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On the endomorphism ring of the canonical module

By

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Introduction.

A ring will always mean a commutative ring with unit. Let R be a noetherian ring, M a finitely generated R-module and N a submodule of M. We denote by $\mathrm{Min}_R(M)$ the set of all minimal elements in $\mathrm{Supp}_R(M)$. In the case where M is of finite dimension, we put $\mathrm{Assh}_R(M)=\{\mathfrak{p}\in\mathrm{Ass}_R(M)\mid \dim R/\mathfrak{p}=\dim M\}$ and $U_M(N)=\cap Q$ where Q runs through all the primary components of N in M such that $\dim M/Q=\dim M/N$. Let T be an R-module and $\mathfrak{a}$ an ideal of R. $E_R(T)$ denotes the injective envelope of T and $H_{\mathfrak{a}}^i(T)$ is the i-th local cohomology module of T with respect to $\mathfrak{a}$. A semi-local ring means a noetherian ring with a finite number of maximal ideals and a local ring is a semi-local ring with unique maximal ideal. We denote by ^ the Jacobson radical adic completion over a semi-local ring. For a ring R, $Q(R)$ denotes the total quotient ring of R and we define $\dim_R 0$ to be $-\infty$ and height R to be $+\infty$.

First we recall the definition of the canonical module.

Definition 0.1 ([6, Definition 5.6]). Let R be an n-dimensional local ring with maximal ideal $\mathfrak{n}$. An R-module C is called *the canonical module* of R if $C\otimes_R\hat{R}\cong\mathrm{Hom}_R(H_{\mathfrak{n}}^n(R), E_R(R/\mathfrak{n}))$.

When R is complete, the canonical module C of R exists and is the module which represents the functor $\mathrm{Hom}_R(H_{\mathfrak{n}}^n(\ ), E_R(R/\mathfrak{n}))$, that is, $\mathrm{Hom}_R(H_{\mathfrak{n}}^n(M), E_R(R/\mathfrak{n}))\cong\mathrm{Hom}_R(M, C)$ (functorial) for any R-module M ([6, Satz 5.2]). For elementary properties of the canonical module, we refer the reader to [5, §6], [6, 5 und 6 Vorträge] and [2, §1]. If R is a homomorphic image of a Gorenstein ring, R has the canonical module C and it is well known that $C_{\mathfrak{p}}$ is the canonical module of $R_{\mathfrak{p}}$ for every $\mathfrak{p}$ in $\mathrm{Supp}_R(C)$ ([6, Korollar 5.25]). On the other hand, as was shown by Ogoma [7, §6], there exists a local ring with canonical module and non-Gorenstein formal fibre, hence not a homomorphic image of a Gorenstein ring. But the following fact holds in general and our consideration largely depends on it.

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Theorem 0.2 ([2, Corollary 4.3]). *Let R be a local ring with canonical module C and let $\mathfrak{p}$ be in* $\mathrm{Supp}_R(C)$. *Then $C_{\mathfrak{p}}$ is the canonical module of $R_{\mathfrak{p}}$.*

Here we state the definitions of the condition (S_t) and a quasi-Gorenstein ring, which are important in our research.

Definition 0.3. Let R be a noetherian ring, M a finitely generated R-module and t an integer. We say that M is (S_t) if depth $M_{\mathfrak{p}} \geqq \min\{t, \dim M_{\mathfrak{p}}\}$ for every $\mathfrak{p}$ in $\mathrm{Supp}_R(M)$.

Definition 0.4 (Platte and Storch). A local ring is said to be *quasi-Gorenstein* if it has a free canonical module. A noetherian ring R is called a *quasi-Gorenstein ring* if $R_{\mathfrak{p}}$ is a quasi-Gorenstein local ring for every prime ideal $\mathfrak{p}$.

A local ring is quasi-Gorenstein if and only if so is the completion. A noetherian ring R is quasi-Gorenstein if and only if $R_{\mathfrak{n}}$ is a quasi-Gorenstein local ring for every maximal ideal $\mathfrak{n}$ by [2, Corollary 2.4]. A noetherian ring is a Gorenstein ring if and only if it is a quasi-Gorenstein Cohen-Macaulay ring.

Throughout the paper, A denotes a d-dimensional local ring with maximal ideal $\mathfrak{m}$ and canonical module K. We put $H=\mathrm{End}_A(K)$ and let h be the natural map from A to H.

In the previous paper [2], the following properties of H were shown:

(0.5.1) H is a finite (S_2) over-ring of $A/U_A(0)$ contained in $Q(A/U_A(0))$. ([2, Theorem 3.2])

(0.5.2) $\dim_A \mathrm{Coker}(h) \leqq d-2$. ([2, Proof of Theorem 4.2])

The main purpose of this paper is to show that H is characterized by the above properties (Theorem 1.6). In section 1, first we show that the map h is an isomorphism if and only if A is (S_2) using Theorem 0.2, and then we prove Theorem 1.6, by which we can consider H as the unique (S_2)-fication of A in a certain sense. As a corollary, we have a remark on the existence of the canonical module (Corollary 1.8). In connection with this, it was recently found out that A is a homomorphic image of a Gorenstein ring if A is an equidimensional local ring of dimension 2 or $H_{\mathfrak{m}}^i(A)$ is of finite length for $i \neq d$. Now we assume $U_A(0)=0$ and put $\mathfrak{c}=A:H$. Let T be the $\mathfrak{c}$-transform of A, i.e., $T=\{x \in Q(A) \mid x\mathfrak{c}^t \subseteq A$ for some $t\}$. Then we show that $T \cong H$ as A-algebras. In section 2 we show that H is a Cohen-Macaulay ring if and only if K is a Cohen-Macaulay module and, as a corollary, that A is Cohen-Macaulay if and only if A is (S_2) and K is Cohen-Macaulay (a result of Schenzel). In section 3 we consider the ideal $\mathfrak{g}_A=\mathrm{Im}(K \otimes_A \mathrm{Hom}_A(K, A) \to A)$. The ideal $\mathfrak{g}_A$ is closely related to Gorensteinness in the case where A is Cohen-Macaulay ([6, 6 Vortrag]) and in general related to quasi-Gorensteinness, that is, A is quasi-Gorenstein if and only if $\mathfrak{g}_A=A$ (Proposition 3.3). The proofs of results in section 3 essentially depend on Theorem 0.2. We also show that the quasi-Gorensteinness of H implies

$\mathfrak{g}_A=\mathfrak{c}$ if $U_A(0)=0$ and the converse does not hold. In the appendix we give a generalization of [6, Satz 6.14] and [2, Proposition 4.1]. Let B be a faithfully flat local A-algebra. Then we prove that $K\otimes_A B$ is the canonical module of B if and only if $B/\mathfrak{m}B$ is Gorenstein under the condition that $B/\mathfrak{m}B$ is Cohen-Macaulay. This result is related to the existence problem of the canonical module for certain local rings.

1. A characterization of H.

We begin with the following

Lemma 1.1 ([7, Lemma 4.1]). *Assume that* depth $A_{\mathfrak{p}}\geqq\min\{2, \dim A_{\mathfrak{p}}\}$ *for every* $\mathfrak{p}$ *in* $\mathrm{Supp}_A(K)$. *Then* $\mathrm{Ass}(A)=\mathrm{Assh}(A)$, *that is,* $U_A(0)=0$.

Proof. Here we give a proof using Theorem 0.2. We proceed by induction on d. If $d\leqq 2$, then A is Cohen-Macaulay and the assertion is obvious. Let $d>2$ and let $(0)=\mathfrak{q}_1\cap\cdots\cap\mathfrak{q}_t$ be a primary decomposition of the zero ideal in A such that $\dim A/\mathfrak{q}_i=d$ if and only if $i\leqq s\,(1\leqq s\leqq t)$. We put $\mathfrak{a}=\mathfrak{q}_1\cap\cdots\cap\mathfrak{q}_s$ and $\mathfrak{b}=\mathfrak{q}_{s+1}\cap\cdots\cap\mathfrak{q}_t$. Note that $\mathfrak{a}=U_A(0)=\mathrm{ann}_A(K)$ (cf. [2, (1.8)]). Let $\mathfrak{p}$ be a non-maximal prime ideal in $\mathrm{Supp}_A(K)$. Then $U_{A_{\mathfrak{p}}}(0)=0$ by the induction hypothesis because $K_{\mathfrak{p}}$ is the canonical module of $A_{\mathfrak{p}}$. Since $U_{A_{\mathfrak{p}}}(0)=(U_A(0))_{\mathfrak{p}}$ by [2, (1.9)], we have $\mathfrak{p}\not\supseteq\mathfrak{b}$. Suppose that $s<t$. Then $\mathfrak{a}+\mathfrak{b}$ is an $\mathfrak{m}$-primary ideal. Since depth $A\geqq 2$ and depth $A/\mathfrak{a}\oplus A/\mathfrak{b}\geqq 1$, we have depth $A/\mathfrak{a}+\mathfrak{b}>0$ from the exact sequence $0\to A\to A/\mathfrak{a}\oplus A/\mathfrak{b}\to A/\mathfrak{a}+\mathfrak{b}\to 0$. This is a contradiction. Hence we have $s=t$, that is, $\mathfrak{a}=0$. q. e. d.

Proposition 1.2 (cf. [1, Proposition 2] and [7, Proposition 4.2]). *The following are equivalent*:

(a) *The map* h *is an isomorphism.*

(b) $\hat{A}$ *is* (S_2).

(b′) *For every* $\mathfrak{q}$ *in* $\mathrm{Supp}_{\hat{A}}(\hat{K})$, depth $\hat{A}_{\mathfrak{q}}\geqq\min\{2, \dim\hat{A}_{\mathfrak{q}}\}$.

(c) A *is* (S_2).

(c′) *For every* $\mathfrak{p}$ *in* $\mathrm{Supp}_A(K)$, depth $A_{\mathfrak{p}}\geqq\min\{2, \dim A_{\mathfrak{p}}\}$.

Proof. (a)⇒(b) and (a)⇒(c) follow from [2, (1.10)]. (b)⇒(b′) and (c)⇒(c′) are obvious, and (*b*′)⇒(*c*′) is well known. Hence it is sufficient to prove (c′)⇒(a). We proceed by induction on d. If $d\leqq 2$, then A is Cohen-Macaulay and the assertion is known (cf. [6, 6 Vortrag]). Let $d>2$. By the induction hypothesis and Theorem 0.2, $\mathrm{Coker}(h_{\mathfrak{p}})=0$ for every non-maximal prime ideal $\mathfrak{p}$. By Lemma 1.1, we have $\mathrm{Ker}(h)=\mathrm{ann}_A(K)=U_A(0)=0$. Since depth $A\geqq 2$, depth $H\geqq 2$ and $\mathrm{Coker}(h)$ is of finite length, we have $\mathrm{Coker}(h)=0$. Hence h is an isomorphism. q. e. d.

Corollary 1.3. *Assume* $\mathrm{Min}(A)=\mathrm{Assh}(A)$. *Then the* (S_2)*-locus* $\{\mathfrak{p}\in\mathrm{Spec}(A)\mid A_{\mathfrak{p}}$ *is* $(S_2)\}$ *is open in* $\mathrm{Spec}(A)$.

Remark 1.4. The following are equivalent:

(a) A is (S_2).

(b) $H_{\mathfrak{m}}^d(K)\cong E_A(A/\mathfrak{m})$.

(c) There is a finitely generated A-module M such that $H_{\mathfrak{m}}^d(M)\cong E_A(A/\mathfrak{m})$.

Proof. We may assume that A is complete by virtue of Proposition 1.2.

(a)$\Rightarrow$(b): Since $A\cong H=\mathrm{Hom}_A(K, K)\cong\mathrm{Hom}_A(H_{\mathfrak{m}}^d(K), E_A(A/\mathfrak{m}))$ (cf. [6, Satz 5.2]), we have $H_{\mathfrak{m}}^d(K)\cong E_A(A/\mathfrak{m})$.

(b)$\Rightarrow$(c): Trivial.

(c)$\Rightarrow$(a): Since $\mathrm{Hom}_A(M, K)\cong\mathrm{Hom}_A(H_{\mathfrak{m}}^d(M), E_A(A/\mathfrak{m}))\cong\mathrm{Hom}_A(E_A(A/\mathfrak{m}), E_A(A/\mathfrak{m}))\cong A$ and K is (S_2), we have the assertion. q.e.d.

Remark 1.5. Let M be a finitely generated (S_2) A-module such that $H_{\mathfrak{m}}^d(M)\cong E_A(A/\mathfrak{m})$ and $\mathrm{Min}_A(M)=\mathrm{Assh}_A(M)$. Then $M\cong K$. In this case A is (S_2). (This gives another proof of the case (I) of [2, Theorem 4.2]).

Proof. By [2, Proposition 4.4], we have $M\cong\mathrm{Hom}_A(\mathrm{Hom}_A(M, K), K)$. Hence we have $M\cong K$ because $\mathrm{Hom}_A(M, K)\cong A$. (Note that $\mathrm{Hom}_A(N, K)\cong A$ if and only if $H_{\mathfrak{m}}^d(N)\cong E_A(A/\mathfrak{m})$ for a finitely generated A-module N). q.e.d.

Now we state and prove our main result.

Theorem 1.6. *Let R be an A-algebra with structure homomorphism f. Then the following are equivalent:*

(a) *$R\cong H$ as A-algebras.*

(b) *R satisfies the following conditions*

(i) *R is (S_2) and finitely generated as an A-module,*

(ii) *For every maximal ideal $\mathfrak{n}$ of R, $\dim R_{\mathfrak{n}}=d$, and*

(iii) $\dim_A \mathrm{Coker}(f)\leqq d-2$ *and* $\dim_A \mathrm{Ker}(f)\leqq d-1$.

Proof. By virtue of [2, Theorem 3.2], it is sufficient to prove (b)$\Rightarrow$(a). First we see $\mathrm{Ker}(f)=U_A(0)$. By [6, Satz 5.12] and the condition (ii), $\mathrm{Hom}_A(R, K)_{\mathfrak{n}}$ is the canonical module of $R_{\mathfrak{n}}$ for every maximal ideal $\mathfrak{n}$ of R. Since $R_{\mathfrak{n}}$ is (S_2), we have $\mathrm{Ass}(R_{\mathfrak{n}})=\mathrm{Assh}(R_{\mathfrak{n}})$ by Lemma 1.1. Let $\mathfrak{q}$ be in $\mathrm{Ass}(R)$ and $\mathfrak{n}$ a maximal ideal containing $\mathfrak{q}$. Then we have $\dim R_{\mathfrak{n}}/\mathfrak{q}R_{\mathfrak{n}}=d$ and $\dim R/\mathfrak{q}=d$. Hence we have $\mathfrak{q}\cap A\in\mathrm{Assh}(A)$. Let s be an element of $A\backslash\bigcup_{\mathfrak{p}\in\mathrm{Assh}(A)}\mathfrak{p}$. Then $f(s)$ is not a zero divisor in R. Hence we have $U_A(0)\subseteq\mathrm{Ker}(f)$ because $sU_A(0)=0$ for some s in $A\backslash\bigcup_{\mathfrak{p}\in\mathrm{Assh}(A)}\mathfrak{p}$. By the condition (iii), we have $\mathrm{Ker}(f)_{\mathfrak{p}}=0$ for every $\mathfrak{p}$ in $\mathrm{Assh}(A)$. Hence we have $U_A(0)=\mathrm{Ker}(f)$. We may assume $U_A(0)=0$ because K is the canonical module of $A/U_A(0)$ and $H=\mathrm{End}_{A/U_A(0)}(K)$ (cf. [2, (1.8)]). We put $L=\mathrm{Hom}_A(R, K)$. Note that $L_{\mathfrak{n}}$ is the canonical module of $R_{\mathfrak{n}}$ for every maximal ideal $\mathfrak{n}$ of R. Since $\dim_A R/A\leqq d-2$, $\mathrm{Hom}_A(R/A, K)=0$ and $\mathrm{Ext}_A^1(R/A, K)=0$ by [2, (1.10)]. Hence we have an isomorphism $L=\mathrm{Hom}_A(R, K)\xrightarrow{\sim}\mathrm{Hom}_A(A, K)\cong K$ from the exact sequence $0\to A\to R\to R/A\to 0$. From this isomorphism, we obtain an A-algebra isomorphism from H to $\mathrm{End}_A(L)$. Because H is commutative, so is $\mathrm{End}_A(L)$ and therefore $\mathrm{End}_A(L)=\mathrm{End}_R(L)$. Since R is (S_2), $R\cong\mathrm{End}_R(L)$

by Proposition 1.2. Hence we have $R \cong H$ as A-algebras. Finally we note that, if $R \subseteq Q(A/U_A(0))$, the condition (ii) holds (cf. [2, Proof of Theorem 3.2(2)]).

q. e. d.

As a corollary to the above proof, we have the following corollary which is an essential part of the proof of [2, Theorem 4.2].

Corollary 1.7. *Let B be a local ring and assume that there is a ring R satisfying the following conditions:*

(i) *R is a finite over-ring of B,*

(ii) *For every maximal ideal $\mathfrak{n}$ of R, $\dim R_{\mathfrak{n}} = \dim B$,*

(iii) *R has the canonical module T, i.e., $T_{\mathfrak{n}}$ is the canonical module of $R_{\mathfrak{n}}$ for every maximal ideal $\mathfrak{n}$ of R, and*

(iv) $\dim_B R/B \leqq \dim B - 2$.

Then T, as a B-module, is the canonical module of B. Furthermore if R is (S_2), then $U_B(0)=0$ and $R \cong \mathrm{End}_B(T)$ as B-algebras.

From the above results, we have the following corollaries concerning the existence of the canonical module.

Corollary 1.8. *Let B be a local ring of dimension n. Then the following are equivalent:*

(a) *B has the canonical module.*

(b) *There is a finite B-algebra R with structure homomorphism g such that*

(i) *R is (S_2), $\dim_B \mathrm{Ker}(g) \leqq n-1$ and $\dim_B \mathrm{Coker}(g) \leqq n-2$,*

(ii) *For every maximal ideal $\mathfrak{n}$ of R, $\dim R_{\mathfrak{n}} = n$, and*

(iii) *R is a homomorphic image of an n-dimensional quasi-Gorenstein ring.*

(c) *There is a finite over-ring R of $B/U_B(0)$ satisfying*

(i) $\dim_B \mathrm{Coker}(B \to R) \leqq n-2$,

(ii) *For every maximal ideal $\mathfrak{n}$ of R, $\dim R_{\mathfrak{n}} = n$, and*

(iii) *R is a homomorphic image of an n-dimensional quasi-Gorenstein ring.*

Proof. (a)$\Rightarrow$(b): Let L be the canonical module of B and $R = \mathrm{End}_B(L)$. Then R satisfies (i) and (ii) (cf. [2, Theorem 3.2]). By [2, Theorem 3.2 and Theorem 2.11], $R \ltimes L$, the idealization, is an n-dimensional quasi-Gorenstein ring, hence R also satisfies (iii).

(b)$\Rightarrow$(c): Obvious (cf. Proof of Theorem 1.6).

(c)$\Rightarrow$(a): R satisfies the conditions in Corollary 1.7 with respect to $B/U_B(0)$ (cf. [6, Satz 5.12]). Therefore B has the canonical module by virtue of [2, (1.12)].

q. e. d.

Corollary 1.9. *Let B be a local ring of dimension 2. Then the following are equivalent:*

(a) *B has the canonical module.*

(b) *There is a finite B-algebra R with structure homomorphism g such that*

(i) *R is a Cohen-Macaulay ring which is a homomorphic image of a*

Gorenstein ring,

(ii) *For every maximal ideal* $\mathfrak{n}$ *of* R, $\dim R_{\mathfrak{n}}=2$, *and*

(iii) $\dim_B \operatorname{Ker}(g) \leqq 1$ *and* $\operatorname{Coker}(g)$ *is of finite length.*

As was seen in [2, Example 3.3], H is not necessarily a local ring. With this we remark the following proposition. The proof is not difficult, so we leave it to the reader.

Proposition 1.10. *Let* $\mathfrak{n}_1, \cdots, \mathfrak{n}_r$ *be the maximal ideals of* H. *Then* $\hat{K}$ *has a decomposition* $\hat{K}=\bigoplus_{i=1}^{r} K_i$ *by indecomposable* $\hat{A}$*-modules* $K_1, \cdots, K_r$ *such that* $\hat{H}_{\mathfrak{n}_i} \cong \operatorname{Hom}_{\hat{A}}(K_i, K_i)$ *for* $i=1, \cdots, r$ *and* $\operatorname{Hom}_{\hat{A}}(K_i, K_j)=0$ *for* $i \neq j$. *In this case* $K_i \cong \widehat{K_{\mathfrak{n}_i}}$ *for* $i=1, \cdots, r$. *In particular,* H *is a local ring if and only if* $\hat{K}$ *is an indecomposable* $\hat{A}$*-module.*

Next we consider a relation between H and ideal transforms.

Let R be a ring and I an ideal containing a non zero divisor. From the exact sequence $0 \to I^t \to R \to R/I^t \to 0$, we have the exact sequence $0 \to R \to \operatorname{Hom}_R(I^t, R) \to \operatorname{Ext}_R^1(R/I^t, R) \to 0$. Taking the direct limits, we have the exact sequence $0 \to R \to \operatorname{ind\,lim}_t \operatorname{Hom}_R(I^t, R) \to H_I^1(R) \to 0$. For an ideal J of R, we put $R(J)=\{x \in Q(R) \mid xJ^t \subseteq R$ for some $t\}$, the J-transform of R, which is an R-subalgebra of $Q(R)$. $R:_{Q(R)} I^t$ is naturally isomorphic to $\operatorname{Hom}_R(I^t, R)$. Hence, from the above argument, we have the following

Lemma 1.11. *There is an exact sequence of* R*-modules* $0 \to R \to R(I) \to H_I^1(R) \to 0$ *and* $R(I)$ *is an* R*-subalgebra of* $Q(R)$.

We put $\mathfrak{c}=\{a \in A \mid aH \subseteq h(A)\}$. The ideal $\mathfrak{c}$ is uniquely determined.

In the remainder of this section, we assume that $d \geqq 2$ and $U_A(0)=0$.

Since $K_{\mathfrak{p}}$ is the canonical module of $A_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p}$, $A_{\mathfrak{p}}$ is (S_2) if and only if $\mathfrak{p} \not\supseteq \mathfrak{c}$ by Proposition 1.2.

Proposition 1.12. *There is a unique intermediate ring* R *between* A *and* $Q(A)$ *such that* $R \cong H$ *as* A*-algebras. In this case* $R=A(\mathfrak{c})=A:_{Q(A)}\mathfrak{c}$.

Proof. The existence of such a ring is due to [2, Theorem 3.2]. Let R be a ring such that $A \subseteq R \subseteq Q(A)$ and $R \cong H$ as A-algebras. We must show $R=A(\mathfrak{c})=A:_{Q(A)}\mathfrak{c}$. If $\mathfrak{c}=A$, the assertion is obvious. Let $\mathfrak{c} \neq A$. Since height $\mathfrak{c} \geqq 2$ (we assume $U_A(0)=0$), there is a subsystem x, y of parameters contained in $\mathfrak{c}$. Because x, y is a K-regular sequence ([2, (1.10)]), x, y is also an R-regular sequence. Hence we have $H_{\mathfrak{c}}^0(R)=0$ and $H_{\mathfrak{c}}^1(R)=0$. From the exact sequence $0 \to R \to Q(A) \to Q(A)/R \to 0$, we have $H_{\mathfrak{c}}^0(Q(A)/R)=0$. Hence from the exact sequence $0 \to R/A \to Q(A)/A \to Q(A)/R \to 0$, we have $R/A \supseteq H_{\mathfrak{c}}^0(R/A)=H_{\mathfrak{c}}^0(Q(A)/A)=A(\mathfrak{c})/A$ and therefore $R \supseteq A(\mathfrak{c})$. On the other hand, we have $\mathfrak{c}R=\mathfrak{c} \subseteq A$ because $\mathfrak{c}=A:_A R$. Hence we have $R \subseteq A:\mathfrak{c} \subseteq A(\mathfrak{c})$. q.e.d.

Corollary 1.13. *The following are equivalent*:
(a) $A(\mathfrak{m})\cong H$ *as A-algebras.*
(b) *For every non-maximal prime ideal* $\mathfrak{p}$, depth $A_{\mathfrak{p}}\geqq\min\{2, \dim A_{\mathfrak{p}}\}$.
(c) $\mathfrak{c}\supseteqq\mathfrak{m}^t$ *for some t.*

Proof. (a)$\Rightarrow$(b): By Lemma 1.11, there is an exact sequence $0\to A\to H\to H_{\mathfrak{m}}^1(A)\to 0$. Since $H_{\mathfrak{m}}^1(A)_{\mathfrak{p}}=0$ for every non-maximal prime ideal $\mathfrak{p}$ and H is (S_2), we have the assertion.

(b)$\Rightarrow$(c): Because $\mathfrak{p}\not\supseteqq\mathfrak{c}$ for every non-maximal prime ideal $\mathfrak{p}$.

(c)$\Rightarrow$(a): If $\mathfrak{c}\neq A$, $A(\mathfrak{m})=A(\mathfrak{c})\cong H$. If $\mathfrak{c}=A$, $A=A(\mathfrak{c})\cong H$. On the other hand $A(\mathfrak{m})=A$ because depth $A\geqq 2$. q.e.d.

Corollary 1.14. (1) *If* $d=2$, *then* $A(\mathfrak{m})\cong H$ *as A-algebras.*
(2) *If* $H_{\mathfrak{m}}^i(A)$ *is of finite length for* $i\neq d$, *then* $A(\mathfrak{m})\cong H$ *as A-algebras.*

Remark 1.15. Assume that $H_{\mathfrak{m}}^i(A)=0$ for $i\neq 1$, d and $H_{\mathfrak{m}}^1(A)$ is of finite length. Then $A(\mathfrak{m})\cong H$ is just the Cohen-Macaulayfication of A due to the second author [3]. (cf. Example 2.4(3))

2. The Cohen-Macaulayness of H.

For a finitely generated A-module M of dimension d, we put $K_M=\operatorname{Hom}_A(M, K)$. Note that $K_M\otimes_A\hat{A}\cong\operatorname{Hom}_A(H_{\mathfrak{m}}^d(M), E_A(A/\mathfrak{m}))$ and that in the case where A is complete K_M is the module representing the functor $\operatorname{Hom}_A(H_{\mathfrak{m}}^d(-\otimes_A M), E_A(A/\mathfrak{m}))$ (cf. [6, Satz 5.2]). By the same argument as in [1, Proof of Lemma 1], we have the following

Lemma 2.1. *Let M be a finitely generated A-module of dimension d and depth t.*

(1) *If M is a Cohen-Macaulay module, then* K_M *is also a Cohen-Macaulay module.*

(2) *Assume that M is not a Cohen-Macaulay module and put* $s=\max\{i\mid i<d$ *and* $H_{\mathfrak{m}}^i(M)\neq 0\}$.

(i) *If* $\operatorname{depth}_{\hat{A}}\operatorname{Hom}_A(H_{\mathfrak{m}}^s(M), E_A(A/\mathfrak{m}))=0$, *then*

$$\operatorname{depth} K_M=\begin{cases} d-s+1 & \text{if } s>0, \\ d & \text{if } s=0. \end{cases}$$

(ii) *If* $s=t$ *and* $\operatorname{depth}_{\hat{A}}\operatorname{Hom}_A(H_{\mathfrak{m}}^t(M), E_A(A/\mathfrak{m}))=u$, *then*

$$\operatorname{depth} K_M=\begin{cases} d-t+u+1 & \text{if } u<t, \\ d & \text{if } u=t. \end{cases}$$

Proposition 2.2. *H is a Cohen-Macaulay ring if and only if K is a Cohen-Macaulay module.*

Proof. Since $H=\operatorname{Hom}_A(K, K)$ and $K\cong\operatorname{Hom}_A(H, K)$, the assertion immediately follows from Lemma 2.1(1). q.e.d.

Corollary 2.3 (Schenzel). *A is a Cohen-Macaulay ring if and only if A is* (S_2) *and K is a Cohen-Macaulay module.*

Example 2.4. (1) If $d \leqq 2$, then H is always Cohen-Macaulay.

(2) Let t, n be integers such that $0 \leqq t < n$. Then there is a local ring B with Cohen-Macaulay canonical module L such that depth $B = t$ and dim $B = n$ ([1, Theorem 1]) and $\mathrm{End}_B(L)$ is a Cohen-Macaulay ring.

(3) If $H^i_{\mathfrak{m}}(A) = 0$ for $1 < i < d$, then H is a Cohen-Macaulay ring. (cf. Lemma 2.1 and [1, Proof of Lemma 1])

(4) If A is an approximately Cohen-Macaulay ring, then H is a Cohen-Macaulay ring. (See [4]).

3. The quasi-Gorensteinness and the ideal $\mathfrak{g}_A$.

We begin with the following two facts which are slight generalizations of results in [6]. The proofs are parallel to those given in [6] by virtue of Theorem 0.2, so we omit them.

(3.1) (cf. [6, Korollar 6.7]). *Assume* $\mathrm{Ass}(A) = \mathrm{Assh}(A)$. *Then the following are equivalent:*

(a) *For every minimal prime ideal* $\mathfrak{p}$, $A_{\mathfrak{p}}$ *is a Gorenstein ring.*

(b) *K is a fractional ideal of A.*

(c) *K is a fractional ideal of A containing a non zero divisor.*

(3.2) (cf. [6, Korollar 7.29]). *Assume that* $d \geqq 1$ *and* $\mathrm{Min}(A) = \mathrm{Assh}(A)$. *Then the following are equivalent:*

(a) *K is a reflexive A-module.*

(b) *A is* (S_1) *and* $A_{\mathfrak{p}}$ *is a Gorenstein ring for every prime ideal* $\mathfrak{p}$ *of height one.*

Let $\mathfrak{g}_A$ be the image of the natural map from $K \otimes_A \mathrm{Hom}_A(K, A)$ to A. The ideal $\mathfrak{g}_A$ is uniquely determined (cf. [6, p. 83]). By Theorem 0.2, we have $\mathfrak{g}_A A_{\mathfrak{p}} = \mathfrak{g}_{A_{\mathfrak{p}}}$ for every $\mathfrak{p}$ in $\mathrm{Supp}_A(K)$.

Proposition 3.3. *A is a quasi-Gorenstein ring if and only if* $\mathfrak{g}_A = A$. (cf. [6, Korollar 6.20]).

Proof. It is sufficient to show the "if" part. Since $\mathfrak{g}_A = A$, there is a surjection from K to A. Hence A is a direct summand of K. Since K is (S_2) (cf. [2, (1.10)]), so is A and $H \cong A$ by Proposition 1.2. Hence K is indecomposable by Proposition 1.10 and therefore $K \cong A$. q.e.d.

Corollary 3.4. *For a prime ideal* $\mathfrak{p}$ *in* $\mathrm{Supp}_A(K)$, $A_{\mathfrak{p}}$ *is a quasi-Gorenstein ring if and only if* $\mathfrak{p} \not\supseteq \mathfrak{g}_A$. *Consequently, if* $\mathrm{Min}(A) = \mathrm{Assh}(A)$, $\{\mathfrak{p} \in \mathrm{Spec}(A) \mid A_{\mathfrak{p}}$

is quasi-Gorenstein} is open in Spec(A).

Corollary 3.5. *A is a Gorenstein ring if and only if K is a Cohen-Macaulay module and $\mathfrak{g}_A=A$.*

Corollary 3.6. *Assume* $\mathrm{Ass}(A)=\mathrm{Assh}(A)$. *Then K is a fractional ideal of A if and only if* height $\mathfrak{g}_A\geqq 1$.

Corollary 3.7. *Assume that $d\geqq 1$ and* $\mathrm{Min}(A)=\mathrm{Assh}(A)$. *Then K is a reflexive A-module if and only if A is (S_1) and* height $\mathfrak{g}_A\geqq 2$.

In the remainder of this section, we assume $U_A(0)=0$.

Since K is an H-module by the usual way, $\mathfrak{g}_A$ is also an ideal of H. The ideal $\mathfrak{c}$ is just the conductor $A:_A H$, the largest common ideal. Hence we have the following inclusion

$$\mathfrak{g}_A\subseteqq\mathfrak{c}. \tag{3.8}$$

Of course the equality $\mathfrak{g}_A=\mathfrak{c}$ does not hold in general, for example, $\mathfrak{g}_A\neq\mathfrak{c}$ if A is a non-Gorenstein Cohen-Macaulay ring.

Proposition 3.9. *If H is a quasi-Gorenstein ring, then $\mathfrak{g}_A=\mathfrak{c}$.*

Proof. Since $\mathrm{Hom}_A(K, A)\cong\mathrm{Hom}_A(H, A)\cong\mathfrak{c}$, we have $\mathfrak{g}_A=\mathrm{Im}(K\otimes_A\mathrm{Hom}_A(K, A)\to A)=\mathrm{Im}(H\otimes_A\mathfrak{c}\to A)=\mathfrak{c}$. q.e.d.

The converse to Proposition 3.9 does not hold.

Example 3.10. Let k be a field and let x, y be indeterminates. We put $B=k[\![x^6, x^9, x^2y, x^5y, xy^2, y^3]\!]$, $\mathfrak{n}=$the maximal ideal of B, $R=k[\![x^3, x^2y, xy^2, y^3]\!]$ and $L=(x^2y, xy^2)R$. Then it is known that R is a non-Gorenstein Cohen-Macaulay ring of dimension 2 and L is the canonical module of R. It is obvious that R is finitely generated as a B-module and $B:_B R=\mathfrak{n}$, especially $\dim_B R/B=0$. Hence $L=(x^2y, x^5y, xy^2)B$ is the canonical module of B and $R\cong\mathrm{End}_B(L)$ by Corollary 1.7. It is easy to see $\mathfrak{g}_B=\mathfrak{n}$ because y/x and x^4/y are in $\mathrm{Hom}_B(L, B)$.

Remark 3.11. It is easy to see that H is a reflexive A-module (e.g., by induction on d using Theorem 0.2). Hence we have that, if height $\mathfrak{g}_A\geqq 2$ and $\mathrm{Hom}_A(K, A)\cong\mathfrak{c}$, then H is a quasi-Gorenstein ring.

Appendix.

In this appendix we give a generalization of [6, Satz 6.14] and [2, Proposition 4.1].

In the following let B denote a faithfully flat local A-algebra.

Theorem 4.1. *The following are equivalent:*

(a) *$B/\mathfrak{m}B$ is a Gorenstein ring.*

(b) *$K\otimes_A B$ is the canonical of B and $B/\mathfrak{m}B$ is a Cohen-Macaulay ring.*

Proof. Suppose that $B/\mathfrak{m}B$ is a Cohen-Macaulay ring and let $y_1, \cdots, y_r$ be a system of elements in the maximal ideal of B which forms a maximal $B/\mathfrak{m}B$-regular sequence ($r=\dim B/\mathfrak{m}B$). Let $R=A[X_1, \cdots, X_r]_{(\mathfrak{m}, X_1, \cdots, X_r)}$ with indeterminates $X_1, \cdots, X_r$ over A and let f be the natural A-algebra homomorphism from R to B such that $f(X_i)=y_i$ for $i=1, \cdots, r$. Then it is known that the map f is a flat local homomorphism by the local criterion of flatness. By [6, Korollar 5.12], $L=K\otimes_A R$ is the canonical module of R. Let $\mathfrak{n}$ be the maximal ideal of R. Since $L\otimes_R B=K\otimes_A B$ and $B/\mathfrak{n}B\cong B/(\mathfrak{m}, y_1, \cdots, y_r)B$ is an artinian ring, the assertion follows from [2, Proposition 4.1]. q. e. d.

Corollary 4.2. *The following are equivalent:*

(a) *A is a quasi-Gorenstein ring and $B/\mathfrak{m}B$ is a Gorenstein ring.*

(b) *B is a quasi-Gorenstein ring and $B/\mathfrak{m}B$ is a Cohen-Macaulay ring.*

Corollary 4.3. *Assume that $B/\mathfrak{m}B$ is a Gorenstein ring.*

(1) *If A is (S_2), then B is also (S_2).*

(2) *If M is a finitely generated (S_2) A-module of dimension d such that* $\mathrm{Min}_A(M)=\mathrm{Assh}_A(M)$, *then* $M\otimes_A B$ *is* (S_2) *and* $\dim B/\mathfrak{q}=\dim B$ *for every* $\mathfrak{q}$ *in* $\mathrm{Min}_B(M\otimes_A B)$.

Proof. The assertion (1) follows from Proposition 1.2 and Theorem 4.1, and (2) from [2, Proposition 4.4] and Theorem 4.1. q. e. d.

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References

[1] Y. Aoyama, On the depth and the projective dimension of the canonical module, Japan. J. Math., **6** (1980), 61-66.

[2] Y. Aoyama, Some basic results on canonical modules, J. Math. Kyoto Univ., **23** (1983), 85-94.

[3] S. Goto, On the Cohen-Macaulayfication of certain Buchsbaum rings, Nagoya Math. J., **80** (1980), 107-116.

[4] S. Goto, Approximately Cohen-Macaulay rings, J. Algebra, **76** (1982), 214-225.

[5] A. Grothendieck, Local cohomology, Lect. Notes Math. **41**, Springer Verlag, 1967,

[6] J. Herzog, E. Kunz et al., Der kanonische Modul eines Cohen-Macaulay-Rings, Lect. Notes Math. **238**, Springer Verlag, 1971.

[7] T. Ogoma, Existence of dualizing complexes, J. Math. Kyoto Univ., **24** (1984), 27-48.