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A comparison theorem for Riccati equations and its application to simple hyperbolic boundary value problems

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§1. Introduction.

In this paper we deal with a certain comparison theorem for Riccati equations and apply it to construct an integral representation of forward progressing solutions of simple hyperbolic boundary value problems. This integral representation enables us to see that the forward progressing solution preserves its order of singularities along characteristic curves. (See (P.1) and (P.2) below.)

Let us explain our comparison theorem (Theorem 2.1). Let $q(x)$ be a smooth function defined on $[0, \infty)$ taking its values in $C-(-\infty, 0]$. By $\sqrt{q(x)}$ we denote a root of $q(x)$ with positive real part. It is evident that, if $q(x)$ is constant q, the equation $w'=q-w^2$ has solutions $\sqrt{q}$ and $-\sqrt{q}$. Here $-\sqrt{q}$ is an unstable solution, $\sqrt{q}$ being stable. In general case it is interesting to seek for the solution $w(x)$ of $w'=q(x)-w^2$, which stays close to $-\sqrt{q(x)}$ for all $x\in[0, \infty)$. We suppose that $(\log q(x))'$ is not so large as compared with Re $\sqrt{q(x)}$. More precisely under the assumption

$$D(q)=\sup_{0\leq x}\left\{\left|\frac{q'(x)}{q(x)}\right| / \operatorname{Re}\sqrt{q(x)}\right\}<4, \tag{1.1}$$

we can show that there exists a solution $w(x)$ of $w'=q(x)-w^2$ satisfying

$$\left|\frac{w(x)-(-\sqrt{q(x)})}{w(x)-\sqrt{q(x)}}\right|<r_1, \text{ for all } x\in[0, \infty), \tag{1.2}$$

where r_1 stands for a root of $r+\dfrac{1}{r}=\dfrac{8}{D(q)}$ less than 1, i.e.

$$r_1+\frac{1}{r_1}=\frac{8}{D(q)}, \quad r_1<\frac{D(q)}{4}<1. \tag{1.3}$$

In order to apply the above result to the boundary problem stated below, we take $q(x)=-\tau^2a(x)$, where $a(x)$ is a positive valued smooth bounded function and $\tau=\sigma-i\gamma$, $\sigma\in R$, $\gamma>0$. Then $D(q)$ can be made arbitrarily small if we take the

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parameter γ sufficiently large. Hence from (1.2) and (1.3) the equation $w'=-\tau^2 a(x)-w^2$ has a solution $w(x, \tau)$ close to $-i\tau\sqrt{a(x)}$. Put

$$v(x, \tau)=\exp\left(\int_0^x w(y, \tau)\, dy\right). \tag{1.4}$$

Then $v(x, \tau)$ will give a unique solution of

$$\begin{cases} v''=-\tau^2 a(x)v & \text{in } (0, \infty) \\ v(0, \tau)=1, & \lim_{x\to\infty} v(x, \tau)=0. \end{cases} \tag{1.5}$$

Now we can make a solution of

$$\begin{cases} \dfrac{\partial^2}{\partial x^2}u-a(x)\dfrac{\partial^2}{\partial t^2}u=0, \ x>0, \ t\in R, \\ u(0, t)=g(t), \qquad t\in R, \end{cases} \tag{P}$$

in the following form;

$$u(x, t)=e^{\gamma t}\bar{F}[v(x, \tau)F[e^{-\gamma s}g(s)]], \tag{1.6}$$

which is independent of $\gamma>\gamma_0\geq 0$, if $e^{-\gamma s}g(s)$ belongs to $\mathscr{S}'$ for $\gamma'>\gamma$. We can say that the solution $u(x, t)$ defined by (1.6) is the forward solution, since the following (P.1) can be verified. We can prove also (P.2). (See Theorem 3.1)

(P.1) $\inf_t \operatorname{supp} u(x, t)$ propagates along the forward characteristic issued from $(0, \inf_t \operatorname{supp} g(t))$.

(P.2) Singularities of $u(x, \cdot)$ propagate along forward characteristics, keeping the order of singularities which is defined below by (1.9). More precisely it holds

(1.7) $\quad Ord\ Sing\ (g; s)=Ord\ Sing\ (u(x, \cdot); t_x(s))$, for all $s\in R$ and $x\in(0, \infty)$,

where $t_x(s)=s+\int_0^x\sqrt{a(y)}\,dy$.

Here the order of singularity of $f(t)$ at a point t_0 is defined as follows. First we put

$$Ord\ Sing(f)=\inf\ \{r\in R:\ \lim_{|\sigma|\to\infty}\hat{f}(\sigma)(1+|\sigma|)^{-r}=0\}, \tag{1.8}$$

for $f\in\mathscr{S}'$ satisfying $\hat{f}(\xi)\in L^1_{loc}$. Using (1.8) we define for $f\in\mathscr{D}'$

$$Ord\ Sing(f;\ t)=\lim_{n\to\infty}\ Ord\ Sing(\alpha_n f), \tag{1.9}$$

where $\{\alpha_n\}$ is a sequence of functions in $\mathscr{D}$ satisfying $\alpha_n(t_0)\neq 0$ and $\operatorname{supp}\alpha_n\to\{t_0\}$. In Section 4 we show that a finite number or $-\infty$ corresponds to the right hand side of (1.8) independently of the choice of $\{\alpha_n\}$. Here we point out some of the properties of $Ord\ Sing\ (f;\ t_0)$:

(S.1) Tan^{-1} $(Ord\ Sing(f;\ t))$ is an upper semi-continuous function and satisfies

$$Ord\ Sing(f)=\max_{t\in \operatorname{snpp} f}\ Ord\ Sing(f;\ t),\ \text{if } f\in\mathscr{E}'.$$

(S.2) If $Ord\ Sing(f;\ t_0)<-(k+1)$ holds, then $f(t_0)$ is k times continuously differentiable in a neighbourhood of t_0. On the other hand *Ord Sing* $(f;\ t_0)>-k_1$ implies that $f(t)$ is not k_1 times continuously differentiable in any neighbourhood of t_0.

(S.3) *Ord Sing*$(f;\ t)<-(k+1)$ on a compact set K implies that $f(t)$ is k times continuously differentiable in a neighbourhood Ω_k of K.

(S.4) $f(t)$ is a C^∞ function in an open set Ω if and only if *Ord Sing* $(f;\ t)=-\infty$ holds for all $t\in\Omega$. And we have

$$\textit{sing supp} f=\overline{\{t;\ \textit{Ord Sing}(f;\ t)>-\infty\}}.$$

(S.5) If *Ord Sing*$(f;\ t)=-\infty$ holds on a compact set K, then we can say that $f(t)$ is a C^∞ function on K since (S.3) holds for all integer $k>0$.

Now remark that *Ord Sing* $(f;\ t)=-\infty$ does not imply $t\notin$ *sing supp f*. Denote the set

$$\{t\colon t\in \textit{sing supp } f,\ \textit{Ord Sing}(f;\ t)=-\infty\}$$

by $CR(f)$, (critical regular points of f). From (S.4) we can verify

$$CR(f)=\{t\colon \max_{|s-t|\le\varepsilon} \textit{Ord Sing}\ (f;\ s)>-\infty \text{ for } \varepsilon>0,\ \lim_{\varepsilon\to 0}\ \max_{|s-t|\le\varepsilon}\ \textit{Ord Sing}(f;\ s)=-\infty\}.$$

Hence we can point out easily examples of functions satisfying $CR(f)\neq\phi$, (see Section 4).

Now we consider an example of the forward solution in a simple case where $q(x)=1$. Remark that $g_1(t-x)+g_2(t+x)$ satisfies (P) whenever $g(t)=g_1(t)+g_2(t)$ holds. In this case the forward solution $u(x,\ t)$ is equal to $g(t-x)$. We can also say that $u(x,\ t)$ is uniquely determined by Cauchy data $u(0,\ t)=g(t)$ and $\frac{\partial}{\partial x}u(0,\ t)$ $=-g'(t)$. However it is not so easy to find the value $\frac{\partial}{\partial x}u(0,\ t)$ in general case. In fact its Laplace-Fourier image is $w(0,\ \tau)\ g(\tau)$, where $w(0,\ \tau)$ is determined depending on all the value $a(x)$, $x>0$.

Historically speaking, the propagation of singularities of this type was considered in [11] in the case where $\frac{\partial}{\partial t}g(t)$ has the discontinuity of the first kind. However if singularities are more general, it is suitable to consider them using Laplace-Fourier inversion formula. In fact we write the solution $u(x,t)$ given by (1.6) in the form

$$u(x,\ t)=\lim_{\varepsilon\to 0}\ \lim_{\varepsilon_1\to 0}\iint e^{-it\tau(t-s)+\int_0^x w(y,\tau)dy}e^{-\varepsilon_1\tau^2}g_\varepsilon(s)ds d\tau, \tag{1.10}$$

where $g_\varepsilon(s)\in\mathcal{D}$ and $e^{-\gamma t}g_\varepsilon(t)$ converges to $e^{-\gamma t}g(t)$ in $\mathcal{S}'$ as ε tends to 0. Especially we remark that there exist positive constants C, γ_0 and θ satisfying

$$C^{-1}e^{-\theta x}\le|v(x,\ \tau)|\le C \quad \text{for Im } \tau<-\gamma^0 \text{ and } x\in[0,\ \infty]. \tag{1.11}$$

In the proof of (P.2) we will see that the estimate (1.11) plays an important role together with the definition of the order of singularities (1.9).

The author consulted Friedlander [3], Hörmander [4], Nirenberg [5] and

Melrose-Shöstrand [6] to justify the interest of the problem studied here and the method used here, and refered also to Mizohata-Yamaguti [10] and Picard [11].

§2. Comparison theorem for Riccati equations.

As is well known, $v''=q(x)v$ is reduced to

$$w'=q(x)-w^2, \tag{2.1}$$

if we put $w=v'/v$ in the case $v\neq 0$. conversely for any w satisfying (2.1),

$$v(x)=\exp\int_0^x w(y)\ dy$$

satisfies $v''=q(x)v$ and $v(0)=1$. Now let us consider another transformation of (2.1) supposing the following condition

$(C)_0$ $q(x)$ is a continuous function defined on $[0, \infty]$ with values in $C-(-\infty, 0]$. Denoting by $\sqrt{q(x)}$ the root of $q(x)$ with positive real part, we put

$$z(x)=\frac{w(x)+\sqrt{q(x)}}{w(x)-\sqrt{q(x)}}. \tag{2.2}$$

If $q(x)$ is constant q, then (2.1) is equal to $z'=2\sqrt{q}\,z$. In this case we have $(\log|z|)'=2\mathrm{Re}\sqrt{q}>0$, if $z\neq 0$ i.e. $w\neq -\sqrt{q}$. This implies that $w=-\sqrt{q}$ is an unstable solution. $v=\exp(-\sqrt{q}x)$ is the unique solution of $v''=qv$ in $(0, \infty)$, $v(0)=1$ and $\lim_{x\to\infty} v(x)=0$. We will extend this argument to general $q(x)$. At first let us introduce

Notation. $w(x;\ x_0,\ w_0)$ stands for the solution of (2.1) satisfying $w(x_0)=w_0$. We put

$$\Omega(x,\ r)=\left\{w\in C,\ \left|\frac{w+\sqrt{q(x)}}{w-\sqrt{q(x)}}\right|<r\right\}.$$

Then we have the following theorem, an ameliolation of some results in [5] and [6].

Theorem 2.1. *Suppose that $q(x)$ satisfies $(C)_0$ and (1.1). Then there exists a solution of (2.1) satisfying $w(x)\in\overline{\Omega(x,\ r_1)}$ for all $x\geq 0$, where r_1 is given in (1.3).*

In the process of the proof of Theorem 2.1 we use

Proposition 2.1. *Assume the same condition as in Theorem 2.1. Let $x_0\in(0, \infty)$ and $w_0\in C$ satisfy $w_0\in\partial\Omega(x_0,\ r_1)$. Then we have $w(x;\ x_0,\ w_0)\in\overline{\Omega(x,\ r_1)}$ for $0<x<x_0$, and $w(x;\ x_0,\ w_0)\notin\Omega(x,\ r_1)$ for $x_0<x$. It holds the same result replaced r_1 by $1/r_1$.*

Remark 2.1. From (1.3) and (2.2), $|z(x)|\leq r_1$ yields

$$\left|\frac{w(x)+\sqrt{q(x)}}{\sqrt{q(x)}}\right|\leq\frac{2r_1}{1-r_1}<\frac{2D(q)}{4-D(q)}$$

Example. Let $q(x)=-\tau^2 a(x)$, $\tau=\sigma-i\gamma$, $\sigma\in R$ and $\gamma>0$. Suppose that $0<m$

$<a(x)<M$ and $|a'(x)|<M_1$ hold for all $x\geq 0$. Then we have $D(q)<(M/m^{3/2})\frac{1}{\gamma}$ and

$$(2.3)\qquad \left|\frac{w(x)+i\tau\sqrt{a(x)}}{\tau\sqrt{a(x)}}\right|<C\frac{1}{\gamma},\quad \text{for } \gamma>\gamma_0=2(M_1/m^{3/2}),$$

which tends to zero as γ tends to ∞.

The estimate (2.3) will be used later.

Proof of Proposition 2.1. First remark that (2.1) and (2.2) make

$$(2.4)\qquad \frac{d}{dx}\log z=2\sqrt{q(x)}-\frac{1}{4}(z-z^{-1})\frac{q'(x)}{q(x)}.$$

Using $|z-z^{-1}|\leq|z|+|z|^{-1}$ and $\mathrm{Re}\frac{d}{dx}\log z(x)=\frac{d}{dx}\log|z(x)|$, we have

$$(2.5)\qquad \frac{d}{dx}\log|z(x)|\geq 2\mathrm{Re}\sqrt{q(x)}-\frac{1}{4}(|z|+|z|^{-1})\left|\frac{q'(x)}{q(x)}\right|.$$

The condition (1.1) implies that the right hand side of (2.5) is non-negative if z satisfies $|z|=r_1$ or $|z|=\frac{1}{r_1}$. Hence we have Proposition 2.1.

Proof of Theorem 2.1. Let us put

$$F_n=\}w(0,\ n,\ w_0);\ w_0\in\overline{\Omega(n,\ r_1)}\}.$$

The family of set $\{F_n\}$ satisfies $F_n=F_n$, $F_n\in\overline{\Omega(0,\ r_1)}$ and $F_n\supset F_{n+1}$, for any integer $n>0$. Therefore F_n converges to a non empty set $F=\bigcap_{n=1}^{\infty}F_n$. Let w_0 belong to F, then $w_0\in F_n$ for all $n>0$. Therefore in view of Proposition 2.1 the solution $w(x)=w(x;\ 0,\ w_0)$ belongs to $\overline{\Omega(x,\ r_1)}$ for all $x\in[0,\ n]$, $n=1,\ 2,\ 3,\ldots$ Hence $w(x)$ belongs to $\overline{\Omega(x,\ r_1)}$ for all $x\in[0,\ \infty)$.

§3. Forward progressing solution of the problem (P).

In this section we state our results concerning forward progressing solution of (P) described in Introduction. The proof is given in later sections. First let us introduce

Definition 3.1. $g(t)$ is said to belong $\mathcal{S}'_{\gamma_0}$, if $e^{-\gamma t}g(t)$ belong to $\mathcal{S}'$ for all $\gamma>\gamma_0\in R$. $\lim_{\varepsilon\to 0}g_\varepsilon=g$ in $\mathcal{S}'_{\gamma_0}$ means $\lim_{\varepsilon\to 0}e^{-\gamma t}g_\varepsilon(t)=e^{-\gamma t}g(t)$ in $\mathcal{S}'$ for all $\gamma>\gamma_0$.

Remark 3.1. Let $\chi_\varepsilon(t)$ and $\varphi_\varepsilon(t)$ be functions in $\mathcal{D}$ satisfying $\lim_{\varepsilon\to 0}\chi_\varepsilon(t)=1$ and $\lim_{\varepsilon\to 0}\varphi_\varepsilon(t)=\delta$ in $\mathcal{S}'$. Then $g_\varepsilon=\varphi_\varepsilon*(\chi_\varepsilon g)$ satisfies $\lim_{\varepsilon\to 0}g_\varepsilon(t)=g(t)$ in $\mathcal{S}'_{\gamma_0}$ for any $g(t)\in\mathcal{S}'_{\gamma_0}$.

We suppose that the boundary data of (P) belongs to $\mathcal{S}'_{\gamma_0}$. Now we can state a characterization of $\mathcal{S}'_{\gamma_0}$.

Lemma 3.1. *g belongs to* $\mathcal{S}'_{\gamma_0}$, *if and only if the following* (i) *and* (ii) *hold.*

(i) $\hat{g}(\tau)=F[e^{-\gamma t}g]$ *is analytic in* $\{\tau\in C\colon \mathrm{Im}\,\tau<-\gamma_0\}$.
(ii) $\hat{g}(\sigma-i\gamma)$ *belongs to* $\mathscr{S}'(\sigma)$ *for all* $\gamma>\gamma$. *For given* $\varepsilon>0$ *there exist positive constants* C *and* k *satisfying*

$$|g(\sigma-i\gamma)|\leq C(|\sigma|+1)^k, \textit{ for all } \sigma\in R \textit{ and } \gamma_0+\varepsilon<\gamma<\gamma_0+\frac{1}{\varepsilon}.$$

Now the coefficient $a(x)$ in (P) is assumed to satisfy the following condition.
(C_{k_1}) $a(x)$ is a C^∞-function satisfying $0<m<a(x)<M<\infty$ and $\left|\frac{d^k}{dx^k}a(x)\right|<M_k<\infty$ for all $x>0$ and all integer $k\in[1, k_1]$, where k_1 is a given positive integer.
(C) $a(x)$ satisfies the above conditions for all $k_1\geq 1$.

Proposition 3.1. *Suppose that* $a(x)$ *satisfies the above condition* (C_1), *i.e.* (C_{k_1}) *with* $k_1=1$. *Then the problem* (1.5) *has a unique solution* $v(x, \tau)$ *for all* τ *belonging to* $\Sigma_{\gamma_0}=\{\tau\in C\colon \mathrm{Im}\,\tau<-\gamma_0\}$, *where* γ_0 *is a positive number. The solution* $v(x, \tau)$ *is analytic in* τ *and satisfies the following estimates: There exist positive constants* C *and* θ *such that we have for all* $x\geq 0$ *and all* $\tau\in\Sigma_{\gamma_0}$

(i) $$C^{-1}e^{-\theta x}<|v(x, \tau)|<C,$$

(ii) $$\frac{1}{|v(x, \tau)|^2}\int_x^\infty |v(y, \tau)|^2\, dy\leq C\frac{1}{|\mathrm{Im}\,\tau|}.$$

In view of Lemma 3.1 and the bove estimate (i) we see that $g\in\mathscr{S}'_{\gamma_0}$ implies $u(x, \cdot)\in\mathscr{S}'_{\gamma_0}$ for all $x>0$, where $u(x, t)$ is defined by (1.6). Now we can state Theorem 3.1 using the above $v(x, \tau)$.

Theorem 3.1. *Assume* (C). *Suppose that* g *belongs to* $\mathscr{S}'_{\gamma_0}$, $\gamma_0=2(M/m^{3/2})$. *Then* $u(x, t)$ *defined by* (1.6) *is a smooth bounded function in* $x\in[0, \infty)$ *with values in* $\mathscr{S}'_{\gamma_0}$, *together with all its dervatives.* $u(x, t)$ *is the forward progressing solution of the problem* (P) *in the sense that* (P.1) *holds.* $u(x, t)$ *satisfies also* (P.2) *if inf supp* $g>-\infty$ *holds.*

In order to verify (P.1) and (P.2) we use some detailed properties of the solution $w(x, \tau)$ of the following problems.

$$(3.1)\qquad \begin{cases} w'=-\tau^2 a(x)-w^2, \ x>0, \\ \mathrm{Re}\displaystyle\int_0^\infty w(y, \tau)\, dy=-\infty. \end{cases}$$

Proposition 3.2. *Assume* (C_{k_1}). *Then the problem* (3.1) *has a unique solution* $w(x, \tau)$ *satisfying the following* (i) *and* (ii):
(i) $w(x, \tau)$ *is analytic in* $\tau\in\Sigma_{\gamma_0}$, *for all* $x\geq 0$.
(ii) *For each integer* $k\in[0, k_1-1]$, *there exist positive constants* C_k *and* γ_k *such that we have, for all* $\tau\in\Sigma_{\gamma_k}=\{\tau\in C;\ \mathrm{Im}\,\tau<-\gamma_k\}$ *and* $x\geq 0$

$$|\partial_\tau^j\partial_x^i(w(x, \tau)+i\sqrt{a(x)}\,\tau)|\leq C_k\frac{|\tau|^{1-j}}{|\mathrm{Im}\,\tau|}, \quad 0\leq i, \ 0\leq j, \ 0\leq i+j\leq k.$$

Remark. we can show that $w(x, \tau)$ has an analytic continuation into $\Sigma=\{\tau\in C;\ \mathrm{Im}\,\tau<0\}$, if we apply the method used in [8].

Incidentally we state another estimate of $u(x, t)$ defined by (1. 6) as a corollary. At first let us introduce

Notation. g is said to belong to H_{γ}^{k} if $e^{-\gamma t}\, g$ belong to H^{k}, for $\gamma\in R$ and $k\in R$. $\|e^{-\gamma t}g\|_{k}$ is equivalent to $\|g\|_{k,\gamma}=\Big(\int_{\mathrm{Im}\,\tau=-\gamma}(1+|\tau|)^{2k}|\hat{g}(\tau)|^{2}\,d\tau\Big)^{1/2}$.

Corollary 3.1. *Suppose $g\in H_{\gamma}^{k}$, $k\in R$, in Theorem* 3.1. *Then the solution $u(x, t)$ is a continuous function in x with values in H_{γ}^{k} satisfying the following estimates*:

$$C_{k}^{-1}e^{-\gamma x}|g\|_{k,\gamma}\leq\sum_{j=0}^{1}\Big\|\frac{\partial}{\partial x^{j}}u(x,\ \cdot)\Big\|_{k-j,\gamma}\leq C_{k}\|g\|_{k,\gamma},\ \textit{for}\ \gamma>\gamma_{0}\ \textit{and}\ x>0. \tag{3.2}$$

$$\int_{0}^{\infty}\Big(\|u(x,\ \cdot)\|_{k,\gamma}^{2}+\Big\|\frac{\partial}{\partial x}u(x,\ \cdot)\Big\|_{k-1,\gamma}^{2}\Big)dx\leq C_{k}\frac{1}{\gamma}\|g\|_{k,\gamma}^{2},\ \textit{for}\ \gamma>\gamma_{0}. \tag{3.3}$$

§4. Order of singularities.

Here Definition (1.9) is shown to be reasonable and we verify some basic properties of *Ord Sing* $(f;\ t)$, which are used later. First we state a fundamental lemma concerning the definition (1.8).

Lemma 4.1. *For any $f\in\mathscr{E}'(R)$ and $\alpha\in\mathscr{D}(R)$, we have*

$$\textit{Ord Sing}(f)\geq \textit{Ord Sing}(\alpha f). \tag{4.1}$$

Proof. For any r satisfying *Ord Sing*$(f)<r$, (1.8) yields

$$|\hat{f}(\sigma)|\leq C_{r}(1+|\sigma|)^{r}\qquad \text{for } \sigma\in R. \tag{4.2}$$

From $\widehat{\alpha f}(\sigma)=\frac{1}{2\pi}\int\hat{f}(\sigma-\rho)\hat{\alpha}(\rho)\,d\rho$ and (4.2) it follows

$$(1+|\sigma|)^{-r}|\widehat{\alpha f}(\sigma)|\leq C_{r}\int\frac{(1+|\sigma|+|\sigma|)^{r}}{(1+|\sigma|)^{r}}|\hat{\alpha}(\rho)|\,d\rho\leq C_{r}\int|\hat{\alpha}(\sigma)|(1+|\rho|)^{r}\,d\rho<\infty.$$

Hence we have *Ord Sing* $(\alpha f)\leq r$, which implies (4.1).

From Lemma 4. 1 we have immediately

(4.3) *Ord Sing*$(\alpha f)\geq$ *Ord Sing* (βf) holds for $f\in\mathscr{D}'(R)$,

if $\alpha\in\mathscr{D}$ and $\beta\in\mathscr{D}(R)$ satisfy $\mathrm{supp}\,\beta\subset\{t;\ \alpha(t)\neq 0\}$.

By virtue of (4.3) we see that the right hand side of (1.9) corresponds to a finite number or $-\infty$ independently of $\{\alpha_{n}(t)\}$. Now remark that Lemma 4.1 is true even if we replace $\mathscr{D}'(R)$ and $\mathscr{D}(R)$ respectively by $\mathscr{D}'(R^{n})$ and $\mathscr{D}(R^{n})$ and extend (1.8) to

(1.8)′ *Ord Sing*$(f)=\inf\{r\in R:\ \lim_{|\xi|\to\infty}f(\xi)(1+|\xi|)^{-r}=0\}$, for $f\in\mathscr{S}'$ satisfying $\hat{f}(\xi)\in L_{loc}^{1}$.

Thus the definition (1.9) can be extended to

Definition 4.1. For $f\in\mathscr{D}'$ (R^{n}) we define the order of singularity of f at a point $x=x_0$ by

$$(4.4)\qquad Ord\ Sing(f;\ x_0)=\lim_{k\to\infty} Ord\ Sing\ (\alpha_k f)=\inf_{\alpha\in\mathscr{D},\alpha(x_0)\neq 0} Ord\ Sing(\alpha f)$$

where $\{\alpha_k\}$ is a sequence of functions in $\mathscr{D}$ satisfying $\alpha_k(x_0)\neq 0$ and supp$\alpha\to\{x_0\}$.

In fact Definition 4.1 is reasonable since it holds Lemma 4.1 replaced the space R by R^n. Now we have the following lemma concerning the order of singularities of distributions in $R,^n$ $n\geq 1$.

Lemma 4.2. *We have for any* $f\in\mathscr{E}'$, $f_1\in\mathscr{E}'$ *and* $f_2\in\mathscr{D}'$

$$(4.5)\qquad Ord\ Sing(f;\ x)\leq Ord\ Sing(f),$$

$$(4.6)\qquad Ord\ Sing(f_1+f_2;\ x)\leq\max_{j=1,2} Ord\ Sing(f_j;\ x),$$

$$(4.7)\qquad Ord\ Sing(f_1+f_2;\ x)=\max_{j=1,2} Ord\ Sing(f_j;\ x),\ if\ Ord\ Sing(f_1;\ x)\neq Ord\ Sing(f_2;\ x).$$

Lemma 4.2 is verified directly from Definition 4.1. We prepare another Lemma concerning *Ord Sing*(f).

Lemma 4.3. *For any* $f\in\mathscr{E}'$ *we have*

$$(4.8)\qquad Ord\ Sing(f)=\max_j Ord\ Sing\ (\alpha_j f),\ if\ \sum_{j=1}^{finite}\alpha_j=1\ on\ supp\ f,\ \alpha_j\in\mathscr{D}.$$

(4.9) *Ord Sing*$(f)<-(k+n)$ *implies that* f *is a* C^k *function for non-negative integer* k.

(4.10) *Ord Sing*$(f)>-k$ *implies that* f *is not a* C^k *function for non-negative integer* k.

Proof. From Lemma 4.1 the left hand side of (4.8) is not less than the right han dside of (4.8). On the other hand the definition (1.9)′ yields the counter estimate. Thus (4.8) holds. If f satisfies the condition of (4.9), then $(1+|\xi|)^r\hat{f}(\xi)$ is bounded, $r=k+n+\varepsilon$. Therefore $(1+|\xi|)^k\hat{f}(\xi)\in L^1$. Hence we have $f(x)\in C^k$. Similarly from *Ord Sing*$(f)<-k$ follows that $|\xi|^k\hat{f}(\xi)$ is not bounded. Therefore since supp f is compact, $\frac{d^k}{dt^k}f(t)$ is not continuous. Thus we have Lemma 4.3.

Now we state

Theorem 4.1. *Let* f *be a distribution in* R^n, $n\geq 1$. *Then Ord Sing*$(f;\ x)$ *has the same properties as we have stated in Introduction in the case of* $n=1$. *Namely we have* (S.1) *replaced* t *by* x, (S.2) *replaced* t_0, $-(k+1)$ *and* t *respectively by* x_0, $-(k+n)$ *and* x, (S.3) *replaced* t *and* $-(k+1)$ *by* x *and* $-(k+n)$ *respectively*, (S.4) *replaced* t *by* x *and* (S.5) *replaced* t *by* x.

Proof. The upper semi-continuity of Tan^{-1} (*Ord Sing* $(f;\ x)$) follows directly from Definition 4.1. Let us take the following series of partitions of unity of supp f: $\sum_j^{finite}\alpha_{k,j}(x)=1$ on supp f, $k=1, 2, 3,\ldots$, where $\alpha_{k,j}$ are functions in $\mathscr{D}$ satisfying $\max_j$ {diameter of supp $\alpha_{k,j}\}\to 0$ as $k\to\infty$. By virtue of (4.8) there exist numbers n_k, $k=1, 2, 3,\ldots$, satisfying

(4.11) $\quad Ord\ Sing(f)=Ord\ Sing(\alpha_{1,n_1}f)=Ord\ Sing(\alpha_k f),\ k\geq 1,$

where $\alpha_k=\prod_{j=1}^{k}\alpha_{j,n_j}$. Remark that supp$\alpha_k$ converges to a point $x_0\in\operatorname{supp}f$. Therefore by the definition (1.8)′ it follows $Ord\ Sing(f;\ x_0)\geq\lim_{k\to\infty} Ord\ Sing\ (\alpha^k f)$. From (4.11) and (4.5) we have $Ord\ Sing(f)=\max_{x\in\operatorname{supp}f} Ord\ Sing(f;\ x)$. Thus (S.1) holds. Now suppose $Ord\ Sing(f;\ x_0)<-(k+n)$. Then $Ord\ Sing(\alpha f)<-(k+n)$ holds for some $\alpha(x)\in\mathscr{D}$ satisfying $\alpha(x_0)\neq 0$. Therefore from (4.9) of Lemma 4.3 we see that $f(x)$ is a C^k functien in a neighbourhood of x_0. Similarly $Ord\ Sing(f;\ x_0)>-k_1$ implies that $f(x)$ is not k_1 times continuously differentiable in any neighbourhood of x_0. We next remark that (S.3) and (S.4) follow from (S.2), the definition of sing supp$f$ and the Heine-Borel theorem. (S.5) is an interpretation of $Ord\ Sing(f;\ x_0)<-(k+n)$ for all $k>0$ in use of (S.2). Thus the proof of Theorem 4.1 is complete.

Now we consider concretely some examples of $Ord\ Sing(f;\ t)$ in R and show an example of the function satisfying $CR(f)=\{t;\ t\in\operatorname{supp}f,\ Ord\ Sing(f;\ t)=-\infty\}\neq\phi$.

Example 4.1. Let $\zeta(t)$ be equal to $e^{-1/t}$ for $t>0$ and zero for $t\leq 0$. Put $\alpha(t)=\zeta(t+1)\zeta(-t+1)$ and $\alpha_n(t)=\alpha(nt)$. Then $\{\alpha_n(t)\}$ is a sequence of functions in $\mathscr{D}$ satisfying the condition in the definition of (1.9) for $t_0=0$, i.e. $\alpha_n(t)\in\mathscr{D}$, $\alpha_n(0)\neq 0$ and $\lim_{n\to\infty}$ supp $\alpha_n=\{0\}$. Denote by t_+^m the function defined by t^m for $t>0$ and zero for $t\leq 0$. From (S.2) $t_+^m\notin C^m$ implies $Ord\,Sing(t_+^m;\ 0)\geq -(m+1)$ and $t_+^m\in C^{m-1}$ implies $Ord\,Sing(t_+^m;\ 0)\leq -(m-1)$. Now we can show by a direct calculation $Ord\,Sing\ (t_+^m;\ 0)=-(m+1)$ as follows. Since it holds $F[\alpha_n t_+^m](\sigma)=\frac{1}{n^{m-1}}F[\alpha t_+^m]\left(\frac{\sigma}{n}\right)$, we have

$$Ord\ Sing(\alpha_n t_+^m)=Ord\ Sing(\alpha t_+^m)=\lim_{n\to\infty} Ord\ Sing(\alpha_n t_+^m)=Ord\ Sing(t_+^m;\ 0).$$

So it suffices to prove $Ord\ Sing(\alpha t_+^m)=-(m+1)$. For $\sigma\neq 0$ it holds

$$F[\alpha t_+^m](\sigma)=\int_0^1 e^{-it\sigma}\alpha(t)t^m dt=\left(\frac{1}{i\sigma}\right)^{m+1}\int_0^1\frac{d}{dt}e^{-it\sigma}\frac{d^m}{dt^m}(\alpha(t)t^m)dt$$

$$=\frac{m!\,\alpha(0)}{(i\sigma)^{m+1}}+\frac{(m+1)!\,\alpha'(0)}{(i\sigma)^{m+2}}+\left(\frac{1}{i\sigma}\right)^{m+2}\int_0^1 e^{-it\sigma}\frac{d^{m+2}}{dt^{m+2}}(\alpha(t)t^m)dt$$

Therefore we have $Ord\ Sing(\alpha t_+^m)=-(m+1)$. Hence $Ord\ Sing(t_+^m;\ 0)=-(m+1)$.

Example 4.2. Using the above function $\alpha(t)$ we denote $f_n(t)=\alpha(4n^2t)t_+^n$. Put $f(t)=e^{-1/t}\sum_{n=2}^{\infty}f_n\left(t-\frac{1}{n}\right)$. Then from Example 4.1 we have $Ord\ Sing\left(f_n\left(t-\frac{1}{n}\right);\frac{1}{n}\right)=-(n+1)$. Remark supp $f_n\left(t-\frac{1}{n}\right)\cap$supp $f_m\left(t-\frac{1}{m}\right)=\phi$ for $n\neq m$ and $|f_n^{(k)}(t)|\leq C_k n^{2k}$ for any fixed integer $k\leq n$. Therefore for any $k>0$, $f(t)$ is C^k in a neighbourhood of the origin. Thus $CR(f)=\{0\}\neq\phi$.

Incidentally from microlocal viewpoints it will be suitable to define

Definition 4.2. Using (1.8)′ we define the order of the singularity of $f\in\mathscr{D}'(R^n)$ at $(x_0, \xi^0)\in R^n\times S^{n-1}$

$$\text{(4.11)} \qquad Ord\ Sing(f;\ x_0,\ \xi^0)=\lim_{k\to\infty} Ord\ Sing(\beta_k(D)\alpha_k f),$$

where $\{\alpha_k\}$ is the same one as in Definition 4.1 and $\{\beta_k\}$ has the following properties; $\beta_k\in C^\infty$ are homogeneous of degree zero in ξ, $|\xi|\geq 1$, satisfying $\beta_k(\xi^0)\neq 0$ and

$$\lim_{k\to\infty} \text{supp}\ \beta_k|_{S^{n-1}}=\{\xi^0\}$$

We can verify that *Ord Sing*$(f;\ x,\ \xi)$ has similar properties as *Ord Sing*$(f;\ x)$. We give some comments in Appendix relating to [1], [4], [9] and [12].

§5. Proof of Theorem 3.1.

In this section we prove Theorem 3.1 admitting Propositions 3.1 and 3.2, which are proved in the next section. First let us confirm

Notations. We say that $u(x,t)$ belongs to $C^k([0,\ \infty);\ H^j_\gamma)$ if $e^{-\gamma t}\ u(x,\ t)$ is k times continuously differentiable in x on $[0,\ \infty)$ with values in H^j. $u(x,\ t)\in C^k\ ([0,\ \infty);\ H^\infty_\gamma)$ means $u(x,\ t)\in C^k([0,\ \infty);\ H^j_\gamma)$ for all $j>0$. $u(x,\ t)$ is said to belong to $C^k([0,\ \infty);\ \mathscr{S}'_{\gamma_0})$ if $e^{-\gamma t}\ u((x,\ t)$ belongs to $C^k([0,\ \infty);\ \mathscr{S}')$ for all $\gamma>\gamma_0$.

From the expression (1.6) and Proposition 3.1 we have directly

Proposition 5.1. *Assume* (C) *in Section* 3. *If* $g\in H^k_\gamma$ *for some* $\gamma>\gamma_0$, *then* $u(x,\ t)$ *is a solution of* (P) *satisfying* (3.2) *and* (3.3). *Moreover* $g\in H^\infty_\gamma$ *implies* $u(x,\ t)\in C^2([0,\ \infty);\ H^\infty_\gamma)$ *for* $\gamma>\gamma_0$, *and* $u(x,\ t)\in C^2([0,\ \infty);\ \mathscr{S}'_{\gamma_0})$ follows from $g\in\mathscr{S}'_{\gamma_0}$.

Proof. From Propositions 3.1 and 3.2 the solution $v(x,\ \tau)$ satisfies

$$\text{(5.1)} \qquad \left|\frac{\partial^k}{\partial x^k}v(x,\ \tau)\right|\leq C_k|\tau|^k, \qquad \tau\in\Sigma_{\gamma_0},\ k=0,\ 1.$$

By virtue of Lemma 3.1, (5.1) and the Lebesgue theorem we have Proposition 5.1.

Let us put

$$\text{(5.2)} \qquad \phi=\phi(t,\ \tau,\ s;\ x)=\tau(t\text{-}s)-i\int_0^x w(y,\ \tau)\ dy.$$

Then the integral representation (1.10) becomes

$$\text{(5.3)} \qquad u(x,\ t)=\lim_{\varepsilon\to 0}\ \lim_{\varepsilon_1\to 0}\iint e^{-i\phi}e^{-\varepsilon_1\tau^2}g_\varepsilon(s)\ dsd\tau,$$

where $g_\varepsilon(s)\in\mathscr{D}$ satisfies $e^{-\gamma t}g_\varepsilon(t)\to e^{-\gamma t}g(t)$ as $\varepsilon\to 0$. Remark that (5.3) follows from (1.6) if we take care of the continuity of Fourier transformation. Use the relation:

(5.4) $$e^{i\phi}=\left(\frac{1}{i\phi_\tau}\frac{\partial}{\partial\tau}\right)^k e^{i\phi},\ k=1,2,3,\dots,\ if\ \phi_\tau\neq 0$$

in (5.3). Then we have

(5.3)′ $$\lim_{\varepsilon\to 0}\lim_{\varepsilon_1\to 0}\iint e^{i\phi}\left(-\frac{1}{i\phi_\tau}\frac{\partial}{\partial\tau}-\frac{\phi_{\tau\tau}}{i\phi_\tau^2}\right)^k e^{-\varepsilon_1\tau^2}g_\varepsilon(s)dsd\tau,$$

if (t, x) satisfies the following condition for some $\gamma>\gamma_0$:

(5.5) $$\inf_{s\in\text{supp }g,\ \text{Im}\,\tau=-\gamma}|\phi_\tau(t,\tau,s;x)|>0.$$

Now remark that ϕ_τ approximates to $t-t_x(s)$ as γ tends to ∞, where $t_x(s)$ equals $s+\int_0^x\sqrt{a(y)}\,dy$. In fact $\phi_\tau-(t-t_x(s))=\int_0^x\partial_\tau(w(y,\tau)+i\tau\sqrt{a(y)})\,dy$ and Proposition 3.2 yield

(5.6) $$|\phi_\tau-(t-t_x(s))|\leq C_1\frac{x}{|\text{Im}\,\tau|},\ \text{for } (t,s,\tau,x)\in R^2\times\Sigma_{\gamma_1}\times(0,\infty).$$

(5.7) $$|\partial_\tau^k\phi(t,\tau,s;x)|\leq C_k x\frac{|\tau|^{1-k}}{|\text{Im}\,\tau|},\ k\geq 2,\ \text{for } (t,s,\tau,x)\in R^2\times\Sigma_{\gamma_k}\times(0,\infty)$$

follows also from Proposition 3.2. Here we put

(5.8) $$p_k(t,\tau,s:x,\varepsilon_1)=e^{\varepsilon_1\tau^2}\left(-\frac{1}{i\phi_\tau}\frac{\partial}{\partial\tau}-\frac{\phi_{\tau\tau}}{i\phi_\tau^2}\right)^k e^{-\varepsilon_1\tau^2}.$$

Here we collect some estimates concerning ϕ_τ and p_k, which are used later.

Proposition 5.2. *Let F be a closed set in R. Suppose that at $(x_0, t_0)\in(0,\infty)\times R$,*

(5.9) $$t_0-t_{x_0}(s)\neq 0,\ for\ all\ s\in F,$$

holds. Then we can find a neighbourhood U of (x_0, t_0), positive constants $\tilde\gamma_k$ and C_k, $k=1, 2, 3,\dots$, such that the following (5.10), (5.11) *and* (5.12) *hold.*

(5.10) $$\inf\{|\phi_\tau(t,\tau,s;x)|;\ (x,t)\in U,\ s\in F,\ \text{Im}\,\tau<-\tilde\gamma_1\}>0,$$

(5.11) $$\sup\{|p_k(t,\tau,s;x,\varepsilon)|;\ (x,t)\in U,\ s\in F,\ \text{Im}\,\tau<-\tilde\gamma_k,\ 0<\varepsilon<1\}\leq C_k\sum_{j=1}^k|\varepsilon\tau|^j|\tau|^{j-k},\quad k\geq 1,$$

(5.12) $$\sup\{|e^{-\varepsilon\tau^2}p_k(t,\tau,s;x,\varepsilon)|;\ (x,t)\in U,\ s\in F,\ \text{Im}\,\tau<-\tilde\gamma_k,\ 0<\varepsilon<1\}\leq C_k\{1+(\varepsilon\gamma^2)^k)e^{\varepsilon\gamma^2}\}\ |\tau|^{-k},\quad k\geq 1.$$

Proof. (5.10) follows from (5.6) and (5.9), if we take $\tilde\gamma_1$ sufficiently large. (5.11) holds with $\tilde\gamma_k=\max\{\gamma_k, \tilde\gamma_1\}$, if we use (5.7), (5.8) and (5.10). In order to obtain (5.12), we employ the following inequality: For $\tau\in\{\tau:-\text{Im}\,\tau=\gamma>0\}$ it holds

(5.13) $$|e^{-\varepsilon\tau^2}(\varepsilon\tau^2)^j|\leq\begin{cases}4^j(j/e)^j, & \text{if } |\text{Re}\,\tau|\geq\sqrt{2}\,\gamma,\\ 4^j(\varepsilon\gamma^2)^je^{\varepsilon\gamma^2}, & \text{if } |\text{Re}\,\tau|<\sqrt{2}\,\gamma.\end{cases}$$

From (5.11) and (5.13) we have (5.12). Thus the proof is complete.

Now we can prove that the solution $u(x, t)$ in Proposition 5.1 has the property (P.1).

Proposition 5.3. *Assume* (*C*). *Suppose that the boundary data g belongs to* $\mathscr{S}'_{\gamma_0}$ *and satisfies* $\inf \operatorname{supp} g=s_0>-\infty$. *Then the solution* $u(x, t)$ *of* (P) *given by* (1.6) *vanishes in* $Z(s_0)\equiv\left\{(x, t): t<t_x(s_0)=s_0+\int_0^x\sqrt{a(y)}\,dy\right\}$.

Proof. In the expression (5.3) we can suppose that supp g_ε converges to supp g as ε tends to zero. Especially lim inf supp $g_\varepsilon=s_0$. Let (x_0, t_0) be an arbitrary point in $Z(s_0)$. Then there exists a positive number ε_0 such that for $F=\overline{\bigcup_{0<\varepsilon<\varepsilon_0}\operatorname{supp} g_\varepsilon}$ we have (5.9). Applying Proposition 5.2 we can find a neighbourhood U of (x_0, t_0) satisfying (5.10), (5.11) and (5.12). Especially we use (5.11) rewriting ε by ε_1. Now in (5.13) put $\varepsilon=\frac{1}{2}\varepsilon_1$, then we have (5.12) replaced ε, $e^{-\varepsilon\tau^2}$ and C^k respectively by ε_1, $e^{-\frac{1}{2}\varepsilon_1\tau^2}$ and $\tilde{C}_k$. From (5.3)′ and (5.8) it holds

$$(5.14)\qquad u(x, t)=\lim_{\varepsilon\to0}\lim_{\varepsilon_1\to0}\iint(e^{i\phi}e^{-\frac{1}{2}\varepsilon_1\tau^2})(e^{-\frac{1}{2}\varepsilon_1\tau^2}p_k)g_\varepsilon(s)ds\,d\tau,\ \gamma>\tilde{\gamma}_k,\ k\geq1,$$

for $(x, t)\in U$. Let us estimate the integrand of the right hand side of (5.14). Now it suffices to consider $\operatorname{Re}\left(i\phi-\frac{1}{2}\varepsilon_1\tau^2\right)$. From Proposition 3.2 we have

$$(5.15)\qquad \operatorname{Re}\left(i\phi-\frac{1}{2}\varepsilon_1\tau^2\right)\leq\gamma(t-t_x(s))+C\frac{x(|\sigma|+\gamma)}{\gamma}-\varepsilon_1\frac{(\sigma^2-\gamma^2)}{2}\leq\gamma(t-t_x(s))$$
$$+\frac{1}{2}(1+Cx)^2+\frac{1}{2}(\varepsilon_1\gamma^2-1)\left(1-\frac{\sigma^2}{\gamma^2}\right).$$

By virtue of (5.15) and the modified (5.12) we can see that it is suitable to put γ as

$$\gamma=\sqrt{\varepsilon_1}\,.$$

Then for $(x, t)\in U$, $\varepsilon\in(0, \varepsilon_0)$ and $s\in F$, there exists a constant $\tilde{\tilde{C}}_k$ such that it holds

$$|\text{integrand of (5.14)}|\leq\tilde{\tilde{C}}_k e^{\gamma(t-t_x(s))}|\tau|^{-k}|g_\varepsilon(s)|,\ k\leq3.$$

Remark $t-t_x(s)<-\delta<0$ in this case and make ε_1 tend to 0, then we see $u(x, t)=0$ in U. Thus the proof is complete.

Proposition 5.4. *Suppose the same conditions as in Proposition* 5.3. *Then* $u(x, t)$ *defined by* (1.6) *satisfies*

(5.16) $t_x(s)\notin$ *sing supp* $u(x, \cdot)$ *if and only if* $s\notin$ *sing supp* g, *for* $x\in(0, \infty)$.

Proof. Suppose $s\notin$ *sing supp* g. We decompose g in the following form: $g=g_1+g_2+g_3$, where $g_1\in C^\infty$, supp $g_1\subset(s-2\delta, s+2\delta)$, supp $g_2\subset(s+\delta, \infty)$ and supp $g_3\subset(-\infty, s-\delta)$ for some positive number δ. Denote by u_k the solution u defined by (1.6) with $g=g_k$, $k=1, 2, 3$. Remark $g_k\in\mathscr{S}'_{\gamma_0}$. It holds $u=u_1+u_2+u_3$. From

Proposition 5.1 u_1 is in C^∞ and Proposition 5.3 says that u_2 vanishes in a neighbourhood of $t_x(s)$. Remark $g_3\in\mathscr{E}'$ from our assumption inf supp $g>-\infty$. So we can write

$$g_3=\sum_{j=1}^{k}\frac{d^j}{dt^j}g_{3j},\ g_{3j}\in L^2\cap L^1,\ \text{supp } g_{3j}\subset(s_0-1,\ s-\delta).$$

Then we have $u_3=\sum_{j=1}^{k}u_{3j}$, where

$$u_{3j}=e^{\gamma t}\bar{F}[v(x,\ \tau)(i\tau)^jF[e^{-\gamma s'}g_{3j}(s')]]=\lim_{\varepsilon\to0}\iint e^{i\phi}e^{-\varepsilon\tau^2}(i\tau)^jg_{3j}(s')ds'd\tau.$$

Differentiate h-times in t, then it holds

$$\left(\frac{\partial}{\partial t}\right)^h u_{3j}(x,\ t)=\lim_{\varepsilon\to0}\iint e^{i\phi}q_{k,j+h}(t,\ \tau,\ s';\ x,\ \varepsilon)g_{3j}(s')ds'd\tau,\qquad h>0,\ k-j-h>1,$$

where $q_{k,j+h}(t,\ \tau,\ s';\ x,\ \varepsilon)=\left(-\frac{1}{i\phi_\tau}\frac{\partial}{\partial\tau}-\frac{\phi_{\tau\tau}}{i\phi_\tau^2}\right)^k(e^{-\varepsilon\tau^2}(i\tau)^{j+h})$. Similarly to the proof of (5.12), we have the following estimate; For any fixed x there exists a neighbourhood V of $t_x(s)$ such that for $k\geq0$ it holds

$$|q_{k,j+h}|\leq C_{k,j+h}\{(1+(\varepsilon\gamma^2))^{k+j+h}e^{\varepsilon\gamma^2}\}|\tau|^{-k+j+h},\qquad \gamma>\tilde{\gamma}_k,$$

where $C_{k,j+h}$ and $\tilde{\gamma}_k$ are certain positive constants. Therefore we can see that u_{3j} is infinitely differenciable in t in a neighbourhood of $t_x(s)$. Thus $s\notin$supp g implies $t_x(s)\notin$*sing supp* $u(x,\ \cdot)$ for all $(s,\ x)\in R\times(0,\ \infty)$. Similarly we can prove the converse, if we use the expression:

$$(5.17)\qquad g(s)=e^{\gamma s}\bar{F}\left[\frac{1}{v(x,\ \tau)}F[e^{-\gamma t}u(x,\ t)]\right]=\lim_{\varepsilon\to0}\iint e^{-i\phi}e^{-\varepsilon\tau^2}u_\varepsilon(x,\ t)dtd\tau,$$

where $u(x,\ t)$ satisfies lim $e^{-\gamma t}u_\varepsilon(x,\ t)=e^{-\gamma t}u(x,\ t)$ in $\mathscr{S}'$ and $u_\varepsilon(x,\ t)\in\mathscr{D}$ for fixed $x>0$.

Using Proposition 5.4 and the estimate (i) in Proposition 3.1 we can exhibit

Proposition 5.5. *Suppose the same conditions as in Proposition* 5, 3. *Then we have* (P.2) *in Inroduction.*

Proof. For any fixed $x\in(0,\ \infty)$ denote $u(x,\ t)$ simply by

$$u(x,\ t)=Tg=e^{\gamma t}\bar{F}[v(x,\ t)F[e^{-\gamma s}g(s)]].$$

The definition (1.8) and the estimate (i) in Proposition 3.1 yield

$$(5.18)\qquad \textit{Ord Sing}(e^{-\gamma t}\ Tg)=\textit{Ord Sing}(e^{-\gamma t}g),\ \text{for } g\in\mathscr{S}'_{\gamma_0} \text{ and } \gamma>\gamma_0.$$

Note $e^{-\gamma t}$ g simply by g_γ, then it holds

$$(5.19)\qquad \textit{Ord Sing }(g_\gamma;\ s)=\textit{Ord Sing }(g;\ s).$$

Let functions β_n be identically 1 in a neighbourhood of s, satisfying lim supp $\beta_n=\{0\}$. Put $g=\beta_ng+(1-\beta_n)g=g_{n,1}+g_{n,2}$. Proposition 5.4, (4.5), (4.7) and (5.18) yield

$$Ord\ Sing\ (Tg;\ t_x(s))=Ord\ Sing\ ((Tg_{n,1})_\gamma;\ t_x(s))\leq Ord\ Sing\ ((T\beta_n g)\gamma)$$
$$=Ord\ Sing\ ((\beta_n g)_\gamma),$$

which converges to *Ord Sing* $(g;\ s)$ in view of (5.19). Therefore we have

$$Ord\ Sing\ (u(x,\ \cdot);\ t_x(s))\leq Ord\ Sing\ (g;\ s),\ \text{for}\ x\in(0,\ \infty),\ s\in R.$$

If we take care (5.17) we can also prove the reverse inequality. Thus the proof of Proposition 5.5 is complete,

We obtain Theorem 3.1 and Corollary 3.1 from Propositions 3.1, 3.2, 5.1, 5.3 and 5.5.

§6. Proof of Lemma 3.1, Propositions 3.1 and 3.2.

Proof of Lemma 3.1. Let $\alpha(t)$ be a C^∞ function with value 1 in $(1,\ \infty)$ and 0 in $(-\infty,\ -1)$. Let γ belong to $(\gamma_1,\ \gamma_2)$, where $\gamma_1=\gamma_0+\varepsilon$ and $\gamma_2=\gamma_0+\frac{1}{\varepsilon}$ for any fixed $\varepsilon\in(0,\ 1)$. Put $g_1=\alpha(t)g$ and $g_2=(1-\alpha(t))g$. We consider $\hat{g}(\tau)=F[e^{-\gamma t}g(t)]$ corresponding to the decomposition:

$$e^{-\gamma t}g=e^{-(\gamma-\gamma_1)t}e^{-\gamma_1 t}g_1+e^{-(\gamma-\gamma_2)t}e^{-\gamma_2 t}g_2.$$

Remark $g_k\in\mathscr{S}'_{\gamma_0}$ and $e^{-\gamma_k t}g_k\in\mathscr{S}'$, $k=1,2$. By virtue of Riesz Theorem we have

$$e^{-\gamma_k t}g_k=\sum_{j=0}^{m}\partial_t^j(1+t^2)^{2m}\partial_t^j f_k,\ k=1,\ 2,$$

where $f_k\in L^2$, supp $f_1\subset(-2,\ \infty)$ and supp $f_2\subset(-\infty,\ 2)$. Rewrite this equality as

$$e^{-\gamma_k t}g_k=\sum_{j=0}^{2m}\partial_t^j(1+t^2)^{2m}f_{kj},$$

where f_{kj} have the same properties as f_k and satisfy

$$e^{-(\gamma-\gamma_k)t}(1+t^2)^{2m}f_{kj}\in L^1\ \text{for}\ \gamma\in(\gamma_1,\ \gamma_2),\ k=1,\ 2\ \text{and}\ j=1,\ 2,\dots,\ 2m.$$

Remark the relation:

$$F[e^{-(\gamma-\gamma_k)t}\partial_t^j(1+t^2)^{2m}f_{kj}]=\{i(\sigma-i\gamma)-\gamma_k\}^jF[e^{-(\gamma-\gamma_k)t}(1+t^2)^{2m}f_{kj}].$$

Then we can see that $\hat{g}(\tau)$ is analytic in $\{\tau:\ \mathrm{Im}\,\tau<-\gamma_0\}$.

Now suppose that $\hat{g}(\tau)$ satisfies (i) and (ii) in Lemma 3.1. Put $g_\gamma=e^{\gamma t}F[\hat{g}(\tau)]$ for $\gamma>\gamma_0$. Then it holds

$$\langle g_\gamma,\ \phi\rangle=\langle e^{-\gamma t}g_\gamma(t),\ e^{\gamma t}\phi(t)\rangle=\langle\hat{g}(\sigma-i\gamma),\ \hat{\phi}(-\sigma-i\gamma)\rangle,\ \text{for}\ \phi\in\mathscr{D}.$$

Since the last expression is independent of γ, we have $g(t)=g_\gamma(t)\in\mathscr{S}'_{\gamma_0}$. Thus the proof of Lemma 3.1 is complete.

Here we introduce a notation on projection mappings concerning oblique coordinates in complex plane C.

Notation. Let α and β be a pair of complex numbers satisfying $\mathrm{Im}(\alpha\bar{\beta})\neq 0$. Then any $w\in C$ can be represented as $w=P_{\alpha,\beta}(w)\alpha+Q_{\alpha,\alpha}(w)\beta$, where

$$P_{\alpha,\beta}(w)=\frac{\mathrm{Im}(w\bar{\beta})}{\mathrm{Im}(\alpha\bar{\beta})} \text{ and } Q_{\alpha,\beta}(w)=\frac{\mathrm{Im}(w\bar{\alpha})}{\mathrm{Im}(\beta\bar{\alpha})}.$$

$Q_{\alpha,\beta}(w)$ as well as $P_{\alpha,\beta}(w)$ satisfies

$$Q_{\alpha,\beta}(\sum r_k w_k)=\sum r_k Q_{\alpha,\beta}(w), \text{ for } r_k\in R,\ w_k\in C. \tag{6.1}$$

Especially we denote $Q_{\tau^2,-i\tau}(w)$, $\mathrm{Re}\,\tau\neq 0$, simply by $Q_\tau(w)$. Then it holds

$$Q_\tau(w)=\frac{-1}{(\mathrm{Re}\,\tau)|\tau|^2}\mathrm{Im}(w\bar{\tau}^2), \tag{6.2}$$

$$Q_\tau(\tau^2)=0,\ Q_\tau(-i\tau)=1 \text{ and } Q_\tau(1)=2(\mathrm{Im}\,\tau)/|\tau|^2. \tag{6.3}$$

Using the above notation we can state the following lemma, which is applied repeatedly in this section.

Lemma 6.1. *Suppose that* $w(x)$ *satisfies* $w'=q(x)-w^2$, $q(x)$ *being a continuous function defined on* $[0, \infty)$ *with values in C. Put* $v(x)=v_0 \exp\int_0^x w(y)\,dy$, $v_0\neq 0$, *which satisfies* $v''=q(x)\,v$ *and* $v(0)=v_0$. *For some* $\tau\in C$ *satisfying* $\mathrm{Re}\,\tau\neq 0$ and $\mathrm{Im}\,\tau<0$, *we assume* $Q_\tau(q(x))\leq 0$ *on* $[0, \infty)$ *and that* $w(x)$ *is bounded on* $[0, \infty)$. *Then* $\inf_x Q_\tau(w(x))>0$ *implies* $\lim_{x\to\infty} v(x)=0$ *and* $\sup_x Q_\tau(w(x))<0$ *implies* $\lim_{x\to\infty}|v(x)|=\infty$.

Proof. Integrate $v''\bar{v}=q(x)|v|^2$ by parts on $[0, x]$, then

$$w(x)|v(x)|^2-w(0)|v(0)|^2=\int_0^x|v'(y)|^2dy+\int_0^x q(y)|v(y)|^2dy. \tag{6.4}$$

Operate Q_τ to both side of (6.4), then

$$Q_\tau(w(x))|v(x)|^2-Q_\tau(w(0))|v(0)|^2=Q_\tau(1)\int_0^x|v'(y)|^2dy+\int_0^x Q_\tau(q(y))|v(y)|^2dy. \tag{6.5}$$

Remark $Q_\tau(1)<0$ from (6.3), $v'(y)=w(x)v(x)$ and $0<c_1<|w(x)|<c_2$ on $[0, \infty)$. Then we can see that (6.5) gives the desired results. This completes the proof of lemma 6.1.

Now we can show

Proposition 6.1. *Assume* (C_1). *Then the solution* $v(x, \tau)$ *of the problem* (1.5) *replaced* $\lim_{x\to\infty} v(x, \tau)=0$ *by* $\overline{\lim_{x\to\infty}}|v(x, \tau)|<\infty$ *is unique for* $\tau\in\left\{\tau\in C;\ -\mathrm{Im}\,\tau>\gamma_0=\frac{2M_1}{m^{3/2}}\right\}$.

Corollary 6.1. *Suppose* (C_1). *Then the solutions* $v(x, \tau)$ *of* (1.5) *and* $w(x, \tau)$ *of* (3.1) *are unique.*

Proof of Proposition 6.1. Denote by $\tilde{w}(x, \tau)$ the solution of $w'=-\tau^2 a(x)-w^2$ and $w(o, \tau)=i\tau\sqrt{a(0)}\ \notin\ \Omega(0, 1/r_1)$. Proposition 2.1 says $\tilde{w}(x, \tau)\notin\Omega(x, 1/r_1)$ for all $x>0$.

Namely we have from (1.3) and Example in Section 2

$$\left|\frac{\tilde{w}(x,\tau)+i\tau\sqrt{a(x)}}{\tilde{w}(x,\tau)-i\tau\sqrt{a(x)}}\right|\geq\frac{1}{r_1}\qquad \text{if } -\mathrm{Im}\tau=\gamma>\gamma_0,\ \text{where } r_1<\frac{\gamma_0}{8}\ \frac{1}{\gamma}. \tag{6.6}$$

From (6.6), similarly to (2.3) we have

$$|\tilde{w}(x,\ \tau)-i\tau\sqrt{a(x)}|\leq\frac{2}{7}\gamma_0\sqrt{a(x)}\frac{|\tau|}{\gamma},\qquad \text{for } \gamma>\gamma_0. \tag{6.6)'$$

Let us operate Q_τ to (6.6)′ taking account of (6.3) and $\max_{|w|\leq 1} Q_\tau(w)=\frac{1}{|\mathrm{Re}\,\tau|}$, then we have

$$-\frac{3}{2}\sqrt{a(x)}<Q_\tau(\tilde{w}(x,\ \tau))<-\frac{1}{2}\sqrt{a(x)},\qquad \text{if} |\mathrm{Re}\,\tau|\geq\gamma>\gamma_0. \tag{6.7}$$

Remark $Q_\tau(-\tau^2a(x))=0$. Then from Lemma 6.1 $\tilde{v}(x,\ \tau)$ satisfies $\lim_{x\to\infty}|\tilde{v}(x,\ \tau)|=\infty$ in this case if we put $\tilde{v}(x,\ \tau)=\exp\int_0^x\tilde{w}(y,\ \tau)\ dy$. If $|\mathrm{Re}\,\tau|<\gamma$, we take the real part of (6.6)′. Then it follows

$$|\mathrm{Re}\ \tilde{w}(x,\ \tau)-\gamma\sqrt{a(x)}|<\frac{2\sqrt{2}}{7}\gamma_0\sqrt{a(x)},\qquad \text{If } |\mathrm{Re}\,\tau|<\gamma,\quad \gamma>\gamma_0.$$

This implies $\sqrt{a(x)}\gamma/2<\mathrm{Re}\,\tilde{w}(x,\ \tau)<3\sqrt{a(x)}\gamma/2$, thus $v(x,\ \tau)>\exp\left(\frac{\sqrt{m}}{2}\gamma x\right)$ This completes the proof of proposition 6. 1.

As for the existence of solutions $v(x,\ \tau)$ of (1.5) and $w(x,\ \tau)$ of (3.1) we state

Proposition 6.2. *The solutions $w(x,\ \tau)$ of* (3.1) *and $v(x,\ \tau)$ of* (1.5) *exist uniquely and satisfy the following estimates for $x\in[0,\ \infty)$ and $-\mathrm{Im}\,\tau>\gamma_0$, if (C_1) is supposed.*

$$|w(x,\ \tau)+i\tau\sqrt{a(x)}|<\frac{2}{7}\ \frac{\gamma_0}{\gamma}|\tau|, \tag{6.8}$$

$$|v(x,\tau)|<3\sqrt{a(0)/a(x)}, \tag{6.9}$$

$$\frac{1}{|v(x,\ \tau)|^2}\int_x^\infty|v(y,\ \tau)|^2dy\ <\ 3\frac{\sqrt{a(x)}}{m}\ \frac{1}{\gamma}. \tag{6.10}$$

Proof. Apply Theorem 2.1 to the equation $w'=-\tau^2a(x)-w^2$ and denote by $w(x,\ \tau)$ the solution belonging to $\Omega(x,\ r_1)$ for all $x\geq 0$. Namely

$$|(w+i\tau\sqrt{a(x)})/(w-i\tau\sqrt{a(x)})|\leq r_1<\frac{\gamma_0}{8\gamma},\qquad \gamma>\gamma,\qquad x\geq 0. \tag{6.11}$$

Similarly to the proof of Proposition 6.1, we have (6.8) and

$$\sqrt{a(x)}|\tau|/2<|w(x,\ \tau)|<3\sqrt{a(x)}|\tau|/2, \tag{6.12}$$

$$\sqrt{a(x)}/2<Q_\tau(w(x,\ \tau))<3\sqrt{a(x)}/2,\qquad \text{if } |\mathrm{Re}\,\tau|\geq\gamma, \tag{6.13}$$

$$-3\sqrt{a(x)}\gamma/2<\mathrm{Re}\ w(x,\ \tau)<-\sqrt{a(x)}\gamma/2,\qquad \text{if } |\mathrm{Re}\,\tau|<\gamma. \tag{6.14}$$

From (6.4) and (6.14) we have (6.9) and (6.10) if $|\mathrm{Re}\,\tau|<\gamma$. In the case where $|\mathrm{Re}\,\tau|\geq\gamma$, we integrate by parts of $v''\bar{v}=-\tau^2a(x)|v|^2$ and operate Q_τ to obtain

$$(6.15)\quad Q_\tau(w(x_0,\tau))|v(x_0,\tau)|^2=Q_\tau(w(x,\tau))|v(x,\tau)|^2-Q_\tau(1)\int_{x0}^{x}|v'(y,\tau)|^2dy,$$

for any x_0 and x satisfying $0\leq x_0<x$. Here we have used $Q_\tau(\tau^2)=0$ in (6.3). Put $x_0=0$. Then (6.15) and (6.13) yield (6.9), because $Q_\tau(1)$ is negative. (6.10) holds from (6.15), (6.12), (6.13) and (6.3). The proof of Proposition 6.2 is complete.

As for the analyticity of $w(x,\tau)$ and $v(x,\tau)$ with respect to τ we state

Proposition 6.3. *Suppose* (C_1). *Then the solution* $w(x,\tau)$ *of* (3.1) *is analytic with repect to* τ *for every fixed* x. *The solution* $v(x,\tau)$ *of* (1.5) *has the same property.*

Proof. First we verify the continuity in τ of $w(x,\tau)$. Suppose $-\mathrm{Im}\,\tau_0>\gamma_0$ and $\lim\tau_j=\tau_0$. Since the set $\{w(0,\tau_j)\}$ is bounded, we can suppose that $w(0,\tau_j)$ converges to w_0 if we replace $\{\tau_j\}$ by a subsequence. Denote by $w_0(x,\tau_0)$ the solution of $w'=-\tau_0^2a(x)-w^2$ and $w(0)=w_0$. Put $v_0(x,\tau_0)=\int_0^x w_0(y,\tau_0)dy$. Then (6.9) yields

$$|v_0(x,\tau_0)|\leq 3M/m,\qquad \text{for all } x>0.$$

Proposition 6.1 says that $v_0(x,\tau_0)=v(x,\tau_0)$ for all $x\geq0$. Hence $w_0=w(x,\tau_0)$. Thus $w(x,\tau)$ is continuous in τ for all fixed $x\geq0$. Therefore $v(x,\tau)$ is also continuous in τ. In order to prove the analyticity, we consider $w_h(x,\tau)=(w(x,\tau+h)-w(x,\tau))/h$, where h is a complex number in a neighbourhood of the origin. From the equation

$$\frac{d}{dx}w_h=-(2\tau+h)a(x)-(w(x,\tau+h)+w(x,\tau))w_h,$$

it follows

$$(6.15)\qquad \frac{d}{dx}(v(x,\tau+h)v(x,\tau)w_h(x,\tau))=-(2\tau+h)v(x,\tau+h)v(x,\tau)a(x).$$

Integrate (6.15) from x to ∞ and tend h to zero. Then from Proposition 6.2 we have

$$\frac{\partial}{\partial\tau}w(x,\tau)=2\tau v(x,\tau)^{-2}\int_x^\infty a(y)v(y,\tau)^2dy.$$

Hence we have Proposition 6.3.

By virtue of Propositions 6.1, 6.2 and 6.3 the proof of Proposition 3.1 is complete if we verify the following

Proposition 6.4. *Suppose* (C_1). *Then the solution* $v(x,\tau)$ *of* (1.5) *satisfies*

$$(6.16)\quad |v(x,\tau)|>\frac{13}{5}\sqrt{a(0)/a(x)}\,e^{-\theta x},\ \textit{for } x>0,\ \gamma>\gamma_0,\ \textit{where } \theta=\frac{M_1+\sqrt{M_1^2+16M^3\gamma^2}}{2m}.$$

Proof. Let us describe in short $L=\frac{d^2}{dx^2}+\tau^2a(x)$ and $v_\theta=e^{\theta x}v$. Then $Lv=0$

means $(Lv)_\theta \equiv L_\theta v_\theta \equiv \left(\frac{d}{dx}-\theta\right)^2 v_\theta+\tau^2 a(x)v_\theta=0$. Then we have

$$I=2\mathrm{Re}\left\{(L_\theta v_\theta)\overline{\left(\frac{d}{dx}-\theta\right)v_\theta}\right\}=\left(\frac{d}{dx}-\theta\right)\{e^{2\theta x}J_\tau(x)\}-e^{2\theta}H_\tau(x)=0, \tag{6.17}$$

where $J_\tau(x)=|v'|^2+a(x)|\tau v|^2$, $H_\tau(x)=a'(x)|\tau v|^2-2\mathrm{Im}\{2\gamma a(x)\tau v\bar{v}'\}$.
Put $X=\sqrt{a(x)}\,\tau v$ and $Y=v'$, then $J_\tau(x)=X\bar{X}+y\bar{y}$ and

$$H_\tau(x)=\frac{a'}{a}X\bar{X}+2\gamma\sqrt{a(x)}\,(X\bar{Y}-Y\bar{X})=(XY)H\begin{pmatrix}\bar{X}\\ \bar{Y}\end{pmatrix},\quad H=\begin{pmatrix}a'/a & 2i\sqrt{a}\,\gamma\\ -2i\sqrt{a}\,\gamma & 0\end{pmatrix}.$$

Since the eigen-values of H are $(a'\pm\sqrt{a'^2+16a^3\gamma^2})/2a$, we have $\frac{d}{dx}\{e^{2\theta x}J_\tau(x)\}>0$ from (6.17) if we take θ as in (6.16). Hence it holds $J_\tau(x)>e^{-2\theta x}J_\tau(0)$, which yields (6.16) from (6.12). Thus the proof of Proposition 6.4 is complete.

Finally we show the following Propositions 6.5 and 6.6, which give Proposition 3.2.

Proposition 6.5. *Suppose* (C_{k_1}). *Then there exist positive constants* C_k *and* γ_k, $k=0, 1, 2,\dots, k_1-1$ *such that we have*

$$|\partial_x^k(w(x,\tau)+i\tau\sqrt{a(x)}\,)|\le C_k|\tau|/\gamma,\ \textit{for}\ x\in[0,\infty),\ -\mathrm{Im}\tau=\gamma>\gamma_k,\ 0\le k\le k_1-1. \tag{6.18}$$

Proposition 6.6. *Suppose that for any fixed integers* $k\ge 0$ *and* $h>0$ *there exist positive constants* C_{ij} *and* γ_{ij}, *for* $i=0, 1,\dots, k+1$, $j=0, 1,\dots, h-1$ *and also for* $i=0, 1,\dots, k-1$, $j=h$ *in the case of* $k\ge 1$, *satisfying*

$$|\partial_x^i\partial_\tau^j(w(x,\tau)+i\tau\sqrt{a(x)}\,)|\le C_{ij}|\tau|^{1-j}/\gamma,\ \textit{for}\ x\in[0,\infty),\ -\mathrm{Im}\tau=\gamma>\gamma_{ij}, \tag{6.19}$$

for all above (i, j). *Then we can find positive numbers* C_{kh} *and* γ_{kh} *satisfying* (6.19) *for* $i=k$ *and* $j=h$.

Proof of Proposition 6.5. For $k=0$ (6.18) corresponds to (6.8). Suppose that (6.18) is true for $k=0, 1, 2,\dots, j-1\le k_1-2$. For simplicity let us put

$$\begin{cases} w_-=w_-(x,\tau)=-i\tau\sqrt{a(x)}\,, & W=W(x,\tau)=w(x,\tau)-w_-(x,\tau).\\ w_-^{(k)}=\partial_x^k w_- &, W^{(k)}=\partial_x^k W\end{cases} \tag{6.20}$$

Then we can use

$$|w_-^{(k)}|\le C_k|\tau|,\ \text{for}\ x\in[0,\infty),\ -\mathrm{Im}\tau>\gamma_0,\ k\ge 0, \tag{6.21}$$

$$|W^{(k)}|\le C_k\frac{|\tau|}{\gamma},\ \text{for}\ x\in[0,\infty),\ -\mathrm{Im}\tau>\gamma_k,\ 0\le k\le j-1. \tag{6.22}$$

From the equation $w'=-\tau^2a(x)-w^2$ it follows

$$\frac{d}{dx}W^{(j)}=-\partial_x^j(w^2-w_-^2)-\partial_x^{j+1}w_-. \tag{6.23}$$

Using $w^2-w_-^2=W^2+2Ww_-$ we can represent $\partial_x^j(w^2-w_-^2)$ as follows.

$$\partial_x^j(w^2-w_-^2)=2wW^{(j)}+Q_j+R_j, \tag{6.24}$$

where Q_j does not contain $W^{(j)}$ and R_j excludes both $W^{(j)}$ and $W^{(j-1)}$. More concretely from

$$\partial_x^j(W^2)=\sum_{k=0}^{j}{}_jC_k W^{(k)}W^{(j-k)}=2(WW^{(j)}+C_{j1}W^{(1)}W^{(j-1)})+\sum_{k=2}^{j-2}{}_jC_k W^{(k)}W^{(j-k)},$$

$$\partial_x^j(2Ww_-)=2(W^{(j)}w_-+{}_jC_1W^{(j-1)}w_-^{(1)})+2\sum_{k=2}^{j}{}_jC_k W^{(k)}w_-^{(k)},$$

we get

$$(6.25)\qquad \begin{cases} Q_1=2Ww_-^{(1)},\ Q_2=(2W^{(1)}+4w_-^{(1)})W^{(1)},\ Q_3=6(W^{(1)}+w_-^{(1)})W^{(2)},\\ Q_j=2j(W^{(1)}+w_-^{(1)})W^{(j-1)},\ j\geq 4,\end{cases}$$

$$(6.26)\qquad \begin{cases} R_1=0,\ R_2=2Ww_-^{(2)},\ R_3=2(3W^{(1)}w_-^{(2)}+Ww_-^{(3)})\ \text{and}\\ R_j=\sum_{k=2}^{j-2}{}_jC_k(W^{(k)}+2w_-^{(k)})W^{(j-k)}+2\sum_{k=j-1}^{j}{}_jC_kw_-^{(k)}W^{(j-k)},\ j\geq 4.\end{cases}$$

Now put

$$(6.27)\qquad \tilde{R}_j=R_j+\partial_x^{j+1}w_-.$$

Then the equation (6.23) is written in the form

$$(6.23)'\qquad \frac{d}{dx}W^{(j)}=-2wW^{(j)}-Q_j-\tilde{R}_j.$$

Taking account of $\frac{d}{dx}v^2=2wv^2$, $\lim_{x\to\infty}v(x,\tau)=0$ and (ii) in Proposition 3.1, we have

$$(6.28)\qquad W^{(j)}(x,\tau)=\frac{1}{v(x,\tau)^2}\int_x^\infty v(y,\tau)^2(Q_j+\tilde{R}_j)dy,$$

since (6.21), (6.22), (6.25), (6.26) and (6.27) yield

$$(6.29)\quad |Q_j|\leq C_j|\tau|^2/\gamma,\ |\tilde{R}_j|\leq C_j|\tau|^2/\gamma,\ \text{for } x\in[0,\infty)\ \text{and}\ \gamma>\tilde{\gamma}_{j-1}=\max_{0\leq k\leq j-1}\gamma_k.$$

Hereafter for convenience we denote by C_j various constants depending on j and $a(x)$. Substitute $v(y,\tau)^2=\frac{1}{2w(y,\tau)}\partial_y v(y,\tau)^2$ into (6.28) and integrate by parts to have

$$(6.30)\qquad W^{(j)}(x,\tau)=-\frac{(Q_j+\tilde{R}_j)}{2w}-\frac{1}{v(x,\tau)^2}\int_x^\infty v(y,\tau)^2\partial_y\Big(\frac{Q_j}{2w}+\frac{\tilde{R}_j}{2w}\Big)dy.$$

From (6.12), (6.21), (6.22), (6.25) and (6.26) the first term of the right hand side of (6.30) is estimated as follows:

$$(6.31)\qquad \left|\frac{Q_j+\tilde{R}_j}{2w}\right|\leq C_j\frac{|\tau|}{\gamma},\ \text{for } x\in[0,\infty),\ -\mathrm{Im}\,\tau>\tilde{\gamma}_{j-1}.$$

As for the integrand of the second term of (6.30) we have

$$\partial_y\Big(\frac{\tilde{R}_j}{2w}\Big)=\frac{\partial_y\tilde{R}_j}{2w}-\frac{(W^{(1)}+w_-^{(1)})}{2w^2}\tilde{R}_j,$$

$$\partial_y\left(\frac{Q_j}{2w}\right)=-\frac{(W^{(1)}+w_-^{(1)})}{2w^2}Q_j+\frac{\partial_y Q_j}{2w}.$$

Similarly to (6.31) the following estimates hold:

(6.32) $$\left|\partial_y\left(\frac{\tilde{R}_j}{2w}\right)\right|\le C_j\frac{|\tau|}{\gamma},\quad \left|\frac{Q_j(W^{(1)}+w_-^{(1)})}{2w^2}\right|\le C_j\frac{|\tau|}{\gamma},\ \text{for } x\in[0,\infty),\ -\text{Im}\,\tau>\gamma_{j-1}.$$

We rewrite $\frac{\partial_y Q_j}{2w}$ as

$$\frac{\partial_y Q_j}{2w}=r_jW^{(j)}+\tilde{r}_j,$$

where r_j and $\tilde{r}_j$ do not contain $W^{(j)}$ and from (6.25) they satisfy

(6.33) $$|r_j|\le C_j,\ |\tilde{r}_j|\le C_j\frac{|\tau|}{\gamma},\ \text{for } x\in[0,\infty),\ -\text{Im}\,\tau>\tilde{\gamma}_{j-1}.$$

Now we put

(6.34) $$\partial_y\left(\frac{Q_j}{2w}+\frac{\tilde{R}_j}{2w}\right)=r_jW^{(j)}+\tilde{\tilde{R}}_j$$

Then it holds

(6.35) $$|\tilde{\tilde{R}}_j|\le C_j\frac{|\tau|}{\gamma},\ \text{for } x\in[0,\infty),\ -\text{Im}\,\tau>\tilde{\gamma}_{j-1}.$$

Substitute (6.31), (6.34) and (6.35) to (6.30) and remark that

(6.36) $$K_j(x,\tau)=\sup_{x\le y}|W^{(j)}(y,\tau)|$$

is a monotone function in x, then it follows

$$K_j(x,\tau)\le C_j\frac{|\tau|}{\gamma}+\left(K_j(x,\tau)+C_j\frac{|\tau|}{\gamma}\right)\frac{1}{|v(x,\tau)|^2}\int_x^\infty|v(y,\tau)|^2dy.$$

Here we use (ii) in Proposition 3.1. Then we see that

$$K_j(x,\tau)\le\tilde{C}\frac{|\tau|}{\gamma},\ \text{for } x\in[0,\infty),\ -\text{Im}\,\tau=\gamma>\tilde{\gamma},$$

holds for sufficiently large $\tilde{C}$ and $\tilde{\gamma}$. Therefore we have (6.18) for $k=j$. Thus step by step we can prove (6.18) for any $j\le k_1-1$.

Proof of Proposition 6. 6. In order to calculate the right hand side of

(6.37) $$\frac{d}{dx}\partial_\tau^h\partial_x^k(w-w_-)=-\partial_\tau^h\partial_x^k(w^2-w_-^2)-\partial_\tau^h\partial_x^{k+1}w_-,$$

we use $w^2-w_-^1=W^2+2w_-W$ and

$(6.38)_1$ $$\partial_\tau^h\partial_x^k(W^2)=\sum_{i=0}^k\sum_{j=0}^h {}_kC_i\ {}_hC_j\partial_\tau^jW^{(i)}\partial_\tau^{h-j}W^{(k-i)},$$

$(6.38)_2$ $$\partial_\tau^h\partial_x^k(w_-W)=\sum_{i=0}^k\sum_{j=0}^h {}_kC_i\ {}_hC_j\partial_\tau^jw_-^{(i)}\partial_\tau^{h-j}W^{(k-i)}.$$

We can denote

$$(6.39)\qquad \partial_\tau^h\partial_x^k(w^2-w_-^2)=2w\partial_\tau^h W^{(k)}+Q_{kh}+R_{kh},$$

where Q_{kh} contains $\partial_\tau^h W^{(k-1)}$ and R_{kh} involves neither $\partial_\tau^h W^{(k)}$ nor $\partial_\tau^h W^{(k-1)}$. Denote

$$Q_{kh}=r_{kh}\partial_\tau^h W^{(k-1)},\quad \tilde{R}_{kh}=R_{kh}+\partial_\tau^h\partial_x^{h+1}w_-.$$

Then there exists a positive constant C_k such that we have

$$(6.40)\qquad |r_{kh}|\leq C_{kh},\ |\tilde{R}_{kh}|\leq C_{kh},\qquad \text{for } x\in[0,\infty),\ -\mathrm{Im}\ \tau>\tilde{\gamma}_{kh}=\max_{(i,j)\in S_{kh}}\gamma_{ij},$$

where S_{kh} is the set of all pair of integers belonging to

$$\{[0,\ k+1]\times[0,\ h-1]\}\cup\{[0,\ k-1]\times\{h\}\}.$$

For convenience let us write r_{kh} precisely using the notation (6.20)

$$(6.41)\qquad \begin{cases} r_{0h}=0,\ r_{1h}=2(W^{(1)}+w^{(1)}), \\ r_{kh}=2k(W^{(1)}+w_-^{(1)}), \qquad \text{for } k\geq 2. \end{cases}$$

Similarly to (6.28) and (6.30) we have from (6.37), (6.38), (6.39) and (6.40)

$$(6.42)\qquad \partial^h W^{(k)}=-\frac{Q_{kh}+\tilde{R}_{kh}}{2w}-\int_x^\infty v(y,\ \tau)^2\left\{\frac{r_{kh}}{2w}\partial_\tau^h W^{(k)}+\tilde{\tilde{R}}_{kh}\right\}dy,$$

where $\tilde{\tilde{R}}_{kh}=\partial_y\left(\frac{r_{kh}}{2w}\right)\partial_\tau^h W_{(k-1)}+\partial_y\left(\frac{\tilde{R}_{kh}}{2w}\right)$. Similarly to (6.31) and (6.35) we have

$$(6.43)\qquad \begin{cases} \left|\frac{Q_{kh}+\tilde{R}_{kh}}{2w}\right|\leq C_{kh}\frac{|\tau|^{1-h}}{\gamma},\ \left|\frac{r_{kh}}{2w}\right|\leq C_{kh} \text{ and} \\ |\tilde{\tilde{R}}_{kh}|\leq C_{kh}(|\tau|^{1-h}/\gamma),\ \text{for } x\in[0,\ \infty),\ -\mathrm{Im}\,\tau>\tilde{\gamma}_{kh}. \end{cases}$$

Here C_{kh} stands for suitable positive constants depending on k and h. Now put

$$K_{kh}(x,\ \tau)=\sup_{x\leq y}|\partial_\tau^h W^{(k)}(y,\ \tau)|.$$

Then we have from (6.42) and (6.43)

$$K_{kh}(x,\ \tau)\leq C_{kh}\frac{|\tau|^{1-h}}{\gamma},\qquad \text{for } x\in[0,\ \infty),\ -\mathrm{Im}\tau>\gamma_{kh},$$

if we choose positive constants C_{kh} and γ_{kh} sufficiently large. Thus we have Proposition 6.6.

Appendix.

Here we give some comments on the order of singularity at $(x,\ \xi^0)$ and its historical back-ground. First we remark that the following theorem holds similarly to Theorem 4.1.

Theorem A. *Let* $f\in\mathscr{D}'(R^n)$, $n\geq 1$. *Then Ord Sing* $(f; x, \xi)$ *defined on* $R^n\times S^{n-1}$ *satisfies the following* (A.1) ~ (A.3).

(A.1) Tan^{-1} (*Ord Sing* $(f; x, \xi)$) *is an upper semi-continuous function on* $R^n\times S^{n-1}$,

(A.2) $$\textit{Ord Sing}\ (f;\ x)=\max_{\xi\in S^{n-1}} \textit{Ord Sing}\ (f;\ x,\ \xi),$$

(A.3) $$WF(f)=\overline{\{(x,\ \xi):\ \textit{Ord Sing}\ (f;\ x,\ \xi)>-\infty\}},\ \textit{for}\ f\in\mathscr{D}'.$$

Remark A.1. For $f\in\mathscr{D}'(R^n)$, let us denote

$$CR(f)_{mic}=\{(x,\ \xi)\in R^n\times S^{n-1}:\ (x,\ \xi)\in WF(f),\ \textit{Ord Sing}\ (f;\ x,\ \xi)=-\infty\}.$$

Then as in Example 4.2, there are examples of $f\in\mathscr{D}'$ satisfying $CR(f)_{mic}\neq\phi$.

In order to discuss further *Ord Sing* $(f; x, \xi)$ we use

Notation. Let us denote $\alpha(x, \xi)\in\tilde{S}^0_{1,0}$, if $\alpha(x, \xi)\in S^0_{1,0}$ is homogeneous in ξ, for $|\xi|\geq 1$ and supp $\alpha(x, \xi)|_{R^n\times S^{n-1}}$ is compact.

Remark A.2. For $f\in\mathscr{D}'(R^n)$ *Ord Sing* $(f; x_0, \xi^0)$ is equal to

(A.4) $$\lim_{k\to\infty} \textit{Ord Sing}\ (\alpha_k^* f)=\inf_{\alpha\in\tilde{S}^0_{1,0},\ \alpha(x_0,\ \xi^0)\neq 0} \textit{Ord Sing}(\alpha^* f),$$

where $\alpha_k\in S^0_{1,0}$, $\alpha_k(x, \xi^0)\neq 0$, $\lim_{k\to\infty} \operatorname{supp}_{|\xi|=1} \alpha_k=(x_0, \xi^0)$ and $\alpha^* f$ is defined by

$$\alpha^* f=\frac{1}{(2\pi)^{n/2}}\iint e^{i(x-y)\xi}\alpha(y,\ \xi)f(y)dyd\xi,$$

this integral being interpreted as an oscillatory integral as in (1,10). Moreover we have for $f\in\mathscr{E}'$

(A.5) $$\textit{Ord Sing}\ (f,\ x_0,\ \xi^0)=\lim_{k\to\infty} \textit{Ord Sing}\ (\alpha_k(x,\ D)f=\inf_{\alpha\in\tilde{S}^0_{1,0},\ \alpha(x_0,\ \xi^0)\neq 0} \textit{Ord Sing}\ (\alpha(x,\ D)f)$$

where $\alpha_k(x, D)$ are the usual pseudo-differencial operators with the same symbols.

We shall prove elsewhere Remark A.2 and other properties of *Ord Sing* $(f; x, \xi)$ together with suitable applications.

Finally we give a historical comment on the micro-localization and the order of singularity in order to clarify the viewpoint of Definition 4.2 and Remark A.2, although it is limited to author's knowledge and references are not complete in any sense. In Mizohata [9] the following micro-localizer $\beta_n(D)\alpha(x)$ was introduced in order to discuss the necessary condition for Cauchy problem to be well-posed. $\alpha(x)$ is a function in C_0^∞ with support contained in $\{x;\ |x-x_0|\leq r_0\}$ and 1 on $\{x;\ |x-x_0|\leq r_0/2\}$. $\alpha(\xi)$ is another function of the same type with its center x_0 replaced by ξ^0, and $\alpha_n(\xi)=\alpha(\xi/n)$. It is remarkable that not only $\beta_n(D)\ \alpha(x)$ but also $\beta_n^{(p)}(D)\alpha_{(q)}(x)$ are used in [9] to prove Lax-Mizohata theorem. Hörmander [4] used pseudo-differential operators as localizers in order to define the wave front set, whose projection to R^n is the singular support. This micro-localization makes it clear to discuss the propagations of singularities of solutions to equations with variable coefficients. Now we can observe that the notion of the order of singularity exists surely behind that of the singular support, since sing supp $f \not\ni x_0$ means that for any $N>0$ there exist C_N and $a(x)\in C_0^\infty$, $a(x)\neq 0$ satisfying $|\widehat{af}|\leq C_n(1+|\xi|)^N$. As *sing supp f* is the compliment set of an open set indicated above, there appears a little difference between *sing supp f* and the set

$\{x: f(x)$ is not infinitely differenciable at $x\}=\{x:$ *Ord Sing* $(f;\ x)>-\infty\}$. The former is strictly larger than the latter if $CR(f)\neq\phi$. However from the proof of Proposition 5.5 we can guess that the notion of the singular support is indispensable in order to discuss the order of singularities for hyperbolic equations with variable coefficients. On the other hand as for equations with constant coefficients, the fundamental solution for Cauthy problem were considered by many authors, relationg to the order of singularity. For example Duff [2] showed the order of singularities of fundamental solution in a neighbourhood of the wave surface. Though the exact definition of the order is not given in [2], we can recognize that the order of $\delta^{(n-1)}$ and $v.p.\ \frac{1}{x^n}$ in R^1 are said in [2] both to be $n-1$, which coincides the order given in Definition 4.1. (Remark that the order of $v.p.\frac{1}{x}$ as distribution is not zero.) Tsuji [12] developed the argument in [2] and introduced the definition of the order of singularity in the following L^2 sence:

$$\text{(A.6)}\qquad s.o.(f;\ x_0)=-\lim_{\varepsilon\to 0}\ \text{supp}\ \{o(u\phi);\phi\in C_0^\infty(x,\ \varepsilon)\},\ \text{for}\ f\in\mathscr{D}'(R^n),$$

where $o(v)=\text{supp}\ \{k;\ v(x)\in H^k(R^n)\}$ and $C_0^\infty(x,\ \varepsilon)=\{\phi\in C_0^\infty;\ \text{supp}\ \phi\subset\{x;\ |x-x_0|<\varepsilon\},\ \phi(x_0)\neq 0\}$. Now we notice that the above $s.o.(f;\ x)$ is equivalent to

$$\text{(A.6)}'\qquad s.o.(f;\ x_0)=\inf\{-r;\ \phi f\in H^r,\ \phi\in C_0^\infty(R^n),\phi(x_0)\neq 0\},$$

since it follows that $u\in H^r(R^n)$ implies $\phi u\in H^r(R^n)$ for $\phi\in C_0^\infty$. As L^2 is transformed isometrically to the dual L^2 space, (A.6) can be extended in the micro-local sence. Bony tried to considered it in the following way: u is micro-locally of class H^2 at the point $(x_0,\ \xi^0)$ if there exists a classical pseudo-differential operator R of order zero whose principal symbol is non zero at $(x_0,\ \xi^0)$ such that $Ru\in H^s$. Similarly to the proof of Remark A.2 we will be able to show that

$$\text{(A.7)}\qquad s.o.(f;\ x,\ \xi_0)=\inf\{-s;\ Ru\in H^s,\sigma(R)\in\tilde{S}^0_{1,0},\ \sigma(R)(x_0,\ \xi^0)\neq 0\}$$

is well-defined and has the similar properties to those of *Ord Sing* $(f;\ x_0,\ \xi^0)$. It seems to the author that *Ord Sing* $(f;\ x_0,\ \xi^0)$ is more natural in the sence that it can be directly related to the properties of C^k as was seen in (S.2) and (S.3) in Introduction. However it is the problem to be solved that will discuss the relative merits.

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