J. Math. Kyoto Univ. (JMKYAZ)
27-4 (1987) 621-633

On the law of entropy increasing of a one-dimensional infinite system

By

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Introduction.

It has been, and is, one of the most fundamental problems of statistical mechanics how one can "explain" the irreversible behaviour of macroscopic system from the reversible mechanical model [1]. The law of entropy increasing is such a typical problem. As is well known, one of the serious conflicts comes from the fact that entropy is invariant under the velocity reversal mapping and microscopic dynamics is time reversible. This conflict cannot resolve even if we assume that macroscopic states are represented by the ensembles, as is usually done, when we require that entropy should increase monotonically instead asymptotically.

We gave one of the possibilities of mathematically rigorous explanation to this problem in the paper [3].

In this paper we consider a one-dimensional hard-points system with several colors whose particles have integral positions and velocities v of unit magnitude $|v|=1$[2]. We show that under some conditions entropy increases for the initial states which have no spatial correlation, that is, the states which are represented by direct product probability measures on the phase space. It should be mentioned that it is impossible to assert that entropy increases for all initial states. We mention also that the densities of the particles with the same color of our system obey the heat equation and our entropy coincides with the usual Boltzmann entropy, when we take a hydrodynamic limit. Our system can be interpreted as a quotient system of a two-dimensional "hydro-dynamical" system [2, 5]. So our results can be interpreted as the results of special form for a two-dimensional hydrodynamical system. For example, above mentioned result says that the hydrodynamical limit of our densities (which are essentially one-dimensional in the sens that they are homogeneous in one direction) represents the field of the velocities of the flow and obey the Navior-Stokes equation.

In section 1, we describe the model in detail. In section 2, we define entropies of various types and related concepts. In section 3 we give our main results. In the last section, we give the proofs of the main results.

The author wish to acknowledge the hospitality of Universität München,

Received and communicated by Prof. Yoshizawa, on May 28, 1986 where much of the manuscript was developed. He also wish to express his sincere gratitude to Prof. H. Spohn and Dr. Mizutani for fruitful and friendly discussions.

1. Description of dynamical system.

1.1. Let $\boldsymbol{Z}$ be the set of all integers: $\boldsymbol{Z}=\{\cdots, -1, 0, 1, 2, \cdots\}$, and S be a "color" space of different k ($k\geqq 2$) colors:

$$S=\{\emptyset, c_1, c_2, \cdots, c_k\}.$$

Let

$$X=\{\omega ; \omega : Z\to S\times S\}.$$

We write

$$\omega(n)=(\omega(n, -), \omega(n, +))\in S\times S \qquad (n\in \boldsymbol{Z}).$$

Then X can be identified with the product space

$$X=\prod_{n\in \boldsymbol{Z}} X_n, \qquad \text{where} \quad X_n=S_n^-\times S_n^+,\ S_n^-=S_n^+=S$$

$\omega\in X$ represents a configuration of particles with colors $c_1, c_2, \cdots, c_k$ on the one-dimensional lattice $\boldsymbol{Z}$. More exactly, $\omega(n, -)$ represent that there exists a particle with color $\omega(n, -)\in S$ on the site $n\in \boldsymbol{Z}$ which has velocity -1, if $\omega(n, -)\neq\emptyset$. If $\omega(n, -)=\emptyset$, then this means that there exists no particle with velocity -1 on the site n. Similarly $\omega(n, +)$ represents the color of the particle with velocity $+1$ on the site n, if $\omega(n, +)\neq\emptyset$. And if $\omega(n, +)=\emptyset$ then there exists no particle with velocity $+1$ on the site n.

We call the X phase space of the system.

1.2. The time evolution mapping T on X of our system is defined as follows:

$$T: X\longrightarrow X$$

is made up of the "free motion" T_0 of X and the "collision" C:

$$T=CT_0.$$

Free motion T_0 is merely a translation of X:

$$(T_0\omega)(n, -)=\omega(n+1, -),$$

$$(T_0\omega)(n, +)=\omega(n-1, +).$$

Collision C is defined as follows:

$$((C\omega)(n, -), (C\omega)(n, +))=\begin{cases}(\omega(n, +), \omega(n, -)) & \text{if} \quad \omega(n, -)\neq\emptyset \ \& \ \omega(n, +)\neq\emptyset \\ (\omega(n, -), \omega(n, +)) & \text{otherwise}.\end{cases}$$

We remark that time evolution defined by T is not exactly time-reversible but is "essentially" time-reversible. Namely, let R be the velocity reversal mapping of X:

$$((R\omega)(n, -), (R\omega)(n, +))=(\omega(n, +), \omega(n, -)).$$

Then T does not satisfy the following relation which represents the time-reversibility:

$$RTR=T^{-1}.$$

But we can get following

Proposition 1. *$R'TR'=T^{-1}$, where $R'=RT_0$.*

This proposition can be proven from the following two lemmas which follow easily from the definitions.

Lemma 1. $RT_0R=T_0^{-1}$

Lemma 2. $CR=RC$

Proposition 1 means that $T'=T_0T=T_0CT_0$ is a time-reversible mapping. In the following we can discuss and get similar results for the T' in place of T, but we do not discuss for the T' for the simplicity.

1.3. We represent by $\mathfrak{M}$, the set of states of the dynamical system (X, T), namely $\mathfrak{M}$ is the set of probability measures on X. We call $\mathfrak{M}$ state space of (X, T).

Let

$$\bar{\mathfrak{M}}=\{\mu\in\mathfrak{M}\,;\,\mu=\bigotimes_{n\in\boldsymbol{Z}}(\mu_n^-\times\mu_n^+)\}$$

where μ_n^ε is a probability measure on S_n^ε ($\varepsilon=\pm$), namely, $\bar{\mathfrak{M}}$ is the space of direct product probability measures on $X=\prod_{n\in\boldsymbol{Z}}(S_n^-\times S_n^+)$. The elements of $\bar{\mathfrak{M}}$ can be considered as "locally equilibrium states" on X. So we call $\bar{\mathfrak{M}}$ locally equilibrium state space.

2. Definition of entropies.

2.1. In the following we use following notations:

Definition 1. For $\mu\in\mathfrak{M}$, $c\in S$, $\varepsilon=\pm$, n, $m\in\boldsymbol{Z}$,

$$P_\varepsilon^c(n, m)=P_\varepsilon^c(n, m\,;\,\mu)=\mu\{\omega\,;\,(T^m\omega)(n, \varepsilon)=c\},$$

$$d_\varepsilon(n, m)=d_\varepsilon(n, m\,;\,\mu)=\sum_{c\neq\emptyset}P_\varepsilon^c(n, m\,;\,\mu)=1-P_\varepsilon^\phi(n, m\,;\,\mu)$$

(Briefly $P_\varepsilon^c(n, 0)=P_\varepsilon^c(n)=P_\varepsilon^c(n\,;\,\mu)$

$d_\varepsilon(n, 0)=d_\varepsilon(n)=d_\varepsilon(n\,;\,\mu)$.)

Definition 2. $\mu\in\bar{\mathfrak{M}}$ is said to be of constant density, iff $d_\varepsilon(n\,;\,\mu)$ are independent of $n\in \boldsymbol{Z}$ for both $\varepsilon=\pm$ respectively, in this case we write

$$d_{\varepsilon}(n\,;\,\mu)=d_{\varepsilon}=d_{\varepsilon}(\mu)\,.$$

The set of the $\mu\in\bar{\mathfrak{M}}$ of constant density is denoted by $\bar{\mathfrak{M}}_{\text{const}}$.

Now we define two types of entropies and a related concept

Definition 3. Let

$$h(\mu):=\lim_{N\to\infty}\frac{-1}{2N+1}\sum_{A\in\mathfrak{A}_N}\mu(A)\log\mu(A)\,,$$

wher $\mathfrak{A}_N$ is the partition of X on $[-N, N]$, i.e. $\mathfrak{A}_N$ is a partition into the sets A of following form:

$$A=\{\omega\,;\,\omega(n,\,\varepsilon)=c(n,\,\varepsilon\,;\,A)\quad\text{for}\quad\varepsilon=\pm\ \text{and}\ -N\leqq\forall n\leqq N\},$$

here $c(n,\,\varepsilon\,;\,A)\in S$.

We call $h(\mu)$ (if exists) K-S type entropy of $\mu\in\mathfrak{M}$.

Definition 4. Let for $c\in S$

$$H^c(\mu):=\lim_{N\to\infty}\frac{-1}{2N+1}\sum_{\substack{|n|\leq N\\ \varepsilon=\pm}}P_{\varepsilon}^{c}(n\,;\,\mu)\log P_{\varepsilon}^{c}(n\,;\,\mu)$$

and $H(\mu):=\sum_{c\in S}H_c(\mu)$

We call $H(\mu)$ and $H_c(\mu)$ $(c\in S)$ Boltzmann type entropy of μ and Boltzmann type entropy of μ w.r.t. $c\in S$ (if exists) respectively.

Proposition 2. *For $\mu\in\bar{\mathfrak{M}}$, we have*

$$h(\mu)=H(\mu)\,.$$

Definition 5. Let

$$I(\mu):=-h(\mu)+H(\mu)$$

We call $I(\mu)$ information of inner structure of $\mu\in\mathfrak{M}$ or complexity of hidden structure of μ.

$I(\mu)$ measures a degree of the strength of correlation between $\omega(n,\,\varepsilon)$ and $\omega(n',\,\varepsilon')$ with respect to the μ.

By this definition, proposition 2 can be restated as,

Proposition 2′. $I(\mu)=0$ *for* $\mu\in\bar{\mathfrak{M}}$

Proof of proposition 2. For

$$A=\{\omega\,;\,\omega(n,\,\varepsilon)=c(n,\,\varepsilon\,;\,A),\ \varepsilon=\pm,\ |n|\leqq N\},\ \mu\in\bar{\mathfrak{M}}$$

$$\mu(A)=\prod_{\varepsilon=\pm,\,|n|\leq N}P_{\varepsilon}^{c(n,\,\varepsilon;\,A)}(n\,;\,\mu)\,.$$

Therefore

$$\sum_A \mu(A)\log\mu(A)=\sum_A\Big\{\prod_{\substack{\varepsilon=\pm\\|n|\leq N}} P_\varepsilon^{c(n,\varepsilon;A)}(n)\sum_{\substack{\varepsilon=\pm\\|m|\leq N}}\log P_{\varepsilon'}^{c,(m,\varepsilon';A)}(m)\Big\}$$

$$=\sum_A\sum_{\substack{\varepsilon'=\pm\\|m|\leq N}}\log P_{\varepsilon'}^{c,(m,\varepsilon';A)}(m)\cdot P_{\varepsilon'}^{c,(m,\varepsilon';A)}\prod_{(n,\varepsilon)\neq(m,\varepsilon')}P^{c(n,\varepsilon;A)}(n)$$

$$=\sum_{\substack{\varepsilon'=\pm\\|m|\leq N}}\Big(\sum_{c\in S}(\log P_{\varepsilon'}^{c}(m)\cdot P_{\varepsilon'}^{c}(m)\Big)\Big(\sum_{A'\in\mathfrak{A}_N:\,c(m,\varepsilon';A')=c}\;\prod_{(n,\varepsilon)\neq(m,\varepsilon')}P^{c(n,\varepsilon;A')}(n)\Big)\Big)$$

$$=\sum_{\substack{\varepsilon'=\pm\\|m|\leq N}}\sum_{c\in S}P_{\varepsilon'}^{c}(m)\log P_{\varepsilon'}^{c}(m).$$

Hence

$$h(\mu)=\lim_{N\to\infty}\frac{-1}{2N+1}\sum_{c\in S}\sum_{\substack{\varepsilon=\pm\\|n|\leq N}}P_\varepsilon^c(n)\log P_\varepsilon^c(n)$$

$$=\sum_{c\in S}H_c(\mu)=H(\mu). \qquad \text{Q.E.D.}$$

3. Main results.

3.1. Now we can state our first main result.

Theorem 1. *("Entropy increasing law") Assume that $\mu\in\bar{\mathfrak{M}}_{\rm const}$ and $d_+ + d_- = 1$. Then we have*

$$H_c(T^{m+1}\mu)\geqq H_c(T^m\mu) \qquad \textit{for} \quad \forall m\geqq 1,\ \forall c\neq\emptyset\in S$$

hence

$$H(T^{m+1}\mu)\geqq H(T^m\mu) \qquad \textit{for} \quad \forall m\geqq 1$$

Remark. Under the assumptions of theorem 1, we have

$$P_+^c(n,m)/P_-^c(n,m)=d_+/d_- \qquad \textit{for} \quad \forall m\geqq 1.$$

So it is natural to assume that

$$P_+^c(n)/P_-^c(n)=d_+/d_-,$$

in this case inequality of the theorem holds for $\forall m\geqq 0$.

As for the strictness of the inequality, we have following:

Definition 6.

$$D_c(m)=D_c(m;\mu):=\lim_{N\to\infty}\frac{1}{2N+1}\sum_{\substack{|n|\leq N\\ \varepsilon=\pm}}(P_\varepsilon^c(n+1,m;\mu)-P_\varepsilon^c(n-1,m;\mu))^2\geqq 0$$

Theorem 2. *Assume that $\mu\in\bar{\mathfrak{M}}_{\rm const}$ and $d_+ + d_- = 1$, and $0<d_1\leqq P_\varepsilon^c(n)\leqq d_2<1$ for $\forall n\in\boldsymbol{Z}$, $\varepsilon=\pm$ $(c\neq\emptyset)$ then we have for $\forall m\geqq 1$*

$$a\cdot D_c(m;\mu)\leqq H_c(T^{m+1}\mu)-H_c(T^m\mu)\leqq A\cdot D_c(m;\mu) \qquad \textit{for some} \quad A>a>0$$

Proposition 3. *$D_c(m;\mu)=0$ means $D_c(m+k;\mu)=0$ $\forall k\geqq 0$ $(m\geqq 1)$ under the* *same assumptions of theorem* **1.**

Remark. Same remark as to theorem 1 holds for theorem 2 and proposition 3.

3.2. The mapping R and T_0 do not change the value of entropy $H_c(\mu)$, namely we have

Proposition 4. $H_c(\mu)=H_c(R\mu)$, $H_c(\mu)=H_c(T_0\mu)$ *for* $\forall\mu\in\mathfrak{M}$

Proof. Note that

$$\begin{aligned} P_\varepsilon^c(n\,;R\mu)&=R\mu\{\omega\,;\omega(n,\varepsilon)=c\}\\ &=\mu(R^{-1}\{\omega\,;\omega(n,\varepsilon)=c\})=\mu\{\omega\,;R\omega(n,\varepsilon)=c\}\\ &=\mu\{\omega\,;\omega(n,-\varepsilon)=c\}=P_{-\varepsilon}^c(n\,;\mu)\,. \end{aligned}$$

Hence

$$H_c(R\mu)=\lim_{N\to\infty}\frac{-1}{2N+1}\sum_{\substack{|n|\leq H\\ \varepsilon}}P_{-\varepsilon}^c(n\,;\mu)\log P_{-\varepsilon}^c(n\,;\mu)=H_c(\mu)\,.$$

In the same way we have

$$P_\varepsilon^c(n\,;T_0\mu)=P_\varepsilon^c(n-\varepsilon\,;\mu)\,.$$

Hence

$$H_c(T_0\mu)=H_c(\mu)\,.$$

Proposition 5. $R'=R\cdot T_0$ *is involutive and leaves* $\bar{\mathfrak{M}}$ *invaiant, i.e.* $(R')^2=id.$ *and* $R'\bar{\mathfrak{M}}=\bar{\mathfrak{M}}$

Proposition 5, whose proof is easy, and proposition 4 mean that theorem 1 holds not only for the future-direction but also for the past-direction, namely we have

Theorem 1′. *Under the same assumption as in theorem* 1,

$$H_c(T^{-(m+1)}\mu)\geqq H_c(T^{-m}\mu) \qquad \textit{for} \quad \forall m\geqq 1\,,\quad c\neq\emptyset$$

$$H(T^{-(m+1)}\mu)\geqq H(T^{-m}\mu) \qquad \textit{for} \quad \forall m\geqq 1$$

Here we make important remarks:

Remark 1. Proposition 4 means also that we can not expect to get the same result for all $\mu\in\mathfrak{M}$.

Because if

$$H_c(T\mu)\geqq H_c(\mu)$$

holds for all $\mu\in\mathfrak{M}$, then, as can be easily seen from propositions 4 and 1

$$H_c(T\mu)=H_c(\mu)$$

holds for all $\mu\in\mathfrak{M}$, but this is impossible from theorem 2.

Remark 2. Theorem 1′ shows that entropy H_c can decrease actually for some $\mu\in\mathfrak{M}$; $H_c(T\mu)<H_c(\mu)$.

3.3. For the *K-S* type entropy $h(\mu)$, we have

Theorem 3. *$h(T^m\mu)=h(\mu)$ for $\forall m\in\boldsymbol{Z}$, $\forall\mu\in\mathfrak{M}$*

Proof.

$$h(T^m\mu)=\lim_{N\to\infty}\frac{-1}{2N+1}\sum_{A\in\mathfrak{A}_N}(T^m\mu)(A)\log(T^m\mu)(A)$$

$$=\lim_{N}\frac{-1}{2N+1}\sum_{A\in T^{-m}(\mathfrak{A}_N)}\mu(A)\log\mu(A)$$

Now, as can be easily seen by the definition of T, for $\forall\omega\in X$, $T^m\omega(n,\varepsilon)$ is determined by the $\{\omega(n+k,+),\omega(n+k,-)\}_{|k|\leq|m|}$. This means

$$\mathfrak{A}_{N+|m|}\succ T^{-m}(\mathfrak{A}_N)\succ\mathfrak{A}_{N-|m|}.$$

Here $\mathfrak{A}\succ\mathfrak{B}$ ($\mathfrak{A}$, $\mathfrak{B}$ are partitions of X) means that for $\forall A\in\mathfrak{A}$ there exists a $B\in\mathfrak{B}$ such that $A\subset B$.

It is well known that if $\mathfrak{A}\succ\mathfrak{B}$, then

$$-\sum_{A\in\mathfrak{A}}\mu(A)\log\mu(A)\geqq-\sum_{B\in\mathfrak{B}}\mu(B)\log\mu(B).$$

Therefore

$$-\sum_{A'\in\mathfrak{A}_{N+|m|}}\mu(A')\log\mu(A')\geqq\sum_{A\in T^{-m}(\mathfrak{A}_N)}\mu(A)\log\mu(A)$$

$$\geqq-\sum_{A''\in\mathfrak{A}_{N-|m|}}\mu(A'')\log\mu(A'').$$

Hence

$$h(\mu)\geqq h(T^m\mu)\geqq h(\mu). \qquad \text{Q.E.D}$$

A part of the theorem 1 is restated in the words of $I(\mu)$ by theorem 3:

Theorem 1″. *Under the same assumptions of theorem 1 we have*

$$I(T^{m+1}\mu)\geqq I(T^m\mu) \qquad \text{for} \quad \forall m\geqq 1$$

Remark. Our "entropy increasing law" says that for the states $\mu(\in\bar{\mathfrak{M}})$ outer structure (or "observable" structure) of the state becomes simpler as time passes in the future. In the place of it, "complexity of hidden structure" of the state becomes more complex. This explains partially why entropy can increase as the time passes in one direction, even though the dynamics of the system is reversible.

3.4. Definition 6. We define the projection π from $\mathfrak{M}$ to $\bar{\mathfrak{M}}$ $\pi:\mathfrak{M}\to\bar{\mathfrak{M}}$ by

$$P_\varepsilon^c(n;\pi\mu)=P_\varepsilon^c(n;\mu) \quad \forall c\in S, \quad \varepsilon=\pm, \quad n\in\boldsymbol{Z}$$

Theorem 4. *For* $\mu \in \bar{\mathfrak{M}}$, *we have*

$$\pi(T^m(\pi(T^{m'}\mu)))=\pi(T^{m+m'}\mu) \qquad for \quad \forall m, m' \geqq 0$$

Definition 7. We define a family of mappings $\{\bar{T}_m\}_{m\geq 0}$ on the space $\bar{\mathfrak{M}}$ by

$$\bar{T}_m\mu=\pi(T^m\mu) \qquad \text{for} \quad \mu \in \bar{\mathfrak{M}}$$

By this definition, theorem 4 can be restated as follows;

Theorem 4′. $\{\bar{T}_m\}_{m\geq 0}$ *forms a semi-group of the mappings on the space* $\bar{\mathfrak{M}}$, *that is,*

$$\bar{T}_{m+m'}=\bar{T}_m \circ \bar{T}_{m'} \qquad for \quad \forall m, m' \geqq 0$$

4. Proofs of the main results.

4.1. At first we define a notion of the fundamental path of the system (X, T) which plays an essential role in the proofs [2].

Definition 8. We call a sequence of integers

$$\sigma=\{\cdots, n_{-1}, n_0, n_1, n_2, \cdots\}=\{n_m\}_{m\in Z}$$

a fundamental path (or briefly a path) of $\omega \in X$, iff

a) $((T^m\omega)(n_m, -), (T^m\omega)(n_m, +)) \neq (\emptyset, \emptyset)$ for $\forall m \in \boldsymbol{Z}$

b) $n_{m+1}-n_m=\pm 1$

c) $n_{m+1}-n_m=1$ if $T^m\omega(n_m, -)=\emptyset$
   $n_{m+1}-n_m=-1$ if $T^m\omega(n_m, +)=\emptyset$

d) $(n_{m+1}-n_m)(n_m-n_{m-1})=-1$ if $T^m\omega(n_m, -)\neq\emptyset$ and $T^m\omega(n_m, +)\neq\emptyset$
   $(n_{m+1}-n_m)(n_m-n_{m-1})=1$ if $T^m\omega(n_m, -)=\emptyset$ or $T^m\omega(n_m, +)=\emptyset$

A path σ of ω is uniquely determined, if we give such a n_m that satisfies the property a) and $n_{m+1}-n_m$ $(=\pm 1)$ or n_m-n_{m-1} $(=\pm 1)$ when $T^m\omega(n_m, -)\neq \emptyset$ and $T^m\omega(n_m, +)\neq\emptyset$. When $T^m\omega(n_m, -)=\emptyset$ or $T^m\omega(n_m, +)=\emptyset$ holds, then $n_{m+1}-n_m$ or $n_{m+1}-n_m$ is automatically determined by c) and d), so n_m determines the path σ in this case. A path represents a trajectory of a particle of ω.

Definition 9. We say that a path $\sigma=\{n_m\}_{m\in Z}$ passes through (n, m) to the $-$(respectively $+$)-direction, if $n_m=n$ and $n_{m+1}-n_m=-1$ (respectively$=1$).

Note that such a path σ is unique if exists, and is denoted by

$$\sigma(n, m; \varepsilon)=\sigma(n, m; \varepsilon; \omega) \quad (\varepsilon=\pm).$$

Definition 10. We fix $\mu \in \mathfrak{M}$. For $n, m, k \in \boldsymbol{Z}$, $0 \leqq k \leqq m$ and $\varepsilon=\pm$, $\varepsilon'=\pm$, we denote by $q^{\varepsilon}_{(n,m)}(k, \varepsilon')$ the probability measured by μ that the path which passes through (n, m) to the ε-direction passes through $(n-m+2k, 0)$ to the ε'-direction, under the condition that there exists a path which passes through

(n, m) to the ε-direction, i.e.

$q^{\varepsilon}_{(n,m)}(k, \varepsilon')=\mu\{\omega; \exists\sigma(n, m; \varepsilon; \omega)=\sigma$ and σ passes through $(n-m+2k, 0)$ to the ε'-direction$\}/\mu\{\omega; \exists\sigma(n, m; \varepsilon; \omega)\}$

Lemma 3. *Assume that* $\mu\in\bar{\mathfrak{M}}$.

$$q^{+}_{(n,m)}(k, \varepsilon)=(1-d_{-}(n+m))q^{+}_{(n-1,m-1)}(k, \varepsilon)+d_{-}(n+m)q^{-}_{(n+1,m-1)}(k-1, \varepsilon)$$

$$q^{-}_{(n,m)}(k, \varepsilon)=d_{+}(n-m)q^{+}_{(n-1,m-1)}(k, \varepsilon)+(1-d_{+}(n-m))q^{-}_{(n+1,m-1)}(k-1, \varepsilon)$$

for $0<k<m$

$$q^{+}_{(n,m)}(0, +)=\prod_{k=1}^{m}(1-d_{-}(n-m+2k)),\quad q^{+}_{(n,m)}(0, -)=0$$

$$q^{+}_{(n,m)}(m, +)=0,\quad q^{+}_{(n,m)}(m, -)=d_{-}(n+m)\prod_{k=1}^{m-1}(1-d_{+}(n-m+2k))$$

$$q^{-}_{(n,m)}(0, +)=d_{+}(n-m)\cdot\prod_{k=1}^{m-1}(1-d_{-}(n-m+2k)),\quad q^{-}_{(n,m)}(0, -)=0$$

$$q^{-}_{(n,m)}(m, +)=0,\quad q^{-}_{(n,m)}(m, 1)=\prod_{k=0}^{m-1}(1-d_{+}(n-m+2k))$$

Proof of Lemma 3. For $0<k<m$,

$A=\{\omega; \exists\sigma(n, m; +; \omega)=\sigma$ and σ passes through $(n-m+2k, 0)$ to the ε-direction$\}$
$=(A\cap\{\omega; \neg\exists\sigma(n+m, 0; -; \omega)\})\cup(A\cap\{\omega; \exists\sigma(n+m, 0; -; \omega)\})$
$=(\{\omega; \exists\sigma(n-1, m-1; +; \omega)=\sigma$ and σ passes through $(n-m+2k, 0)$ to the ε-direction$\}\cap\{\omega; \neg\exists\sigma(n+m, 0; -; \omega)\})\cup(\{\omega; \exists\sigma(n-m, 0; +; \omega)\}\cap\{\omega; \exists\sigma(n+1, m-1; -; \omega)=\sigma$ and σ passes through $(n-m+2k, 0)$ to the ε-direction$\})$

Therefore, as $\mu\in\bar{\mathfrak{M}}$

$$\mu\{\omega; \exists\sigma(n, m; +; \omega)\}q^{+}_{(n,m)}(k, \varepsilon)=\mu(A)=$$

$$=\mu\{\omega; \exists\sigma(n-1, m-1; +; \omega)\}q^{+}_{(n-1,m-1)}(k, \varepsilon)\cdot\mu\{\omega; \neg\exists\sigma(n+m, 0; -; \omega)\}$$

$$+\mu\{\omega; \exists\sigma(n-m, 0; +; \omega)\}\cdot q^{-}_{(n+1,m-1)}(k-1, \varepsilon)\cdot\mu\{\omega; \exists\sigma(n+1, m-1; -; \omega)\}$$

As

$$\{\omega; \exists\sigma(n, m; +; \omega)\}=\{\omega; \exists\sigma(n-m, 0; +; \omega)\},\quad \text{etc.}$$

we have

$$d_{+}(n-m)q^{+}_{(n,m)}(k, \varepsilon)=d_{+}(n-m)(1-d_{-}(n+m))q^{+}_{(n-1,m-1)}(k, \varepsilon)$$
$$+d_{+}(n-m)d_{-}(n+m)q^{-}_{(n+1,m-1)}(k-1, \varepsilon)$$

that is

$$q^{+}_{(n,m)}(k, \varepsilon)=(1-d_{-}(n+m))q^{+}_{(n-1,m-1)}(k, \varepsilon)+d_{-}(n+m)q^{-}_{(n+1,m-1)}(k-1, \varepsilon)$$

Similarly we have

$$q^{-}_{(n,m)}(k, \varepsilon)=d_{+}(n-m)q^{+}_{(n-1,m-1)}(k, \varepsilon)+(1-d_{+}(n-m))q^{-}_{(n+1,m-1)}(k-1, \varepsilon)$$

For $k=0$ or m, the formulas are easily obtained. Q.E.D.

4.2. Fundamental Lemma 4. *Assume that* $\mu\in\bar{\mathfrak{M}}$, *then we have following recursive formula*:

$$P_+^c(n, m)=(1-d_-(n+m))P_+^c(n-1, m-1)+d_+(n-m)P_-^c(n+1, m-1)$$

$$P_-^c(n, m)=d_-(n+m)P_+^c(n-1, m-1)+(1-d_+(n-m))P_-^c(n+1, m-1),$$

$$m\geqq 1,\ c\neq\emptyset$$

Proof of Fundamental Lemma 4.

$$P_+^c(n, m)=d_+(n-m)\sum_{\substack{0\leq k\leq m\\ \varepsilon=\pm}} q_{(n,m)}^+(k, \varepsilon)P_\varepsilon^c(n-m+2k)/d_\varepsilon(n-m+2k)$$

So by lemma 3,

$$P_+^c(n, m)$$

$$=d_+(n-m)\sum_{\substack{0<k<m\\ \varepsilon=\pm}}\{(1-d_-(n+m))q_{(n-1,m-1)}^+(k, \varepsilon)+d_-(n+m)$$

$$\cdot q_{(n+1,m-1)}^-(k-1, \varepsilon)\}\times P_\varepsilon^c(n-m+2k)/d_\varepsilon(n-m+2k)$$

$$+d_+(n-m)\cdot\sum_\varepsilon q_{(n,m)}^+(0, \varepsilon)\cdot P_\varepsilon^c(n-m)/d_\varepsilon(n-m)$$

$$+d_+(n-m)\cdot\sum_\varepsilon q_{(n,m)}^+(m, \varepsilon)P_\varepsilon^c(n+m)/d_\varepsilon(n+m)$$

$$=d_+(n-m)(1-d_-(n+m))\sum_{\substack{0\leq k<m\\ \varepsilon}} q_{(n-1,m-1)}^+(k, \varepsilon)\cdot P_\varepsilon^c(n-m+2k)/d_\varepsilon(n-m+2k)$$

$$+d_+(n-m)d_-(n+m)\cdot\sum_{\substack{0\leq k<m\\ \varepsilon}} q_{(n+1,m-1)}^-(k, \varepsilon)\cdot P_\varepsilon^c(n-m+2k+2)/d_\varepsilon(n-m+2k+2)$$

$$-d_+(n-m)(1-d_-(n+m))\sum_\varepsilon q_{(n-1,m-1)}^+(0, \varepsilon)\cdot P_\varepsilon^c(n-m)/d_\varepsilon(n-m)$$

$$-d_+(n-m)d_-(n+m)\sum_\varepsilon q_{(n+1,m-1)}^-(m-1, \varepsilon)\cdot P_\varepsilon^c(n+m)/d_\varepsilon(n+m)$$

$$+d_+(n-m)\sum_\varepsilon q_{(n,m)}^+(0, \varepsilon)\cdot P_\varepsilon^c(n-m)/d_\varepsilon(n-m)$$

$$+d_+(n-m)\sum_\varepsilon q_{(n,m)}^+(m, \varepsilon)\cdot P_\varepsilon^c(n+m)/d_\varepsilon(n+m)$$

Again by lemma 3 we get

$$=(1-d_-(n+m))d_+(n-m)\sum_{\substack{0\leq k\leq m-1\\ \varepsilon}} q_{(n-1,m-1)}^+(k, \varepsilon)\cdot P_\varepsilon^c(n-m+2k)/d_\varepsilon(n-m+2k)$$

$$+d_+(n-m)d_-(n+m)\sum_{\substack{0\leq k\leq m-1\\ \varepsilon}} q_{(n+1,m-1)}^-(k, \varepsilon)\cdot P_\varepsilon^c(n-m+2k+2)/d_\varepsilon(n-m+2k+2)$$

Similarly we get

$$P_-^c(n, m)$$

$$=d_-(n+m)\sum_{\substack{0\leq k\leq m\\ \varepsilon}} q_{(n,m)}^-(k, \varepsilon)\cdot P_\varepsilon^c(n-m+2k)/d_\varepsilon(n-m+2k)$$

$$=d_-(n+m)d_+(n-m)\sum_{\substack{0\leq k\leq m-1\\ \varepsilon}} q^+_{(n-1,m-1)}(k,\varepsilon)\cdot P^c_\varepsilon(n-m+2k)/d_\varepsilon(n-m+2k)$$

$$+d_-(n+m)(1-d_+(n-m))\sum_{\substack{0\leq k\leq m-1\\ \varepsilon}} q^-_{(n+1,m-1)}(k,\varepsilon)P^c_\varepsilon(n-m+2k+2)/d_\varepsilon(n-m+2k+2)$$

Therefore

$$P^c_+(n,m)=(1-d_-(n+m))P^c_+(n-1,m-1)+d_+(n-m)P^c_-(n+1,m-1)$$

$$P^c_-(n,m)=d_-(n+m)P^c_+(n-1,m-1)+(1-d_+(n-m))P^c_-(n+1,m-1)$$

Q. E. D.

4.3. We prove now theorem 1. For that sake we need some more lemmas.

Lemma 5. *For* $\mu\in\bar{\mathfrak{M}}_{\text{const}}$, *if* $d_++d_-=1$, *then*

$$\frac{P^c_+(n,m)}{P^c_-(n,m)}=\frac{d_+}{d^-}\quad \textit{for}\quad \forall m\geqq 1,\quad n\in\boldsymbol{Z},\quad c\neq\emptyset$$

Proof. From lemma 4, for $m\geqq 1$

$$d_-P^c_+(n,m)-d_+\cdot P^c_-(n,m)=d_-\{(1-d_-)P^c_+(n-1,m-1)+d_+P^c_-(n+1,m-1)\}$$
$$-d_+\{d_-P^c_+(n-1,m-1)+(1-d_+)P^{c'}_-(n+1,m-1)\}$$
$$=(d_-d_+-d_+d_-)P^c_+(n-1,m-1)+(d_-d_+-d_+d_-)P^c_-(n-1,m-1)=0$$

Lemma 6. *Assume that* $\mu\in\bar{\mathfrak{M}}_{\text{const}}$ *and* $d_++d_-=1$, *then we have*

$$P^c_\varepsilon(n,m+1)=d_+P^c_\varepsilon(n-1,m)+d_-P^c_\varepsilon(n+1,m)$$

$$\textit{for}\quad m\geqq 1,\quad c\neq\emptyset,\quad n\in\boldsymbol{Z},\quad \varepsilon=\pm,$$

Proof. It follows easily from lemmas 4 and 5.

Lemma 7. 1) $\log(1+x)\leqq x$ *for* $-1<x$
2) *for* $-1<b\leqq x\leqq B<\infty$, *there exist* $0<a<A$, *such that* $x-Ax^2\leqq\log(1+x)\leqq x-ax^2$

Proof of theorem 1. As is easily seen

$$P^c_\varepsilon(n;T^m\mu)=P^c_\varepsilon(n,m;\mu)$$

Therefore, for $m\geqq 1$

$$H_c(T^{m+1}\mu)-H_c(T^m\mu)$$

$$=\lim_{N\to\infty}\frac{-1}{2N+1}\sum_{|n|\leq N,\,\varepsilon=\pm} P^c_\varepsilon(n,m+1;\mu)\log P^c_\varepsilon(n,m+1;\mu)$$

$$-\lim_{N\to\infty}\frac{-1}{2N+1}\sum_{|n|\leq N,\,\ =\pm} P^c_\varepsilon(n,m;\mu)\log P^c_\varepsilon(n,m;\mu)$$

Hence from lemma 6 (we omit the suffix c and ε)

$$
\begin{aligned}
&H_c(T^{m+}{}_c\mu)-H_c(T^m\mu)\\
&=\lim\frac{-1}{2N+1}\sum_{n,\varepsilon}\{d_+P(n-1,\ m)+d_-P(n+1,\ m))\log(d_+P(n-1,\ m)+d_-P(n+1,\ m))\\
&\quad-(d_++d_-)P(n,\ m)\log P(n,\ m)\}\\
&=\lim\frac{-1}{2N+1}\sum_{n,\varepsilon}\left\{d_+P(n-1,\ m)\log\frac{d_+P(n-1,\ m)+d_-P(n+1,\ m)}{P(n-1,\ m)}\right.\\
&\quad\left.+d_-P(n+1,\ m)\log\frac{d_+P(n-1,\ m)+d_-P(n+1,\ m)}{P(n+1,\ m)}\right\}\\
&=\lim\frac{-1}{2N+1}\sum_{n,\varepsilon}\left\{d_+P(n-1,\ m)\log\left(1+\frac{d_-(P(n+1,\ m)-P(n-1,\ m))}{P(n-1,\ m)}\right)\right.\\
&\quad\left.+d_-P(n+1,\ m)\log\left(1+\frac{d_+(P(n-1,\ m)-P(n+1,\ m))}{P(n+1,\ m)}\right)\right\}
\end{aligned}
$$

Hence from lemma 7, 1)

$$
\begin{aligned}
&H_c(T^{m+1}\mu)-H_c(T^m\mu)\\
&\geqq\lim\frac{-1}{2N+1}\sum\left\{d_+P(n-1,\ m)\frac{d_-(P(n+1,\ m)-P(n-1,\ m))}{P(n-1,\ m)}\right.\\
&\quad\left.+d_-P(n+1,\ m)\frac{d_+(P(n-1,\ m)-P(n+1,\ m))}{P(n+1,\ m)}\right\}=0 \qquad \text{Q.E.D.}
\end{aligned}
$$

4.4. *Proof of theorem* 2. As can be easily seen from lemma 4 the assumption

$$0<d_1\leqq P_\varepsilon^c(n)\leqq d_2<1$$

means that the same inequality holds for

$$0<d_1'\leqq P_\varepsilon^c(n,\ m)\leqq d_2'<1 \qquad n\in\boldsymbol{Z},\quad m\geqq 0.$$

Therefore, from lemma 7, 2) we get as in the proof of theorem 1,

$$
\begin{aligned}
&H_c(T^{m+1}\mu)-H_c(T^m\mu)\\
&\geqq\lim_N\frac{a}{2N+1}\sum_{n,\varepsilon}\left\{d_+P(n-1,\ m)\left(\frac{d_-(P(n+1,\ m)-P(n-1,\ m))}{P(n-1,\ m)}\right)^2\right.\\
&\quad\left.+d_-P(n+1,\ m)\left(\frac{d_+(P(n-1,\ m)-P(n+1,\ m))}{P(n+1,\ m)}\right)^2\right\}\\
&=a\cdot\lim_N\frac{1}{2N+1}\sum_{n,\varepsilon}d_+d_-(P_\varepsilon^c(n+1,\ m)-P_\varepsilon^c(n-1,\ m))^2\left(\frac{d_-}{P_\varepsilon^c(n-1,\ m)}+\frac{d_+}{P_\varepsilon^c(n+1,\ m)}\right)\\
&\geqq a\cdot d_+d_-\lim_N\frac{1}{2N+1}\sum_{\substack{|n|\leqq N\\ \varepsilon}}(P_\varepsilon^c(n+1,\ m)-P_\varepsilon^c(n-1,\ m))^2=a\cdot d_+d_-D_c(m\,;\ \mu)
\end{aligned}
$$

In the same way, we have

$$H_c(T^{m+1}\mu)-H_c(T^m\mu)\leqq A\cdot d_+d_-D_c(m\,;\ \mu) \qquad \text{Q.E.D.}$$

Proposition 3 follows easily from the inequality which we can get easily from lemma 6:

$$(P_\varepsilon^c(n+1, m+1)-P_\varepsilon^c(n-1, m+1))^2$$
$$\leqq 2d_+^2(P_\varepsilon^c(n, m)-P_\varepsilon^c(n-2, m))^2+2d_-^2(P_\varepsilon^c(n+2, m)-P_\varepsilon^c(n, m))^2$$

4.5. *Proof of theorem* 4. Note that a measure $\mu\in\mathfrak{M}$ is uniquely determined by $\{P_\varepsilon^c(n;\mu)\}_{n\in Z}$, $\varepsilon=\pm$, $c\in S$.

$$P_\varepsilon^c(n;\pi(T^{m+m'}\mu))=P_\varepsilon^c(n;T^{m+m'}\mu)=P_\varepsilon^c(n, m+m';\mu)$$

On the other hand

$$P_\varepsilon^c(n;\pi(T^m(\pi T^{m'}\mu)))=P_\varepsilon^c(n;T^m(\pi(T^{m'}\mu)))=P_\varepsilon^c(n, m;\pi(T^{m'}\mu)).$$

Note that also

$$d_-(n+m+m';\mu)=d_-(n+m;\pi(T^{m'}\mu))$$
$$d_+(n-(m+m'), \mu)=d_+(n-m;\pi(T^{m'}\mu)).$$

Hence $P_\varepsilon^c(n, m+m';\mu)$ and $P_\varepsilon^c(n, m;\pi(T^{m'}\mu))$ satisfy the same recursive formula w.r.t. m (m': fixed) as one in fundamental lemma 4. They have also the same initial values:

$$P_\varepsilon^c(n, m';\mu)=P_\varepsilon^c(n, 0;\pi(T^{m'}\mu)).$$

Therefore

$$P_\varepsilon^c(n, m+m';\mu)=P_\varepsilon^c(n, m;\pi(T^{m'}\mu)) \quad \text{for } \forall m\geqq 0.. \qquad \text{Q.E.D.}$$

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