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Metastable homotopy groups of Sp (n)

Dedicated to Professor Hiroshi Toda on his 60th Birthday

By

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§ 0. Introduction and the statements of results

Let $Sp(n)$ be the n-th symplectic group. The homotopy groups of $Sp(n)$, $\pi_i(Sp(n))$, have been studied by various authors. If $i<4n+2$, then $\pi_i(Sp(n))$ is well known by the Bott periodicity theorem [2]. For $4n+2\leqq i\leqq 4n+8$, $\pi_i(Sp(n))$ are determined in [2], [3], [8], [6]. For $i=4n+9$, $\pi_i(Sp(n))$ is determined by Ōshima [12]. In this paper we determine the 2-primary component of the group $\pi_i(Sp(n))$ for $4n+10\leqq i\leqq 4n+15$. In the previous paper [10], we reduced the calculation of $\pi_i(Sp(n))$ to that of $\pi_{i+1}(Sp/Sp(n))$ for some range of i, where $Sp=\varinjlim Sp(n)$ and $Sp/Sp(n)$ is the orbit space. Since in the metastable range of i, $4n+2\leqq i\leqq 8n+4$, $\pi_i(Sp/Sp(n))$ is isomorphic to the $\pi_i^s(Q_{n+1}^\infty)$, the stable homotopy group of the stunted quasi-quaternionic projective space Q_{n+1}^∞, we carry out the calculation of $\pi_i^s(Q_{n+1}^\infty)$ for the range $4n+11\leqq i\leqq 4n+16$.

Before the statement of the main result, we prepare some notation. For $n\geqq 1$ and $s\geqq 1$, define a number $M(n, s)$ by the following equation [16]:

$$(e^t+e^{-t}-2)^s=\sum_{n\geqq 1}\frac{(2s)!}{(2n)!}M(n, s)t^{2n}.$$

Then it is easy to see that $M(n, s)$ is an integer [16]. Define a number $d^A(n, m)$ by

$$d^A(n, m)=\underset{s\geqq m+1}{\text{g.c.d.}}\left\{\frac{a(n-1)}{a(n-s)}(2s-1)!M(n, s)\right\},$$

where $a(k)$ is 1 or 2 according as k is even or odd. Let $d_2^A(n, m)$ be the index of 2 in the prime decomposition of the integer $d^A(n, m)$. In the following theorem, $\pi_*(\ )$ means the 2-component of homotopy groups, the symbol + means the direct sum and (l, k) means the greatest common divisor of integers l and k. Our main results are as follows;

Main theorem.

1) *If* $n\geqq 2$, *then*

$$\pi_{4n+10}(Sp(n))\cong\begin{cases} Z/2+Z/2+Z/2^{d_2^A(n+3,n)} & \text{if } n\equiv 1(4),\\ Z/2+Z/2^{d_2^A(n+3,n)}, & \text{otherwise}.\end{cases}$$

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2) *If $n \geqq 2$, then*

$$\pi_{4n+11}(Sp(n)) \cong \begin{cases} Z/2+Z/2+Z/2 & \text{if} \quad n \equiv 1(4), \\ Z/2+Z/2 & \text{otherwise}. \end{cases}$$

3) *If $n \geqq 2$, then*

$$\pi_{4n+12}(Sp(n)) \cong \begin{cases} Z/2+Z/2+Z/2 & \text{if} \quad n \equiv 0(2), \\ Z/2+Z/2 & \text{if} \quad n \equiv 1(2). \end{cases}$$

4) *If $n \geqq 3$, then*

$$\pi_{4n+13}(Sp(n)) \cong \begin{cases} Z/16 & \text{if} \quad n \equiv 1(2), \\ Z/2+Z/8+Z/64 & \text{if} \quad n \equiv 6(8), \\ Z/2+Z/32 & \text{if} \quad n \equiv 2(8), \\ Z/2+Z/2+Z/8 & \text{if} \quad n \equiv 0(8), \\ Z/2+Z/2+Z/4+Z/k & \text{if} \quad n \equiv 4(8), \end{cases}$$

where $k=16/(16, (n+4)/8)$.

5) *If $n \geqq 3$, then*

$$\pi_{4n+14}(Sp(n)) \cong \begin{cases} Z/2+Z/2+Z/2^{d_2^A(n+4,n)}, & \text{if} \quad n \equiv 0(4), \\ Z/2+Z/2^{d_2^A(n+4,n)}, & \text{otherwise}. \end{cases}$$

6) *If $n \geqq 3$, then*

$$\pi_{4n+15}(Sp(n)) \cong \begin{cases} Z/2+Z/2+Z/2 & \text{if} \quad n \equiv 0(4), \\ Z/2+Z/2 & \text{otherwise}. \end{cases}$$

Note that for exceptional value of n in the above theorem, those homotopy groups are already known by [14] and [7].

Since our methods for calculation of $\pi_{4n+i}^s(Q_{n+1}^\infty)$ are e-invariant methods, in §1 we recall the basic facts about e-invariants. In §2 we apply the e-invariants to the stunted (quasi-) projective spaces HP_{n+1}^{n+k} or Q_{n+1}^{n+k} and investigate the properties of the attaching maps of their top cells, which we need in §§4–5. In §3 we recall the relations among $\pi_i(Sp/Sp(n))$, $\pi_i^s(Q_{n+1}^\infty)$ and $\pi_i(Sp(n))$. In §§4–5 we carry out the calculation of the 2-component of $\pi_i^s(Q_{n+1}^\infty)$ for $4n+11 \leqq i \leqq 4n+16$.

§ 1. The e-invariant

In this section we recall the basic properties of e-invariant (Cf. [1], [15], [16]). Let $K^*(\ )$ be the reduced complex K theory and $KO^*(\ )$ be the reduced real K theory. We denote its representative spectrum by K or KO. Let HQ be the representative spectrum of the cohomology theory with rational coefficients, $H^*(\ ;Q)$. For $i \in Z$ (integers), there is a stable map

$$CH_i\colon K \longrightarrow \Sigma^{2i}HQ,$$

such that for a large j the $2j$-th component of CH_i,

$$(CH_i)_{2j}\colon BU \longrightarrow K(Q, 2i+2j),$$

is the usual universal $(i+j)$-th chern character. We denote the composite

$$KO \xrightarrow{c} K \xrightarrow{CH_{2i}} \Sigma^{4i}HQ,$$

by PH_i and call it the i-th Pontrjagin character, where $c\colon KO \to K$ is the complexification map.

Let $n \geqq 1$. Let X be a spectrum such that

$$H^{4n}(X; Q) = H^{4n-1}(X; Q) = 0. \tag{1.1}$$

For $\alpha \in \pi_{4n-1}(X)$, there exists a homomorphism

$$e(\alpha)\colon KO^{4*}(X) \longrightarrow Q/Z,$$

defined by

$$e(\alpha)(\beta) = \langle \Sigma^{4s}PH_{n-s}, \beta, \alpha \rangle / a(n-s),$$

where $\beta \in KO^{4s}(X)$, $\langle \Sigma^{4s}PH_{n-s}, \beta, \alpha \rangle \subseteq \pi_{4n}(\Sigma^{4n}HQ) = Q$ is the stable Toda bracket [14] associated with the sequence

$$S^{4n-1} \xrightarrow{\alpha} X \xrightarrow{\beta} \Sigma^{4s}KO \xrightarrow{\Sigma^{4s}PH_{n-s}} \Sigma^{4n}HQ,$$

and $a(i) = 1$ or 2 according as i is even or odd. It is easy to see that under the assumption (1.1) the e-invariant $e(\alpha)$ is well defined. Now the following proposition is well known (See [1], [15] and [16]).

Proposition 1.1. *Let X be a spectrum such that $H^{4n}(X; Q) = H^{4n-1}(X; Q) = 0$. Let $\alpha \in \pi_{4n-1}(X)$. Then*

1) *$e(\alpha)\colon KO^{4s}(X) \to Q/Z$ is a homomorphism.*
2) *$e(\alpha + \alpha') = e(\alpha) + e(\alpha')$, where $\alpha' \in \pi_{4n-1}(X)$.*
3) *When $X = S^{4s}$, $e(\alpha)(g)$ is equal to the Adams e'_R invariant [1] up to sign, where $g \in KO^{4s}(S^{4s})$ is the standard generator.*
4) *$e(\alpha)(\gamma\beta) = PH_k(\gamma)(e(\alpha)(\beta))$, where $\gamma \in \pi_{4k}(KO)$ and $\beta \in KO^{4s}(X)$.*
5) *(Naturality) Let Y be a spectrum which satisfies that $H^{4n}(Y; Q) = H^{4n-1}(Y; Q) = 0$. Let $f\colon X \to Y$ be a map. Then for any $\beta \in KO^{4s}(Y)$, $e(f_*\alpha)(\beta) = e(\alpha)(f^*\beta)$.*
6) *Let $\gamma\colon S^{4(n+m)-2} \to S^{4m-1}$ and $\alpha\colon S^{4m-1} \to X$ such that $q\alpha = q\gamma = 0$ for some integer q. Then*

$$e(\langle \alpha, q, \gamma \rangle)(\beta) = (q \cdot e(\alpha)(\beta)) e(\Sigma\gamma)(g_{m-s}),$$

where g_{m-s} is a generator of $KO^{4s}(S^{4m})$.

§ 2. Stunted quaternionic (quasi-) projective spaces

Let HP^n be the quaternionic n dimensional projective space. We denote the stunted projective space HP^n/HP^{m-1} by HP^n_m $(n\geqq m)$. Let Q^n be the $4n-1$ dimensional quaternionic quasi-projective space [5]. We denote the stunted quasi-projective space Q^n/Q^{m-1} by Q^n_m. We denote the attaching map of the top cell in HP^{n+k}_{n+1} (resp. Q^{n+k}_{n+1}) by ${}_H\varphi^{n+k}_{n+1}$ (resp. ${}_Q\varphi^{n+k}_{n+1}$). Recall that $KO^*(HP^{n+k}_{n+1})$ is the free $KO^*(S^0)$ module generated by x^{n+i} for $1\leqq i\leqq k$ and $KO^*(Q^{n+k}_{n+1})$ is the free $KO^*(S^0)$ module generated by x_{n+i} for $1\leqq i\leqq k$. Originally $x^i\in KO^{4i}(HP^\infty_+)$ is the i-fold iterated product of the first KO theoretic Pontrjagin class $x\in KO^4(HP^\infty_+)$, where X_+ means a space with a disjoint base point. These generators can be chosen so that $x^{i-1}\in KO^{4(i-1)}(HP^{l-1}_+)$ corresponds to $x_i\in KO^{4i-1}(Q^l)$ under the Thom isomorphism (Cf. [4]). The following theorem is essential in our later calculation and has been proved in [4] or [16].

Theorem 2.1. 1) *PH_{n-s}: $KO^{4s}(HP^\infty_+)\to H^{4n}(HP^\infty_+;Q)$ is given by*

$$PH_{n-s}(x^s)=\frac{(2s)!}{(2n)!}M(n,s)\cdot(x^H)^n,$$

where $x^H\in H^4(HP^\infty_+;Z)$ is the standard generator. Similarly PH_{n-s}: $KO^{4s-1}(Q^\infty)\to H^{4n-1}(Q^\infty;Q)$ is given by

$$PH_{n-s}(x_s)=\frac{(2s-1)!}{(2n-1)!}M(n,s)(x^H)_n,$$

where $(x^H)_n\in H^{4n-1}(Q^\infty;Z)$ corresponds to $(x^H)^{n-1}\in H^{4(n-1)}(HP^\infty_+;Z)$ under the Thom isomorphism.
2) *Let $n\geqq m+1$ and $m\geqq 1$. For any s such that $m\leqq s\leqq n-1$,*

$$e({}_H\varphi^n_m)(x^s)=\frac{(2s)!M(n,s)}{(2n)!a(n-s)},\quad \text{for}\quad x^s\in KO^{4s}(HP^n_m),$$

$$e({}_Q\varphi^n_m)(x_s)=\frac{(2s-1)!M(n,s)}{(2n-1)!a(n-s)},\quad \text{for}\quad x_s\in KO^{4s-1}(Q^n_m).$$

Examples 2.2.
1) $e({}_H\varphi^{n+2}_{n+1})(x^{n+1})=(n+1)/24$,
2) $e({}_H\varphi^{n+3}_{n+1})(x^{n+1})=(5n+4)(n+1)/(2\cdot 6!)$,
3) $e({}_H\varphi^{n+4}_{n+1})(x^{n+1})=(35n^2+49n+18)(n+1)/(2\cdot 9!)$,
4) $e({}_Q\varphi^{n+2}_{n+1})(x_{n+1})=(n+2)/24$,
5) $e({}_Q\varphi^{n+3}_{n+1})(x_{n+1})=(5n+4)(n+3)/(2\cdot 6!)$,
6) $e({}_Q\varphi^{n+4}_{n+1})(x_{n+1})=(35n^2+49n+18)(n+4)/(2\cdot 9!)$,

Proof. By definition of the number $M(n,s)$ the following is a permanent equation with respect to a variable z;

$$\left(\sum_{i\leqq 1}\frac{2z}{(2i)!}\right)^s=\sum_{n\geqq 1}\frac{(2s)!}{(2n)!}M(n,s)z^n.$$

Comparing the both sides in the above equation, the assertions are easily verified by direct calculation and Theorem 2.1.

Proposition 2.3. *For* $1 \leqq i \leqq 3$ *let* X^{n+3}_{n+i} *be* HP^{n+3}_{n+i} *or* ΣQ^{n+3}_{n+i}. *Then the e-invariant*

$$e\colon \pi^s_{4n+15}(X^{n+3}_{n+i}) \longrightarrow \mathrm{Hom}\,(KO^{4*}(X^{n+3}_{n+i}), Q/Z)$$

is monomorphic.

Proof. Consider the cofiber sequence:

$$X^{n+2}_{n+2} \longrightarrow X^{n+3}_{n+2} \longrightarrow X^{n+3}_{n+3}.$$

Then we have the following commutative diagram:

$$\begin{array}{ccccc} \pi^s_{4n+15}(X^{n+2}_{n+2}) & \longrightarrow & \pi^s_{4n+15}(X^{n+3}_{n+2}) & \longrightarrow & \pi^s_{4n+15}(X^{n+3}_{n+3}) \\ \downarrow{\scriptstyle e_1} & & \downarrow{\scriptstyle e_2} & & \downarrow{\scriptstyle e_3} \\ 0\to \mathrm{Hom}(KO^{4*}(X^{n+2}_{n+2}), Q/Z) & \to & \mathrm{Hom}(KO^{4*}(X^{n+3}_{n+2}), Q/Z) & \to & \mathrm{Hom}(KO^{4*}(X^{n+3}_{n+3}), Q/Z), \end{array}$$

where e_i is the e-invariant and horizontal sequences are exact. Since e_1 and e_3 are equal (up to sign) to the usual e'_R-invariants, both e_1 and e_3 are monomorphic. Therefore so is e_2. Similarly, considering the cofiber sequence:

$$X^{n+1}_{n+1} \longrightarrow X^{n+3}_{n+1} \longrightarrow X^{n+3}_{n+2},$$

we have the desired results.

Let j_3 (resp. j_7) be a generator of $\pi^s_3(S^0) \cong Z/24$ (resp. $\pi^s_7(S^0) \cong Z/240$) such that $e'_R(j_3) = 1/24$ (resp. $e'_R(j_7) = 1/240$). Then both HP^{n+2}_{n+1} and $\Sigma^5 Q^{n+1}_n$ are homotopy equivalent to the mapping cone of $(n+1)j_3$, that is, $S^{4n+4} \cup_{(n+1)j_3} e^{4n+8}$. Thus we identify them.

Proposition 2.4.

$${}_H\varphi^{n+3}_{n+1} = \Sigma^5_Q \varphi^{n+2}_n + i_{0*} j_7,$$

where i_0 *is the inclusion map of the bottom sphere.*

Proof. From Proposition 2.3 it is enough to show that

$$e(\Sigma^5_Q \varphi^{n+2}_n + i_{0*} j_7) = e({}_H\varphi^{n+3}_{n+1}),$$

under the identification $HP^{n+2}_{n+1} \approx \Sigma^5 Q^{n+1}_n$. Note that under this identification the element $x^i \in KO^{4i}(HP^{n+2}_{n+1})$ corresponds to the element $x_{i-1} \in KO^{4i}(\Sigma^5 Q^{n+1}_n)$ for $i = n+1$ or $n+2$. Then the above equation easily follows by Proposition 1.1 and Example 2.2.

Proposition 2.5. *For* $1 \leqq j \leqq 4$, *there exist stable maps,*

$$f_j\colon HP^{n+j}_{n+1} \longrightarrow \Sigma Q^{n+j}_{n+1},$$

such that for $j \leqq 3$ *the following diagram commutes:*

$$\begin{array}{ccccc} S^{4n+4j+3} & \xrightarrow{H\varphi_{n+1}^{n+j+1}} & HP_{n+1}^{n+j} & \longrightarrow & HP_{n+1}^{n+j+1} \\ \Big\downarrow{\scriptstyle (n+1)\cdots(n+j)} \; {\scriptstyle (\mathrm{I})_j} & & \Big\downarrow{\scriptstyle (n+j+1)f_j} \; {\scriptstyle (\mathrm{II})_j} & & \Big\downarrow{\scriptstyle f_{j+1}} \\ S^{4n+4j+3} & \xrightarrow{\Sigma_Q\varphi_{n+1}^{n+j+1}} & \Sigma Q_{n+1}^{n+j} & \longrightarrow & \Sigma Q_{n+1}^{n+j+1}, \end{array} \tag{2.6}$$

where the horizontal lines are cofiber sequences. In particular, for $1 \leqq i \leqq j$,

$$f_j^*(x_{n+i}) = ((n+1)\cdots(n+j)/(n+i))x^{n+i}, \tag{2.7}$$

where $f_j^*\colon KO^{4(n+i)}(\Sigma Q_{n+1}^{n+j}) \to KO^{4(n+i)}(HP_{n+1}^{n+j})$ *is the homomorphism induced by* f_j.

Proof. By induction on j. For $j=1$, we take the identity map of S^{4n+4} as f_1 because $HP_{n+1}^{n+1} = \Sigma Q_{n+1}^{n+1} = S^{4n+4}$. Clearly for $j=1$ (2.7) holds and the diagram $(\mathrm{I})_1$ commutes. Suppose that for some k there exists a map f_k such that the diagram $(\mathrm{I})_k$ commutes and (2.7) holds. Then clearly there exists a map $f_{k+1}\colon HP_{n+1}^{n+k+1} \to \Sigma Q_{n+1}^{n+k+1}$ such that the diagram $(\mathrm{II})_k$ commutes. Then from Theorem 2.1, investigating the Pontrjagin character, it follows that for $j=k+1$, (2.7) holds. Now using (2.7) for $j=k+1$, by easy computation we have

$$e((n+k+1)f_k \circ {}_H\varphi_{n+1}^{n+k+1}) = e((n+1)\cdots(n+k)\Sigma_Q\varphi_{n+1}^{n+k+1}).$$

Therefore, when $k+1 \leqq 4$, by Proposition 2.3, we see that the diagram $(\mathrm{I})_{k+1}$ commutes. This completes the proof of Proposition 2.5.

§ 3. Metastable homotopy groups of Sp(n)

The following Proposition are proved in [10, Proposition 2.4].

Proposition 3.1. *Let* $i > 4n+1$. *If* $i \equiv 0$, 1, 3 *or* 7 mod 8, *then* $\pi_i(Sp(n))$ *is isomorphic to* $\pi_{i+1}(Sp/Sp(n))$. *If* $i \equiv 4$ *or* 5 mod 8, *then* $\pi_i(Sp(n))$ *is isomorphic to* $\pi_{i+1}(Sp/Sp(n)) + Z/2$.

Except the case $m=5$, the following theorem is proved in [10, Theorem II].

Theorem 3.2. *Let* $n \geqq 1$ *and* $1 \leqq m \leqq 5$. *Then, in the 2-component,* $\pi_{4(n+m)-2}(Sp(n))$ *is isomorphic to the direct sum of* Tor $(\pi_{4(n+m)-1}(Sp/Sp(n)))$ *and a cyclic group* $Z/2^{d_2^A(n+m,n)}$.

Since Q_{n+1}^{n+m} is a subcomplex of the Stiefel manifold $Sp(n+m)/Sp(n)$, and since the pair $(Sp(n+m)/Sp(n), Q_{n+1}^{n+m})$ is $(8n+9)$-connected [5], by the suspension theorem it is obvious that $\pi_{4n+i+1}(Sp/Sp(n))$ is isomorphic to $\pi_{4n+i+1}^s(Q_{n+1}^\infty)$ for $i \leqq 4n+3$. So from Proposition 3.1 and Theorem 3.2, in the metastable range for our purpose it is enough to compute the group $\pi_{4n+i+1}^s(Q_{n+1}^\infty)$. This can be done in the following sections.

Proof of Theorem 3.2. For the proof it is enough (see [10]) to show that for $m=5$, the stable quaternionic James number $X^s\{n+m, m\}$ ([11] or [10]) is equal

to the order of e-invariant of ${}_Q\varphi_{n+1}^{n+m}$ which is easily obtained by Theorem 2.1 and we denote it by $X^A\{n+m, m\}$ [10].
Consider the following Atiyah-Hirzebruch spectral sequence:

$$E_{p,q}^2 = H_p(Q^\infty; Z) \otimes \pi_q^s(S^0),$$

which converges to $\pi_*^s(Q^\infty)$, where all spectra are localized at (2). We denote the generator of $H_{4p-1}(Q^\infty; Z)$ by γ_p. Then we have

Lemma 3.3.
$d^8(\gamma_{n+1} \otimes \eta\kappa) = \gamma_{n-1} \otimes \varepsilon\kappa$ *if* $n \equiv 1$ *or* $2 \bmod 4$ *and* $= 0$, *otherwise.*
$d^8(\gamma_{n+3} \otimes \bar{\nu}) = \gamma_{n+1} \otimes \eta\kappa$, *if* $n \equiv 0$ *or* $3 \bmod 4$ *and* $= 0$, *otherwise.*
Here $\eta \in \pi_1^s(S^0)$, $\kappa \in \pi_{14}^s(S^0)$, $\varepsilon \in \pi_8^s(S^0)$ *and* $\bar{\nu} \in \pi_8^s(S^0)$ *are some generators of the 2-primary component of* $\pi_*^s(S^0)$ (*see* [14]).

Proof. Consider the following spectral sequence;

$$E_{p,q}^2(X) = H_p(X; Z) \otimes \pi_q^s(S^0) \Longrightarrow \pi_*^s(X),$$

for $X = Q^\infty$, HP^∞ or MSp (the symplectic Thom spectrum). As is well known there is a stable map j: $HP^\infty \to \Sigma^4 MSp$ such that $j_*\beta_{n+1} = b_n$, where $\beta_{n+1} \in H_{4n+4}(HP^\infty; Z) \cong Z\{\beta_1, \beta_2, \ldots\}$ and $b_n \in H_{4n}(MSp; Z) \cong Z[b_1, b_2, \ldots]$ are standard generators. Let $\alpha \in \pi_k^s(S^0)$ and ν be a generator of $\pi_3^s(S^0) \cong Z/8$. Now under the assumption that $(n+1)\nu\alpha = 0$, Proposition 2.4 implies that $d^8(\gamma_{n+1} \otimes \alpha) = \gamma_{n-1} \otimes \delta$ for some $\delta \in \pi_{k+7}^s(S^0)$ if and only if $d^8(\beta_{n+2} \otimes \alpha) = \beta_n \otimes (\delta + \alpha\sigma)$, where $\sigma \in$ is a generator of $\pi_7^s(S^0) \cong Z/16$. On the other hand, if $d^8(\beta_{n+2}) \otimes \alpha) = \beta_n \otimes (\delta + \alpha\sigma)$ then it holds that in $E_{*,*}^*(MSp)$ $d^8(b_{n+1} \otimes \alpha) = b_{n-1} \otimes (\delta + \alpha\sigma)$, moreover, $d^8(S^{\Delta_{n-1}}(b_{n+1}) \otimes \alpha) = \delta + \alpha\sigma$, where $S^{\Delta_{n-1}}$ is a certain Landweber Novikov operation in MSp-theory (See, for example [9]). Now it is not difficult [9] to see that

$$d^8(S^{\Delta_{n-1}}(b_{n+1}) \otimes \eta\kappa) = d^8\left(\left(nb_2 + \binom{n}{2} b_1^2\right) \otimes \eta\kappa\right) = (n(n+1)/2)\varepsilon\kappa.$$

This proves the first assertion of Lemma 3.3. Similarly the second assertion follows.

Since $X^s\{n+5, 4\} = X\{n+5, 4\} = X^A\{n+5, 4\}$, there is an element $\delta_n \in \pi_{4n+18}^s(S^{4n+3})$ such that $i_*\delta_n = X^A\{n+5, 4\}_Q\varphi_{n+1}^{n+5}$ (see the diagram below).

$$\begin{array}{ccccccc}
 & & S^{4n+18} & \xrightarrow{X^A\{n+5,4\}} & S^{4n+18} & \searrow^{{}_Q\varphi_{n+2}^{n+5}} & \\
 & & \downarrow{\scriptstyle \delta_n} & & \downarrow{\scriptstyle {}_Q\varphi_{n+1}^{n+5}} & & \\
\Sigma^{-1}Q_{n+2}^{n+4} & \xrightarrow{\bar{\varphi}} & S^{4n+3} & \xrightarrow{i_0} & Q_{n+1}^{n+4} & \longrightarrow & Q_{n+2}^{n+4},
\end{array}$$

It is not difficult to see that $X^s\{n+5, 5\} = X^A\{n+5, 5\}$ if and only if the order of δ_n in $\pi_{4n+18}^s(S^{4n+3})/\mathrm{Im}\,\bar{\varphi}_*$ is equal to the order of the e-invariant of δ_n. In terms of the spectral sequence, the above diagram implies that

$$d^{16}(X\{n+5, 4\} \otimes \gamma_{n+5}) = \gamma_{n+1} \otimes \delta_n \quad \text{in} \quad E^{16}\text{-term}.$$

Since $\delta_n \in \pi_{15}^s(S^0) \cong Z/16\{\rho\} + Z/2\{\eta\kappa\}$, Lemma 3.3 implies that if $n \equiv 0$ or $3 \bmod 4$

then $\eta\kappa \in \operatorname{Im}\bar{\varphi}_*$ and that if $n \equiv 1$ or 2 mod 4 then δ_n can not be equal to $\eta\kappa$. Thus the order of δ_n in $\pi^s_{4n+18}(S^{4n+3})/\operatorname{Im}\bar{\varphi}_*$ is equal to the order of the e-invariant of δ_n and consequently the differential $d^{16}\colon E^{16}_{4n+19,0} \to E^{16}_{4n+3,15}$ can be completely determined by e-invariants. This proves that $X^A\{n+5, 5\} = X^s\{n+5, 5\}$.

§ 4. Computation

From now on all spaces or spectra should be considered to be localized at (2) since our interest are in the 2-component. We freely use the structure or notations in [14] about the 2-component of the stable homotopy groups of spheres.

1) $\operatorname{Tor}(\pi^s_{4n+11}(Q^\infty_{n+1}))$.

Consider the following diagram:

$$\begin{array}{ccccccc}
 & & S^{4n+6} & & & & \\
 & & \downarrow{\scriptstyle (n+2)\nu} & & & & \\
 & & S^{4n+3} & & & & \\
 & & \downarrow{\scriptstyle i_0} & & & & \\
S^{4n+10} & \xrightarrow{{}_Q\varphi^{n+3}_{n+1}} & Q^{n+2}_{n+1} & \xrightarrow{i} & Q^{n+3}_{n+1} & \xrightarrow{p} & S^{4n+11}. \\
 & & \downarrow & & & & \\
 & & S^{4n+7} & & & &
\end{array}$$

Applying $\pi^s_{4n+11}(\ )$, we have

$$\operatorname{Tor}(\pi^s_{4n+11}(Q^{n+3}_{n+1})) \cong \pi^s_{4n+11}(Q^{n+2}_{n+1})/({}_Q\varphi^{n+3}_{n+1}\circ\eta).$$

It is easy to see that $i_{0*}\colon \pi^s_{4n+11}(S^{4n+3}) \to \pi^s_{4n+11}(Q^{n+2}_{n+1})$ is an isomorphism.

Lemma 4.1.

$${}_Q\varphi^{n+3}_{n+1}\circ\eta = \begin{cases} i_{0*}\bar{\nu} & \text{if } n \equiv 3(4), \\ i_{0*}\eta\sigma & \text{if } n \equiv 2(4), \\ 0 & \text{if } n \equiv 1(4). \\ i_{0*}\varepsilon & \text{if } n \equiv 0(4). \end{cases}$$

Proof. By Proposition 2.4 it is enough to show that

$${}_H\varphi^{n+4}_{n+2}\circ\eta = (n+2)\varepsilon + \binom{n+2}{2}\bar{\nu}.$$

But the above equation has already obtained in [13], and also follows from the second assertion of Lemma 3.3.

Therefore we have

$$\mathrm{Tor}\,(\pi^s_{4n+11}(Q^\infty_{n+1}))\cong\begin{cases} Z/2+Z/2 & \textit{if}\quad n\equiv 1(4)\,,\\ Z/2 & \textit{otherwise}\,.\end{cases}$$

2) $\mathrm{Tor}\,(\pi^s_{4n+15}(Q^\infty_{n+1}))$ for $n\geqq 3$.

Now consider the following diagram:

$$\begin{array}{ccccccc} S^{4n+3} & \xrightarrow{i_0} & Q^\infty_{n+1} & \longrightarrow & Q^\infty_{n+2} & \longrightarrow & S^{4n+4} \\ & & & & \uparrow{\scriptstyle i_0} & \nearrow{\scriptstyle (n+2)\nu} & \\ & & & & S^{4n+7}. & & \end{array}$$

Since $\pi^s_{12}(S^0)=0$, $(n+2)\nu_*\colon \pi^s_{4n+15}(S^{4n+7})\to\pi^s_{4n+15}(S^{4n+4})$ is trivial and i_{0*}: $\pi^s_{4n+15}(S^{4n+7})\to\mathrm{Tor}\,(\pi^s_{4n+15}(Q^{n+4}_{n+2}))$ is epic, applying $\pi^s_{4n+15}(\ )$ to the above diagram we have

$$\mathrm{Tor}\,(\pi^s_{4n+15}(Q^\infty_{n+1}))\cong\mathrm{Tor}\,(\pi^s_{4n+15}(Q^\infty_{n+2}))\,.$$

This completes the proof of 1) and 5) in Main Theorem in §0.

3) $\pi^s_{4n+12}(Q^\infty_{n+1})$ for $n\geqq 2$.

We shall prove

Theorem 4.2.

If $n\geqq 2$, *then* $\pi^s_{4n+12}(Q^{n+3}_{n+1})$ *is isomorphic to* $Z/2+Z/2$ *if* $n\not\equiv 1 \bmod 4$ *and is isomorphic to* $Z/2+Z/2+Z/2$, *otherwise.*

Consider the homotopy exact sequence associated with the cofiber sequence;

$$S^{4n+3}\xrightarrow{i_0} Q^{n+2}_{n+1}\xrightarrow{p} S^{4n+7}.$$

Then we have

Lemma 4.3. $\pi^s_{4n+12}(Q^{n+2}_{n+1})\cong Z/2+Z/2$ *if n is odd, and* $Z/2+Z/2+Z/2$ *if n is even.*

Now consider the cofiber sequence

$$S^{4n+10}\xrightarrow{{}_Q\varphi^{n+3}_{n+1}} Q^{n+2}_{n+1}\longrightarrow Q^{n+3}_{n+1}\,.$$

Using Lemma 4.1 it is easy to see that if $n\not\equiv 1$ mod 4, then $\pi^s_{4n+12}(Q^{n+3}_{n+1})\cong Z/2+Z/2$ and that if $n\equiv 1$ mod 4, there is a short exact sequence;

$$0\longrightarrow Z/2+Z/2\longrightarrow \pi^s_{4n+12}(Q^{n+3}_{n+1})\longrightarrow Z/2\longrightarrow 0\,,$$

where the last $Z/2$ is generated by $\eta\in\pi^s_{4n+12}(S^{4n+11})\cong Z/2$. In order to see that the above group extension is trivial, as is well-known, it is enough to show that the Toda bracket $\langle {}_Q\varphi^{n+3}_{n+1}, \eta, 2\rangle\ni 0$.

Lemma 4.4. *If* $n\equiv 1$ mod 4 *then the Toda bracket* $\langle {}_Q\varphi^{n+3}_{n+1}, \eta, 2\rangle$ *contains* 0.

Proof. In Proposition 2.5 we proved that there is a map $f_2\colon HP^{n+2}_{n+1}\to\Sigma Q^{n+2}_{n+1}$ such that $(n+1)(n+2)\Sigma_Q\varphi^{n+3}_{n+1}=(n+3)f_2\circ_H\varphi^{n+3}_{n+1}$. But if $n\equiv 1$ mod 2, then by the similar method it can be shown that

$$((n+1)(n+2)/2)\Sigma_Q\varphi_{n+1}^{n+3}=((n+3)/2)f_2\circ_H\varphi_{n+1}^{n+3}.$$

This implies that if $n\equiv 1 \bmod 4$, then $\Sigma_Q\varphi_{n+1}^{n+3}=2lf_2\circ_H\varphi_{n+1}^{n+3}$ for some l in $Z_{(2)}$ (integers localized at (2)). Thus in order to prove Lemma 4.4 it is enough to show that $\langle f_2\circ_H\varphi_{n+1}^{n+3}\circ 2, \eta, 2\rangle$ contains 0. But the bracket

$$\langle f_2\circ_H\varphi_{n+1}^{n+3}\circ 2, \eta, 2\rangle \ni f_2\circ_H\varphi_{n+1}^{n+3}\circ\eta^2.$$

On the other hand, from Lemma 4.1 and Proposition 2.5,

$$f_2\circ_H\varphi_{n+1}^{n+3}\circ\eta^2=f_2\circ i_{0*}\nu^3=(n+2)i_{0*}\nu^3=0,$$

in $Q_{n+1}^{n+2}=S^{4n+3}\cup_{(n+2)\nu}e^{4n+8}$.

This completes the proof of Theorem 4.2 and 2) of Main theorem.

4) $\pi_{4n+16}^s(Q_{n+1}^\infty)$ for $n\geqq 3$.
Consider the cofiber sequence:

$$S^{4n+3}\xrightarrow{i_0}Q_{n+1}^\infty\xrightarrow{p}Q_{n+2}^\infty\longrightarrow S^{4n+4}.$$

Since $\pi_{4n+16}^s(S^{4n+3})=\pi_{4n+16}^s(S^{4n+4})=0$, we have

$$\pi_{4n+16}^s(Q_{n+1}^\infty)\cong\pi_{4n+16}^s(Q_{n+2}^\infty).$$

Therefore from Theorem 4.3 we have proved 6) of Main Theorem.

5) $\pi_{4n+13}^s(Q_{n+1}^\infty)$ for $n\geqq 2$.
Consider the following cofiber sequence:

(4.5) $$S^{4n+10}\xrightarrow{{}_Q\varphi_{n+1}^{n+3}}Q_{n+1}^{n+2}\longrightarrow Q_{n+1}^{n+3}\longrightarrow S^{4n+11}\xrightarrow{\Sigma_Q\varphi_{n+1}^{n+3}}.$$

Applying $\pi_{4n+13}^s(\ )$ to the above sequence, from Lemma 4.1 we have

(4.6) $$(\Sigma_Q\varphi_{n+1}^{n+3})_*\eta^2=\begin{cases}0, & \textit{if } n \textit{ is odd},\\ i_{0*}\eta^2\sigma & \textit{if } n \textit{ is even}.\end{cases}$$

On the other hand,

Lemma 4.7. coker $({}_Q\varphi_{n+1}^{n+3})_*$ *is isomorphic to* $Z/2$ *which is generated by* $i_{0*}\eta\mu$.

Proof. Consider the cofiber sequence:

$$S^{4n+3}\xrightarrow{i_0}Q_{n+1}^{n+2}\xrightarrow{p}S^{4n+7}\xrightarrow{(n+2)\nu}S^{4n+4}.$$

Apply $\pi_{4n+13}^s(\ )$ to the above sequence. Since $p\circ_Q\varphi_{n+1}^{n+3}=(n+3)\nu$, it is easy to see that if n is even then Lemma 4.7 holds, and that if n is odd then $\pi_{4n+13}^s(Q_{n+1}^{n+2})$ is isomorphic to $Z/2$ generated by $i_{0*}\eta\mu$. When n is odd, the image of $({}_Q\varphi_{n+1}^{n+3})_*$ is trivial. Because, if the image of $({}_Q\varphi_{n+1}^{n+3})_*$ is non-trivial, it follows that ${}_Q\varphi_{n+1}^{n+3}\circ\nu=i_{0*}\eta\mu$. But this contradicts with the fact that $\eta\mu$ can be detected by d-invariant in KO-theory [1]. Thus Lemma 4.7 has been proved.

From (4.6) and Lemma 4.7, if n is even then we see that $\pi^s_{4n+13}(Q^{n+3}_{n+1})$ is isomorphic to $Z/2$ generated $i_{0*}\eta\mu$. If n is odd, then $\pi^s_{4n+13}(Q^{n+3}_{n+1})$ is a certain group extension of $Z/2$ and $Z/2$. However this group extension is trivial. Because, if not, then $\langle {}_Q\varphi^{n+3}_{n+1}, \eta^2, 2\rangle = i_{0*}\eta\mu$. But this also contradicts with the fact that $\eta\mu$ can be detected by d-invariant in KO-theory [1]. This completes the proof of 3) of Main theorem.

§5. $\pi^s_{4n+14}(Q^{n+4}_{n+1})$.

Consider the following cofiber sequence:

$$S^{4n+14} \xrightarrow{{}_Q\varphi^{n+4}_{n+1}} Q^{n+3}_{n+1} \longrightarrow Q^{n+4}_{n+1} \longrightarrow S^{4n+15}.$$

From the above sequence we have

$$\pi^s_{4n+14}(Q^{\infty}_{n+1}) \cong \pi^s_{4n+14}(Q^{n+4}_{n+1}) \cong \pi^s_{4n+14}(Q^{n+3}_{n+1})/\mathrm{Im}({}_Q\varphi^{n+4}_{n+1*}).$$

First we calculate the group $\pi^s_{4n+14}(Q^{n+2}_{n+1})$. In the next Proposition, $Z/a\{\alpha\}$ means the cyclic group of order a, generated by α and the symbol $+$ between groups means the direct sum. And e-invariant $e(\ )$ should be considered as

$$e\colon \pi^s_{4*-1}(X) \longrightarrow \mathrm{Hom}\,(KO^{4*}(X), Q/Z_{(2)}),$$

where a spectrum X is localized at (2).
Considering the homotopy exact sequence associated with the cofiber sequence:

$$S^{4n+6} \xrightarrow{(n+2)\nu} S^{4n+3} \xrightarrow{i_0} Q^{n+2}_{n+1} \longrightarrow S^{4n+7},$$

it is easy to see that there exists a coextension $\tilde{\sigma} \in \pi^s_{4n+14}(Q^{n+2}_{n+1})$ of $\sigma \in \pi^s_{4n+14}(S^{4n+7}) \cong \pi^s_7(S^0)$. Then we have

Proposition 5.1.

$$\pi^s_{4n+14}(Q^{n+2}_{n+1}) \cong \begin{cases} Z/8\{i_{0*}\xi\} + Z/16\{\tilde{\sigma}\}, & \text{if } n \equiv 6(8), \\ Z/4\{i_{0*}\xi - 4\tilde{\sigma}\} + Z/32\{\tilde{\sigma}\}, & \text{if } n \equiv 2(8), \\ Z/2\{i_{0*}\xi - 8\tilde{\sigma}\} + Z/64\{\tilde{\sigma}\}, & \text{if } n \equiv 0(4), \\ Z/128\{\tilde{\sigma}\}, & \text{if } n \text{ is odd}. \end{cases}$$

Moreover generators can be taken so that they satisfy

$$e(\tilde{\sigma})(x_{n+2}) = 1/16,$$

$$e(i_{0*}\xi)(x_{n+2}) = 0,$$

$$e(i_{0*}\xi)(x_{n+1}) = 1/8,$$

and $e(\tilde{\sigma})(x_{n+1}) = (n+2)l/128$, *where l is a certain unit in $Z_{(2)}$.*

Proof. It is easy to see that there exists a short exact sequence of homotopy groups;

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi^s_{4n+14}(S^{4n+3}) & \xrightarrow{i_{0*}} & \pi^s_{4n+14}(Q^{n+2}_{n+1}) & \xrightarrow{p_*} & \pi^s_{4n+14}(S^{4n+7}) & \longrightarrow & 0. \\ & & \wr\| & & & & \wr\| & & \\ & & Z/8\{\xi\} & & & & Z/16\{\sigma\} & & \end{array}$$

Since $\langle (n+2)\nu, \sigma, 16\rangle = (n+2)\xi$, there is only one relation that

$$(n+2)i_{0*}\xi = 16\tilde{\sigma},$$

for any choice of $\tilde{\sigma}$. This proves the first assertion of Proposition 5.1. The rest of the assertion follows easily from the properties of e-invariant (See §1).

Next we investigate the structure of the group $\pi^s_{4n+14}(Q^{n+3}_{n+1})$. Considering the cofiber sequence:

$$S^{4n+10} \xrightarrow{{}_Q\varphi^{n+3}_{n+1}} Q^{n+2}_{n+1} \xrightarrow{i} Q^{n+3}_{n+1} \xrightarrow{p} S^{4n+11},$$

we have the following commutative diagram:

$$\begin{array}{ccccccccc} & & & & & & & & 0 \\ & & & & & & & & \downarrow \\ & & & & & & & & \pi^s_k(S^{4n+4}) \\ & & & & & & & & \downarrow \\ 0 & \longrightarrow & \pi^s_k(Q^{n+2}_{n+1}) & \xrightarrow{i_*} & \pi^s_k(Q^{n+3}_{n+1}) & \xrightarrow{p_*} & \pi^s_k(S^{4n+11}) & \xrightarrow{(\Sigma_Q\varphi^{n+3}_{n+1})_*} & \pi^s_k(\Sigma Q^{n+2}_{n+1}) \\ & & & & & & & \searrow^{(n+3)\nu_*} & \downarrow \\ & & & & & & & & \pi^s_k(S^{4n+8}) \\ & & & & & & & & \downarrow \\ & & & & & & & & 0, \end{array}$$

where straight sequences are exact and $k = 4n+14$.

Lemma 5.2. Im $p_* \subset \pi^s_{4n+14}(S^{4n+11}) \cong Z/8\{\nu\}$ *is generated by* 2ν *if* n *is even and is generated by* ν *if* n *is odd.*

Proof. If is odd, then ${}_Q\varphi^{n+3}_{n+1}\circ(n+4)\nu = 0$. This can be seen from the existence of the spectrum Q^{n+4}_{n+1}. Thus ${}_Q\varphi^{n+3}_{n+1}\circ\nu = 0$. If n is even, then clearly ${}_Q\varphi^{n+3}_{n+1}\circ\nu \neq 0$. On the other hand, since $\langle 2\nu, \nu, 2\nu\rangle = 0$, we see ${}_Q\varphi^{n+3}_{n+1}\circ(2\nu) = 0$.

Since there exists a short exact sequence:

$$0 \longrightarrow \pi^s_{4n+14}(Q^{n+2}_{n+1}) \xrightarrow{i_*} \pi^s_{4n+14}(Q^{n+3}_{n+1}) \xrightarrow{p_*} \mathrm{Im}\, p_* \longrightarrow 0,$$

and since $p_*({}_Q\varphi^{n+4}_{n+1}) = (n+4)\nu$, if $n \not\equiv 0 \bmod 4$, then $p_*({}_Q\varphi^{n+4}_{n+1})$ is a generator of Im $p_* \cong Z/4(a(n))$. Thus if $n \not\equiv 0 \bmod 4$ then there exists a unique element $\alpha_n \in \pi^s_{4n+14}(Q^{n+2}_{n+1})$ such that

$$4(a(n))_Q\varphi^{n+4}_{n+1} = i_*\alpha_n.$$

Then clearly we have

$$\pi^s_{4n+14}(Q^{n+4}_{n+1}) \cong \pi^s_{4n+14}(Q^{n+3}_{n+1})/\mathrm{Im}\,(\varphi^{n+4}_{n+1})_*$$
$$\cong \pi^s_{4n+14}(Q^{n+2}_{n+1})/(\alpha_n).$$

Therefore, if $n \not\equiv 0 \bmod 4$, in order to determine $\pi^s_{4n+14}(Q^{n+4}_{n+1})$, it is sufficient to describe α_n in terms of generators of $\pi^s_{4n+14}(Q^{n+2}_{n+1})$ (See Lemma 5.1). This can be done using the e-invariant.

Case n: odd.

In this case from Lemma 5.1 we can put $\alpha_n = l\tilde{\sigma}$ for some integer l, that is, $l(i_*\tilde{\sigma}) = 8_Q\varphi^{n+4}_{n+1}$. Evaluating the e-invariant by $x_{n+1} \in KO^{4n+3}(Q^{n+3}_{n+1})$ at the both sides of the last equation, we have

$$l/128 = 8(n+4)(35n^2+49n+18)/2\cdot 9!, \quad \mathrm{mod}\ Z_{(2)}.$$

Solving the above equation we obtain that $l \equiv 8 \bmod 128$ up to some multiple of units in $Z_{(2)}$. Therefore we see that if n is odd then $\pi^s_{4n+14}(Q^{n+4}_{n+1}) \cong Z/16$.

Case $n \equiv 6 \bmod 8$.

From Lemma 5.1, in this case we can put $\alpha_n = ki_{0*}\xi + l\tilde{\sigma}$ for some integers k and l. Then we have the following equation:

$$e(ki_{0*}\xi + l\tilde{\sigma})(x_{n+2}) = e(4_Q\varphi^{n+4}_{n+1})(x_{n+2}),$$
$$e(ki_{0*}\xi + l\tilde{\sigma})(x_{n+1}) = e(4_Q\varphi^{n+4}_{n+1})(x_{n+1}),$$

where $x_{n+2} \in KO^{4n+7}(Q^{n+3}_{n+1})$ and $x_{n+1} \in KO^{4n+3}(Q^{n+3}_{n+1})$. Using Example 2.2, we see that $k \equiv 4 \bmod 8$ and $y \equiv \pm 4 \bmod 16$. Therefore in this case $\pi^s_{4n+14}(Q^{n+4}_{n+1}) \cong Z/8 + Z/4$.

Case $n \equiv 2 \bmod 8$.

By similar method, we have

$$\alpha_n = k(i_{0*}\xi - 4\tilde{\sigma}) + l\tilde{\sigma},$$

where $k \equiv \pm 1 \bmod 4$ and $l \equiv 0 \bmod 8$. Thus, if $n \equiv 2 \bmod 8$, then $\pi^s_{4n+14}(Q^{n+4}_{n+1}) \cong Z/32$.

Now we shall treat the case $n \equiv 0 \bmod 4$. Consider the element $f_3 \circ {}_H\varphi^{n+4}_{n+2} \in \pi^s_{4n+14}(Q^{n+3}_{n+1})$, where f_3: $HP^{n+3}_{n+1} \to \Sigma Q^{n+3}_{n+1}$ is the map in Proposition 2.5. Since $p_*(f_3 \circ {}_H\varphi^{n+4}_{n+1}) = (n+1)(n+2)(n+3)\nu$, if $n \equiv 0 \bmod 4$ then the element $f_3 \circ {}_H\varphi^{n+4}_{n+1}$ is a coextension of a generator of $\mathrm{Im}\, p_* \cong Z/4\{2\nu\}$. Since $(n+4)f_3 \circ {}_H\varphi^{n+4}_{n+1} = (n+1)\cdot(n+2)(n+3)_Q\varphi^{n+4}_{n+1}$ and since $n \equiv 0 \bmod 4$, it is easy to see that if $n \equiv 0 \bmod 8$, then ${}_Q\varphi^{n+4}_{n+1} = 2 \circ f_3 \circ {}_H\varphi^{n+4}_{n+1}$ up to some multiple of units in $Z_{(2)}$ and that if $n \equiv 4 \bmod 8$, then ${}_Q\varphi^{n+4}_{n+1} = 4(n+4/8)f_3 \circ {}_H\varphi^{n+4}_{n+1}$ up to some multiple of units. Let $4f_3 \circ {}_H\varphi^{n+4}_{n+1} = i_*\alpha_n$. Then investigating the e-invariant we see that if $n \equiv 0 \bmod 8$, then $\alpha_n = i_{0*}\xi - 8\tilde{\sigma} + 4l\tilde{\sigma}$ for some unit l and that if $n \equiv 4 \bmod 8$, then $\alpha_n = 4l\tilde{\sigma}$ for some unit l. Therefore we have

$$\pi^s_{4n+14}(Q^{n+3}_{n+1}) \cong \begin{cases} Z/2+Z/4+Z/64 & \text{if} \quad n \equiv 4 \bmod 8, \\ Z/8+Z/64 & \text{if} \quad n \equiv 0 \bmod 8, \end{cases}$$

where $Z/2$, $Z/4$, $Z/8$ and $Z/64$ are generated by $i_*(i_{0*}\xi - 8\tilde{\sigma})$, $f_3 \circ_H \varphi^{n+3}_{n+1} - i_{0*}\tilde{\sigma}$, $f_3 \circ_H \varphi^{n+3}_{n+1} - i_{0*}\tilde{\sigma}$ and $i_{0*}\tilde{\sigma}$ respectively. Therefore, by calculating $\pi^s_{4n+14}(Q^{n+3}_{n+1})/({}_Q\varphi^{n+4}_{n+1})$, we have

$$\pi^s_{4n+14}(Q^{n+4}_{n+1}) \cong \begin{cases} Z/2+Z/8 & \text{if} \quad n \equiv 0 \bmod 8, \\ Z/2+Z/4+Z/(16/(16, (n+4)/8)) & \text{if} \quad n \equiv 4 \bmod 8. \end{cases}$$

Thus we have completed the proof of Main theorem.

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References

[1] J. F. Adams, On the group $J(X)$, IV, Topology, **5** (1966), 21–71.
[2] R. Bott, The space of loops on a Lie group, Michigan Math. J., **5** (1958), 35–61.
[3] B. Harris, Some calculations of homotopy groups of symmetric spaces, Trans. of Amer. Math. Soc., **106** (1963), 174–184.
[4] M. Imaoka and K. Morisugi, On the stable Hurewicz image of some stunted projective spaces, III, Mem. Fac. Sci., Kyushu Univ. Ser. A, **39** (1985), 197–208.
[5] I. M. James, The topology of Stiefel manifolds, London Math. Soc. Lecture Note Series, **24**, Cambridge U.P. 1976.
[6] M. Mimura, Quelques groupes d'homotopie metastables des espaces symetriques $Sp(n)$ et $U(2n)/Sp(n)$, C.R. Acad. Sci. Paris, **263** (1966).
[7] M. Mimura and H. Toda, Homotopy groups of $SU(3)$, $SU(4)$ and $Sp(2)$, J. Math. Kyoto Univ., **3** (1964), 217–250.
[8] M. Mimura and H. Toda, Homotopy groups of symplectic groups, J. Math. Kyoto Univ., **3** (1964), 251–273.
[9] K. Morisugi, Massey products in MSP_* and its application, J. Math. of Kyoto Univ., **23**-2 (1983), 239–263.
[10] K. Morisugi, Homotopy groups of symplectic groups and the quaternionic James numbers, Osaka J. Math. 23 (1986), 867–880.
[11] H. Ōshima, On stable James numbers of stunted complex or quaternionic projective spaces, Osaka J. Math., **16** (1979), 479–504.
[12] H. Ōshima, Some James numbers of Stiefel manifolds, Math. Proc. Phil. Soc. **92** (1982), 139–161.
[13] H. Ōshima, On the homotopy group $\pi_{2n+9}(U(n))$ for $n \geqq 6$, Osaka J. Math., **17**-2 (1980), 495–511.
[14] H. Toda, Composition methods in homotopy groups of spheres, Annals of Math. studies, **49** (1962), Princeton.
[15] H. Toda, A survey of homotopy theory, Sugaku **15** (1963/64), 141–155.
[16] G. Walker, Estimates for the complex and quaternionic James numbers, Quart. J. Math. Oxford (2), **32** (1981), 467–489.