On injectivity and p-injectivity

Dedicated to Professor Hisao Tominaga on his sixtieth birthday

By

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Introduction.

In this paper, certain results known for semiprime left Goldie rings are shown to have their analogs for rings with von Neumann regular classical left quotient rings. Conditions for von Neumann regularity and connections between injective modules over certain rings are studied. The following are among the results proved here: (1) Let A be a ring having a classical left quotient ring Q. Then (a) if A is left hereditary and Q is regular, then A is left Noetherian iff every essential left ideal of A is essentially finitely generated; (b) Q is regular iff every divisible torsionfree left A-module is p-injective; (c) Q is semisimple Artinian iff every divisible torsionfree quasi-injective left A-module is injective; (2) The following conditions are equivalent: 1) A is von Neumann regular; 2) A has a classical left quotient ring which is a projective left A-module and every finitely generated divisible singular left A-module is flat; 3) for any non-zero proper principal right ideal I of A, there exist a non-trivial idempotent e and a left regular element d such that $I=edA$; (3) Let A be a left $p.p.$ ring (every principal left ideal is projective) having a classical left quotient ring Q. If Q is left p-injective then it is von Neumann regular; (4) Let A be a left YJ-ring with Jacobson radical J and satisfying the maximum condition on left annihilators. If E, Q are injective left A-module such that $r(J)$ is isomorphic to $r_E(J)$ (as left A-modules), then Q is isomorphic to E.

Throughout, A will represent an associative ring with identity and A-modules considered will be unital. J, Z denote respectively the Jacobson radical and the left singular ideal of A. An ideal of A will always mean a two-sided ideal and A is called left duo if every left ideal is an ideal. A left (right) ideal of A is called reduced if it contains no non-zero nilpotent element. A left A-module M is called p-injective if, for any principal left ideal P of A, every left A-homomorphism of P into M extends to one of A into M. In general, flatness and p-injectivity are distinct concepts. However, if K is a maximal left ideal of A which is an ideal, then $_AA/K$ is flat iff A/K_A is injective iff A/K_A is p-injective [23]. Also if I is a p-injective left ideal of A, then $_AA/I$ is flat. For results on p-injectivity and allied concepts, consult [1], [3], [7], [8], [9], [11], [13], [16], [17], [18]. Recall that for any left A-module M, $Z(M)=$

$\{z \in M \mid Ez = 0$ for some essential left ideal E of $A\}$ is the singular submodule of M. $_AM$ is called singular (resp. non-singular) if $Z(M) = M$ (resp. $Z(M) = 0$). Thus A is left non-singular iff $Z = 0$. As usual, a ring Q is called a classical left quotient ring of A if (a) $A \subseteq Q$; (b) every non-zero-divisor of A is invertible in Q; (c) for any $q \in Q$, $q = b^{-1}a$, a, $b \in A$, b being a non-zero-divisor. It is well-known that A has a classical left quotient ring iff for any a, $b \in A$, where b is a non-zero-divisor, there exist d, $c \in A$, c non-zero-divisor, such that $ca = db$. A theorem of A. W. GOLDIE [5, Theorem 3.35] asserts that A has a semi-simple Artinian classical left quotient ring iff A is semi-prime satisfying the maximum condition on left annihilator ideals and complement left ideals (such rings are called semi-prime left Goldie). A left A-module M is called torsionfree if, for any $0 \neq y \in M$, $cy \neq 0$ for every non-zero-divisor c of A. M is divisible iff $M = cM$ for each non-zero-divisor c of A.

Rings having semi-simple Artinian classical quotient rings have been extensively studied. We shall here be essentially concerned with rings having von Neumann regular classical quotient rings. The connection between p-injectivity and von Neumann regularity is similar to that between injectivity and being semi-simple Artinian : A is semi-simple Artinian iff every left (right) A-module is injective iff every cyclic left (right) A-module is injective (B. Osofsky).

First we quote [14, Propositions II. 4.2 and 4.3].

Proposition 1. *Suppose that A has a classical left quotient ring Q. Then a left A-submodule L of Q is projective and contains a non-zero-divisor of A if, and only if, there exist $u_j \in L$ and $y_j \in Q$ with $Ly_j \subseteq A$ $(1 \leq j \leq m)$ such that $\sum_{j=1}^{m} y_j u_j = 1$.*

When this is so, $_AL$ is finitely generated.

Recall that a left ideal I of A is essentially finitely generated if I contains a finitely generated left subideal which is essential in $_AI$ [5, p. 70].

Proposition 2. *If A has a regular classical left quotient ring Q, then for any projective essential left ideal E of A, $_AE$ is finitely generated if (and only if) E is essentially finitely generated.*

Proof. Assume that there exists a finitely generated left ideal L which is essential in $_AE$. Since $_AL$ is essential in $_AA$, then $_QQL$ is essential in $_QQ$. Now Q being regular implies that $_QQL$ (being finitely generated) is a direct summand of $_QQ$, whence $QL = QE = Q$. Since $_AE$ is projective, there exists a family $\{u_i\}$ of elements of E and a family of left A-homomorphisms $f_i : E \to A$ such that for each $u \in E, u = \sum_i f_i(u) . u_i$, where the coefficients $f_i(u)$ are zero for all but finitely many indices i. Now by [26, Theorem 1.2], E contains a non-zero-divisor c of A. Hence, $_AE$ is finitely generated by Proposition 1.

Corollary 2.1. *If A is a left hereditary ring having a regular classical ledt quotient ring, then A is left Noetherian iff every essential left ideal of A is essentially finitely*

Every left duo, left hereditary ring has a strongly regular classical quotient ring (see, e.g., [7, Theorem 1]). Therefore, combining Corollary 2.1 together with [5, Theorem 3.14] and [21, Lemma 1], we get

Corollary 2.2. *The following conditions are equivalent for a left duo left hereditary ring A: (a) A is left Noetherian; (b) A satisfies the maximum condition on left annihilators; (c) every essential left ideal of A is essentially finitely generated.*

A theorem of L. LEVY [10, Theorem 3.3] states that if A has a classical left quotient ring Q, then Q is semi-simple Artinian iff every divisible torsionfree left A-module is injective. We now give an analogous result for regular classical quotient rings.

Theorem 3. *Suppose that A has a classical left quotient ring Q. Then Q is von Neumann regular if, and only if, every divisible torsionfree left A-module is p-injective.*

Proof. First assume that every divisible torsionfree left A-module is p-injective. For any $y \in Q$, the left A-module $C = Qy$ is divisible torsionfree and hence p-injective. If $y = b^{-1}d$, b, $d \in A$, b a non-zero-divisor, then $Ad \subseteq C$ and $d = dqy$ for some $q \in Q$. Hence $y = yqy$, showing that Q is von Neumann regular. Conversely, assume that Q is von Neumann regular. Let M be a divisible torsionfree left A-module, $P = Aa$, $a \in A$ and $f : {}_A P \to {}_A M$ a homomorphism. As is claimed in the proof of [10, Theorem 3.3], M is canonically a left Q-module. Then f extends to a Q-homomorphism $h : Qa \to M$. Since Q is regular, there exists $w \in M$ such that $h(u) = uw$ for all $u \in Qa$. In particular, $f(p) = h(p) = pw$ for all $p \in P$. This proves that ${}_A M$ is p-injective.

Applying [26, Lemma 1.1], we get a p-injective analog of [5, p. 102 ex. 18].

Corollary 3.1. *If A has a regular classical left quotient ring, then a non-singular left A-module is p-injective iff it is divisible.*

Corollary 3.2. *A left duo ring whose divisible torsionfree left A-modules are p-injective possesses a strongly regular classical left quotient ring.*

Corollary 3.3. *If A has a regular classical left quotient ring Q, then Q is semi-simple Artinian iff every p-injective torsionfree left A-module is injective.*

Following [25], A is called a left *YJ*-ring if, for any $0 \neq a \in A$, there exists a positive integer n such that $a^n A$ is a non-zero right annihilator ideal. Note that A is a left *YJ*-ring iff for any $0 \neq a \in A$, there exists a positive integer n such that $a^n \neq 0$ and any left A-homomorphism of Aa^n into A extends to an endomorphism of ${}_A A$. Left *YJ*-rings generalize effectively left self-injective rings and von Neumann regular rings. A left A-module M is called *YJ*-injective if, for any $0 \neq a \in A$, there exists a positive integer n such that $a^n \neq 0$ and any left A-homomorphism of Aa^n into M extends to one of A into M. Thus A is a left *YJ*-ring iff ${}_A A$ is *YJ*-injective. Although reduced left *YJ*-rings are strongly regular [24], we do not know whether arbitrary von Neumann regular rings may be characterized in terms of *YJ*-injective modules.

generated.

Following [4], an element c of A is called left (resp. right) regular if $l(c)=0$ (resp. $r(c)=0$). Thus c is a non-zero-divisor iff c is left and right regular. We now give some new characteristic properties of regular rings.

Theorem 4. *The following conditions are equivalent*:

1) *A is von Neumann regular*;

2) *A has a classical left quotient ring which is a projective left A-module and every finitely generated divisible singular left A-module is flat*;

3) *A is either a left or right YJ-ring whose divisible torsionfree left modules are p-injective*;

4) *For any non-zero proper principal right ideal P of A, there exists a non-trivial idempotent e and a left regular element d such that $P=edA$.*

Proof. Obviously, 1) implies 2)–4).

2)$\Rightarrow$1). Let Q be a classical left quotient ring of A which is a projective left A-module. Then $_AQ$ is finitely generated by Proposition 1, and it is easy to see that $_AQ/A$ is finitely related. Now $_AQ$ is divisible, which implies that $_AQ/A$ is divisible singular and hence flat. It follows that $_AQ/A$ is projective, which implies that $_AA$ is a direct summand of $_AQ$. But $_AA$ is essential in $_AQ$, which yields $A=Q$. In as much as every left A-module is now divisible, we conclude that every cyclic singular left A-module is flat and therefore A is von Neumann regular by [8, Corollary 5].

3)$\Rightarrow$1). By Theorem 3.

4)$\Rightarrow$1). Let c be a left regular element of A. If we assume that $cA \neq A$, then $cA=edA$ for some non-trivial idempotent e and left regular element d. Now $0=l(c)=l(e)=A(1-e)$, which contradicts $e \neq 1$. This shows that every left regular element is right invertible in A. For any $0 \neq a \in A$ such that $aA \neq A$, $aA=ubA$, where u is a non-trivial idempotent and b is right invertible, which yields $aA=uA$, proving 1).

Now, we improve [10, Theorem 3.3] as follows:

Theorem 5. *Let A be a ring having a classical left quotient ring Q. Then the following conditions are equivalent*:

1) *Q is semi-simple Artinian*;

2) *Every divisible torsionfree left A-module is injective*;

3) *Every divisible torsionfree quasi-injective left A-module is injective.*

Proof. Obviously, 1)$\Rightarrow$2)$\Rightarrow$3), by [10, Theorem 3.3].

3)$\Rightarrow$1). Let M be a quasi-injective left Q-module. Then $_AM$ is divisible and torsionfree. Now, let U be an A-submodule of M, and $f: {}_AU \to {}_AM$ a homomorphism. Since $_AU$ is torsionfree, we can define $g: {}_QQU \to {}_QM$ by $g(c^{-1}u)=c^{-1}f(u)$, c a non-zero-divisor and $u \in U$. Then g extends to $h: {}_QM \to {}_QM$. Since h is an extension of f, this implies that $_AM$ is quasi-injective, and therefore injective by 3). Let I be a left ideal of Q, and $s: {}_QI \to {}_QM$ a homomorphism. Since s induces $r: {}_A(I \cap A) \to {}_AM$, there exists $y \in M$ such that $r(k)=ky$ for all $k \in I \cap A$. Then, for any non-zero-

divisor d in A and $k \in I \cap A$, we have $s(d^{-1} k) = d^{-1} r(k) = d^{-1} ky$. Since $I = Q(I \cap A)$, this proves that s extends to a homomorphism of Q to M. Therefore $_Q M$ is injective, in particular, every simple left Q-module is injective, and so Q is left Noetherian by [4, Proposition 20.4 B]. Since the left V-ring Q is semi-prime and is a classical left quotient ring of its own, Q is semi-simple Artinian.

Corollary 5.1. *Let A be a ring having a classical left quotient ring Q. Suppose that* (1) *every divisible torsionfree quasi-injective left A-module is injective and* (2) *every divisible singular left A-module is injective. Then A is semi-prime, left Noetherian and left hereditary. If, furthermore, Q is a two-sided classical quotient ring of A, then every finitely generated left A-module contains its singular submodule as a direct summand.*

Proof. For any left A-module M with injective hull $\hat{M}$, $_A\hat{M}/M$ is divisible singular and hence injective. This enables us to see that the homomorphic image of any injective left A-module is injective, which proves that A is left hereditary. Therefore A is semi-prime left Noetherian by [10, Theorem 3.11] and Theorem 5. The latter part follows from [10, Theorem 6.1].

We can restate [10, Theorem 3.4] as follows:

Corollary 5.2. *The following conditions are equivalent for a ring having a two-sided classical quotient ring:* (1) *Every divisible left A-module is injective;* (2) *A is a left hereditary ring such that every divisible torsionfree quasi-injective left A-module is injective.*

Corollary 5.3. *If A is a left duo ring such that every divisible torsionfree quasi-injective left A-module is injective, then A is a reduced left and right Goldie ring(see e.g. [14, Propasition XII. 5.5]).*

The next remark is motivated by [20, Remark 2].

Remark 1. If every p-injective left A-module is injective and every principal left ideal of A is projective, then A is left hereditary left Noetherian. Consequently, by [14, Proposition XV. 4.7], A must possess a left Artinian classical left quotient ring.

The next lemma extends [10, Theorem 3.1].

Lemma 6. *Every YJ-injective left A-module is divisible.*

Proof. Let M be a YJ-injective left A-module. If c is a non-zero-divisor of A, there exists a positive integer n such that any left A-homomorphism of Ac into M extends to one of A into M. For any element $y \in M$, define a left A-homomorphism $f: Ac^n \to M$ by $f(ac^n) = ay$ for all $a \in A$. Then $y = f(c^n) = c^n d$ for some $d \in M$, which implies that $M = cM$, showing that $_A M$ is divisible.

Corollary 6.1. *Let A be a ring having a classical left quotient ring Q. Suppose*

that every torsionfree YJ-injective quasi-injective left A-module is a projective left Q-module. Then Q is quasi-Frobenius, and any torsionfree YJ-injective quasi-injective left A-module is injective.

Proof. As is claimed in the proof of [10, Theorem 3.3], every divisible torsionfree left A-module is canonically a left Q-module. It is easy to see that if $_QM$ is injective then so is $_AM$. This together with Lemma 6 and [4, Theorem 24.20] enables us to complete the proof.

Remark 2. By Lemma 6 and Theorem 3, we see that if A has a von Neumann regular classical left quotient ring, then every YJ-injective torsionfree left A-module is p-injective.

Remark 3. Again by [14, Proposition XV. 4.7], every commutative hereditary Noetherian ring has a semi-simple Artinian classical quotient ring. So, by making use of [10, Theorem 3.4], Remark 1 and Corollary 5.1, we can see that the following conditions are equivalent for a commutative ring A: (a) A is hereditary Noetherian; (b) every p-injective A-module is injective and every principal ideal of A is projective; (c) both divisible singular A-modules and divisible torsionfree quasi-injective A-modules are injective.

Following [12], a left A-module M is called semi-simple if the intersection of all the maximal left submodules is zero.

Theorem 7. *The following conditions are equivalent:*

1) *A is semi-simple Artinian;*

2) *A is of finite left Goldie dimension and each proper principal left ideal is YJ-injective;*

3) *A is a left YJ-ring whose divisible torsionfree quasi-injective left modules are injective;*

4) *Every cyclic torsionfree left A-module is injective;*

5) *A is a left or right perfect ring whose divisible torsionfree left A-modules are YJ-injective;*

6) *Every semi-simple left A-module is flat and quasi-injective;*

7) *For any non-zero proper right ideal I of A, there exist a non-trivial idempotent e and a left regular element d such that I=edA.*

Proof. Obviously, 1) implies 2)–7).

2)$\Rightarrow$1). As is easily seen, $l(1-v)=0$ for every $v \in Z$. We claim first that $Z=0$. Suppose, to the contrary, that Z contains $u \neq 0$. Since $Au \neq A$, by hypothesis there exists a positive integer n and $b \in A$ such that $u^n \neq 0$ and $u^n = u^n bu$ (consider the inclusion map $Au^n \to Au$). Then $u^n(1-bu)=0$, which forces a contradiction $u^n = 0$. Hence $Z=0$. Similarly, we can prove that $J=0$. Therefore A is a semi-prime left Goldie ring. By Lemma 6, any proper principal left ideal P is divisible and non-singular, and therefore $_AP$ is injective by [10, Theorem 3.3]. Thus the left Goldie ring A is von Neumann regular and hence semi-simple Artinian.

3)$\Rightarrow$1). By Lemma 6 and Proposition 5.

4)$\Rightarrow$1). Since $J=0$, the left self-injective ring A is regular by [4, Theorem 19.27] and every (cyclic) left A-module is torsionfree. Therefore, every cyclic left A-module is injective, and so A is semi-simple Artinian by a result of Osofsky.

5)$\Rightarrow$1). Since A is left or right perfect, every non-zero-divisor is invertible in A. Then every left ideal is a divisible torsionfree left A-module and so YJ-injective. Thus $J=0$ as before, and hence A is semi-simple Artinian.

6)$\Rightarrow$1). Since $_A A/J$ is semi-simple, $_A A/J$ is flat, and therefore $J=0$. Now, A is left self-injective and regular, by [4, Theorem 19.27]. Therefore, every left ideal of A is a semi-simple left A-module and hence quasi-injective. Then, by [22, Lemma 1.1], every left A-module is semi-simple and therefore quasi-injective. Hence A is semi-simple Artinian by [24, Theorem 6].

7)$\Rightarrow$1). By Theorem 4(4).

Rings whose simple left modules are either YJ-injective or projective need not be semi-prime (cf. for example, [15, p. 297]). However, the next proposition (motivated by [12, Lemma 2.3]) holds.

Proposition 8. *Suppose that every simple left A-module is either YJ-injective or projective.*

(1) $Z \cap J=0$;

(2) $AdA=A$ *for each non-zero-divisor d of A*;

(3) *If A is semi-prime, the centre of A is von Neumann regular.*

Proof. (1) Suppose that $Z \cap J \neq 0$. Then by [24, Lemma 7], there exists $0 \neq z \in Z \cap J$ such that $z^2=0$. We claim that $AzA+l(z)=A$. If not, let M be a maximal left ideal containing $AzA+l(z)$. Since $l(z)$ is an essential left ideal, then $_A A/M$ cannot be projective, whence it is YJ-injective. Let $f: Az \to A/M$ be the left A-homomorphism defined by $f(az)=a+M$ for all $a \in A$. Then there exists $c \in A$ such that $1+M=f(z)=zc+M$, which yields $1 \in M$, a contradiction! Therefore $AzA+l(z)=A$ as claimed, and if $1=u+v, u \in AzA, v \in l(z)$, then $d(1-u)=1$ for some $d \in A$ (because $u \in J$), which implies $z=dvz=0$, a contradiction. This proves that $Z \cap J=0$.

(2) Let d be a non-zero-divisor of A. If $AdA \neq A$, let L be a maximal left ideal containing AdA. Then $_A A/L$ must be YJ-injective. If n is a positive integer such that any left A-homomorphism of Ad^n into A/L extends to one of A into A/L, since d is a non-zero-divisor, we have a well-defined left A-homomorphism $g: Ad^n \to A/L$ given by $g(ad^n)=a+L$ for all $a \in A$. Then $1+L=g(d^n)=d^n c+L$ for some $c \in A$, which implies $1 \in L$, contradicting $L \neq A$. This proves (b).

(3) Let C denote the centre of A. If $c \in C$ such that $c^2=0$, then $Ac \subseteq J \cap Z=0$ by (a) which shows that C is a reduced ring. Suppose that $d \in C$ such that $Ad+l(d) \neq A$. If M is a maximal left ideal of A containing $Ad+l(d)$, then M must be an essential left ideal, which implies $_A A/M$ YJ-injective. Since A is semi-prime, this leads to a contradiction. Therefore, for any $d \in C$, $d=bd^2$ for some $b \in A$. Then, as is well-known, $d=vd^2$ for some $v \in C$.

It is well-known that every simple left A-module is injective iff every proper left ideal of A is an intersection of maximal left ideals [12].

Proposition 9. *The following conditions are equivalent*:

1) *Every relative complement of any minimal projective left ideal of A is a maximal left ideal*;

2) *Every minimal projective left ideal of A is injective and its relative complement is an intersection of maximal left ideals.*

When this is the case, every simple projective left A-module is injective.

Proof. 1)⇒2): It is sufficient to show that if S is a simple projective left A-module, then S must be injective. Let L be a non-zero left ideal of A, $f: L \rightarrow S$ a non-zero left A-homomorphism. Then $F = \ker f$ is a maximal left subideal of L and $L/F \approx S$. Since $_A S$ is projective, then $L = F \oplus I$, where $I(\approx S)$ is a minimal projective left ideal of A. Let M be a relative complement of $_A I$ containing F in $_A A$. Then by hypothesis, M is maximal, whence $A = M \oplus I$. This shows that f may be extended to $g: A \rightarrow S$, proving that $_A S$ is injective.

2)⇒1): Let U be a minimal projective left ideal of A, K a relative complement of U. Since K is an intersection of maximal left ideals, there exists a maximal left ideal M of A such that $K \subseteq M$ but $K \oplus U \nsubseteq M$. Therefore $M \cap U = 0$ implies that $K = M$.

Corollary 9.1. *Let A be a prime ring whose essential one-sided ideals are two-sided and idempotent. If every relative complement of any minimal one-sided ideal of A is maximal, then A is simple Artinian.*

Proof. By [22, Remark 3], A is a primitive ring whose simple one-sided modules are injective or projective. Since every simple projective one-sided A-module is injective by Proposition 9, we see that A is a left and right V-ring. Hence A is Artinian by [22, Theorem 1.13].

Remark 4. If A is left Noetherian and each prime factor ring of A is a left YJ-ring, then A is left Artinian.

Remark 5. Let A be commutative. Then (1) the following conditions are equivalent: (a) every factor ring of A is quasi-Frobeniusean; (b) every factor ring of A is a YJ-injective Goldie ring. (2) A is Artinian iff each non-zero factor ring of A is a Goldie ring with non-zero socle (see [2], [4, Theorem 24.4] and apply a theorem of Lanski and Lemma 6).

If T is an ideal of A, M a left A-module, then $r_M(T) = \{y \in M \mid Ty = 0\}$ is a submodule of M. Consequently, $r_M(J)$, $r_M(Z)$ are submodules of M. We now turn to sufficient conditions for injective modules to be isomorphic.

Proposition 10. *Let A be a left YJ-ring satisfying the maximum condition on left annihilators. If K_1, K_2 are injective left A-modules, $g_1: M_1 \rightarrow K_1$, $g_2: M_2 \rightarrow K_2$ left A-monomorphisms of left A-modules M_1, M_2 into K_1, K_2 respectively, and $f: M_1 \rightarrow$*

M_2 *a left A-homomorphism, then there exists a left A-homomorphism* $h\colon K_1 \to K_2$ *such that* $hg_1 = g_2 f$. *If* $g_i(r_{M_i}(J)) = r_{K_i}(J)$ $(i=1, 2)$, *and f is a monomorphism (resp. isomorphism), then h is a monomorphism (resp. isomorphism).*

Proof. Note that, in general, knowing that the g_i are monomorphisms, we always have $g_i(r_{M_i}(J)) = r_{K_i}(J) \cap g_i(M_i)$ for $i=1, 2$. The fact that K_2 is an injective left A-module implies the existence of a left A-homomorphism $h\colon K_1 \to K_2$ such that $hg_1 = g_2 f$. Now our hypothesis says that $g_i(r_{M_i}(J)) = r_{K_i}(J)$ $(i=1, 2)$. Let us first suppose that f is a monomorphism. Then $f^{-1}(g_2^{-1}(0)) = 0$ and $\ker h \cap \operatorname{Im} g_1 = 0$. Since $r_{K_1}(J) = g_1(r_{M_1}(J)) = \operatorname{Im} g_1 \cap r_{K_1}(J)$, then $r_{K_1}(J) \subseteq \operatorname{Im} g_1$, which implies that $\ker h \cap r_{K_1}(J) = 0$. If $J=0$, the proposition clearly holds. Now assume that $J \neq 0$. Since A is a left YJ-ring satisfying the maximum condition on left annihilators, then by [25, Proposition 1] and [5, Proposition 3.31], $J=Z$ is nilpotent. If n is the smallest positive integer such that $J^n = 0$, then $J(J^{n-1} \ker h) = 0$ implies that $J^{n-1} \ker h \subseteq r_{K_1}(J) \cap \ker h = 0$. If $n-1 > 1$, we obtain similarly $J^{n-2} \ker h = 0$, and so on. Finally, we reach $\ker h = 0$, showing that h is a monomorphism. Now suppose that f is an isomorphism. Then ${}_A K_1 \approx {}_A h(K_1) \subseteq K_2$, which yields $K_2 = h(K_1) \oplus K$ for some submodule K. If $s\colon K_2 \to K_1$ is the A-homomorphism defined by $s(b) = h^{-1}(d)$, where $b = d+c$, $d \in h(K_1)$, $c \in K$, then s is well-defined (because h is a monomorphism). Since $K \cap h(K_1) = 0$, for any $y \in M_2$, $hg_1 f^{-1}(y) = hsg_2(y)$, which implies that $g_1 f^{-1} = sg_2$. In this case, the preceding argument shows that s is a monomorphism (because $g_2(r_{M_2}(J)) = r_{K_2}(J)$). Now $K = \ker s = 0$, and therefore h is an isomorphism from K_1 onto K_2.

Corollary 10.1. *If A is a left YJ-ring satisfying the maximum condition on left annihilators, E, Q injective left A-modules such that* $r_E(J)$ *is isomorphic to* $r_Q(J)$ *(as left A-modules), then* ${}_A E \approx {}_A Q$.

Recall that A is left GQ-injective [24] if, for any left ideal I which is isomorphic to a complement left ideal of A, every left A-homomorphism of I into A extends to an endomorphism of ${}_A A$. In view of [24, Proposition 1], Proposition 10 and Corollary 10.1 are valid for left GQ-injective rings satisfying the maximum condition on left annihilators.

Proposition 11. *Let A be a left GQ-injective ring satisfying any one of the following conditions*: (a) *every simple left A-module is either YJ-injective or projective*; (b) *A is a semi-prime ring with a maximal right ideal M such that for any* $u \in M$, *A/uM is a flat right A-module. Then A is regular.*

Proof. If A satisfies (a), then A is von Neumann regular by [24, Proposition 1] and Proposition 8(a). Now assume that (b) holds. It is sufficient to show that $J=0$ and then A will again be regular. Suppose that J contains a non-zero element v. Since A/vM_A is flat, we get $vM=0$. Therefore $M=r(v)$ and $vA (\approx A/M)$ is a minimal right ideal which is generated by an idempotent. But this contradicts $vA \subseteq J$.

We now prove a result on left duo rings.

Proposition 12. *If A is left duo, then A/Z is a reduced ring.*

Proof. Suppose that A/Z is not reduced. Then there exists $a \in A \setminus Z$ such that $a^2 \in Z$. If K is a left subideal of $l(a^2)$ such that $K \nsubseteq l(a)$, then $0 \neq Ka \subseteq l(a) \cap K$, which implies that $l(a)$ is essential in $l(a^2)$, and therefore so in A. But this contradicts $a \notin Z$.

Corollary 12.1. *Let A be a left duo ring.*

(1) *The following are equivalent*: 1) *A/Z is strongly regular*; 2) *A/Z is a left YJ-ring*; 3) *A/Z is a p-injective left A-module* (*see* [25, Proposition 1]).

(2) *If A is left or right perfect, then A/Z is a finite direct sum of division rings* (*see* [4, Lemma 22.12 B]).

(3) *Every homomorphic image of a quasi-injective left A/Z-module contains its singular submodule as a direct summand* (*see* [4, p. 88] *and* [19, p. 341]).

Left *CM*-rings [22] generalize left duo rings. G. F. Birkenmeier kindly remarked some time ago that if A is left non-singular, left *CM*, then either A is reduced or A has non-zero left socle.

Remark 6. If a left *CM*, left hereditary ring A contains an injective maximal left ideal E, then it is semi-simple Artinian.

Proof. By [22, Lemma 1.6 (3)], A is semi-prime, and every minimal left ideal of A is generated by an idempotent. Let U be a minimal left ideal such that $A = E \oplus U$. We prove first that $_A U$ is injective. Let $f: {}_A L \to {}_A U$ be a non-zero homomorphism, where L is a left ideal of A. Then, since $_A U$ is projective, $L = K \oplus \ker f$ with some left ideal K isomorphic to U. As is well known, there exists an element a in K such that $f(k) = ka$ for all $k \in K$. Let C be a relative complement of K containing $\ker f$. Obviously, if $K \oplus C = A$ then f can be extended to an A-homomorphism of A onto U. We assume henceforth $K \oplus C \neq A$, and choose a maximal left ideal M containing $K \oplus C$. Since A is a left *CM* ring and C is a relative complement of K in $_A M$, C is an ideal of M. Hence $\ker f \cdot a \subseteq \ker f \cdot K \subseteq CK \subseteq C \cap K = 0$, and so $f(x) = xa$ for all $x \in L$. We have thus seen that $A = E \oplus U$ is left self-injective. Recalling here that A is left hereditary, we see that every cyclic left A-module is injective. Hence A is semi-simple Artinian by Osofski's theorem.

It is known that if A is semi-prine left and right Goldie, then every complement one-sided ideal is an annihilator (see, e.g., [5, Theorem 2.38]).

Remark 7. Let A be a semi-prime ring with maximum condition on left annihilators. (Note that A is left non-singular by [5, Proposition 3.31].) If $r(C) \neq 0$ for every proper complement left ideal C of A, then A is a left Goldie ring. Actually, it is easy to see that $r(I) \neq 0$ for every non-essential left ideal I of A, and therefore every complement left ideal of A is a left annihilator (see, e.g., [14, Proposition XII. 4.7]). Hence A is left Goldie by [5, Theorem 3.14].

Remark 8. Careful scrutiny of the proof of [4, Definition and Proposition 19.59, Lemma 19.60, Lemma 19.61 and Theorem 19.62] shows the following generalization of [4, Theorem 19.62]: Let A be semi-prime left self-injective. (1) If T, K are ideals of A containing a prime ideal P and $_AT$, $_AK$ are non-singular, then either $T \subseteq K$ or $K \subseteq T$; (2) Let V be an ideal of A containing a prime ideal. If I_1, I_2 are ideals of A strictly containing V and are left non-singular p-injective, then $I_1 \cap I_2 \nsubseteq V$.

Remark 9. As in Remark 6, we can prove the following: (1) If a minimal left ideal U of a semi-prime ring A is an ideal, then $_AU$ is injective; (2) if a semi-prime ring A contains an injective maximal left ideal which is an ideal, then A is left self-injective. Also in case A is commutative (not necessarily semi-prime), we can prove the following: (3) Every non-nilpotent minimal ideal of A is injective; (4) if A contains an injective maximal ideal, then A is self-injective.

As usual, a left ideal I of A is called von Neumann regular in A if, for any $b \in I$, there exists $a \in A$ such that $b = bab$. Left self-injective regular rings are now characterized as follows:

Theorem 13. *The following conditions are equivalent*:

1) *A is left self-injective regular*;

2) *A is a left self-injective ring containing a maximal left ideal which is von Neumann regular in A*;

3) *There exists a maximal left ideal M of A such that for any $a \in M$, $l(a)$ is an injective left A-module*;

4) *A is a left self-injective ring containing a non-singular maximal left ideal.*

Proof. Obviously, 1)$\Rightarrow$2).

2)$\Rightarrow$3). Let I be a maximal left ideal which is von Neumann regular in A. For any $b \in I$, $_AAb$ is projective as a direct summand of $_AA$. Hence $l(b)$ is a direct summand of $_AA$ and so an injective left A-module.

3)$\Rightarrow$4). Obviously, $_AA = l(0)$ is injective. If $z \in Z(M)$, $l(z)$ is an essential left ideal which is injective, and so $A = l(z)$. Hence $z = 0$.

4)$\Rightarrow$1). Let M be a non-singular maximal left ideal. If M is essential in $_AA$, then A is left non-singular [19, Lemma 2], which implies that A is regular in this case. Now suppose that $M \oplus U = A$, where $M = Ae$, $e = e^2 \in A$, $U = A(1-e)$. If $0 \neq z \in Z$, $z = ze + z(1-e)$, then $ze \in Z \cap M = Z(M) = 0$, which yields $z = z(1-e) \in U$, whence $U = Az$. Since $_AU$ is projective, $l(z)$ is a direct summand of $_AA$, which implies $z = 0$, a contradiction. This proves that, in either case, A is left non-singular, and hence regular.

Remark 10. In view of [24, Proposition 1], the proof of Proposition 13 enables us to see that if A is left GQ-injective with a non-singular maximal left ideal, then A is von Neumann regular.

Remark 11. By making use of Remark 9(4) and Proposition 13, we can see that the following conditions are equivalent for a commutative ring A: 1) A is a self-

injective regular ring with non-zero socle; 2) A contains an injective maximal ideal which is von Neumann regular; 3) A contains a non-singular injective maximal ideal.

Remark 12. By making use of Remark 9(3), [19, Lemma 2] and [20, Remark 2], we can prove the following: Let A be a commutative ring. Then A is a von Neumann regular ring with non-zero socle if (and only if) A is a *p.p.* ring containing a non-singular finitely generated p-injective maximal ideal.

We return to a sufficient condition for a classical left quotient ring to be regular.

Theorem 14. *Suppose that A is a left p.p. ring having a classical left quotient ring Q. If Q is left p-injective, then it is von Neumann regular.*

Proof. In view of [7, Proposition 1], Q is a left *p.p.* ring. Then, every cyclic left Q-module is p-injective by [20, Remark 2], and therefore Q is regular.

Recall that A is a local ring if A has a unique maximal left (right) ideal.

Proposition 15. *Let A be a left p-injective ring. Then the following conditions are equivalent:*

1) A *is a left duo ring with a maximal left ideal M with $M^2=0$;*

2) A *is either a division ring or a quasi-Frobenius local ring with a unique non-trivial left ideal.*

Proof. 1)$\Rightarrow$2). It is enough to consider the case that M is non-zero. Let b, d be arbitrary non-zero elements in M. Then, as is easily seen, $l(b)=M=l(d)$, and there exists an isomorphism $g: Ab\rightarrow Ad$ with $g(b)=d$. Since A is left p-injective, we can find $y\in A$ such that $by=d$. Hence, noting that A is left duo, we get $Ad=Aby\subseteq Ab$; similarly, $Ab\rightarrow Ad$ and therefore $Ab=Ad$. This proves that M is a minimal left ideal of A. Furthermore, if I is a non-zero left ideal of A different from M, then $M=MA=M(M+I)\subseteq I$, whence $I=A$ follows. Thus M is the unique non-trivial left ideal of A, and A is left self-injective; and therefore A is quasi-Frobenius.

2)$\Rightarrow$1). Let $M=J$, which is a maximal left ideal of A. Suppose $M^2\neq 0$. Then M is the unique non-trivial left ideal of A and $JM=M^2=M$. But this forces a contradiction $M=0$.

We add a further result on p-injectivity.

Proposition 16. *If A is semi-prime and left p-injective, then the centre of A is von Neumann regular.*

Proof. Given an element c of the centre C of A, we can define a left A-homomorphism $g: Ac^2\rightarrow A$ by $g(ac^2)=ac$. Then, there exists $d\in A$ such that $c=g(c^2)=c^2d$. Now, as is well-known, $c=c^2v$, for some $v\in C$.

Corollary 16.1. *Let A be semi-prime. If every injective left A-module is flat, then the centre of A is von Neumann regular.*

Proof. By [9, Corollary 2.5 and Theorem 3.3], we see that $l(r(Aa)) = Aa$ for any $a \in A$. Then A is right p-injective by a result of Ikeda-Nakayama (see, e.g. [8, Corollary 1]), and hence the centre of A is regular by Proposition 16.

Remark 13. From the proof of Proposition 16, we can easily see that if A is semi-prime and if Ac is a left annihilator for any c in the centre C of A, then C is von Neumann regular.

Remark 14. Rings whose cyclic singular left modules are flat must be von Neumann regular [8, Corollary 5] but rings whose cyclic singular left modules are injective need not be semi-simple Artinian. It seems fitting to mention the last parallel between injectivity and p-injectivity: (1) A left self-injective ring whose cyclic singular left modules are injective is semi-simple Artinian (apply [4, Theorem 19.46 A and Lemma 19.60]; (2) A left p-injective ring whose cyclic singular left modules are p-injective is von Neumann regular ([8, Corollary 5]).

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