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Generalized divisors on Gorenstein curves and a theorem of Noether

By

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§0. Introduction.

Recently while considering the possible special linear systems which can exist on a nonsingular plane curve, we rediscovered an old result of Max Noether. The problem is to find the largest possible dimension of a linear system of degree d on a plane curve of degree k. The answer is that the linear systems of maximal dimension are the ones which "exist naturally" on the curve because of its plane embedding, namely the linear systems cut out by all plane curves of some other fixed degree, plus a few extra points or minus a few assigned base points. (See (2.1) for the exact statement.)

A natural way to approach the problem is by induction on k. If the divisor D on the curve C is nonspecial, then $h^0(\mathcal{O}_C(D))$ can be found by the Riemann-Roch theorem. If on the other hand D is special, then it is contained in a canonical divisor. The canonical divisors are cut out on C by curves of degree $k-3$ in the plane, so D is contained in a curve of degree $k-3$, and one can try to use induction. The trouble is that the new curve of lower degree containing D may not be nonsingular. For this reason we have developed a theory of generalized divisors on Gorenstein curves, which appears in §1. We believe this theory may be useful in other contexts, and by way of example have given a new proof of a theorem of Fujita (1.6) telling when the canonical divisor on a Gorenstein curve is very ample. This result should simplify the beginning theory of divisors on K3 surfaces as given in the paper os Saint-Donat [19].

Since any plane curve is Gorenstein, we can use the theory of §1, combined with Bertini's theorem, to formulate and prove Noether's theorem for generalized divisors on irreducible plane curves. This is done in §2. It turns out that Noether's original proof followed the same method, but we must consider that it is incomplete, because he assumes without justification that a curve of lower degree containing the divisor D can be chosen to be nonsingular. Meanwhile another proof of Noether's theorem, for nonsingular plane curves, has been given by Ciliberto [4] using a different method.

An important application of Noether's theorem, and indeed the reason for

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which he considered this problem in the first place, is to bound the genus of a space curve of degree d contained in an irreducible surface of degree k. Since these ideas are amply documented elsewhere, we have contented ourselves with a few remarks (2.2.2) and some references (see also [13].).

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§1. Generalized divisors on Gorenstein curves.

A Noetherian local ring A of dimension n with maximal ideal m and residue field $k=A/m$ is said to be *Gorenstein* if $\mathrm{Ext}^i_A(k, A)=0$ for $i\neq n$ and $\mathrm{Ext}^n_A(k, A)\cong k$ [7], p. 63.

Let C be an integral, projective, *Gorenstein* curve, i.e. an integral projective curve over an algebraically closed ground field k, all of whose local rings are Gorenstein local rings. For example, any curve contained in a nonsingular surface or any locally complete intersection curve in a projective space is Gorenstein. Let $\mathcal{O}_C$ be the structure sheaf of C, and let $\mathcal{K}$ be the constant sheaf of the function field of C. A *fractional ideal* of C is a nonzero subsheaf of $\mathcal{K}$ which is a coherent $\mathcal{O}_C$-module. We define the set GDiv(C) of *generalized divisors* of C to be the set of fractional ideals of C. In particular, this contains as a subset the set of nonzero coherent sheaves of ideals in $\mathcal{O}_C$. These correspond in a $1-1$ manner to the closed subschemes Z of dimension 0 of C, which we call the *effective generalized divisors* of C.

The set GDiv(C) contains as a subset the group CDiv(C) of locally principal fractional ideals, which are exactly the Cartier divisors of C. If Z is a generalized divisor and D is a Cartier divisor, we define the sum $Z+D$ by multiplying the corresponding fractional ideals. In this way the group CDiv(C) acts on the set GDiv (C). Since any fractional ideal $\mathcal{F}$ can be written locally as $f^{-1}\cdot\mathcal{I}_Z$ for $f\in\mathcal{O}_C$ and $\mathcal{I}_Z\subseteq\mathcal{O}_C$, we see that any generalized divisor can then be written in the form $Z+(-D)$ where Z is an effective generalized divisor and D is an effective Cartier divisor. For an effective generalized divisor Z we define $d=\deg Z$, its *degree*, to be the length of the structure sheaf $\mathcal{O}_Z$ of the corresponding closed subscheme. By linearity this extends to give a *degree* mapping $\mathrm{GDiv}(C)\to\boldsymbol{Z}$, which restricted to CDiv(C) is the usual degree homomorphism for Cartier divisors.

Note that we do not attempt to define addition of two generalized divisors. An examination of the possible closed subschemes of lengths 1, 2, and 3 of an ordinary double point on a curve should convince the reader that there is no group structure on GDiv(C) compatible with the degree function.

For any generalized divisor Z, corresponding to a fractional ideal $\mathcal{I}\subset\mathcal{K}$, we define its *inverse* $-Z$ to be the inverse fractional ideal $\mathcal{I}^{-1}$ which is locally $\{f\in\mathcal{K}\mid f\cdot\mathcal{I}\subset\mathcal{O}_C\}$. Then $-Z$ is another generalized divisor. To establish good properties for this operation, we need a lemma.

Lemma 1.1. *Let C be an integral Gorenstein curve. Let $\mathcal{F}$ be a torsion-free coherent sheaf on C. Then*

a) $\mathcal{E}xt^i_{\mathcal{O}_C}(\mathcal{F}, \mathcal{O}_C)=0$ *for all* $i>0$

b) $\mathcal{F}$ *is reflexive.*

Proof. The question is local, so let us pass to the local ring A of a closed point of C, and let the stalk of $\mathcal{F}$ be the finitely generated torsion-free A-module M. If M has rank r, then we can embed M in K^r, where K is the quotient field of A. Choosing a suitable common denominator for the generators of M, we can find an inclusion $M \subseteq A^r$. Let us write the quotient as R,

$$0 \to M \to A^r \to R \to 0$$

so that R is an A-module of finite length. Now applying the functor $\mathrm{Hom}_A(\cdot, A)$ we obtain an exact sequence

$$0 \to A^r \to M^\vee \to \mathrm{Ext}^1_A(R, A) \to 0$$

and isomorphisms

$$\mathrm{Ext}^i_A(M, A) \to \mathrm{Ext}^{i+1}_A(R, A)$$

for $i \geq 1$. Since A is a 1-dimensional Gorenstein ring, $\mathrm{Ext}^j_A(R, A)=0$ for $j \neq 1$, and $\mathrm{Ext}^1_A(\cdot, A)$ is a dualizing functor for finite length A-modules [7, pp. 63, 64].

In particular $\mathrm{Ext}^i_A(M, A)=0$ for $i>0$, and $R'=\mathrm{Ext}^1_A(R, A)$ is another finite-length A-module with the same length as R. Applying the functor $\mathrm{Hom}(\cdot, A)$ once more we obtain

$$0 \to M^{\vee\vee} \to A^r \to \mathrm{Ext}^1(R', A) \to \mathrm{Ext}^1(M^\vee, A) = 0\,.$$

The last term is 0 by part a) applied to $M^\vee$. On the other hand $R''=\mathrm{Ext}^1(R', A)$ is isomorphic to R. Now comparing with the original sequence, from which there are natural maps to this one, we find $M \cong M^{\vee\vee}$, so M is reflexive.

Proposition 1.2. *On an integral Gorenstein curve C the minus operation $Z \mapsto -Z$ for generalized divisors obeys the usual laws of arithmetic: $-(-Z)=Z$; $-(Z+D)=(-Z)+(-D)$ where D is a Cartier divisor; and* $\deg(-Z)=-\deg Z$.

Proof. This follows immediately from the lemma, which shows that for any fractional ideal $\mathcal{J}$, $(\mathcal{J}^{-1})^{-1}=\mathcal{J}$.

From here we can develop the usual theory of linear systems and associated sheaves, provided we remember never to add two generalized divisors.

The set $\mathrm{GDiv}(C)$ contains as a subset the group $\mathrm{PDiv}(C)$ of principal divisors (f) for nonzero elements $f \in K(C)^*$. We say Z and $Z' \in \mathrm{GDiv}(C)$ are *linearly equivalent* if $Z'=Z+(f)$ for some $f \in K^*$. The set of effective divisors linearly equivalent to Z is denoted $|Z|$ and is called the *complete linear system* of Z.

To conform with the usual convention, we define the sheaf $\mathcal{L}(Z)$ associated to the generalized divisor Z to be the inverse $\mathcal{I}^{-1}$ of the fractional ideal $\mathcal{I}$ corresponding to Z. Thus for a closed subscheme $Z\subset C$, $\mathcal{L}(-Z)=\mathcal{I}_Z$. The generalized divisors Z and Z' are linearly equivalent if and only if $\mathcal{L}(Z)\cong\mathcal{L}(Z')$ as $\mathcal{O}_C$-modules. Thus the set of generalized divisors modulo linear equivalence is in $1-1$ correspondence with the set of isomorphism classes of torsion-free rank 1 coherent $\mathcal{O}_C$-modules.

If $s\in H^0(\mathcal{L}(Z))$ is a non-zero section of $\mathcal{L}(Z)$, it defines an inclusion $\mathcal{O}_C\to\mathcal{L}(Z)$ and by dualizing an inclusion $\mathcal{L}(-Z)\to\mathcal{O}_C$. Thus $\mathcal{L}(-Z)$ is identified with the ideal sheaf $\mathcal{I}_{Z'}$ of some effective generalized divisor Z', which we denote also $Z(s)$. The mapping $s\mapsto Z(s)$ induces a bijection from the projective space of lines in $H^0(\mathcal{L}(Z))$ to the complete linear system $|Z|$.

Finally, we have the Riemann-Roch theorem and Serre duality.

Theorem 1.3 (Riemann-Roch). *For any generalized divisor Z on the integral Gorenstein curve C,*

$$h^0(\mathcal{L}(Z))-h^1(\mathcal{L}(Z)) = d+1-p_a$$

where $d=\deg Z$ and p_a is the arithmetic genus of C.

Proof. Write $-Z=Z'-D$ where Z' is an effective generalized divisor and D is a Cartier divisor. Then $\mathcal{L}(Z)\cong\mathcal{L}(D-Z')=\mathcal{I}_{Z'}(D)$. Now use the cohomology sequence associated to the exact sequence

$$0\to\mathcal{I}_{Z'}(D)\to\mathcal{L}(D)\to\mathcal{O}_{Z'}\to 0$$

and the usual Riemann-Roch theorem for $\mathcal{L}(D)$.

For duality we must introduce the canonical divisor on C. Recall that any projective variety X has a *dualizing sheaf* ω_X in the sense of [11, III, §7]. For an integral curve C, this dualizing sheaf is torsion-free of rank 1 on C. Furthermore, C is Gorenstein if and only if ω_C is invertible [10, V, §9]. So in our case of an integral Gorenstein curve C, ω_C is invertible, and the associated Cartier divisor K_C, defined up to linear equivalence, is the *canonical divisor* on C.

Theorem 1.4 (Serre duality). *For any generalized divisor Z on C, $H^i(\mathcal{L}(Z))$ is dual to $H^{1-i}(\mathcal{L}(K_C-Z))$ for $i=0, 1$.*

Proof. First observe that the statement makes sense, because K_C is a Cartier divisor, so we can consider the generalized divisor K_C-Z. By symmetry it is enough to consider the case $i=1$. Then by the duality theorem [11, III, 7.6], or simply by the definition of a dualizing sheaf [ibid, p. 241], $H^1(\mathcal{L}(Z))$ is dual to $\mathrm{Hom}(\mathcal{L}(Z), \omega_C)=H^0(\mathcal{H}om(\mathcal{L}(Z), \omega_C))=H^0(\mathcal{L}(K_C-Z))$.

Now we will see how some of the standard results about nonsingular curves can be adapted to the case of an integral projective Gorenstein curve. A point $P\in C$ is a *base point* of a linear system $\mathfrak{d}$ on C if $P\in\mathrm{Supp}Z$ for every divisor $Z\in\mathfrak{d}$.

If a linear system $|Z|$ has no base points, then its general member does not contain any singular point of C in its support. Thus Z is a Cartier divisor, and $\mathcal{L}(Z)$ is generated by global sections. Conversely, if Z is a Cartier divisor and $\mathcal{L}(Z)$ is generated by global sections, then $|Z|$ has no base points [11, II, 7.8]. However in general $\mathcal{L}(Z)$ can be generated by global sections even though $|Z|$ has base points. If $|Z|$ has base points, we can define the base locus B of $|Z|$ as the scheme-theoretic intersection of all effective $Z'\in|Z|$. Then B is an effective generalized divisor. However we cannot in general consider $|Z-B|$, and so we cannot define an associated base-point-free linear system (see example (1.6.1) below).

Proposition 1.5. *Let D be a Cartier divisor on the integral projective Gorenstein curve C. Then:*

(a) *The complete linear system $|D|$ has no base points if and only if for every point $P\in C$,*

$$\dim|D-P| = \dim|D|-1;$$

(b) *D is very ample if and only if for every subscheme $Z\subseteq C$ of length* 2,

$$\dim|D-Z| = \dim|D|-2.$$

Proof. The proof is the same as the proof of [11, IV, 3.1], using the criterion that $|D|$ is very ample if and only if it "separates points and tangent vectors". The only difference is that instead of considering divisors of the form $Z=P+Q$ and $Z=2P$ for a smooth point P, we must consider all possible schemes of length 2 supported at a point P.

We say that an integral Gorenstein curve C of arithmetic genus $p_a\geq 2$ is *hyperelliptic* if there exists a finite morphism $f\colon C\to \boldsymbol{P}^1$ of degree 2. Considering the associated linear system on C, we see that C is hyperelliptic if and only if there is a linear system g^1_2 of Cartier divisors, of dimension 1 and degree 2, without base points.

Theorem 1.6 (*Fujita* [6]). *Let C be an integral projective Gorenstein curve of arithmetic genus p_a.*

(a) *If $p_a\geq 1$, the canonical linear system $|K|$ has no base points.*

(b) *If $p_a\geq 2$, then K is very ample if and only if C is not hyperelliptic.*

Proof. Using (1.5) the proofs are almost the same as for nonsingular curves [11, IV 5.1 and 5.2]. For (a), to show $|K|$ has no base points we must show for each $P\in C$ that $\dim|K-P|=\dim|K|-1$. By Riemann-Roch and duality, this is equivalent to $\dim|P|=0$. If on the other hand for some P, $\dim|P|=1$, then necessarily $|P|$ has no base points. Hence P is a smooth point of C, and the linear system $|P|$ determines a morphism of degree 1 from C to $\boldsymbol{P}^1$. It follows that $C\cong\boldsymbol{P}^1$ which has $p_a=0$, a contradiction.

To prove (b), suppose that $|K|$ is not very ample. Then for some subscheme $Z\subseteq C$ of length 2, $\dim|K-Z|=\dim|K|-1$. Using Riemann-Roch and duality we find $\dim|Z|=1$. If $|Z|$ has no base points, then it defines a morphism of degree 2 to $\boldsymbol{P}^1$, so C is hyperelliptic.

Now suppose $|Z|$ has a base point P. It cannot have two distinct base points since it has degree 2 and positive dimension. For any smooth point $Q\in C$, with $Q\neq P$ we consider the exact sequence

$$0 \to \mathcal{L}(Z-Q) \to \mathcal{L}(Z) \to k(Q) \to 0$$

and its cohomology sequence

$$0 \to H^0(\mathcal{L}(Z-Q)) \to H^0(\mathcal{L}(Z)) \to k\,.$$

Since $h^0(\mathcal{L}(Z))=2$, we see that $h^0(\mathcal{L}(Z-Q))\geq 1$. Hence $Z-Q$ is effective, so there is a divisor in $|Z|$ whose support contains Q. Since P is a base point, we see that for any smooth point $Q\in C$, the divisor $P+Q\in|Z|$.

Now consider the canonical morphism ρ: $C\to\boldsymbol{P}^{p_a-1}$, which is well-defined by (a) since $p_a\geq 2$. By construction, for the given point P, and for any smooth point Q,

$$\dim|K-P-Q| = \dim|K|-1\,.$$

This means that $\rho(P)=\rho(Q)$, so $\rho(C)$ is a point. But $\rho(C)$ spans $\boldsymbol{P}^{p_a-1}$ by construction, so $p_a=1$, a contradiction. This completes the proof.

Example 1.6.1. It can happen that an integral Gorenstein curve C with $p_a=1$ has a linear system $|Z|$ of dimension 1 and degree 2 with a base point. Let C be a nodal plane cubic curve, with node P. Let $\mathfrak{d}$ be the linear system of Cartier divisors $C\cap L$ where L is a line passing through P. This has dimension 1 and degree 3. It is Cartier and has P for a base point, so we can consider $Z=C\cap L-P$. Then $|Z|$ is such an example. $|Z|$ contains all divisors of the form $P+Q$ where P is the node and Q is any other point. In particular, for Q, Q' two smooth points of C, we have $P+Q\sim P+Q'$. This does not imply $Q\sim Q'$, because it is not allowed to subtract P from the generalized divisor $P+Q$.

Remark 1.6.2. Our proof of (1.6) is completely different from Fujita's, since he deduces (b) from the deeper theorem of Noether which states that if C is not hyperelliptic, then its canonical embedding $\rho(C)$ is projectively Cohen-Macaulay [6], p. 39. Another proof of (1.6), with a slight restrictive hypothesis, is given by Catanese [3], p. 91, who studies more generally canonical and pluricanonical morphisms of reducible Gorenstein curves.

Example 1.6.3. Any irreducible plane curve C of degree 4 si a non-hyperelliptic integral Gorenstein curve of arithmetic genus $p_a=3$. Indeed, the canonical divisor is $\mathcal{O}_C(1)$ which is very ample. Hence C is not hyperelliptic.

Remark 1.6.4. One can extend some of this theory of generalized divisors to varieties of higher dimension. Let X be an integral scheme of dimension n, and let GDiv(X) be the set of reflexive fractional ideals. Then the group CDiv(X)

of Cartier divisors acts on this set, and taking the inverse of a fractional ideal gives an operation $Z \to -Z$ having the properties of (1.2). So one can develop the theory of linear equivalence, sheaves $\mathcal{L}(Z)$, and linear systems as above.

The problem is, to what do these generalized divisors correspond geometrically? Put another way, under what conditions is the ideal sheaf $\mathcal{I}_Z$ of a closed subscheme Z of X reflexive? If one examines carefully the proof of [14, 1.3] one sees that instead of X normal, it would be sufficient to assume X satisfies Serre's condition S_2, and is Gorenstein in codimension 1. Indeed, one only needs to know that for $\dim \mathcal{O}_x = 1$, any torsion-free sheaf is reflexive, which is our Lemma (1.1) above. So for example if X is a Gorenstein variety, the result [loc. cit. 1.3] and its corollary [loc. cit. 1.5] will hold, and we find that $\mathcal{I}_Z$ is reflexive if and only if Z is a closed subscheme of X all of whose associated primes have codimension 1.

This gives a reasonable notion of generalized divisors on an integral Gorenstein scheme of higher dimension.

Remark 1.6.5. Various authors [2], [5], [16], [18] have considered compactifications of the generalized Jacobian of an integral curve by considering torsion-free rank 1 sheaves in addition to invertible sheaves. Thus the space of linear equivalence classes of generalized divisors is represented by a projective scheme $\bar{P}$, containing as an open subset the generalized Jacobian P, which is a group scheme parametrizing linear equivalence classes of Cartier divisors. $\bar{P}$ is irreducible (and hence P is dense in $\bar{P}$) if and only if the curve has planar singularities [1], [16].

§2. Noether's theorem.

Theorem 2.1. (Noether) *Let C be an irreducible plane curve of degree k, and let Z be a closed subscheme of finite length $d \geq 0$ of C. Let $\mathcal{L}(Z)$ be the associated torsion-free sheaf $\mathcal{I}^{\vee}_{Z,C}$. Let $p_a = \frac{1}{2}(k-1)(k-2)$ be the arithmetic genus of C.*

1) *If $d > k(k-3)$ then*

$$h^0(\mathcal{L}(Z)) = d+1-p_a$$

2) *If $d \leq k(k-3)$, write $d = kr-e$ with $0 \leq r \leq k-3$ and $0 \leq e < k$. Then*

$$h^0(\mathcal{L}(Z)) \leq \begin{cases} \frac{1}{2} r(r+1) & \text{if } e > r+1 \\ \frac{1}{2}(r+1)(r+2)-e & \text{if } e \leq r+1\,. \end{cases}$$

Furthermore, equality occurs if and only if

a) *$Z = C \cap C' + Z_0$ where C' is a curve of degree $r-1$ and Z_0 is a subscheme of length $k-e$, in the first case, or*

b) *$Z = C \cap C'' - E$ where E is a subscheme of length e and C'' is a curve of degree r containing E, in the second case.*

Proof. For the first statement, we use Serre duality and the Riemann-Roch theorem. Let K_C denote a canonical divisor on C. Then $h^1(\mathcal{L}(Z))=h^0(\mathcal{L}(K_C-Z))$ by duality. On the other hand for a plane curve C of degiee k, the dualizing sheaf ω_C is isomorphic to $\mathcal{O}_C(k-3)$, so K_C has degree $k(k-3)$. Hence K_C-Z has negative degree, so $h^0(\mathcal{L}(K_C-Z))=0$ and $h^1(\mathcal{L}(Z))=0$. Now statement 1) follows from Riemann-Roch.

For the second statement we use induction on k. For $k\leq 3$ the statement is vacuous so there is nothing to prove. So we may assume that $k\geq 4$ and that the theorem has already been proved for curves of degree $<k$.

Lemma 2.2. *Suppose that Z is contained in an irreducible curve D of degree $k-1$. Then the theorem holds on C for any $Z'=Z+C\cap F$ where F is a curve of degree $a\geq 0$. In particular (taking $a=0$) it holds for Z on C.*

Proof. Since the divisor $C\cap F$ corresponds to the invertible sheaf $\mathcal{O}_C(a)$, we may write $\mathcal{L}(Z')=\mathcal{L}(Z+a)$. Then since the canonical divisor is $\mathcal{O}_C(k-3)$, using Serre duality we find

$$(1) \qquad h^0(\mathcal{L}(Z')) = h^1(\mathcal{L}(k-3-a-Z)) = h^1(\mathcal{J}_{Z,C}(k-3-a)),$$

where $\mathcal{J}_{Z,C}$ is the ideal sheaf of Z on C.

Let $\mathcal{J}_Z$ denote the ideal sheaf of Z as a subscheme of $\boldsymbol{P}^2$, and let $\mathcal{J}_C$ be the ideal sheaf of C. Then from the exact sequence of sheaves

$$0 \to \mathcal{J}_C \to \mathcal{J}_Z \to \mathcal{J}_{Z,C} \to 0,$$

twisted by $k-3-a$, we obtain an exact sequence of cohomology groups

$$0 = H^1(\mathcal{J}_C(k-3-a)) \to H^1(\mathcal{J}_Z(k-3-a)) \to H^1(\mathcal{J}_{Z,C}(k-3-a)) \to$$
$$H^2(\mathcal{J}_C(k-3-a)) \to H^2(\mathcal{J}_Z(k-3-a)) \to 0.$$

Note first that if $a\geq k-3$, then $\deg Z'>k(k-3)$ so the result is true by 1). Hence we may assume $a<k-3$. Next observe that $H^2(\mathcal{J}_Z(k-3-a))=H^2(\mathcal{O}_{P^2}(k-3-a))=0$ since $k-3-a>0$. On the other hand, since $\mathcal{J}_C\cong\mathcal{O}_{P^2}(-k)$, $H^2(\mathcal{J}_C(k-3-a))=H^2(\mathcal{O}_{P^2}(-a-3))$ which has dimension $\frac{1}{2}(a+1)(a+2)$. Thus we find

$$(2) \qquad h^1(\mathcal{J}_{Z,C}(k-3-a)) = h^1(\mathcal{J}_Z(k-3-a))+\frac{1}{2}(a+1)(a+2).$$

Now we consider Z as a closed subscheme of the curve D of degree $k-1$. The same arguments, taking into account $\mathcal{J}_D\cong\mathcal{O}_{P^2}(-k+1)$ and $\mathcal{L}_D(K_D)\cong\mathcal{O}_D(k-4)$ give

$$(3) \qquad h^1(\mathcal{J}_{Z,D}(k-3-a)) = h^1(\mathcal{J}_Z(k-3-a))+\frac{1}{2}a(a+1)$$

and

$$(4) \qquad h^1(\mathcal{J}_{Z,D}(k-3-a)) = h^0(\mathcal{L}_D(Z+a-1)),$$

where $\mathcal{L}_D$ denotes the torsion-free sheaf on D associated to a generalized divisor. Combining, we find

$$h^0(\mathcal{L}_C(Z')) = h^0(\mathcal{L}_D(Z+a-1))+a+1\,. \tag{5}$$

It is useful also to use the same method to compute $h^1(\mathcal{L}(Z'))$. Again using Serre duality and the lower part of the exact cohomology sequence above, one finds

$$h^1(\mathcal{L}_C(Z')) = h^1(\mathcal{L}_D(Z+a-1))\,. \tag{6}$$

Thus we are in a position to apply the induction hypothesis to a generalized divisor Z'' associated to the sheaf $\mathcal{L}_D(Z+a-1)$ on D. If $\deg Z''>(k-1)(k-4)$ then $h^1(\mathcal{L}_D(Z''))=0$, so $h^1(\mathcal{L}_C(Z'))=0$. Therefore the equality of 1) in the theorem holds, and this is better than the inequalities of 2), for generalized divisors of any degree. So we may assume $\deg Z''\leq(k-1)(k-4)$. If Z'' is not effective, then $a=1$ and $h^0(\mathcal{L}_C(Z'))=1$ in which case the bounds of the theorem hold. Thus we may assume Z'' is effective, and apply the induction hypothesis to Z'' on D.

As functions of d (keeping a fixed), the bound on $h^0(\mathcal{L}_C(Z'))$ we wish to prove, and the bound on $h^0(\mathcal{L}_D(Z+a+1))$ given by the induction hypothesis, are both functions made of line segments of slopes 0 and 1. To prove that one graph lies below the other, it is sufficient to consider the highest corners of the lower graph. In other words, if we write $\deg Z'=d'=kr'-e$ and $\deg Z''=d''=(k-1)r''-e''$, it is sufficient to consider the case $e''=0$.

So let $d=kr-e$ as above. Then

$$d' = d+ak = k(r+a)-e$$

and

$$d'' = d+(a-1)(k-1) = (k-1)(a+r-1)-(e-r)\,.$$

It is thus sufficient to consider the case $e-r=0$, and $r''=a+r-1$. The induction hypothesis then gives

$$h^0(\mathcal{L}_D(Z''))\leq\frac{1}{2}(a+r)(a+r+1)\,.$$

Then by (5)

$$h^0(\mathcal{L}_C(Z'))\leq\frac{1}{2}(a+r)(a+r+1)+a+1\,.$$

On the other hand, $r'=a+r$, and $e'=e=r$ so what we want to prove is

$$h^0(\mathcal{L}_C(Z'))\leq\frac{1}{2}(a+r+1)(a+r+2)-r\,,$$

which is the same.

In case of equality, by induction we must have $Z''=D\cap D''$ where D'' has degree $a+r-1$. But $\mathcal{L}_D(Z'')=\mathcal{L}_D(Z+a-1)$ so $\mathcal{L}_D(Z)\cong\mathcal{O}_D(r)$. By completeness of the linear system of curve sections of any degree, $Z=D\cap C'$ for some curve

C' of degree r. Thus Z is contained in a curve C' of degree r, so we can write $Z = C \cap C' - E$ where $E = C \cap C' - Z$ is a generalized divisor of degree $e = r$.

Here we have been discussing the case $e'' = 0$. From the proof above we see that $e'' = 0$ corresponds to $e' = r$. Hence the only other case where equality is possible is for $e'' = 1$, $e = r+1$, in which case the same argument applies to show that Z is of the form $C \cap C' - E$, so $Z' = Z + C \cap F$ is also of the same form. This completes the proof of the lemma.

Proof of theorem (continued). Let Z be a subscheme of C of degree $d \leq k(k-3)$. By the Riemann-Roch theorem, $h^0(\mathcal{L}_C(Z))$ is larger for a *special* divisor (i.e. one for which $h^1(\mathcal{L}_C(Z)) \neq 0$) than for a nonspecial divisor, so we may assume Z is special. Then $h^1(\mathcal{L}_C(Z)) \neq 0$. By duality $h^1(\mathcal{L}_C(Z)) = h^0(\mathcal{L}_C(k-3-Z)) = h^0(\mathcal{J}_{Z,C}(k-3)) \neq 0$. The exact sequence used in the proof of the lemma shows also that $h^0(\mathcal{J}_{Z,C}(k-3)) = h^0(\mathcal{J}_Z(k-3))$. So this is non-zero, hence there exist curves (possibly singular, reducible, ...) of degree $k-3$ containing Z. Let $|\mathcal{J}_Z(k-3)|$ denote the linear system of all curves (meaning Cartier divisors) on $\boldsymbol{P}^2$ of degree $k-3$ containing Z. Let the fixed component of this linear system be a curve F of a degree $a \geq 0$. Let Z' be the scheme-theoretic union of Z and $C \cap F$. Then Z' is another closed subscheme of C, of degree $d' \geq d$. On the other hand, by construction, all curves of degree $k-3$ in $\boldsymbol{P}^2$ containing Z contain Z', so $h^0(\mathcal{J}_Z(k-3)) = h^0(\mathcal{J}_{Z'}(k-3))$. This implies $h^1(\mathcal{L}_C(Z)) = h^1(\mathcal{L}_C(Z'))$ and so by Riemann-Roch $h^0(\mathcal{L}_C(Z)) = h^0(\mathcal{L}_C(Z')) - d' + d$. Therefore, since the bound we wish to prove, as a function of d, consists of line segments of slopes 0 and 1, it is sufficient to prove the theorem for Z'.

Now let $Z'' = Z' - C \cap F$. This is an effective generalized divisor on C since Z' contains $C \cap F$. If $f = 0$ is the equation of F, there is an exact sequence

$$0 \to \mathcal{J}_{Z''}(-a) \xrightarrow{f} \mathcal{J}_{Z'} \to \mathcal{J}_{C \cap F, F} \to 0 .$$

In other words, Z'' is the *residual intersection* of Z' and F, in the sense of [15], p. 381. Twisting by $k-3$ we obtain an exact sequence

$$0 \to H^0(\mathcal{J}_{Z''}(k-3-a)) \xrightarrow{f} H^0(\mathcal{J}_{Z'}(k-3)) \xrightarrow{\alpha} H^0(\mathcal{J}_{C \cap F, F}(k-3)) .$$

Since F is the fixed component of the linear system $|\mathcal{J}_{Z'}(k-3)|$, the map α is the zero map, and so f is an isomorphism. It follows that the linear system $|\mathcal{J}_{Z''}(k-3-a)|$ of curves of degree $k-3-a$ containing Z'' has no fixed component. Adding to these curves arbitrary curves of degree $a+2$ we see that the linear system $|\mathcal{J}_{Z''}(k-1)|$ has no fixed component and is not composed with a pensil. Hence by one of the Bertini theorems [20], p. 30, a general member of this linear system is either irreducible or is of the form $p^r D$ with D irreducible and $p = \text{char.} k > 0$. But some of the divisors in our linear system contain an arbitrary curve of degree $a+2$, so this latter case cannot occur.

We conclude that Z'' is contained in an irreducible curve of degree $k-1$. Since $Z' = Z'' + C \cap F$ the lemma implies that the theorem holds for Z'. Note this argument works also in the two extreme cases when $F=\emptyset$, so Z itself is contained in an irreducible curve of degree $k-1$, or $\deg F=k-3$, so $Z''=0$, in which case the lemma is trivially true. So the theorem is proved.

Remark 2.2.1. This theorem was stated by Noether [17, §5] for a nonsingular plane curve. His idea, like ours, is to use induction on k. However, we cannot accept his proof today, because he assumes without justification, that the movable part of the linear system of curves of degree $k-3$ containing Z (which we called $|\mathcal{J}_{Z''}(k-3-a)|$) contains an *irreducible nonsingular* curve of degree $k-3-a$. Recently a different proof, in the case C nonsingular, has been given by Ciliberto [4]. Note that our proof is more general, in that it holds also for singular curves, and in arbitrary characteristic.

Remark 2.2.2. One can use this result, as did Noether, to obtain an upper bound for the genus of a curve Y of degree d lying on an irreducible surface F of degree k in $\boldsymbol{P}^3$. First we use the theorem (2.1) to bound $h^1(\mathcal{J}_{Z,H}(l))$ where Z is a subscheme of length d of an irreducible curve C in a plane H. Thus we obtain a new proof of [12, 5.4], under slightly more general hypotheses: Z need not consist of d distinct points. To apply this result, we let H be a general plane in $\boldsymbol{P}^3$, let $Z=Y\cap H$ and let $C=F\cap H$. Then, as in [12, §6], we obtain an upper bound for the arithmetic genus of a curve Y of degree d, which may be reducible or non-reduced, lying on an irreducible surface F of degree k in $\boldsymbol{P}^3$, provided $d>k(k-1)$:

$$p_a(Y)\leq\frac{d^2}{2k}+\frac{1}{2}d(k-4)+1+\frac{1}{2}f(f+1-k-\frac{f}{k})$$

where $d\equiv f\pmod{k}$ and $0\leq f<k$.

This is a new proof, closest to Noether's original proof [17, §6] of this bound. Other modern proofs of this result are given by Harris [9], Hartshorne [12, 6.1], Gruson and Peskine [8] (cf. Remarque 3.7), and Ciliberto [4, 3.31]. Note that Gruson and Peskine prove more, since they establish this same bound under the weaker hypothesis that Y is not contained in any surface of degree $<k$.

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