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# Varieties which have two projective space bundle structures

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## Introduction.

In this article we study the structure of varieties which have two bundle structures whose fibers are projective spaces. Well-known examples are $\boldsymbol{P}^r\times\boldsymbol{P}^s$, $\boldsymbol{P}(T_{\boldsymbol{P}^n})$ and fiber product $R_1\times_Y R_2$ of $R_1$ and $R_2$ over $Y$ where $R_1$ and $R_2$ are ruled varieties over $Y$. The aim of this work is to classify such varieties under some additional conditions.

Let $M$ and $M_i (i=1, 2)$ be varieties over an algebraically closed field $k$ and let $p$ and $q$ be proper surjective morphisms $M\to M_1$ and $M\to M_2$, respectively, where every closed fiber of $p$ and $q$ is isomorphic to $\boldsymbol{P}^r$ and $\boldsymbol{P}^s$ respectively. To fix the idea let us introduce the following notion:

$(P)$ We say that $M$ has two projective space bundle structures $(M_1, \boldsymbol{P}^r, p; M_2, \boldsymbol{P}^s, q)$ if there are two varieties $M_1$, $M_2$ and two morphisms $p$, $q$ as above and if $\dim\Phi(M)>\max\{\dim M_1, \dim M_2\}$, where $\Phi$ is the morphism $M\to M_1\times M_2$ induced by $p$ and $q$ (see Remark 1.6 about the second condition).

Under this notation, we have

**Theorem A.** *Let $M$ be a non-singular projective variety over an algebraically closed field $k$. Assume $M$ has two projective space bundle structures $(\boldsymbol{P}^l, \boldsymbol{P}^r, p; \boldsymbol{P}^m, \boldsymbol{P}^s, q)$.*

1) *If the characteristic of $k$ is zero, then $M$ is isomorphic to either* a) $\boldsymbol{P}^l\times\boldsymbol{P}^m$ *($p$ and $q$ are the first and the second projections, respectively), or,* b) $\boldsymbol{P}(T_{\boldsymbol{P}^l})$, *where $T_{\boldsymbol{P}^l}$ is the tangent bundle of $\boldsymbol{P}^l$. (See Lemma 1.15). In the case of* (b), $l=m=r+1=s+1$.

2) *If the characteristic of $k$ is positive, additionally, assume that $p$ (or, $q$) is $\boldsymbol{P}^r$-bundle on $\boldsymbol{P}^l$(or, $\boldsymbol{P}^s$-bundle on $\boldsymbol{P}^m$, resp.) in the Zariski topology. Then we have the same conclusion as in* 1).

**Theorem B.** *Let $M$ be a non-singular projective 3-fold over an algebraically closed field of characteristic 0. Assume that $M$ has two projective space bundle structures $(S_1, \boldsymbol{P}^1, p; S_2, \boldsymbol{P}^1, q)$ with non-singular surfaces $S_1$, $S_2$. Then $M$ is one of the following*

1) $S_1\times_C S_2$, *where $S_i$ is a $\boldsymbol{P}^1$-bundle over a non-singular complete curve $C$.*

2) *$\boldsymbol{P}(T_{\boldsymbol{P}^2})$, where $T_{\boldsymbol{P}^2}$ is the tangent bundle of $\boldsymbol{P}^2$ and $S_i \cong \boldsymbol{P}^2$.*

In order to show Theorem A, we shall compute the Chow ring of $M$ and study the property of the tangent bundle $T_M$ of $M$ and two vector bundles $E_1$, $E_2$ which determine $M$. As for Theorem B, we shall use the properties of a ruled surface especially, a rational ruled surface.

**Notation and conventions.**

We work over an algebraically closed field $k$ of any characteristic unless stated. A variety means an irreducible and reduced algebraic $k$-scheme. $\boldsymbol{P}^n$ denotes an $n$-dimensional projective space and $\mathcal{O}_{\boldsymbol{P}^n}(1)$ is the line bundle corresponding to the divisor class of hyperplanes in $\boldsymbol{P}^n$. We use the terms vector bundle and locally free sheaf interchangeably. For a vector bundle $E$ on a variety $X$, $\mathcal{O}_{\boldsymbol{P}(E)}(1)$ denotes the tautological line bundle of $E$. $\check{E}$ denotes the dual vector bundle of $E$. Moreover, when $Y$ is a subvariety of $X$, $E|_Y$ denotes $i^*E$ with $i$ the natural immersion of $Y$ to $X$.

## § 1. Preliminaries.

Let $M$, $S$ be non-singular projective varieties and $p: M \to S$ be a surjective morphism such that for every closed point $s$ in $S$, $p^{-1}(s)$ is isomorphic to $\boldsymbol{P}^r_k$. At first let us study the conditions under which $p$ is a $\boldsymbol{P}^r$-bundle in the Zariski topology.

Let us consider the exact sequence of algebraic groups as follows:

$$0 \longrightarrow G_m \longrightarrow GL(u) \longrightarrow PGL(u) \longrightarrow 0. \tag{1.1}$$

Then we have an exact sequence of étale cohomologies;

$$H^1(S_{\text{ét}}, GL(u)) \xrightarrow{f} H^1(S_{\text{ét}}, PGL(u)) \xrightarrow{g} H^2(S_{\text{ét}}, G_m). \tag{1.2}$$

Under the above notation we have

**Lemma 1.3.** *Let $p: M \to S$ be as above. Then $p$ is a $\boldsymbol{P}^r$-bundle in the étale topology. Moreover assume that $f$ is surjective. Then $p$ is a $\boldsymbol{P}^r$-bundle in the Zariski topology and there exists a vector bundle $E$ of rank $r+1$ on $S$ such that $M$ is isomorphic to $\boldsymbol{P}(E)$ and $p$ corresponds to the canonical projection $\boldsymbol{P}(E) \to S$.*

For a proof, see Theorem 0.1. in [5] and Lemma 1.2. in [6].

**Corollary 1.4.** *Under the same conditions as above assume $S$ is one of the following:*

1) *curve,*
2) *rational surface, and*
3) *projective space.*

*Then $f$ in (1.2) is surjective.*

*Proof.* In the case of 1) and 2), it is known that $H^2(S_{ét}, G_m)=0$([3], [1]). Therefore assume that $S$ is a projective space. Now let us consider two exact sequences:

$$0 \longrightarrow \mu_u \longrightarrow G_m \longrightarrow G_m \longrightarrow 0$$

$$0 \longrightarrow \mu_u \longrightarrow SL(u) \longrightarrow PGL(u) \longrightarrow 0.$$

Then we obtain exact sequences of étale cohomologies as follows:

$$\longrightarrow H^2(\boldsymbol{P}^n{}_{ét}, \mu_u) \xrightarrow{h} H^2(\boldsymbol{P}^n{}_{ét}, G_m) \longrightarrow$$

$$\longrightarrow H^1(\boldsymbol{P}^n{}_{ét}, PGL(u)) \xrightarrow{k} H^2(\boldsymbol{P}^n{}_{ét}, \mu_u) \longrightarrow .$$

Then $g$ in (1.2) is the composition of $k$ and $h$ (see IV of [7]). To prove $f$ is surjective, it suffices to check that $h$ is a zero map. Since it is known that $k$ is surjective (9 of VI [7]), we complete the proof. q. e. d.

In the next place let us study the second condition appearing in $(P)$ of Introduction. Let $h$ and $k$ be very ample line bundles on $M_1$ and $M_2$ respectively and let us consider the following conditions:

1) $\dim \Phi(M)=\max\{\dim M_1, \dim M_2\}$.

1′) $\dim \Phi(M)=\dim M_1=\dim M_2$.

2) For a closed point $x$ in $M_1$, $q^*k|_{p^{-1}(x)}$ is a trivial line bundle, or for a closed point $y$ in $M_2$, $p^*h|_{q^{-1}(y)}$ is so. Consequently both hold.

3) There is an isomorphism $\sigma: M_1 \cong M_2$ such that $\sigma p=q$.

Then we have

**Lemma 1.5.** *Under the above notations, assume that both $M_1$ and $M_2$ are normal varieties and that both $r$ and $s$ are positive. Then we have the following:*

A) *Conditions* 1), 1)′ *and* 2) *are equivalent to each other and condition* 3) *implies the other conditions.*

B) *If the characteristic of $k$ is zero, all the conditions are equivalent to each other.*

*Proof.* It is clear that 3) implies 1)′ and 1)′ implies 1). Therefore we show that 1) implies 2). Note that a representation of the morphism $\Phi$: $M \to M_1 \times M_2$ is given by the morphism $\phi$: $M \to P^{\dim|p^*h \otimes q^*k|}$, which the line bundle $p^*h \otimes q^*k$ yields, where $|*|$ is the complete linear system of a line bundle $*$. As $p^*h$ is generated by its global sections, we see that, for a point $y$ in $M_2$, $p^*h|_{q^{-1}(y)} \cong \mathcal{O}_{P^s}(\alpha)$ and $\alpha$ is non-negative. Now suppose that $\alpha$ is positive. Then since $p^*h \otimes q^*k|_{q^{-1}(y)} \cong \mathcal{O}_{P^s}(\alpha)$, the restricted map of $\Phi$ to $q^{-1}(y)$ is finite, which implies that $\Phi$ is finite. Consequently we see that $\dim M=\dim \Phi(M)>\max\{\dim M_1, \dim M_2\}$ because of positive integers $r$, $s$. This result contradicts condition 1). Therefore, we could prove that 1) means 2). In the next place, we show that 2) implies 1)′. Noting the above proof and the fact that a morphism from a projective space to another projective space is constant or finite, we see that the morphism $pr_i|_{\Phi(M)}$: $\Phi(M) \to M_i$ is bijective for $i=1, 2$, where $pr_i$ is the projection: $M_1 \times$

$M_2 \to M_i$. Hence 2) means 1)'. Therefore we complete A). Moreover if the characteristic of $k$ is zero, bijective morphism $pr_i|_{\Phi(M)}$ is biregular by Zariski Main Theorem, which implies B). q. e. d.

**Remark 1.6.** By the above lemma, if $r \neq s$, the condition that $\dim \Phi(M) > \max\{\dim M_1, \dim M_2\}$ automatically holds.

Hereafter in this section we shall study the fundamental properties of $M$ in order to show Theorem A.

Let $E_1$ (or $E_2$) be a vector bundle of rank $r+1$ (or, $s+1$, resp.) on $\boldsymbol{P}^l$ (or, $\boldsymbol{P}^m$, resp.). $p$ (or, $q$) denotes the canonical projection $\boldsymbol{P}(E_1) \to \boldsymbol{P}^l$ (or, $\boldsymbol{P}(E_2) \to \boldsymbol{P}^m$, resp.).

(1.7) Assume that $\boldsymbol{P}(E_1)$ is isomorphic to $\boldsymbol{P}(E_2) (\cong M)$ and $\dim \Phi(M) > \max(l, m)$ where the morphism $M \longrightarrow \boldsymbol{P}^l \times \boldsymbol{P}^m (= \Phi)$ is the one induced by $p$ and $q$.

Note that the assumption of $M$ in (1.7) is equivalent to the one in Theorem A. Furthermore put $\xi = \mathcal{O}_{P(E_1)}(1)$, $\eta = \mathcal{O}_{P(E_2)}(1)$, $h = p^* \mathcal{O}_{P^l}(1)$ and $k = q^* \mathcal{O}_{P^m}(1)$.

Now since we know that $Pic\ M \cong \boldsymbol{Z}\xi + \boldsymbol{Z}h \cong \boldsymbol{Z}\eta + \boldsymbol{Z}k$, we obtain two equalities:

$$\eta = a\xi + bh, \qquad k = \bar{a}\xi + \bar{b}h \tag{1.8}$$

where $a$, $\bar{a}$, $b$ and $\bar{b}$ are integers.

Moreover, we have

**Lemma 1.9.** *$\bar{a} > 0$ and $\bar{a}b - a\bar{b} = 1$.*

*Proof.* Let $f_p$ be a fiber of $p (\cong \boldsymbol{P}^r)$. Then we see that the intersection $(k . f_p . \xi^{r-1}) = \bar{a}(\xi^r . f_p) + \bar{b}(h . f_p . \xi^{r-1}) = \bar{a}(\xi^r . f_p)$. This yields $\bar{a} > 0$ because the assumption (1.7) and Lemma 1.5 imply that $(k . f_p . \xi^{r-1}) > 0$. On the other hand, if we $\xi = c\eta + dk$ and $h = \bar{c}\eta + \bar{d}k$ with integers $c$, $d$, $\bar{c}$, and $\bar{d}$, we obtain $\begin{pmatrix} a & b \\ \bar{a} & \bar{b} \end{pmatrix} \begin{pmatrix} c & d \\ \bar{c} & \bar{d} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\bar{c} > 0$ similarly. These show our lemma. q. e. d.

We denote by $A^r(S)$ the group of cycles of codim $r$ on a non-singular variety $S$ modulo rational equivalence and by $A(S) = \bigoplus_{r=0}^{n} A^r(S)$ with $\dim S = n$. Now let $E$ be a vector bundle of rank $r$ on a non-singular variety $Z$ and $\sigma$ the tautological line bundle of $E$. The $i$-th Chern class $c_i(E)$ of $E$ is an element of $A^i(S)$. The following is well-known.

**Theorem 1.10.** *Under the above notation let $f : \boldsymbol{P}(E) \to Z$ be the canonical projection. Then $f^*$ makes $A(\boldsymbol{P}(E))$ into a free $A(Z)$-module generated by $1$, $\sigma$, $\cdots$, $\sigma^{r-1}$. Moreover the following equality holds:*

$$\sum_{i=0}^{r} (-1)^i f^* c_i(E) \sigma^{r-i} = 0 \quad in \quad A^r(\boldsymbol{P}(E)).$$

For a proof, see [2].

Now for the variety $M$ in (1.7), let us study the Chow ring $A(M)$ using the Chern classes of $E_1$, $E_2$ and $\xi$, $\eta$. $c_i(E_1)$ (or, $c_j(E_2)$) is written in the form $c_i h^i$(or, $d_j k^j$, resp.), where $1 \leqq i \leqq \min(l, r+1)(=\gamma)$ (or, $1 \leqq j \leqq \min(m, s+1)(=\delta)$, resp.), $c_i$ and $d_j$ are integers. By virtue of Theorem 1.10, we consider a $\boldsymbol{Z}$-isomorphism of finite $\boldsymbol{Z}$-algebra as follows:

$$(1.11) \quad \boldsymbol{Z}[X, Y]/(Y^{l+1}, X^{r+1}-c_1X^rY+c_2X^{r-1}Y^2+\cdots+(-1)^{\gamma}c_{\gamma}X^{r+1-\gamma}Y^{\gamma})$$
$$\cong \boldsymbol{Z}[U, V]/(V^{m+1}, U^{s+1}-d_1U^sV+d_2U^{s-1}V^2+\cdots+(-1)^{\delta}d_{\delta}U^{s+1-\delta}V^{\delta})$$

under the additional condition that $U=aX+bY$, $V=\bar{a}X+\bar{b}Y$, $\bar{a}>0$ and $\bar{a}b-a\bar{b}=1$ where $X, Y, U$ and $V$ are indeterminates. We denote by $I$ the ideal $(Y^{l+1}, \sum_{i=0}^{r} c_i X^{r+1-i}(-Y)^i)$ of $\boldsymbol{Z}[X, Y]$ and by $J$ the ideal $(V^{m+1}, \sum_{j=0}^{\delta} d_j U^{s+1-j}(-V)^j)$ of $\boldsymbol{Z}[U, V]$ with $c_0=d_0=1$ and put

$$f(X, Y)=\sum_{i=0}^{r} c_i X^{r+1-i}(-Y)^i \quad \text{and} \quad g(U, V)=\sum_{j=0}^{\delta} d_j U^{s+1-j}(-V)^j.$$

From now on we shall investigate equalities about $l$, $m$, $r$ and $s$.

(1.12) Assume that $l \geqq m$.

Firstly we know that $\dim M=l+r=m+s$, since $M$ has two fiber bundle structures. Secondly, noting the degrees of generators in two ideals, $I$, $J$ it is easy to check that

$$(1.13) \qquad \min\{l+1, r+1\}=\min\{m+1, s+1\}.$$

by the isomorphism in (1.11).

Therefore as for $l$, $m$, $r$ and $s$, we obtain four cases as follows:

(1.14) a) $l<r$, $l=m$ and $r=s$,

b) $l=r=m=s$,

c) $l>r$, $l=s$ and $m=r$,

d) $l>r$, $l=m$ and $r=s$.

Finally in this section we shall show a key lemma for the proof of Theorem A.

**Lemma 1.15.** *Under the same notations as above and the assumption* (1.7) *we suppose that* $\bar{a}=1$ *in* (1.8). *Then in the cases* b), c), *the morphism* $\Phi: M \to \boldsymbol{P}^l \times \boldsymbol{P}^m$ *is an isomorphism, where* $\Phi$ *is the morphism induced by the fiber product of maps* $p$ *and* $q$. *In the case* d), *assume additionally that* $l=r+1$. *Then* $\Phi$ *is a closed immersion and* $\Phi(M)$ *is isomorphic to* $\boldsymbol{P}(T_{P^l})$.

*Proof.* In the cases b), c), we see easily that $\Phi$ is a finite birational morphism and therefore, an isomorphism by Zariski Main Theorem.

In the case d) we see similarly that $\Phi: M \to \Phi(M)(\subseteqq \boldsymbol{P}^l \times \boldsymbol{P}^l)$ is a finite birational morphism. On the other hand, since $\Phi(M)$ is a Cartier divisor in $\boldsymbol{P}^{l}\times\boldsymbol{P}^{l}$, we can put the defining equation of $\Phi(M)$ as $F(X_0, \cdots, X_l; Y_0, \cdots, Y_l)$ $(=F)$, where $F$ is a homogeneous polynomial of degree $d_1$(or, $d_2$) with respect to $X_0, \cdots, X_l$(or, $Y_0, \cdots, Y_l$, resp.). The assumption implies that $d_1=d_2=1$ and every fiber of one projection is transformed to a hyperplane in another base space by another projection. Therefore it is easy to see that $F$ is written in the form $\sum_{i=0}^{l} X_iY_i$ after suitable linear transformations of $X_0, \cdots, X_l$ and $Y_0, \cdots, Y_l$, respectively. Hence we see that $\Phi(M)$ is isomorphic to $\boldsymbol{P}(T_{\boldsymbol{P}^l})$ and $\Phi$ is a closed immersion. q. e. d.

## §2. Proof of Theorem A.

Case a) Using a well-known fact that if $r>m$, every morphism $\boldsymbol{P}^r\to\boldsymbol{P}^m$ is a constant map, we see easily that such an $M$ does not exist.

In view of Lemma 1.15, we devote ourselves to showing $\bar{a}=1$.

Case b) The isomorphism (1.11) provides us with the following:

$$(\bar{a}X+\bar{b}Y)^{l+1}=Af(X, Y)+BY^{l+1} \text{ with } A, B \text{ integers.}$$

Therefore we obtain $A=\bar{a}^{l+1}$ and $B=\bar{b}^{l+1}$, namely, $f(X, Y)=(X+\bar{b}Y/\bar{a})^{l+1}-(\bar{b}Y/\bar{a})^{l+1}$. Since the coefficient of $X^lY$ is $(l+1)\bar{b}^l/\bar{a}^l$ and $\bar{a}b-a\bar{b}=1$ says that $\bar{a}$ is prime to $\bar{b}$, we see that $\bar{a}^l$ devides $l+1$, which means $\bar{a}=1$. q. e. d.

Case c) It suffices to prove $\bar{a}=1$. Comparing the degree of generators of ideals $I$ and $J$, we obtain the following:

$$(\bar{a}X+\bar{b}Y)^{r+1}(\cong\bar{Y}^{r+1})=Af(X, Y) \text{ with an integer } A.$$

Since $(X+\bar{b}Y/\bar{a})^{r+1}$ is an elements of $\boldsymbol{Z}[X, Y]$, we get $\bar{a}=1$, for $\bar{a}$ is prime to $\bar{b}$. q. e. d.

In the last case d), we have to show the following facts.

1) $\bar{a}=1$,

2) There exists no $M$ satisfying $l>r+1$.

Then by Lemma 1.15, the proof of Theorem A will be completed.

First let us begin with 1), for which the following is essential.

**Theorem 2.1.** *Let $\sigma$ be a primitive n-th root of unity and $\boldsymbol{Q}(\sigma)$ the field generated by the rational number field $\boldsymbol{Q}$ and $\sigma$. Let $A$ be the ring of integers of $\boldsymbol{Q}(\sigma)$. Then $A=\bigoplus_{i=0}^{\phi(n)}\boldsymbol{Z}\sigma^i$, where $\phi(n)$ denotes Euler's number of n.*

For a proof, see Theorem 4 in [4].

The above theorem yields the following

**Proposition 2.2.** *Let $\Phi_n(X)$ be a cyclotomic polynomial of n-th root of unity $(n\geqq 3)$ and $\alpha$, $\beta$ integers. Assume that $\alpha^{\phi(n)}$ divides $\Phi_n(\alpha X+\beta)$ in $\boldsymbol{Z}[X]$. Then $\alpha=\pm 1$.*

*Proof.* Put $\Phi_n(\alpha X+\beta)=\alpha^{\phi(n)}h(X)$ where $h(X)$ is a monic polynomial in $\boldsymbol{Z}[X]$. For an $n$-th root of unity $(=\theta)$, $\alpha^{-1}(\theta-\beta)$ is a root of $h(X)$. Therefore by Theorem 2.1, we have $\alpha=\pm 1$. q.e.d.

Since $l=m$, we have the following equalities:

$$V^{m+1}=AY^{m+1}+f(X, Y)\bar{f}(X, Y), \tag{2.3}$$

$$Y^{m+1}=BV^{m+1}+g(U, V)\bar{g}(U, V), \tag{2.4}$$

where $\bar{f}(X, Y)\in\boldsymbol{Z}[X, Y]$, $\bar{g}(U, V)\in\boldsymbol{Z}[X, Y]$ and $A, B\in\boldsymbol{Z}$ and $\deg\bar{f}(X, Y)=\deg\bar{g}(U, V)\geqq 1$. Then we have

**Lemma 2.5.** $A=B=1$ *or* $-1$.

*Proof.* Computing $(2.3)\times B+(2.4)$, we get

$$Y^{m+1}=ABY^{m+1}+Bf(X, Y)\bar{f}(X, Y)+g(U, V)\bar{g}(U, V). \tag{2.6}$$

On the other hand, taking the degree of $f(X, Y)$, $g(U, V)$ and $\boldsymbol{Z}$-isomorphism of (1.11) into account, we see that $g(U, V)=\alpha f(X, Y)$ with $\alpha$ an integer. Hence we obtain

$$Y^{m+1}(1-AB)=f(X, Y)(B\bar{f}(X, Y)+\alpha\bar{g}(-\bar{b}X+bY,\ \bar{a}X-aY)).$$

It follows that $AB=1$, which is the desired result. q.e.d.

In the next place, we divide d) into two cases:
(2.7) $m$ is odd and $r=1$,
(2.8) otherwise.
We shall treat the case of (2.8) first and show that (2.7) does not occur at last in this section.

**Lemma 2.9.** *Assume that $M$ satisfies* (1.7) *and the condition in* d) *and that*
1) *$m$ is odd and $r\geqq 2$, or*
2) *$m$ is even.*
*Then, $\bar{a}=1$.*

*Proof.* Assume that $A=1$. Substituting $\bar{a}X+\bar{b}Y$ for $V$ and 1 for $Y$ in (2.3), we obtain the equality:

$$(\bar{a}X+\bar{b})^{m+1}-1=f(X, 1)\bar{f}(X, 1). \tag{2.10}$$

The left hand side in (2.10) is written as the product of $\Phi_j(\bar{a}X+\bar{b})$, where $\Phi_j(X)$ is a cyclotomic polynomial. Note that $\Phi_j(\bar{a}X+\bar{b})$ is irreducible in $\boldsymbol{Q}[X]$. On the other hand, factorize $f(X, 1)$ into the product of prime elements $f_j(X)$ in $\boldsymbol{Z}[X]$. Since $\boldsymbol{Z}[X]$ is UFD, there exist $u$ and $v$ such that $f_u(X)$ and $\Phi_v(\bar{a}X+\bar{b})$ are factors of $f(X, 1)$ and the left hand side of (2.10) respectively and, moreover, $\deg f_u(X)\geqq 2$ and $\Phi_v(\bar{a}X+\bar{b})/f_u(X)$ is an integer $(=\bar{a}^{\phi(v)})$. Therefore, Proposition 2.2 and $\bar{a}>0$ imply that $\bar{a}=1$. In the next place, assume that $A=-1$. We can prove $\bar{a}=1$ in the same way as in the case of $A=1$. q.e.d.

**Remark 2.11.** After determining the structure of $M$, we see that the case of $A=-1$ does not occur.

It is enough for the proof of Theorem A in the case of (2.8) to show the following

(2.12) There exists no $M$ satisfying the conditions that $\bar{a}=1$ and $l>r+1$. (See Lemma 1.15 for $l=r+1$.)

To prove (2.12), we need some results about the tangent bundle on $\boldsymbol{P}^n$, $M$, etc. By the assumption (1.7), we have the following exact sequences of the tangent bundles:

$$(2.13)_p \qquad 0 \longrightarrow T_p \xrightarrow{i} T_M \xrightarrow{j} p^*T_{\boldsymbol{P}^m} \longrightarrow 0,$$

$$(2.13)_q \qquad 0 \longrightarrow T_q \xrightarrow{\bar{i}} T_M \xrightarrow{\bar{j}} q^*T_{\boldsymbol{P}^m} \longrightarrow 0,$$

where $T_p$ and $T_q$ are the relative tangent bundle with respect to the projections $p$ and $q$. Restricting the above exact sequences on a fiber of $p(\cong\boldsymbol{P}^r)$ we have the following:

**Lemma 2.14.**

$$(2.14.1) \qquad 0 \longrightarrow T_{\boldsymbol{P}^r} \xrightarrow{i_p} T_M|_{\boldsymbol{P}^r} \longrightarrow \mathcal{O}_{\boldsymbol{P}^r}^{\oplus m} \longrightarrow 0$$

$$(2.14.2) \qquad 0 \longrightarrow T_q|_{\boldsymbol{P}^r} \longrightarrow T_M|_{\boldsymbol{P}^r} \xrightarrow{\bar{j}_p} T_{\boldsymbol{P}^r} \oplus \mathcal{O}_{\boldsymbol{P}^r}(1)^{\oplus m-r} \longrightarrow 0.$$

*Moreover, by* (2.14.1) *we have* $T_M|_{\boldsymbol{P}^r}\cong\mathcal{O}_{\boldsymbol{P}^r}^{\oplus m}\oplus T_{\boldsymbol{P}^r}$.

*Proof.* For a linear subspace $\boldsymbol{P}^u$ in $\boldsymbol{P}^v (u<v)$, we can easily check that $T_{\boldsymbol{P}^v}|_{\boldsymbol{P}^u}=T_{\boldsymbol{P}^u}\oplus\mathcal{O}_{\boldsymbol{P}^u}(1)^{v-u}$. On the other hand, $\bar{a}=1$ implies that for every fiber $f_p$ of $p$, $q(f_p)$ is a linear subspace of $\boldsymbol{P}^m$. This yields (2.14.1). The last part is obvious because $H^1(\boldsymbol{P}^r, T_{\boldsymbol{P}^r})=0$. q.e.d.

By the above lemma, we immediately have

**Lemma 2.15.** *$i_p\bar{j}_p$ is injective. Therefore in* $(2.13)_p$ *and* $(2.13)_q$, *$i\bar{j}$: $T_p\to q^*T_{\boldsymbol{P}^m}$ is an injective homomorphism of vector bundles. Similarly so is $\bar{i}j$: $T_q\to p^*T_{\boldsymbol{P}^m}$.*

*Proof.* The following are well known:

$$(2.15.1) \qquad H^0(\boldsymbol{P}^r, \mathcal{H}om(T_{\boldsymbol{P}^r}, T_{\boldsymbol{P}^r}))\cong k \quad\text{and}\quad H^0(\boldsymbol{P}^r, \mathcal{H}om(T_{\boldsymbol{P}^r}, \mathcal{O}_{\boldsymbol{P}^r}(1)))=0.$$

Assume that $i_p\bar{j}_p$ is a zero map. Then we have $T_{\boldsymbol{P}^r}\oplus\mathcal{O}_{\boldsymbol{P}^r}(1)^{\oplus m-r}\cong\mathcal{O}_{\boldsymbol{P}^r}^{\oplus m}$ which is absurd. Hence we see that $i_p\bar{j}_p$ is not a zero map. Therefore (2.15.1) completes the proof.

The above argument provides us with the exact sequence:

$$0 \longrightarrow T_p \oplus T_q \xrightarrow{(i,\bar{i})} T_M \longrightarrow (\mathrm{Coker}(i, \bar{i}) = A) \longrightarrow 0.$$

(2.15.2) Remark that $A$ is a vector bundle isomorphic to the quotient bundle $q^*T_{P^m}/i\bar{j}(T_p)$ and to $p^*T_{P^m}/\bar{i}j(T_q)$ in Lemma 2.14.

As for the bundle $A$, we obtain more detailed results.

**Corollary 2.16.** *For every fiber of* $p(=f_p)$, $A|_{f_p} \cong \mathcal{O}_{P^r}(1)^{\oplus m-r}$. *For every fiber of* $q(=f_q)$, $A|_{f_q} \cong \mathcal{O}_{P^r}(1)^{\oplus m-r}$. *Therefore $A$ is isomorphic to* $q^*\mathcal{O}_{P^m}(1) \otimes p^*A_p$ *and also to* $p^*\mathcal{O}_{P^m}(1) \otimes q^*A_q$, *where $A_p$ and $A_q$ are vector bundles of rank $m-r$ on* $\boldsymbol{P}^m$.

*Proof.* The first part is obvious by virtue of Lemma 2.14. Therefore the restriction of $A \otimes q^*\mathcal{O}_{P^m}(-1)$ on $f_p$ is a trivial vector bundle on $\boldsymbol{P}^m$ of rank $m-r$. Using the base change theorem by Grothendieck we immediately obtain $A \otimes q^*\mathcal{O}_{P^m}(-1) \cong p^*p_*A \otimes q^*\mathcal{O}_{P^m}(-1)$. Hence we get the desired vector bundle $A_p = p_*(A \otimes q^*\mathcal{O}_{P^m}(-1))$. q.e.d.

Now we divides the case (2.8) into two cases as follows:

(2.8.1) $m > 2r$,

(2.8.2) $m \leqq 2r$.

Let us show first that

(2.17) there is no $M$ satisfying $\bar{a}=1$ and $m>2r$.

*Proof.* We have to compute the first Chern class of $E_1$. By the assumption (1.7) we obtain the following exact sequences:

$(2.18)_p$ $$0 \longrightarrow T_p \longrightarrow T_M \longrightarrow p^*T_{P^m} \longrightarrow 0,$$

$(2.19)_p$ $$0 \longrightarrow \xi \otimes T_p \longrightarrow p^*E_1 \longrightarrow \xi \longrightarrow 0,$$

$(2.18)_q$ $$0 \longrightarrow T_q \longrightarrow T_M \longrightarrow q^*T_{P^m} \longrightarrow 0,$$

$(2.19)_q$ $$0 \longrightarrow \eta \otimes T_q \longrightarrow q^*E_2 \longrightarrow \eta \longrightarrow 0$$

$(2.19)_p$ and $(2.19)_q$ yield

$$c_1(T_p) = c_1h - (r+1)\xi \quad \text{and} \quad c_1(T_q) = d_1k - (r+1)\eta.$$

Moreover, by virtue of $(2.18)_p$ and $(2.18)_q$ we obtain

(2.18) $$\begin{aligned} c_1(T_M) &= c_1(T_p) + c_1(p^*T_{P^m}) = (m+1-c_1)h + (r+1)\xi \\ &= c_1(T_q) + c_1(q^*T_{P^m}) = (m+1-d_1)k + (r+1)\eta. \end{aligned}$$

Now by the assumption $\bar{a}=1$, we can take $E_1$ (or, $E_2$) as $p_*q^*k$ (or, $q_*p^*h$, resp.). Therefore since $h=\eta$ and $k=\xi$ (namely, $\bar{a}=b=1$, $a=\bar{b}=0$), we obtain $m+1-c_1=r+1$ and $m+1-d_1=r+1$, that is, (2.20) $$c_1=d_1=m-r.$$

On the other hand, (2.10) says that $f(X, 1)(=X^{r+1}+c_1X^r+\cdots+c_{r+1})$ divides $X^{m+1}-1$ or $X^{m+1}+1$. Now let $f_j(X)$ be a prime divisor of $f(X, 1)$ in $\boldsymbol{Z}[X]$. Then there exists a positive integer $n_0$ such that $f_j(X)$ is a cyclotomic polynomial $\Phi_{n_0}(X)=X^{\phi(n_0)}+\bar{c}_1X^{\phi(n_0)-1}+,,,+c_{\phi(n_0)}$ (see proposition 2.2). Moreover, it is well-known that $\bar{c}_1=-\mu(n_0)$, where $\mu(n)$ is the Möbius function. Since $\mu(n)$ takes only 0 and $\pm1$ as its values, $\deg f(X, 1)=r+1(\leqq m-r)$ and since $X^{m+1}-1$ and $X^{m+1}+1$ are not products of linear functions of $X$ for $m\geqq2$, we get $c_1\leqq r$, which contradicts (2.20). q. e. d.

In the next place, let us consider the case (2.8.2). In this case we shall study the Chern classes of two vector bundles $A_p$ and $A_q$ of rank $m-r$. Put the Chern polynomial of $A_p$ as $1+c_1(A_p)t+\cdots+c_{m-r}(A_p)t^{m-r}$ and put $c_i(A_p)=u_ih^i$ and $c_j(A_q)=v_jk^j$ with $u_i$, $v_j$ integers. Now we use a well-known result about the Chern class:

(2.21) For a vector bundle $E$ of rank $r(m\geqq r)$ on an $m$-fold $X$ and a line bundle $L$ on $X$,

$$c_i(E\otimes L)={}_rC_iL^i+{}_{r-1}C_{i-1}L^{i-1}c_1(E)+,\cdots,+{}_{r-i+1}C_1Lc_{i-1}(E)+c_i(E).$$

Let us return to $M$ in question. Since $m\leqq2r$, $h^i$, $h^{i-1}k$, $\cdots hk^{i-1}$, $k^i(1\leqq i\leqq m-r)$ are a free basis of $A^i(M)$. Now applying 2.21 to the vector bundle $A$ in Corollary 2.16 and noting 2.22, we have

**Proposition 2.23.** $u_i=v_i={}_{m-r}C_i(1\leqq i\leqq m-r)$.

Under the above preparations we can show

**Lemma 2.24.** *There is no $M$ satisfying the condition that $\bar{a}=1$ and $m\leqq2r$ $(m>r+1)$.*

*Proof.* By (2.15.2) and Corollary 2.16, we have

$$0\longrightarrow T_p\otimes q^*\mathcal{O}_{P^m}(-1)\longrightarrow q^*T_{P^m}(-1)\longrightarrow p^*A_p\longrightarrow0.$$

Since we know that $T_{P^m}(-1)$ is generated by its $m+1$ global sections, we obtain

$$0\longrightarrow(subbundle=B)\longrightarrow\overset{m+1}{\bigoplus}\mathcal{O}_M\longrightarrow p^*A_p\longrightarrow0.$$

Moreover, taking the direct image $p_*$ of the above sequence, we get

$$0\longrightarrow p_*B\longrightarrow\overset{m+1}{\bigoplus}\mathcal{O}_{P^m}\longrightarrow A_p\longrightarrow0.$$

Since $c(\overset{m+1}{\bigoplus}\mathcal{O}_{P^m})=c(p_*B)c(A_p)$ and $c(A_p)=(1+t)^{m-r}$ by Proposition 2.23. we have $m-r=1$. q. e. d.

Thus Theorem A has been proved in the case (2.8) and the remainder of the proof is to show that (2.7) does not occur.

**Lemma 2.25.** *There is no $M$ such that $m$ is odd and $r=1$.*

*Proof.* Recall the proof of Lemma 2.9. We could not show $\bar{a}=1$ in the case where $f(X, Y)$ is a product of linear forms $X$ and $Y$. Studying the divisors of the left hand side in (2.6), however, we see that $f(X, 1)=\frac{1}{\bar{a}^2}(\bar{a}X+\bar{b}-1)(\bar{a}X+\bar{b}+1)$ in $\boldsymbol{Z}[X]$ and, therefore, $\bar{a}=1$ or 2. Under the condition that $\bar{a}=1$ we have shown non-existence of $M$ with $m-r\geqq 2$. Therefore we can assume that $\bar{a}=2$. Namely we have

$$\eta=a\xi+bh, \qquad k=2\xi+\bar{b}h. \tag{2.26}$$

Since $f(X, 1)=X^2-c_1X+c_2=X^2+\bar{b}X+(\bar{b}^2-1)/4$, we have

(2.27) $c_1$ and $\bar{b}$ are odd, for $\bar{a}(=2)$ is prime to $\bar{b}$.

On the one hand, by (2.18) we have

$$\begin{aligned} c_1(T_M)&=(m+1-c_1)h+2\xi \\ &=(m+1-d_1)k+2\eta \\ &=(2b+(m+1-d_1)\bar{b})h+(2a+2(m+1-d_1))\xi \quad \text{(see (2.26))}. \end{aligned}$$

On the other hand, since we know that $\operatorname{Pic} M\cong\boldsymbol{Z}\xi+\boldsymbol{Z}h$, we obtain

$$2=2a+2(m+1-d_1) \tag{2.28}$$

$$m+1-c_1=2b+(m+1-d_1)\bar{b}. \tag{2.29}$$

Then $(2.28)\times\bar{b}/2-(2.29)$ gives us $c_1+\bar{b}=m$. Since $c_1$, $\bar{b}$ and $m$ are odd, the equality is absurd. q.e.d.

Combining Lemma 1.15, Corollary 2.17, Lemma 2.24 and Lemma 2.25, we complete proof of Theorem A.

## § 3. Proof of Theorem B.

Throughout this section we assume that *char* $k=0$. Let us begin with a simple lemma.

**Lemma 3.1.** *Let $Y$ and $Z$ be non-singular projective surfaces. Assume that $Y$ is a geometrically ruled surface and $f: Y\to Z$ is surjective. Then $Z$ is a geometrically ruled surface or $\boldsymbol{P}^2$.*

*Proof.* It is obvious that $f^*K_Z=K_Y-O_Y(D)$ where $D$ is the ramification divisor of $f$. Since $Y$ is ruled, we know that $H^0(Y, mK_Y)=0$ for every positive integer $m$. Therefore we have

$$\begin{aligned} h^0(Z, mK_Z)&\leqq h^0(Y, f^*mK_Z)=h^0(Y, mK_Y-O_Y(mD)) \\ &\leqq h^0(Y, mK_Y)(\dim H^0(\ ,\ )=h^0(\ ,\ )). \end{aligned}$$

Moreover, since $f$ is surjective, we see $2=b_2(Y)\geqq b_2(Z)$. Hence we get the desired result. q.e.d.

This lemma yields the next.

**Corollary 3.2.** *$S_1$ and $S_2$ are geometrically ruled surface or $\boldsymbol{P}^2$.*

*Proof.* Let $C$ be a general non-singular curve on $S_2$ with the genus of $C$ ($\geqq 1$). Then we see that $p: q^{-1}C\to S_1$ is surjective. q.e.d.

($B$, 1) Now assume that $S_2$ is geometrically ruled. Then let $\bar{q}: S_2\to C$ be the canonical projection where $C$ is the non-singular base curve. Put $\bar{q}^{-1}(\lambda)=l_\lambda$ for a point $\lambda$ of $C$.

**Lemma 3.3.** *Under the above notation, let us assume that there is a point $\lambda$ of $C$ such that $p: q^{-1}(l_\lambda)\to S_1$ is surjective. Then for every point $\lambda$ in $C$, $p: q^{-1}(l_\lambda)\to S_1$ is surjective.*

*Proof.* It is obvious.

Therefore we shall consider the structure of $M$ in two cases as follows:

(3.4) for every point $\lambda$ in $C$, $\dim p(q^{-1}(l_\lambda))=1$,

(3.5) for every point $\lambda$ in $C$, $\dim p(q^{-1}(l_\lambda))=2$.

First let us treat the case (3.4). The next proposition is important for this case.

**Proposition 3.6.** *Let $\phi: F_n\to P^1$ be a rational ruled surface with $F_n\cong \boldsymbol{P}(\mathcal{O}_{P^1}\oplus\mathcal{O}_{P^1}(n))$. Assume that $C$ is an irreducible curve of $F_n$, $\phi: C\to\boldsymbol{P}^1$ is finite and that the self-intersection number $C^2$ of $C$ is zero. Then $n=0$ and $C$ is a section of $\phi$.*

*Proof.* Assume that $n\geqq 1$. Let $C_0$ be the minimal section of $F_n$ and $f$ a fiber. Then $C$ is linearly equivalent to $aC_0+bf$ with $a$, $b$ integers. Then the surjectivity of $\phi: C\to\boldsymbol{P}^1$ implies $a>0$ and $2b=na$ is obtained by the computation of $C^2$. On the other hand $(C.C_0)=b-an=-b<0$, which is a contradiction by virtue of $C\neq C_0$. Hence we see $n=0$ and, therefore, the last part is obvious. q.e.d.

**Proposition 3.7.** *In the case (3.4) $M$ is isomorphic to $S_1\times_C S_2$, where both $S_1$ and $S_2$ are ruled surfaces over a non-singular curve $C$.*

*Proof.* Take a general fiber $l_\lambda$ in $S_2$ and put $p(q^{-1}(l_\lambda))=D_\lambda$. Then by the assumption, $D_\lambda$ is a curve. Choose a general point $A$ on $D_\lambda$. Since $q^{-1}(l_\lambda)$ is a rational ruled surface and the self-intersection number of $p^{-1}(A)$ in $q^{-1}(l_\lambda)$ is 0, we see that $q^{-1}(l_\lambda)$ is isomorphic to $\boldsymbol{P}^1\times\boldsymbol{P}^1$ and $p^{-1}(A)$ is a section of $q: q^{-1}(l_\lambda)\to l_\lambda$ by Proposition 3.6. Moreover, note that two projections of $\boldsymbol{P}^1\times\boldsymbol{P}^1\to\boldsymbol{P}^1$ are equal to the restriction of $p$ and $q$ to $q^{-1}(l_\lambda)$, respectively and $D_\lambda\cap D_\mu=$

$\emptyset(\lambda\neq\eta)$, which implies that $S_1$ is geometrically ruled. Take a general section $C_0$ of $\bar{q}: S_2\rightarrow C$. Then we see $p: q^{-1}(C_0)\rightarrow S_1$ is a birational morphism. As $q^{-1}(C_0)$ and $S_1$ are geometrically ruled, it is isomorphism. Therefore since $S_1$ is a ruled surface over $C$, we get $M\cong S_1\times_C S_2$. q.e.d.

In the next place, let us consider the case (3.5). Then we have

**Proposition 3.8.** *In the case* (3.5). *$M$ is isomorphic to $S_1\times C$ where $S_1$ is a rational ruled surface and $C(=\boldsymbol{P}^1)$ is a base curve of a ruled surface $S_2$.*

*Proof.* As $q^{-1}(l_\lambda)$ is rational, and $p: q^{-1}(l_\lambda)\rightarrow S_1$ is surjective, we see $S_1$ is rational. Similarly taking a general rational curve on $S_1$, we can check that $S_2$ is also rational. By virtue of Lemma 1.3, there are vector bundles $E_1$, $E_2$ of rank 2 on $S_1$ and $S_2$, respectively such that $\boldsymbol{P}(E_1)\cong\boldsymbol{P}(E_2)\cong M$. Hence $b_4(M)=3$ means that $S_1$ is geometrically ruled. Therefore let $\bar{p}: S_1\rightarrow\boldsymbol{P}^1$ be the natural projection and let us take a very ample divisor $C_0$ which yields a section of $\bar{p}$. Then we see that $q: p^{-1}(C_0)\rightarrow S_2$ is surjective and $\bar{l}_\lambda=q^{-1}(C_0)\cap q^{-1}(l_\lambda)$ is an irreducible curve in $q^{-1}(l_\lambda)$ by Bertini's Theorem. Applying Proposition 3.6, to $p$: $p^{-1}(C_0)\rightarrow C_0$ and $\bar{l}_\lambda$ we see that $\bar{l}_\lambda$ is a section of $p$ and therefore $p: q^{-1}(l_\lambda)\rightarrow S_1$ is isomophism. Now since $C$ is a base curve of $S_2$, we get the morphism $g: M \xrightarrow{(p,\,\bar{q}q)} S_1\times C$ by the fiber product. Then it is easy to see that $g$ is finite birational morphism and therefore a biregular morphism. q.e.d.

Looking into the proof (B.1), carefully, in the rest of the proof of Theorem B, we may assume that $S_1$ and $S_2$ are $\boldsymbol{P}^2$. But we have already shown the more generalized Theorem A. Therefore we complete the proof of Theorem B.

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