1 J(C)=\sum_2(J(C)+J(-C))=2\sum_2 J(C)=\sum_2(J(C)+J(wCw))$, where C in $\sum_1$ runs over all hyperbolic Γ-conjugacy classes in Φ and C in $\sum_2$ runs over those such that $a+d>0$.

For C in $\sum_2$, we can take $\beta=(4n)^{-1}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in C$ with $d>0$, $(b,\ d)=1$ and $(b/f,\ (t^2-64n^2)/f^2)=1$, where $t=a+d$ and $f=(a-d,\ b,\ c)$ (cf. Remark (2.7)). Then, from the Remark (2.6), we have:

$$J(\beta)=J(C)=-(1/2)\Big(\frac{d}{b}\Big)\Big(\frac{\varepsilon}{b}\Big)\Big(\frac{a}{M_0}\Big)\Big(\frac{-1}{b}\Big)^{k+1/2}\lambda(\beta)^{-k-1/2}(1-\lambda(\beta)^{-2})^{-1}$$

with

$$\lambda(\beta)=\begin{cases}(4n)^{-1}(a-cz_0), & \text{if } z_0\neq\infty;\\ d/4n, & \text{if } z_0=\infty;\end{cases}$$

where z_0 is the upper fixed point of β which is a cusp of Γ (cf. Remarks (1.4) and (1.5)). Hence,

$$J(\beta)+J(w\beta w)$$
$$=-(1/2)\Big(\frac{d}{b}\Big)\Big(\frac{\varepsilon}{b}\Big)\Big(\frac{a}{M_0}\Big)\lambda(\beta)^{-k-1/2}(1-\lambda(\beta)^{-2})^{-1}\Big(\Big(\frac{-1}{b}\Big)^{k+1/2}+\varepsilon\Big(\frac{-1}{-b}\Big)^{k+1/2}\Big)$$
$$=-((1+\varepsilon(-1)^k\sqrt{-1})/2)\Big(\frac{d}{b}\Big)\Big(\frac{a}{M_0}\Big)\lambda(\beta)^{-k-1/2}(1-\lambda(\beta)^{-2})^{-1}.$$

Let t be an integer such that $t\equiv 0 \pmod 4$ and that $t>8n$ and that t^2-64n^2 is square. Then, we write $t^2-64n^2=m^2$ with $m>0$. Let f be a positive integer such that $f|m$ and that $(f,\ 2n)=1$. We put

$$B_1(t,\ f)=\left\{\begin{array}{r} M_2(\boldsymbol{Z})\ni\begin{pmatrix} a & b\\ c & d\end{pmatrix};\ a+d=t,\ (a-d,\ b,\ c)=f\ ,\\ (a,\ b,\ c,\ d)=1,\ ad-bc=16n^2\end{array}\right\}$$

and, for $A\in B_1(t,\ f)$, let $D(A)$ and $Z(A)$ be the same as in the elliptic case. Then, $SL_2(\boldsymbol{Z})$ operates on $B_1(t,\ f)$ by means of the inner automorphism and a system of all representatives of the $SL_2(\boldsymbol{Z})$-conjugacy classes in $B_1(t,\ f)$ is given by

$$X(t,\ f)=\left\{\begin{array}{l} M_2(\boldsymbol{Z})\ni\begin{pmatrix} \nu' & \tau\\ 0 & \nu\end{pmatrix};\ \nu\nu'=16n^2,\ 0<\nu'<\nu,\\ 0\leqq\tau<\nu-\nu',\ \nu+\nu'=t,\ f=(\nu-\nu',\ \tau)\end{array}\right\}$$

(cf. [K] p. 55). For $A\in X(t,\ f)$, take a representative B of $Z(A)\backslash D(A)/\Gamma$. When $t,\ f,\ A=\begin{pmatrix} \nu' & \tau\\ 0 & \nu\end{pmatrix}$ and B vary under the above conditions, the matrix $(4n)^{-1}B^{-1}AB=(4n)^{-1}\begin{pmatrix} a & b\\ c & d\end{pmatrix}$ forms a complete system of all representatives for the hyperbolic Γ-conjugacy classes in Φ, such that $a+d>0$. Here, by the suitable choice of the representative B, we may assume that $d>0$, $(b,\ d)=1$ and $(b/f,\ (t^2-64n^2)/f^2)=1$ with $t=a+d$ and $f=(a-d,\ b,\ c)$ (cf. Remark (2.7)).

From an elementary calculation, we get $\lambda\left((4n)^{-1}B^{-1}\begin{pmatrix} \nu' & \tau\\ 0 & \nu\end{pmatrix}B\right)=\nu/4n=(\nu'/4n)^{-1}$. Hence, we have:

$$\tilde{h}_0(n)=-((1+\varepsilon(-1)^k\sqrt{-1})/2)(4n)^{-k+3/2}\sum_t\sum_f\sum_A\sum_B\nu'^{k-1/2}(\nu-\nu')^{-1}\left(\frac{d}{b}\right)\left(\frac{a}{M_0}\right).$$

In the same way as in the elliptic case, we may assume that $t+8n$ is square. Since $t^2-64n^2=(t+8n)(t-8n)$ is also square, we can write $t+8n=4s^2$ and $t-8n=4r^2$ with $s>0$ and $r>0$. Since $\nu=(t+(t^2-64n^2)^{1/2})/2$ and $\nu'=(t-(t^2-64n^2)^{1/2})/2$, we have $\nu-\nu'=4sr$ and $\nu'^{1/2}=((t+8n)^{1/2}-(t-8n)^{1/2})/2=s-r$. Therefore, in the same way as in the elliptic case, we obtain:

$$\tilde{h}_0(n)=-((1+\varepsilon(-1)^k\sqrt{-1})/2)n^{-k+3/2}$$
$$\times\sum_s((s-r)/2)^{2k-1}(sr)^{-1}\sum_f\varphi(4sr/f)\prod_{p\mid M}\tilde{c}_p(4s^2-8n,\ f),$$

where s in $\sum_s$ runs over all integers such that $s>2\sqrt{n}$ and that s^2-4n is square, $r=(s^2-4n)^{1/2}$ and f runs over all positive divisors of $4sr$ such that $(f,\ 2n)=1$.

Finally, by applying the same argument as in §1 (hyperbolic case), we can deduce the following result from the above formula of $\tilde{h}_0(n)$:

(2.11) $$n^{k-3/2}\sum_{0<a\mid n_0}a\tilde{h}_0(n/a^2)$$
$$=-(1+\varepsilon(-1)^k\sqrt{-1})\sum^{(h)}((s-r)/2)^{2k-1}\prod_{p\mid M}m_p(\theta_p),$$

where s in $\sum^{(h)}$ runs over all integers such that $s>2\sqrt{n}$ and that s^2-4n is square, $r=(s^2-4n)^{1/2}$, $\theta_p=\mathrm{ord}_p(sr)$ and the constant $m_p(\theta_p)$ is given by the table (*case* (1)) *in the Appendix* 2.

Remark. Any integer s which satisfies the above conditions is always even. Hence, we may consider only the even integer s.

4. The calculation of $\tilde{d}_0(n)$.

We have

$$[\Delta'\tau_0'(n)\Delta']_{1/2}=\left[\Delta'\left(\begin{pmatrix}4n^2 & -1\\ 0 & 4\end{pmatrix}, n^{-1/2}\right)\Delta'\right]_{1/2}$$
$$=\left[\Delta'\left(\begin{pmatrix}4 & -1\\ 0 & 4\end{pmatrix}, 1\right)\Delta'\right]_{1/2}\cdot\left[\Delta'\left(\begin{pmatrix}n^2 & 0\\ 0 & 1\end{pmatrix}, n^{-1/2}\right)\Delta'\right]_{1/2},$$

as operators on $G(1/2, N, \chi)$ (cf. Lemma (2.3)). Then, from the proof of [K] §2 Proposition 1 and the fact that $\{h^0(\psi; tz)|(\psi, t)\in\Omega^0(N, \chi)\}$ is a $\boldsymbol{C}$-basis of $G(1/2, N, \chi)$ (cf. §1), it follows that the operator $\left[\Delta'\left(\begin{pmatrix}4 & -1\\ 0 & 4\end{pmatrix}, 1\right)\Delta'\right]_{1/2}$ acts on $G(1/2, N, \chi)$ as the multiplication by $2(1-\varepsilon\sqrt{-1})$. Similarly, from the Lemma (2.3) and the fact: $U(N, \chi)\subseteq S(3/2, N, \chi)_K$, we have, as operators on $U(N, \chi)$,

$$[\Delta\tau_0(1; n)\Delta]_{3/2}=2(1-\varepsilon\sqrt{-1})\left[\Delta\left(\begin{pmatrix}1 & 0\\ 0 & n^2\end{pmatrix}, n^{3/2}\right)\Delta\right]_{3/2}.$$

Therefore, we get:

(2.12) $\tilde{d}_0(n)=2(1-\varepsilon\sqrt{-1})\tilde{d}(n)$, *where $\tilde{d}(n)$ is the same as in* §1.

Thus, the calculation of the trace of the operator $[\Delta\tau_0(k; n)\Delta]_{k+1/2}$ is completed.

Now, we have from the formulas (2.1), (2.2) and (2.4):

$$\mathrm{trace}\left(\tilde{T}_{k+1/2, N, \chi}(n^2)\middle|S\left(k+\frac{1}{2}, N, \chi\right)_K\right)$$
$$=((1-\varepsilon(-1)^k\sqrt{-1})/6)n^{k-3/2}\sum_{0<a|n_0} a\,\mathrm{trace}\left([\Delta\tau_0(k; n/a^2)\Delta]\middle|S\left(k+\frac{1}{2}, N, \chi\right)\right)$$
$$+(1/3)\mathrm{trace}\left(\tilde{T}_{k+1/2, N, \chi}(n^2)\middle|S\left(k+\frac{1}{2}, N, \chi\right)\right).$$

Hence, by combining the above formulas (2.9)-(2.12) with the results of §1, we obtain the following proposition.

Proposition 2. (1) *Suppose $k\geqq 2$. For all natural numbers n with $(n, N)=1$, we have:*

$$\mathrm{trace}\left(\tilde{T}_{k+1/2, N, \chi}(n^2)\middle|S\left(k+\frac{1}{2}, N, \chi\right)_K\right)=T(s)+T(p)+T(e)+T(h).$$

(2) *Let n be the same as in* (1). *Then, we have:*

$$\mathrm{trace}(\tilde{T}_{3/2,N,\chi}(n^2)\,|\,V(N,\chi)_K)=T(s)+T(p)+T(e)+T(h)+T(d).$$

Here, the term $T(d)$ *is the same as in the Proposition* 1 (*or the formula* (1.12)) *and the terms* $T(s)$, $T(p)$, $T(e)$ *and* $T(h)$ *are given by the following formulas* (2.13), (2.14), (2.15) *and* (2.16) *respectively:*

For any prime divisor p *of* M, *let* $\nu=\nu_p=\mathrm{ord}_p(M)$. *Let* $\delta_0(\sqrt{n})$, $\pi_k(s, n)$, n_0 *and* n_1 *be the same as in the statement of the Proposition* 1.

(2.13) $$T(s)=((2k-1)/12)\delta_0(\sqrt{n})n^{k-1}M\prod_{p|M}(p+1)/p.$$

(2.14) *For* $n\equiv 1 \pmod 4$,

$$T(p)=-(1/2)\delta_0(\sqrt{n})n^{k-1/2}\prod_{p|M}(p^{[\nu/2]}+p^{[(\nu-1)/2]})$$
$$+(\varepsilon(-1)^k/2)\left(\frac{n}{M_0}\right)n^{k-1}\prod_{p|M}\left(p^{[\nu/2]}+\left(\frac{-n}{p}\right)^{\nu}p^{[(\nu-1)/2]}\right)$$
$$\times\sum_{0<a|n_0}h'(-4n/a^2).$$

For $n\equiv 3 \pmod 4$,

$$T(p)=(\varepsilon(-1)^k/2)\left(3-\left(\frac{2}{n}\right)\right)\left(\frac{n}{M_0}\right)n^{k-1}$$
$$\times\prod_{p|M}\left(p^{[\nu/2]}+\left(\frac{-n}{p}\right)^{\nu}p^{[(\nu-1)/2]}\right)\sum_{0<a|n_0}h'(-n/a^2).$$

(2.15) $$T(e)=-\sum\nolimits^{(e)}\pi_k(s, n)h'(u)\alpha_u(t_1)\prod_{p|M}\{p^{-\mathrm{ord}_p(s)}n_p(\theta_p)\},$$

where s *in the sum* $\sum^{(e)}$ *runs over all integers such that* $0<s<2\sqrt{n}$. *The other notations are as follows: Write* $s^2-4n=t^2u$ *with a positive integer* t *and a fundamental discriminant* u. *Put* $t_1=t\prod_{p|M}p^{-\mathrm{ord}_p(t)}$ *and* $\theta_p=\mathrm{ord}_p(st)$. *The constant* $n_p(\theta_p)$ *is given by the table* (*case* (1)-*case* (3)) *in the Appendix* 1.

(2.16) $$T(h)=-\sum\nolimits^{(h)}((s-t)/2)^{2k-1}\prod_{p|M}m_p(\theta_p),$$

where s *in the sum* $\sum^{(h)}$ *runs over all even integers such that* $s>2\sqrt{n}$ *and that* s^2-4n *is square. The other notations are as follows: Put* $t=(s^2-4n)^{1/2}$ *and* $\theta=\theta_p=\mathrm{ord}_p(st)$. *The constant* $m_p(\theta_p)$ *is given by the table* (*case* (1)) *in the Appendix* 2.

§ 3. The relations.

Let N be a positive integer. Then, from [H], we have:

$$\mathrm{trace}(T_{2k,N}(n)\,|\,S(2k, N))=T_0(s)+T_0(p)+T_{00}(e)+T_0(e)+T_0(h)+T_0(d)$$

for all positive integers n with $(n, 2N)=1$. Here, the each term is given by the following formulas (3.1)-(3.6): We use the following notations. For prime numbers p, we write

$$\mathrm{ord}_p(N)=\tilde{\nu}_p=\tilde{\nu}=\begin{cases}\nu_p=\nu, & \text{if } p \text{ is odd};\\ \mu, & \text{if } p=2.\end{cases}$$

Let $\delta_0(\sqrt{n})$ be the same as in §1 and M the odd part of N. We decompose $n=n_0^2n_1$ with a positive integer n_0 and a square-free positive integer n_1.

(3.1) $$T_0(s)=((2k-1)/12)\delta_0(\sqrt{n})n^{k-1}N\prod_{p\mid N}(p+1)/p.$$

(3.2) $$T_0(p)=-(1/2)\delta_0(\sqrt{n})n^{k-1/2}\prod_{p\mid N}(p^{[\tilde{\nu}/2]}+p^{[(\tilde{\nu}-1)/2]}).$$

(3.3) For $n\equiv 1 \pmod 4$,

$$T_{00}(e)=\begin{cases}((-1)^k/2)n^{k-1}\prod_{p\mid M}\left(1+\left(\frac{-n}{p}\right)\right)\sum_{0<a\mid n_0}h'(-4n/a^2), & \text{if } \mu\leqq 1;\\ 0, \text{ if } \mu\geqq 2.\end{cases}$$

For $n\equiv 3 \pmod 4$,

$$T_{00}(e)=((-1)^k/2)n^{k-1}\prod_{p\mid M}\left(1+\left(\frac{-n}{p}\right)\right)\sum_{0<a\mid n_0}h'(-n/a^2)\times C_2$$

with

$$C_2=\begin{cases}3-\left(\frac{2}{n}\right), & \text{if } \mu=0;\\ 5-\left(\frac{2}{n}\right), & \text{if } \mu=1;\\ 6, & \text{if } \mu=2;\\ 4\left(1+\left(\frac{2}{n}\right)\right), & \text{if } \mu\geqq 3.\end{cases}$$

(3.4) $$T_0(e)=-\sum\nolimits_0^{(e)}\pi_k(s,n)h'(u)\alpha_u(t_0)\prod_{p\mid N}n_{0,p}(\theta_{0,p}),$$

where s in the sum $\sum_0^{(e)}$ runs over all integers such that $2\sqrt{n}>s>0$ and besides that s is even if $\mu\geqq 1$. The other notations are as follows: Let $\pi_k(s,n)$ be the same as in §1. $s^2-4n=t^2u$ with a positive integer t and a fundamental discrimenant u. Put $\theta_0=\theta_{0,p}=\mathrm{ord}_p(t)$ and $t_0=t\prod_{p\mid N}p^{-\mathrm{ord}_p(t)}$. The constant $n_{0,p}(\theta_{0,p})$ is given by the table in the Appendix 4.

(3.5) $$T_0(h)=-\sum\nolimits_0^{(h)}((s-t)/2)^{2k-1}\prod_{p\mid N}m_{0,p}(\theta_{0,p}),$$

where s in the sum $\sum_0^{(h)}$ runs over all even integers such that $s>2\sqrt{n}$ and that s^2-4n is square, and put $t=(s^2-4n)^{1/2}$ and $\theta_0=\theta_{0,p}=\mathrm{ord}_p(t)$ and, for any prime number $p\mid N$,

$$m_{0,p}(\theta_{0,p})=\begin{cases}p^{[\tilde{\nu}/2]}+p^{[(\tilde{\nu}-1)/2]}, & \text{if } \theta_0\geqq[(\tilde{\nu}+1)/2];\\ 2p^{\theta_0}, & \text{if } \theta_0\leqq[(\tilde{\nu}-1)/2].\end{cases}$$

$$T_0(d)=\delta(k)\prod_{p\mid n}(p^{\tau+1}-1)/(p-1) \tag{3.6}$$

with $n=\prod_{p\mid n}p^{\tau}$ and $\delta(k)=1$ or 0, according as $k=1$ or not.

Now, let N_0 be a positive divisor of N such that $(N_0, N/N_0)=1$ and that $N_0\neq 1$. Take any element $\gamma(N_0)\in SL_2(\boldsymbol{Z})$ which satisfies the conditions:

$$\gamma(N_0)\equiv\begin{cases}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \pmod{N_0};\\[2ex] \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \pmod{N/N_0}.\end{cases}$$

Put $W(N_0)=\gamma(N_0)\begin{pmatrix}N_0 & 0\\ 0 & 1\end{pmatrix}$. Then, it is well-known that $W(N_0)$ is a normalizer of $\Gamma=\Gamma_0(N)$ and that $[W(N_0)]_{2k}$ induces a $\boldsymbol{C}$-linear automorphism of order 2 on $S(2k, N)$.

In [Y], M. Yamauchi explicitly calculated the trace of the operator $[W(N_0)]_{2k}T_{2k,N}(n)$ with $(n, N)=1$ acting on $S(2k, N)$. But, his formula contains several errors in the hyperbolic and parabolic cases. Therefore, though we need only the trace formula of the operator $[W(N_0)]_{2k}T_{2k,N}(n)$ with $(n, 2N)=1$, we shall give the corrected version of the Yamauchi's formula in all cases.

For all positive integers n with $(n, N)=1$, we have:

$$\mathrm{trace}([W(N_0)]_{2k}T_{2k,N}(n)\mid S(2k, N))=T_1(p)+T_{10}(e)+T_1(e)+T_1(h)+T_1(d).$$

Here, the each term is given by the following formulas (3.7)-(3.11):

Let $\tilde{\nu}$, ν, μ, $\delta_0(\sqrt{n})$ and $\pi_k(s, n)$ be the same as in the trace formula of $T_{2k,N}(n)$.

$$T_1(p)=-(1/2)\delta_2\delta_0(\sqrt{n})n^{k-1/2}\prod_{p\mid(N/N_0)}(p^{[\nu/2]}+p^{[(\nu-1)/2]}) \tag{3.7}$$

with $\delta_2=1$ or 0, according as $N_0=4$ or not.

(3.8) Write $-4nN_0=t^2u$ with a positive integer t and a fundamental discriminant u. Then, we have:

If $\mathrm{ord}_2(N/N_0)=0$, or $\mathrm{ord}_2(N/N_0)=1$ and $nN_0\equiv 1 \pmod 4$,

$$T_{10}(e)=((-1)^k/2)n^{k-1}\prod_{\substack{p\mid(N/N_0)\\ p:\text{ odd}}}\left\{1+\left(\frac{-nN_0}{p}\right)\right\}\sum_{\substack{0<a\mid t\\ (a,N_0)=1}}h'(-4nN_0/a^2).$$

If $\mathrm{ord}_2(N/N_0)\geqq 2$ and $nN_0\equiv 1 \pmod 4$, $T_{10}(e)=0$.

If $\mathrm{ord}_2(N/N_0)\geqq 1$ and $nN_0\equiv 3 \pmod 4$,

$$T_{10}(e)=((-1)^k/2)n^{k-1}\prod_{\substack{p\mid(N/N_0)\\ p:\text{ odd}}}\left\{1+\left(\frac{-nN_0}{p}\right)\right\}$$

$$\times\sum_{\substack{0<a\mid(t/2)\\ (a,N_0)=1}}h'(-nN_0/a^2)\times C_2'$$

with

$$C_2'=\begin{cases}5-\left(\frac{2}{nN_0}\right), & \text{if } \operatorname{ord}_2(N/N_0)=1;\\ 6, & \text{if } \operatorname{ord}_2(N/N_0)=2;\\ 4\left(1+\left(\frac{2}{nN_0}\right)\right), & \text{if } \operatorname{ord}_2(N/N_0)\geqq 3.\end{cases}$$

(3.9)
$$T_1(e)=-N_0^{1-k}\sum\nolimits_1^{(e)}\pi_k(s,\ nN_0)h'(t_1^2u)\alpha_u(t_0)\times\prod_{p\mid(N/N_0)}n_{0,p}(\theta_{0,p}),$$

where s in the sum $\sum_1^{(e)}$ runs over all integers such that $2\sqrt{nN_0}>s>0$ and that $s\equiv 0 \pmod{N_0}$ and besides that s is even if N/N_0 is even. The other notations are as follows: $s^2-4nN_0=t^2u$ with a fundamental discriminant u and a positive integer t. Put $\theta_0=\theta_{0,p}=\operatorname{ord}_p(t)$, $t_0=t\prod_{p\mid N}p^{-\operatorname{ord}_p(t)}$ and $t_1=\prod_{p\mid N_0}p^{\operatorname{ord}_p(t)}$. The constant $n_{0,p}(\theta_{0,p})$ is given by the table in the Appendix 4.

(3.10)
$$T_1(h)=-\delta_0(\sqrt{N_0})\sum\nolimits_1^{(h)}((s-t)/2)^{2k-1}\varphi(\sqrt{N_0}\,t_1)t_1^{-1}\times\prod_{p\mid(N/N_0)}m_{0,p}(\theta_{0,p}),$$

where s in the sum $\sum_1^{(h)}$ runs over all integers such that $s>2\sqrt{n}$ and that $s\equiv 0 \pmod{\sqrt{N_0}}$ and that s^2-4n is square. Put $t=(s^2-4n)^{1/2}$, $\theta_0=\theta_{0,p}=\operatorname{ord}_p(t)$ and $t_1=\prod_{p\mid N_0}p^{\operatorname{ord}_p(t)}$. The constant $m_{0,p}(\theta_{0,p})$ is given by the same table as in the hyperbolic case of the trace formula of $T_{2k,N}(n)$.

(3.11)
$$T_1(d)=\delta(k)\prod_{p\mid n}(p^{\tau+1}-1)/(p-1)$$

with $n=\prod_{p\mid n}p^{\tau}$ and $\delta(k)=1$ or 0, according as $k=1$ or not.

From these trace formulas and the results of the previous sections, we can deduce relations between those traces.

Theorem. *Let N be a positive integer such that $2\leqq\operatorname{ord}_2(N)=\mu\leqq 4$, and put $M=2^{-\mu}N$. Let χ be an even character modulo N such that $\chi^2=1$ and suppose that the conductor of χ is divisible by 8 if $\mu=4$. Then, we have the following relations* (3.12)-(3.15):

(3.12) *Suppose $k\geqq 2$, then, for all positive integers n with $(n, N)=1$, we have:*

$$\begin{aligned}&\operatorname{trace}\left(\tilde{T}_{k+1/2,N,\chi}(n^2)\mid S\left(k+\tfrac{1}{2},\ N,\ \chi\right)\right)\\ &=\operatorname{trace}(T_{2k,N/2}(n)\mid S(2k,\ N/2))\\ &\quad+\textstyle\sum_1 A(n,\ L_0)\operatorname{trace}([W(L_0)]_{2k}T_{2k,2^{\mu-1}L_0L_1}(n)\mid S(2k,\ 2^{\mu-1}L_0L_1)).\end{aligned}$$

(3.13) *Let k and n be the same as in* (3.12) *and suppose $N=4M$, then, we have:*

$$\mathrm{trace}(\tilde{T}_{k+1/2,N,\chi}(n^2)\,|\,S\left(k+\frac{1}{2},N,\chi\right)_K)$$
$$=\mathrm{trace}(T_{2k,M}(n)\,|\,S(2k,M))$$
$$+\textstyle\sum_1 \Lambda(n,L_0)\,\mathrm{trace}([W(L_0)]_{2k}T_{2k,L_0L_1}(n)\,|\,S(2k,L_0L_1)).$$

(3.14) *Let* n *be the same as in* (3.12), *then, we have:*

$$\mathrm{trace}(\tilde{T}_{3/2,N,\chi}(n^2)\,|\,V(N,\chi))$$
$$=\mathrm{trace}(T_{2,N/2}(n)\,|\,S(2,N/2))$$
$$+\textstyle\sum_1 \Lambda(n,L_0)\,\mathrm{trace}([W(L_0)]_2T_{2,2^{\mu-1}L_0L_1}(n)\,|\,S(2,2^{\mu-1}L_0L_1)).$$

(3.15) *Let* n *be the same as in* (3.12) *and suppose* $N=4M$, *then, we have:*

$$\mathrm{trace}(\tilde{T}_{3/2,N,\chi}(n^2)\,|\,V(N,\chi)_K)$$
$$=\mathrm{trace}(T_{2,M}(n)\,|\,S(2,M))$$
$$+\textstyle\sum_1 \Lambda(n,L_0)\,\mathrm{trace}([W(L_0)]_2T_{2,L_0L_1}(n)\,|\,S(2,L_0L_1)).$$

Here, L_0 *in the sum* $\sum_1$ *runs over all square integers such that* $1<L_0|M$. *Put* $L_1=M\prod_{p|L_0}p^{-\mathrm{ord}_p(M)}$. *The constant* $\Lambda(n,L_0)$ *is defined as follows:*

$$\Lambda(n,L_0)=\prod_{p|M}\lambda(p,n;(\mathrm{ord}_p(L_0))/2) \quad \textit{with}$$

$$\lambda(p,n;a)=\begin{cases}1, & \textit{if } a=0;\\ 1+\left(\frac{-n}{p}\right), & \textit{if } 1\leqq a\leqq[(\nu-1)/2];\\ \chi_p(-n), & \textit{if } \nu \textit{ is even and } a=\nu/2;\end{cases}$$

where $\nu=\mathrm{ord}_p(N)$ *and* χ_p *is the p-component of* χ.

Proof. We can easily verify the following claimes (1)-(5):

(1) When $n\equiv 1 \pmod 4$, the first term of $T(p)$ is equal to the contribution from the parts of $T_0(p)$ and $T_1(p)$ to the right-hand side of the above relations. Also the second term of $T(p)$ is equal to the contribution from the parts of $T_{00}(e)$ and $T_{10}(e)$ to the right-hand side of the above relations.

(2) When $n\equiv 3 \pmod 4$, $T(p)$ is equal to the contribution from the parts of $T_{00}(e)$ and $T_{10}(e)$ to the right-hand side of the above relations.

(3) $T(e)$ (resp. $T(h)$) is equal to the contribution from the parts of $T_0(e)$ and $T_1(e)$ (resp. $T_0(h)$ and $T_1(h)$) to the right-hand side of the above relations.

(4) $T(s)$ is equal to the contribution from the part of $T_0(s)$ to the right-hand side of the above relations.

(5) When $k=1$, $T(d)$ is equal to the contribution from the parts of $T_0(d)$ and $T_1(d)$ to the right-hand side of the above relations.

From these claimes, we can easily deduce the assertions of the Theorem.

Now, we introduce some notations for a statement of a corollary of the Theorem. Let N be a positive integer and $M=2^{-\mathrm{ord}_2(N)}N$. Then, let $H(M)$ be a Hecke algebra over $\boldsymbol{C}$ generated by double cosets $\Gamma_0(M)\begin{pmatrix}1 & 0\\0 & n\end{pmatrix}\Gamma_0(M)$ with all natural numbers n such that $(n, 2M)=1$. Then, $H(M)$ has a $\boldsymbol{C}$-basis consisting of elements $\Gamma_0(M)\begin{pmatrix}a & 0\\0 & d\end{pmatrix}\Gamma_0(M)$, where $a, d>0$, $a|d$ and $(d, 2M)=1$. We can define a representation from $H(M)$ to $\mathrm{End}_{\boldsymbol{C}}(S(2k, N))$ by $H(M)\ni\Gamma_0(M)\xi\Gamma_0(M)\to[\Gamma_0(N)\xi\Gamma_0(N)]_{2k}$. Similarly, we can define a representatiom from $H(M)$ to $\mathrm{End}_{\boldsymbol{C}}\left(S\left(k+\frac{1}{2}, N, \chi\right)\right)$ by:

$$H(M)\ni\Gamma_0(M)\begin{pmatrix}a & 0\\0 & d\end{pmatrix}\Gamma_0(M)\longrightarrow a(ad)^{k-3/2}\left[\Delta\left(\begin{pmatrix}a^2 & 0\\0 & d^2\end{pmatrix}, (d/a)^{k+1/2}\right)\Delta\right]_{k+1/2},$$

where $a, d>0$, $a|d$, $(d, 2M)=1$ and $\Delta=\Delta_0(N, \chi)_{k+1/2}$ (cf. [N] Introduction). By using these representations, we consider $S(2k, N)$ and $S\left(k+\frac{1}{2}, N, \chi\right)$ as $H(M)$-modules. Then, from [K] §3 and §4, we can see that the Kohnen subspace $S\left(k+\frac{1}{2}, N, \chi\right)_K$ is a $H(M)$-submodule of $S\left(k+\frac{1}{2}, N, \chi\right)$. Similarly, $V(N, \chi)$ and $V(N, \chi)_K$ become $H(M)$-submodules of $S(3/2, N, \chi)$.

Under these notations, we can easily deduce the following corollary from the Theorem.

Corollary. *Let notations and assumptions be the same as in the above Theorem. Moreover, we suppose that M is square-free. Then, we have the following isomorphisms between $H(M)$-modules:*

For $k\geqq 2$, $S(k+1/2, 4M, \chi)_K\cong S(2k, M)$,

$S(k+1/2, 4M, \chi)\cong S(2k, 2M)$,

$S(k+1/2, 8M, \chi)\cong S(2k, 4M)$

and

$S(k+1/2, 16M, \chi)\cong S(2k, 8M)$.

For $k=1$, $S(3/2, 4M, \chi)_K=V(4M, \chi)_K\cong S(2, M)$,

$S(3/2, 4M, \chi)=V(4M, \chi)\cong S(2, 2M)$,

$S(3/2, 8M, \chi)=V(8M, \chi)\cong S(2, 4M)$

and

$S(3/2, 16M, \chi)=V(16M, \chi)\cong S(2, 8M)$.

§4. Applications.

By using the Theorem in §3, we can give decompositions of $H(M)$-modules $S(k+1/2, N, \chi)$ and $S(k+1/2, N, \chi)_K$. For a simplicity, we shall discuss only the decomposition of $S(k+1/2, 4p^m, \chi)_K$, where k and m are some integers with $k\geqq 1$ and $m\geqq 0$, and p is an odd prime number and χ is an even character modulo $4p^m$, namely, $\chi=\left(\frac{1}{}\right)$ or $\left(\frac{p}{}\right)$.

Before a statement of results, we must introduce some notations. Let δ_ψ be the twisting operator for the character $\psi=\left(\frac{}{p}\right)$ (cf. [S-Y]). By $S^0(2k, p^m)$, we denote a subspace of $S(2k, p^m)$ spaned by all newforms in $S(2k, p^m)$.

For $m\geqq 3$, we define (cf. [S-Y]):

$$S_{\mathrm{I}}(2k, p^m)=\{S^0(2k, p^m)\ni f;\ f|[W]_{2k}=f,\ f|\delta_\psi[W]_{2k}=f|\delta_\psi\},$$

$$S_{\mathrm{II}}(2k, p^m)=\{S^0(2k, p^m)\ni f;\ f|[W]_{2k}=f,\ f|\delta_\psi[W]_{2k}=-f|\delta_\psi\},$$

$$S_{\mathrm{II}_\psi}(2k, p^m)=\{S^0(2k, p^m)\ni f;\ f|[W]_{2k}=-f,\ f|\delta_\psi[W]_{2k}=f|\delta_\psi\},$$

$$S_{\mathrm{III}}(2k, p^m)=\{S^0(2k, p^m)\ni f;\ f|[W]_{2k}=-f,\ f|\delta_\psi[W]_{2k}=-f|\delta_\psi\},$$

where $W=W(p^m)$ (cf. §3).

For $m=2$, let $S^n(2k, p^2)$ be the orthogonal complement of $S^0(2k, p)|\delta_\psi+S^0(2k, 1)|\delta_\psi$ in $S^0(2k, p^2)$ with respect to the Petersson inner product. Then, we define (cf. [S-Y]):

$$S_{\mathrm{I}}(2k, p^2)=\{S^n(2k, p^2)\ni f;\ f|[W]_{2k}=f,\ f|\delta_\psi[W]_{2k}=f|\delta_\psi\},$$

$$S_{\mathrm{II}}(2k, p^2)=\{S^n(2k, p^2)\ni f;\ f|[W]_{2k}=f,\ f|\delta_\psi[W]_{2k}=-f|\delta_\psi\},$$

$$S_{\mathrm{II}_\psi}(2k, p^2)=\{S^n(2k, p^2)\ni f;\ f|[W]_{2k}=-f,\ f|\delta_\psi[W]_{2k}=f|\delta_\psi\},$$

$$S_{\mathrm{III}}(2k, p^2)=\{S^n(2k, p^2)\ni f;\ f|[W]_{2k}=-f,\ f|\delta_\psi[W]_{2k}=-f|\delta_\psi\},$$

where $W=W(p^2)$ (cf. §3).

Under these notations, we have the following decompositions as $H(p^m)$-modules.

Proposition 3. *Suppose $k\geqq 2$. Then, we have the following decompositions as $H(p^m)$-modules:*

(1) $(m=0)$.

$$S\left(k+\frac{1}{2}, 4\right)_K\cong S^0(2k, 1)$$

(2) $(m=1)$.

$$S\left(k+\frac{1}{2}, 4p\right)_K\cong S\left(k+\frac{1}{2}, 4p, \left(\frac{p}{}\right)\right)_K$$
$$\cong S^0(2k, p)\oplus 2S^0(2k, 1).$$

(3) $\left(m=2 \text{ and } \chi=\left(\frac{1}{}\right)\right)$.

$$S\left(k+\frac{1}{2}, 4p^2\right)_K\cong 2\{S_{\mathrm{I}}(2k, p^2)\oplus S_{\mathrm{II}}(2k, p^2)\}$$
$$\oplus\left(1+\left(\frac{-1}{p}\right)\right)\{S^0(2k, p)|\delta_\psi\oplus S^0(2k, 1)|\delta_\psi\}$$

$$\oplus 2S^0(2k,\ p)\oplus 4S^0(2k,\ 1)\,.$$

(4) $\left(m=2 \textit{ and } \chi=\left(\frac{p}{}\right)\right)$.

$$S\left(k+\frac{1}{2},\ 4p^2,\ \left(\frac{p}{}\right)\right)_K$$

$$\cong\left(1+\left(\frac{-1}{p}\right)\right)\{S_{\mathrm{I}}(2k,\ p^2)\oplus S_{\mathrm{II}\,\psi}(2k,\ p^2)\}$$

$$\oplus\left(1-\left(\frac{-1}{p}\right)\right)\{S_{\mathrm{II}}(2k,\ p^2)\oplus S_{\mathrm{III}}(2k,\ p^2)\}$$

$$\oplus S^0(2k,\ p)\,|\,\delta_\psi\oplus\left(1+\left(\frac{-1}{p}\right)\right)S^0(2k,1)\,|\,\delta_\psi\oplus 3S^0(2k,\ p)\oplus 4S^0(2k,1).$$

(5) $(m=2a+3 \textit{ with } a\geqq 0)$.

$$S\left(k+\frac{1}{2},\ 4p^{2a+3}\right)_K\cong S\left(k+\frac{1}{2},\ 4p^{2a+3},\ \left(\frac{p}{}\right)\right)_K$$

$$\cong\bigoplus_{b=1}^{a+1}(2a+3-2b)S^0(2k,\ p^{2b+1})$$

$$\oplus\bigoplus_{b=1}^{a+1}\left(3+\left(\frac{-1}{p}\right)\right)(a+2-b)S_{\mathrm{I}}(2k,\ p^{2b})$$

$$\oplus\bigoplus_{b=1}^{a+1}\left(3-\left(\frac{-1}{p}\right)\right)(a+2-b)S_{\mathrm{II}}(2k,\ p^{2b})$$

$$\oplus\bigoplus_{b=1}^{a+1}\left(1+\left(\frac{-1}{p}\right)\right)(a+2-b)S_{\mathrm{II}\,\psi}(2k,\ p^{2b})$$

$$\oplus\bigoplus_{b=1}^{a+1}\left(1-\left(\frac{-1}{p}\right)\right)(a+2-b)S_{\mathrm{III}}(2k,\ p^{2b})$$

$$\oplus\left(2+\left(\frac{-1}{p}\right)\right)(a+1)S^0(2k,\ p)\,|\,\delta_\psi$$

$$\oplus\left(1+\left(\frac{-1}{p}\right)\right)(2a+2)S^0(2k,\ 1)\,|\,\delta_\psi$$

$$\oplus(3a+4)S^0(2k,\ p)\oplus(4a+6)S^0(2k,\ 1)\,.$$

(6) $\left(m=2a+4 \textit{ with } a\geqq 0 \textit{ and } \chi=\left(\frac{1}{}\right)\right)$.

$$S\left(k+\frac{1}{2},\ 4p^{2a+4}\right)_K\cong 2\{S_{\mathrm{I}}(2k,\ p^{2a+4})\oplus S_{\mathrm{II}}(2k,\ p^{2a+4})\}$$

$$\oplus\bigoplus_{b=1}^{a+1}(2a+4-2b)S^0(2k,\ p^{2b+1})$$

$$\oplus\bigoplus_{b=1}^{a+1}\left\{\left(3+\left(\frac{-1}{p}\right)\right)(a+2-b)+2\right\}S_{\mathrm{I}}(2k,\ p^{2b})$$

$$\oplus \bigoplus_{b=1}^{a+1} \left\{\left(3-\left(\frac{-1}{p}\right)\right)(a+2-b)+2\right\} S_{\mathrm{II}}(2k,\ p^{2b})$$

$$\oplus \bigoplus_{b=1}^{a+1} \left(1+\left(\frac{-1}{p}\right)\right)(a+2-b) S_{\mathrm{II}_\psi}(2k,\ p^{2b})$$

$$\oplus \bigoplus_{b=1}^{a+1} \left(1-\left(\frac{-1}{p}\right)\right)(a+2-b) S_{\mathrm{III}}(2k,\ p^{2b})$$

$$\oplus \left\{\left(1+\left(\frac{-1}{p}\right)\right)(a+2)+a+1\right\} S^0(2k,\ p)|\delta_\psi$$

$$\oplus \left(1+\left(\frac{-1}{p}\right)\right)(2a+3) S^0(2k,\ 1)|\delta_\psi$$

$$\oplus (3a+5) S^0(2k,\ p) \oplus (4a+8) S^0(2k,\ 1)\,.$$

(7) $\left(m=2a+4 \text{ with } a\geqq 0 \text{ and } \chi=\left(\frac{p}{\cdot}\right)\right)$.

$$S\left(k+\frac{1}{2},\ 4p^{2a+4},\ \left(\frac{p}{\cdot}\right)\right)_K$$

$$\cong \left(1+\left(\frac{-1}{p}\right)\right)\{S_{\mathrm{I}}(2k,\ p^{2a+4}) \oplus S_{\mathrm{II}_\psi}(2k,\ p^{2a+4})\}$$

$$\oplus \left(1-\left(\frac{-1}{p}\right)\right)\{S_{\mathrm{II}}(2k,\ p^{2a+4}) \oplus S_{\mathrm{III}}(2k,\ p^{2a+4})\}$$

$$\oplus \bigoplus_{b=1}^{a+1} (2a+4-2b) S^0(2k,\ p^{2b+1})$$

$$\oplus \bigoplus_{b=1}^{a+1} \left\{\left(3+\left(\frac{-1}{p}\right)\right)(a+2-b)+1+\left(\frac{-1}{p}\right)\right\} S_{\mathrm{I}}(2k,\ p^{2b})$$

$$\oplus \bigoplus_{b=1}^{a+1} \left\{\left(3-\left(\frac{-1}{p}\right)\right)(a+2-b)+1-\left(\frac{-1}{p}\right)\right\} S_{\mathrm{II}}(2k,\ p^{2b})$$

$$\oplus \bigoplus_{b=1}^{a+1} \left(1+\left(\frac{-1}{p}\right)\right)(a+3-b) S_{\mathrm{II}_\psi}(2k,\ p^{2b})$$

$$\oplus \bigoplus_{b=1}^{a+1} \left(1-\left(\frac{-1}{p}\right)\right)(a+3-b) S_{\mathrm{III}}(2k,\ p^{2b})$$

$$\oplus \left\{\left(1+\left(\frac{-1}{p}\right)\right)(a+1)+a+2\right\} S^0(2k,\ p)|\delta_\psi$$

$$\oplus \left(1+\left(\frac{-1}{p}\right)\right)(2a+3) S^0(2k,\ 1)|\delta_\psi$$

$$\oplus (3a+6) S^0(2k,\ p) \oplus (4a+8) S^0(2k,\ 1)\,.$$

Here, the coefficient in front of the $H(p^m)$*-modules* $S^0(2k,\ p^n)$ $(0\leqq n\leqq m)$ *etc. is the multiplicity.*

If we replace $S(k+1/2,\ 4p^m,\ \chi)_K$ *by* $V(4p^m,\ \chi)_K$ *and put* $k=1$ *at the right-hand side of the decompositions* (1)-(7), *we have the decomposition of the* $H(p^m)$-

module $V(4p^m, \chi)_K$.

Remark. Let m be a positive integer and s an integer such that $0\leqq s\leqq m$. Then, for all positive integers n with $(n, p)=1$, $T_{2k,p^s}(n)$ coinsides with $T_{2k,p^m}(n)$ as an operator on $S(2k, p^m)$. Therefore, we can naturally consider $S(2k, p^s)$ and $S^0(2k, p^s)$, etc. as $H(p^m)$-modules.

Proof. The decomposition (1) and (2) are immediate consequences of the Corollary in § 3. Hence, in the following, we assume that $m\geqq 1$ and that the letter n means any positive integer prime to $2p$.

Firstly, we note that, when m is odd, by using the Theorem in § 3,

$$\mathrm{trace}\Big(\tilde{T}_{k+1/2,4p^m,(\frac{-1}{\cdot})}(n^2)\Big|S\Big(k+\frac{1}{2}, 4p^m\Big)_K\Big)$$

$$=\mathrm{trace}\Big(\tilde{T}_{k+1/2,4p^m,(\frac{p}{\cdot})}(n^2)\Big|S\Big(k+\frac{1}{2}, 4p^m, \Big(\frac{p}{\cdot}\Big)\Big)_K\Big).$$

Since the operator $\tilde{T}_{k+1/2,4p^m,\chi}(n^2)$ on $S(k+1/2, 4p^m, \chi)_K$ is hermitian with respect to the Petersson inner product, we get, for any odd integer $m\geqq 1$, $S\Big(k+\frac{1}{2}, 4p^m\Big)_K\cong S\Big(k+\frac{1}{2}, 4p^m, \Big(\frac{p}{\cdot}\Big)\Big)_K$ as $H(p^m)$-modules.

Next, from the definitions and [S-Y] Lemma 5.1, we can easily get the following identities:

$$\mathrm{trace}\,(T_{2k,p^m}(n)|S(2k, p^m))=\sum_{a=0}^{m}(m+1-a)\,\mathrm{trace}\,(T_{2k,p^a}(n)|S^0(2k, p^a))$$

and, for any integer $t\geqq 1$,

$$\mathrm{trace}\,([W(p^{2t})]_{2k}T_{2k,p^{2t}}(n)\mid S(2k, p^{2t}))$$

$$=\sum_{a=1}^{t}\mathrm{trace}\,(T_{2k,p^{2a}}(n)\mid S_{\mathrm{I}}(2k, p^{2a})\oplus S_{\mathrm{II}}(2k, p^{2a}))$$

$$-\sum_{a=1}^{t}\mathrm{trace}\,(T_{2k,p^{2a}}(n)\mid S_{\mathrm{II}_\psi}(2k, p^{2a})\oplus S_{\mathrm{III}}(2k, p^{2a}))$$

$$+\Big(\frac{-1}{p}\Big)\mathrm{trace}\,(T_{2k,p^2}(n)\mid S^0(2k, p)|\delta_\psi\oplus S^0(2k, 1)|\delta_\psi)$$

$$+\mathrm{trace}\,(T_{2k,1}(n)|S^0(2k, 1))\,.$$

Moreover, from the Proposition 1.1 (and § 5) of [S-Y], we have:

$$S_{\mathrm{I}}(2k, p^{2a})|\delta_\psi=S_{\mathrm{I}}(2k, p^{2a})\,,$$

$$S_{\mathrm{II}}(2k, p^{2a})|\delta_\psi=S_{\mathrm{II}_\psi}(2k, p^{2a})\,,$$

$$S_{\mathrm{II}_\psi}(2k, p^{2a})|\delta_\psi=S_{\mathrm{II}}(2k, p^{2a}) \quad \text{and}$$

$$S_{\mathrm{III}}(2k, p^{2a})|\delta_\psi=S_{\mathrm{III}}(2k, p^{2a}) \qquad \text{for any integer } a\geqq 1\,.$$

Now, for any integer $t\geqq 1$, we put

$$A_{2t}=\mathrm{trace}\Big(\tilde{T}_{k+1/2,4p^{2t},(\frac{1}{\cdot})}(n^2)\Big|S\Big(k+\frac{1}{2},\,4p^{2t}\Big)_K\Big)$$

$$-\mathrm{trace}\Big(\tilde{T}_{k+1/2,4p^{2t-1},\chi}(n^2)\Big|S\Big(k+\frac{1}{2},\,4p^{2t-1},\,\chi\Big)_K\Big),$$

and

$$B_{2t}=\mathrm{trace}\Big(\tilde{T}_{k+1/2,4p^{2t},(\frac{p}{\cdot})}(n^2)\Big|S\Big(k+\frac{1}{2},\,4p^{2t},\,\Big(\frac{p}{\cdot}\Big)\Big)_K\Big)$$

$$-\mathrm{trace}\Big(\tilde{T}_{k+1/2,4p^{2t-1},\chi}(n^2)\Big|S\Big(k+\frac{1}{2},\,4p^{2t-1},\,\chi\Big)_K\Big).$$

From the Theorem in §3 and the above formulas, we have:

$$\begin{aligned}
A_{2t}=&\,\mathrm{trace}(T_{2k,p^{2t}}(n)\mid S(2k,\,p^{2t}))\\
&+\mathrm{trace}([W(p^{2t})]_{2k}T_{2k,p^{2t}}(n)\mid S(2k,\,p^{2t}))\\
&-\mathrm{trace}(T_{2k,p^{2t-1}}(n)\mid S(2k,\,p^{2t-1}))\\
=&\,2\sum_{a=1}^{t}\mathrm{trace}(T_{2k,p^{2a}}(n)\mid S_{\mathrm{I}}(2k,\,p^{2a})\oplus S_{\mathrm{II}}(2k,\,p^{2a}))\\
&+\sum_{b=1}^{t}\mathrm{trace}(T_{2k,p^{2b-1}}(n)\mid S^0(2k,\,p^{2b-1}))\\
&+\Big(1+\Big(\frac{-1}{p}\Big)\Big)\mathrm{trace}(T_{2k,p^2}(n)\mid S^0(2k,\,p)|\delta_\phi\oplus S^0(2k,\,1)|\delta_\phi)\\
&+2\,\mathrm{trace}(T_{2k,1}(n)\mid S^0(2k,\,1)),
\end{aligned}$$

and

$$\begin{aligned}
B_{2t}=&\,\mathrm{trace}(T_{2k,p^{2t}}(n)\mid S(2k,\,p^{2t}))\\
&+\Big(\frac{-n}{p}\Big)\mathrm{trace}([W(p^{2t})]_{2k}T_{2k,p^{2t}}(n)\mid S(2k,\,p^{2t}))\\
&-\mathrm{trace}(T_{2k,p^{2t-1}}(n)\mid S(2k,\,p^{2t-1}))\\
=&\Big(1+\Big(\frac{-1}{p}\Big)\Big)\sum_{a=1}^{t}\mathrm{trace}(T_{2k,p^{2a}}(n)\mid S_{\mathrm{I}}(2k,\,p^{2a})\oplus S_{\mathrm{II}_\phi}(2k,\,p^{2a}))\\
&+\Big(1-\Big(\frac{-1}{p}\Big)\Big)\sum_{a=1}^{t}\mathrm{trace}(T_{2k,p^{2a}}(n)\mid S_{\mathrm{II}}(2k,\,p^{2a})\oplus S_{\mathrm{III}}(2k,\,p^{2a}))\\
&+\sum_{b=1}^{t}\mathrm{trace}(T_{2k,p^{2b-1}}(n)\mid S^0(2k,\,p^{2b-1}))\\
&+\mathrm{trace}(T_{2k,p^2}(n)\mid S^0(2k,\,p)|\delta_\phi)\\
&+\Big(1+\Big(\frac{-1}{p}\Big)\Big)\mathrm{trace}(T_{2k,p^2}(n)\mid S^0(2k,\,1)|\delta_\phi)\\
&+\mathrm{trace}(T_{2k,p}(n)\mid S^0(2k,\,p))\\
&+2\,\mathrm{trace}(T_{2k,1}(n)\mid S^0(2k,\,1)).
\end{aligned}$$

From this expression of A_2, we have:

$$\operatorname{trace}\left(\tilde{T}_{k+1/2,4p^2,\left(\frac{1}{\cdot}\right)}(n^2)\middle|S\left(k+\frac{1}{2},4p^2\right)_K\right)$$
$$=2\operatorname{trace}(T_{2k,p^2}(n)\mid S_{\mathrm{I}}(2k,p^2)\oplus S_{\mathrm{II}}(2k,p^2))$$
$$+\left(1+\left(\frac{-1}{p}\right)\right)\operatorname{trace}(T_{2k,p^2}(n)\mid S^0(2k,p)|\delta_\psi\oplus S^0(2k,1)|\delta_\psi)$$
$$+2\operatorname{trace}(T_{2k,p}(n)\mid S^0(2k,p))$$
$$+4\operatorname{trace}(T_{2k,1}(n)\mid S^0(2k,1)).$$

The decomposition (3) follows from this equality and we can prove the decomposition (4) in a similar way.

When $m=2a+3$ with $a\geqq 0$, by using the Theorem in §3, we have:

$$\operatorname{trace}\left(\tilde{T}_{k+1/2,4p^{2a+3},\chi}(n^2)\middle|S\left(k+\frac{1}{2},4p^{2a+3},\chi\right)_K\right)$$
$$-\operatorname{trace}\left(\tilde{T}_{k+1/2,4p^{2a+2},\left(\frac{1}{\cdot}\right)}(n^2)\middle|S\left(k+\frac{1}{2},4p^{2a+2}\right)_K\right)$$
$$-\operatorname{trace}\left(\tilde{T}_{k+1/2,4p^{2a+2},\left(\frac{p}{\cdot}\right)}(n^2)\middle|S\left(k+\frac{1}{2},4p^{2a+2},\left(\frac{p}{\cdot}\right)\right)_K\right)$$
$$+\operatorname{trace}\left(\tilde{T}_{k+1/2,4p^{2a+1},\chi}(n^2)\middle|S\left(k+\frac{1}{2},4p^{2a+1},\chi\right)_K\right)$$
$$=\operatorname{trace}(T_{2k,p^{2a+3}}(n)\mid S(2k,p^{2a+3}))$$
$$-2\operatorname{trace}(T_{2k,p^{2a+2}}(n)\mid S(2k,p^{2a+2}))$$
$$+\operatorname{trace}(T_{2k,p^{2a+1}}(n)\mid S(2k,p^{2a+1}))$$
$$=\operatorname{trace}(T_{2k,p^{2a+3}}(n)\mid S^0(2k,p^{2a+3})).$$

Hence, we get inductively:

$$\operatorname{trace}\left(\tilde{T}_{k+1/2,4p^{2a+3},\chi}(n^2)\middle|S\left(k+\frac{1}{2},4p^{2a+3},\chi\right)_K\right)$$
$$=\operatorname{trace}(T_{2k,p^{2a+3}}(n)\mid S^0(2k,p^{2a+3}))+A_{2a+2}+B_{2a+2}$$
$$+\operatorname{trace}\left(\tilde{T}_{k+1/2,4p^{2a+1},\chi}(n^2)\middle|S\left(k+\frac{1}{2},4p^{2a+1},\chi\right)_K\right)$$
$$=\sum_{b=0}^{a}\operatorname{trace}(T_{2k,p^{2b+3}}(n)\mid S^0(2k,p^{2b+3}))$$
$$+\sum_{b=0}^{a}A_{2b+2}+\sum_{b=0}^{a}B_{2b+2}+\operatorname{trace}\left(\tilde{T}_{k+1/2,4p,\chi}(n^2)\middle|S\left(k+\frac{1}{2},4p,\chi\right)_K\right).$$

From this equality and the above expressions of A_{2t} and B_{2t}, we immediately obtain the decomposition (5).

Finally, by using the expressions of A_{2a+4} and B_{2a+4} with $a\geqq 0$, we can deduce the decompositions (6) and (7) from the decomposition (5).

In this Proposition 3, we note the following:
When m is zero or an odd positive integer, the $H(p^m)$-module $S^0(2k, p^m)$ occurs with the multiplicity one in the decompositions of the $H(p^m)$-modules $S(k+1/2, 4p^m, \chi)_K$ (if $k\geqq 2$) and $V(4p^m, \chi)_K$ (if $k=1$). Hence, a non-zero element f of $S(k+1/2, 4p^m, \chi)_K$ or $V(4p^m, \chi)_K$, which corresponds to a primitive form F in $S^0(2k, p^m)$, also becomes a common eigen form with respect to the n-th Hecke operators for $n=2$ and p (cf. [K] Preliminaries (a) and § 3, as in the definitions of the n-th Hecke operators for $n=2$ and p).

Let the Fourier expansion of f (resp. F) be as follows:

$$f=\sum_{n=1}^{\infty} a(n)e(nz) \qquad (\text{resp. } F=\sum_{n=1}^{\infty} A(n)e(nz)).$$

If u is a fundamental discriminant with $\varepsilon(-1)^k u>0$ (cf. § 0 (**d**) for the definition of ε), then, their Fourier expansions are related as follows:

$$L\left(s-k+1, \chi\left(\frac{u}{}\right)\right)\sum_{n=1}^{\infty} a(|u|n^2)n^{-s}=a(|u|)\sum_{n=1}^{\infty} A(n)n^{-s}$$

(cf. [K] § 5 Theorem 2). Therefore, all primitive forms in $S^0(2k, p^m)$ are costructed from some elements of $S(k+1/2, 4p^m, \chi)_K$ or $V(4p^m, \chi)_K$ through the Shimura (—Niwa—Kohnen) correspondence.

In fact, for more general situations, similar results can be proved by using the Theorem in § 3. However, we omit the details.

Appendix 1.

For a simplicity, we shortly write $\theta=\theta_p$. Then, the constant $n_p(\theta_p)$ is given by the following table:

Case (1) ($p\,|\,M$ and $p\,|\,s$).

$$n_p(\theta_p)=\chi_p(-n)p^{\theta}\times\begin{cases} p^{[\nu/2]}+\left(\frac{-n}{p}\right)^{\nu} p^{[(\nu-1)/2]}, & \text{if } \theta\geqq[(\nu+1)/2]; \\ \left(1+\left(\frac{-n}{p}\right)\right)p^{\theta}, & \text{if } \theta\leqq[(\nu-1)/2]. \end{cases}$$

Case (2) ($p\,|\,M$, $p\nmid s$ and $p\,|\,u$).

$$n_p(\theta_p)=\begin{cases} \{p^{\theta+1}(p^{[\nu/2]}+p^{[(\nu-1)/2]})-p^{\nu}-p^{\nu-1}\}/(p-1), & \text{if } \theta\geqq[\nu/2]; \\ 0, & \text{if } \theta<[\nu/2]. \end{cases}$$

Case (3) ($p\,|\,M$, $p\nmid s$ and $p\nmid u$).

$$n_p(\theta_p)=\begin{cases} \left(p-\left(\frac{u}{p}\right)\right)(p^{[\nu/2]}+p^{[(\nu-1)/2]})(p^{\theta}-p^{[\nu/2]})(p-1)^{-1} & \\ \quad +p^{[\nu/2]}\left(p^{[\nu/2]}+\left(\frac{u}{p}\right)^{\nu} p^{[(\nu-1)/2]}\right), & \text{if } \theta\geqq[(\nu+1)/2]; \\ \left(1+\left(\frac{u}{p}\right)\right)p^{2\theta}, & \text{if } \theta\leqq[(\nu-1)/2]. \end{cases}$$

Case (4) ($p=2$ and $\mu=2$).

$$n_2(\theta_2)=\begin{cases}2^{\theta+1}, & \text{if } u\equiv 1 \pmod 8;\\ 3\times 2^{\theta}, & \text{if } u\equiv 5 \pmod 8 \text{ and } s/2 \text{ is even};\\ 3\times 2^{\theta+1}-12, & \text{if } u\equiv 5 \pmod 8 \text{ and } s/2 \text{ is odd};\\ 2^{\theta+2}-6, & \text{if } u\equiv 0 \pmod 4 \text{ and } t \text{ is even};\\ 2^{\theta}, & \text{if } u\equiv 0 \pmod 4 \text{ and } t \text{ is odd}.\end{cases}$$

Case (5) ($p=2$ and $\mu=3$).

$$n_2(\theta_2)=\begin{cases}3\times 2^{\theta}, & \text{if } u\equiv 1 \pmod 8;\\ 3\times 2^{\theta}, & \text{if } u\equiv 5 \pmod 8 \text{ and } s/2 \text{ is even};\\ 9\times 2^{\theta}-24, & \text{if } u\equiv 5 \pmod 8 \text{ and } s/2 \text{ is odd};\\ 3\times 2^{\theta+1}-12, & \text{if } u\equiv 0 \pmod 4 \text{ and } t \text{ is even};\\ 0, & \text{if } u\equiv 0 \pmod 4 \text{ and } t \text{ is odd}.\end{cases}$$

Case (6) ($p=2$ and $\mu=4$).

$$n_2(\theta_2)=\begin{cases}3\times 2^{\theta+1}, & \text{if } u\equiv 1 \pmod 8 \text{ and } f(\chi_2)|4;\\ 2^{\theta+2}, & \text{if } u\equiv 1 \pmod 8 \text{ and } f(\chi_2)=8;\\ 3\times 2^{\theta+1}, & \text{if } u\equiv 5 \pmod 8,\ s/2 \text{ is even and } f(\chi_2)|4;\\ 0, & \text{if } u\equiv 5 \pmod 8,\ s/2 \text{ is even and } f(\chi_2)=8;\\ 9\times 2^{\theta+1}-48, & \text{if } u\equiv 5 \pmod 8,\ s/2 \text{ is odd and } f(\chi_2)|4;\\ 3\times 2^{\theta+2}-48, & \text{if } u\equiv 5 \pmod 8,\ s/2 \text{ is odd and } f(\chi_2)=8;\\ 3\times 2^{\theta+2}-24, & \text{if } u\equiv 0 \pmod 4,\ t \text{ is even and } f(\chi_2)|4;\\ 2^{\theta+3}-24, & \text{if } u\equiv 0 \pmod 4,\ t \text{ is even and } f(\chi_2)=8;\\ 0, & \text{if } u\equiv 0 \pmod 4 \text{ and } t \text{ is odd}.\end{cases}$$

Case (7) ($p=2$ and $\mu=2g+1\geqq 5$).

$$n_2(\theta_2)=\begin{cases}2^{\theta+g+1}, & \text{if } u\equiv 1 \pmod 8 \text{ and } \theta\geqq g+1;\\ 2^{2\theta+1}, & \text{if } u\equiv 1 \pmod 8 \text{ and } \theta\leqq g;\\ 0, & \text{if } u\equiv 5 \pmod 8 \text{ and } s/2 \text{ is even};\\ 3\times 2^{\theta+g+1}-3\times 2^{2g+1}, & \text{if } u\equiv 5 \pmod 8,\ s/2 \text{ is odd and } \theta\geqq g+1;\\ 0, & \text{if } u\equiv 5 \pmod 8,\ s/2 \text{ is odd and } \theta\leqq g;\\ 2^{\theta+g+2}-3\times 2^{2g}, & \text{if } u\equiv 0 \pmod 4,\ t \text{ is even and } \theta\geqq g;\\ 0, & \text{if } u\equiv 0 \pmod 4,\ t \text{ is even and } \theta\leqq g-1;\\ 0, & \text{if } u\equiv 0 \pmod 4 \text{ and } t \text{ is odd}.\end{cases}$$

Case (8) ($p=2$ and $\mu=2g\geqq 6$).

$$n_2(\theta_2)=\begin{cases} 3\times 2^{\theta+g-1}, & \text{if } u\equiv 1 \pmod 8 \text{ and } \theta\geqq g;\\ 2^{2\theta+1}, & \text{if } u\equiv 1 \pmod 8 \text{ and } \theta\leqq g-1;\\ 3\times 2^{\theta+g-1}\chi_2(5), & \text{if } u\equiv 5 \pmod 8,\ s/2 \text{ is even and } \theta\geqq g;\\ 0, & \text{if } u\equiv 5 \pmod 8,\ s/2 \text{ is even and } \theta\leqq g-1;\\ 9\times 2^{\theta+g-1}-3\times 2^{2g}, & \text{if } u\equiv 5 \pmod 8,\ s/2 \text{ is odd and } \theta\geqq g;\\ 0, & \text{if } u\equiv 5 \pmod 8,\ s/2 \text{ is odd and } \theta\leqq g-1;\\ 3\times 2^{\theta+g}-3\times 2^{2g-1}, & \text{if } u\equiv 0 \pmod 4,\ t \text{ is even and } \theta\geqq g,\\ 0, & \text{if } u\equiv 0 \pmod 4,\ t \text{ is even and } \theta\leqq g-1;\\ 0, & \text{if } u\equiv 0 \pmod 4 \text{ and } t \text{ is odd.} \end{cases}$$

Appendix 2.

For a simplicity, we shortly write $\theta=\theta_p$. Then, the constant $m_p(\theta_p)$ is given by the following table:

Case (1) ($p|M$).

$$m_p(\theta_p)=\begin{cases} p^{[\nu/2]}+p^{[(\nu-1)/2]}, & \text{if } \theta\geqq[(\nu+1)/2];\\ 2p^{\theta}, & \text{if } \theta\leqq[(\nu-1)/2]. \end{cases}$$

Case (2) ($p=2$)

$$m_2(\theta_2)=\begin{cases} 2, & \text{if } \mu=2;\\ 3, & \text{if } \mu=3;\\ 6, & \text{if } \mu=4 \text{ and } f(\chi_2)|4;\\ 4, & \text{if } \mu=4 \text{ and } f(\chi_2)=8;\\ 2^{[\mu/2]}+2^{[(\mu-1)/2]}, & \text{if } \mu\geqq 5 \text{ and } \theta\geqq[(\mu+1)/2];\\ 2^{\theta+1}, & \text{if } \mu\geqq 5 \text{ and } \theta\leqq[(\mu-1)/2]. \end{cases}$$

Appendix 3.

Let A, $D(A)$ and Γ be the same as in the Remark (2.8) of §2. We shall calculate $n(A)=\#(D(A)/\Gamma)$.

For a representative x of $(\boldsymbol{Z}/M\boldsymbol{Z})^{\times}$ and a prime divisor p of M, we define sets $V(x)$, $V_2(x)$ and $V_p(x)$ as follows:

$$V(x)=\left\{SL_2(\boldsymbol{Z})\ni B\,;\ B^{-1}AB\equiv\begin{pmatrix}4x+4M\nu & *\\ 0 & *\end{pmatrix} \pmod{16M} \text{ for some } \nu\in\boldsymbol{Z}\right\}.$$

$$V_2(x)=\left\{SL_2(\boldsymbol{Z})\ni B\,;\ B^{-1}AB\equiv\begin{pmatrix}4x+4M\nu & *\\ 0 & *\end{pmatrix} \pmod{16} \text{ for some } \nu\in\boldsymbol{Z}\right\}.$$

$$V_p(x)=\left\{SL_2(\boldsymbol{Z})\ni B\,;\, B^{-1}AB\equiv\begin{pmatrix}4x & *\\ 0 & *\end{pmatrix}(\mathrm{mod}\ p^{\nu_p})\right\}$$

with $\nu_p=\mathrm{ord}_p(M)$.

Then, $\Gamma=\Gamma_0(4M)$, $\Gamma_0(4)$ and $\Gamma_0(p^{\nu_p})$ operate on the sets $V(x)$, $V_2(x)$ and $V_p(x)$ respectively by multiplication from the right, and $D(A)=\bigcup_{x\in(\boldsymbol{Z}/M\boldsymbol{Z})^\times} V(x)$ (disjoint union). Moreover, we can define an isomorphism φ from $\{V_2(x)/\Gamma_0(4)\}\times\prod_{p\mid M}\{V_p(x)/\Gamma_0(p^{\nu_p})\}$ to $V(x)/\Gamma$ as follows:
Take any element $(B_2, (B_p)_{p\mid M})$ of $V_2(x)\times\prod_{p\mid M}V_p(x)$. Then, there exists an element B of $SL_2(\boldsymbol{Z})$ such that $B\equiv B_2$ (mod 16) and that $B\equiv B_p$ (mod p^{ν_p}) for all prime divisors p of M. We define $\varphi((B_2\Gamma_0(4), (B_p\Gamma_0(p^{\nu_p}))_{p\mid M}))=B\Gamma$.

Now, from the discussion in [K] §4 Appendix (proof of the Lemma 5), we know $\#(V_2(x)/\Gamma_0(4))=1$. Therefore, we obtain:

$$\begin{aligned}n(A)&=\sum_{x\in(\boldsymbol{Z}/M\boldsymbol{Z})^\times}\#(V(x)/\Gamma)=\sum_{x\in(\boldsymbol{Z}/M\boldsymbol{Z})^\times}\prod_{p\mid M}\#(V_p(x)/\Gamma_0(p^{\nu_p}))\\ &=\prod_{p\mid M}\sum_{x\in(\boldsymbol{Z}/p^{\nu}p\boldsymbol{Z})^\times}\#(V_p(x)/\Gamma_0(p^{\nu_p}))\,.\end{aligned}$$

In order to calculate $\#(V_p(x)/\Gamma_0(p^{\nu_p}))$, we note the following general facts:

Let L be a positive integer. We denote by $C(L)$ a set consisting of all elements of $(\boldsymbol{Z}/L\boldsymbol{Z})\times(\boldsymbol{Z}/L\boldsymbol{Z})$ whose order is exactly L. We define an equivalence relation $\sim$ of $C(L)$ as follows: For two elements $(\bar{c}_1, \bar{d}_1)$, $(\bar{c}_2, \bar{d}_2)$ of $C(L)$, $(\bar{c}_1, \bar{d}_1)\sim(\bar{c}_2, \bar{d}_2)$ if and only if there exists $\bar{m}\in(\boldsymbol{Z}/L\boldsymbol{Z})^\times$ such that $\bar{m}(\bar{c}_1, \bar{d}_1)=(\bar{c}_2, \bar{d}_2)$. Then, we have the bijection: $SL_2(\boldsymbol{Z})/\Gamma_0(L)\ni\begin{pmatrix}a & b\\ c & d\end{pmatrix}\Gamma_0(L)\rightarrow$ *the equivalence class containing* (a mod L, b mod L)$\in C(L)/\sim$.

Now, we shall calculate $\#(V_p(x)/\Gamma_0(p^{\nu_p}))$.

Let write $A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$. Then, for $B=\begin{pmatrix}u & w\\ v & z\end{pmatrix}\in SL_2(\boldsymbol{Z})$, the condition $B^{-1}AB\equiv\begin{pmatrix}4x & *\\ 0 & *\end{pmatrix}$ (mod p^{ν_p}) is equivalent to $\begin{pmatrix}a-4x & b\\ c & d-4x\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}\equiv\begin{pmatrix}0\\ 0\end{pmatrix}$ (mod p^{ν_p}). By using the elementary divisor theory, there exist U_1 and $U_2\in SL_2(\boldsymbol{Z})$ such that $\begin{pmatrix}a-4x & b\\ c & d-4x\end{pmatrix}=U_1\begin{pmatrix}g_1 & 0\\ 0 & g_2\end{pmatrix}U_2$. Here, $g_1=(a-4x, b, c, d-4x)$ and $g_1g_2=((a-4x)(d-4x)-bc)=((4x)^2-4tx+16n^2)$, where $t=a+d$, $ad-bc=16n^2$ from the assumptions.

Put $\alpha=\mathrm{ord}_p(g_1)$ and $\beta=\mathrm{ord}_p(g_2)$. Since p is odd, $g_1\boldsymbol{Z}_p=(a-d, b, c, t-8x)\boldsymbol{Z}_p$. Hence, $\alpha=\min(\rho_p, \tau_{p,x})$, where $\rho_p=\mathrm{ord}_p(f)$ with $f=(a-d, b, c)$ and $\tau_{p,x}=\mathrm{ord}_p(t-8x)$.

Thus, we have:

$$\#(V_p(x)/\Gamma_0(p^{\nu_p}))=\#\left\{\begin{array}{l}C(p^{\nu_p})/\sim\ni\text{the equivalence class containing}\\ (u, v)\ \text{modulo}\ p^{\nu_p}\ \text{such that}\\ p^\alpha u\equiv p^\beta v\equiv 0\ (\mathrm{mod}\ p^{\nu_p})\end{array}\right\}$$

$$=\begin{cases} p^{\nu_p}+p^{\nu_p-1}, & \text{if } \alpha\geqq\nu_p; \\ p^{\alpha}, & \text{if } \beta\geqq\nu_p>\alpha; \\ 0, & \text{if } \nu_p>\beta. \end{cases}$$

Firstly, we assume $\rho_p\geqq\nu_p$. If $\tau_{p,x}<\nu_p$, we have $\mathrm{ord}_p((t^2-64n^2)/4)\geqq2\rho_p>2\tau_{p,x}=\mathrm{ord}_p((4x-t/2)^2)$. Hence, $\alpha+\beta=\mathrm{ord}_p((4x-t/2)^2-(t^2-64n^2)/4)=2\tau_{p,x}=\alpha+\tau_{p,x}$. Therefore, if $\#(V_p(x)/\Gamma_0(p^{\nu_p}))\neq0$, then, we get $\tau_{p,x}\geqq\nu_p$, namely, $t\equiv8x \pmod{p^{\nu_p}}$.

Thus,

$$\sum_{x\in(\boldsymbol{Z}/p^{\nu_p}\boldsymbol{Z})^{\times}}\#(V_p(x)/\Gamma_0(p^{\nu_p}))=p^{\nu_p}+p^{\nu_p-1}.$$

Next, we assume that $\nu_p>\rho_p$. If $\tau_{p,x}<\rho_p$, we have: $\alpha=\tau_{p,x}$ and

$$\mathrm{ord}_p((4x-t/2)^2)=2\tau_{p,x}<2\rho_p\leqq\mathrm{ord}_p((t^2-64n^2)/4).$$

Hence,

$$\alpha+\beta=\mathrm{ord}_p((4x-t/2)^2-(t^2-64n^2)/4)=2\tau_{p,x}=2\alpha<\alpha+\nu_p.$$

Therefore, if $\#(V_p(x)/\Gamma_0(p^{\nu_p}))\neq0$, we get $\tau_{p,x}\geqq\rho_p$. Thus, we obtain:

$$\sum_{x\in(\boldsymbol{Z}/p^{\nu_p}\boldsymbol{Z})^{\times}}\#(V_p(x)/\Gamma_0(p^{\nu_p}))$$

$$=p^{\rho_p}\times\#\left\{(\boldsymbol{Z}/p^{\nu_p}\boldsymbol{Z})\ni x;\ t\equiv8x \pmod{p^{\rho_p}} \text{ and } x^2-(t/4)x+n^2\equiv0 \pmod{p^{\nu_p+\rho_p}}\right\}.$$

Moreover, we can omit the assumption: $t\equiv8x \pmod{p^{\rho_p}}$ by using the assumption: $\nu_p>\rho_p$ and the fact: $\mathrm{ord}_p((t^2-64n^2)/4)\geqq2\rho_p$.

Appendix 4.

For a simplicity, we shortly write $\theta_0=\theta_{0,p}$. Then, the constant $n_{0,p}(\theta_{0,p})$ is given by the following table:

Case (1) ($p|M$ and $p|u$).

$$n_{0,p}(\theta_{0,p})=\begin{cases} \{p^{\theta_0+1}(p^{[\nu/2]}+p^{[(\nu-1)/2]})-p^{\nu}-p^{\nu-1}\}/(p-1), & \text{if } \theta_0\geqq[\nu/2]; \\ 0, & \text{if } \theta_0<[\nu/2]. \end{cases}$$

Case (2) ($p|M$ and $p\nmid u$).

$$n_{0,p}(\theta_{0,p})=\begin{cases} \left(p-\left(\frac{u}{p}\right)\right)(p^{[\nu/2]}+p^{[(\nu-1)/2]})(p^{\theta_0}-p^{[\nu/2]})/(p-1) \\ \quad +p^{[\nu/2]}\left(p^{[\nu/2]}+\left(\frac{u}{p}\right)^{\nu}p^{[(\nu-1)/2]}\right), & \text{if } \theta_0\geqq[(\nu+1)/2]; \\ \left(1+\left(\frac{u}{p}\right)\right)p^{2\theta_0}, & \text{if } \theta_0\leqq[(\nu-1)/2]. \end{cases}$$

Case (3) ($p=2$).

$$n_{0,2}(\theta_{0,2})=\begin{cases} 2^{\theta_0}(2^{[\mu/2]}+2^{[(\mu-1)/2]}), & \text{if } u\equiv 1 \pmod 8 \text{ and } \theta_0 \geqq [(\mu+1)/2]; \\ 2^{2\theta_0+1}, & \text{if } u\equiv 1 \pmod 8 \text{ and } \theta_0 \leqq [(\mu-1)/2]; \\ 3\times 2^{\theta_0}(2^{[\mu/2]}+2^{[(\mu-1)/2]})-3\times 2^{\mu}, & \\ & \text{if } u\equiv 5 \pmod 8 \text{ and } \theta_0 \geqq [(\mu+1)/2]; \\ 0, & \text{if } u\equiv 5 \pmod 8 \text{ and } \theta_0 \leqq [(\mu-1)/2]; \\ 2^{\theta_0+1}(2^{[\mu/2]}+2^{[(\mu-1)/2]})-3\times 2^{\mu-1}, & \\ & \text{if } u\equiv 0 \pmod 4 \text{ and } \theta_0 \geqq [(\mu-1)/2]; \\ 0, & \text{if } u\equiv 0 \pmod 4 \text{ and } \theta_0 < [(\mu-1)/2]. \end{cases}$$

DEPARTMENT OF MATHEMATICS
KYOTO UNIVERSITY

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